A characterization of rich c-partite ($c \ge 8$) tournaments without (c + 2)-cycles

Jie Zhang

Zhilan Wang

Jin Yan*

School of Mathematics, Shandong University, Jinan, China

revisions 23rd June 2022, 26th Dec. 2022, 2nd Aug. 2023; accepted 24th Sep. 2023.

Let c be an integer. A c-partite tournament is an orientation of a complete c-partite graph. A c-partite tournament is rich if it is strong and each partite set has at least two vertices. In 1996, Guo and Volkmann characterized the structure of all rich c-partite tournaments without (c+1)-cycles, which solved a problem by Bondy. They also put forward a problem that what the structure of rich c-partite tournaments without (c+k)-cycles for some $k \geq 2$ is. In this paper, we answer the question of Guo and Volkmann for k=2.

Keywords: Multipartite tournaments; cycles; strong

1 Introduction

In this paper, we consider only finite digraphs without loops or multiple arcs. For a digraph D, we denote its vertex set by V(D) and its arc set by A(D). A digraph is strong if, for every pair x,y of distinct vertices in D, there exist a path from x to y and a path from y to x. The notation q-cycle (q-path) means a cycle (path) with q arcs. We will use (A,B)-arc to denote an arc from a vertex in A to a vertex in B. A c-partite tournament is an orientation of a complete c-partite graph and is rich if it is strong and each partite set has at least two vertices. We denote by D the family of all rich c-partite $(c \ge 5)$ tournaments. It is clear that tournaments are special c-partite tournaments on c vertices with exactly one vertex in each partite set.

An increasing interest is to generalize results in tournaments to larger classes of digraphs, such as multipartite tournaments. For results on tournaments and multipartite tournaments, we refer the readers to Bang-Jensen and Gutin (1998, 2001, 2018); Beineke and Reid (1978); Volkmann (2007). Many researchers have done a lot of work on the study of cycles whose length do not exceed the number of partite sets. In 1976, Bondy (1976) proved that every strong c-partite ($c \ge 3$) tournament contains a k-cycle for all $k \in \{3, 4, \ldots, c\}$. He also showed that every c-partite tournament in $\mathcal D$ contains a k-cycle for some k0, and asked the following question: does every multipartite tournament of k1 contains a k2 contains a k3 negative answer to this question was obtained by Gutin (1982). The same counterexample was found independently by Balakrishnan and Paulraja (1984). Further in Gutin (1984), the following result was proved.

^{*}The author's work is supported by NNSF of China (No.12071260).

Theorem 1.1 Gutin (1984) Every multipartite tournament in \mathcal{D} has a (c+1)-cycle or a (c+2)-cycle.

In Guo and Volkmann (1996), \mathcal{W}_m is defined as follows. Let $c\ (\geq 5)$ be an integer and $P=x_1\cdots x_m$ be a path with $m\geq c$. The c-partite tournament consisting of the vertex set $\{x_1,\ldots,x_m\}$ and the arc set $A(P)\cup\{x_ix_j:i-j>1\text{ and }i\not\equiv j(\text{mod }c)\text{ where }i,j\in[m]\}$ is denoted by W_m . The set of all c-partite tournaments obtained from W_m by replacing x_i by a vertex set A_i with $|A_i|\geq 2$ for $i\in\{1,2,m-1,m\}$ is denoted by \mathcal{W}_m .

In 1996, Guo and Volkmann (1996) gave a complete solution of this problem of Bondy and determined the structure of all c-partite ($c \ge 5$) tournaments of \mathcal{D} , that have no (c + 1)-cycle.

Theorem 1.2 Guo and Volkmann (1996) Let D be a c-partite tournament in \mathcal{D} . Then D has no (c+1)-cycle if and only if D is isomorphic to a member of \mathcal{W}_m .

In this paper, we characterize all c-partite ($c \geq 8$) tournaments in \mathcal{D} without (c + 2)-cycles. Before defining families \mathcal{Q}_m and \mathcal{H} , we present the main theorem.

Theorem 1.3 Let D be a c-partite ($c \ge 8$) tournament in D. Then D has no (c+2)-cycle if and only if D is isomorphic to a member of Q_m or H.

The families Q_m and \mathcal{H} are described as follows.

• Let i be a given integer with 2 < i < c-1. Define \mathcal{H}' the set of (c+1)-partite tournaments whose partite sets are V_1, \ldots, V_{c+1} , where $V_1 = \{v_1\}, \ |V_i| \ge 1$ and $|V_j| \ge 2$ for $j \in [c+1] \setminus \{1, i\}$, and the arc set consists of arcs from each vertex of V_{j_1} to each vertex of V_{j_2} , where $2 \le j_1 < j_2 \le c+1$, and arcs between v_1 and vertices in other partite sets with arbitrary directions. The family of all c-partite tournaments obtained from a member of \mathcal{H}' by deleting all arcs between v_1 and V_i and merging V_1 and V_i into a partite set is denoted by \mathcal{H} .

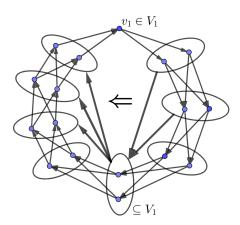


Fig. 1: An example of the 8-partite tournament with $|V_1| = 3$ and $|V_j| = 2$ for $2 \le j \le 8$. Here, the arcs between v_1 and other vertices are arbitrary.

• Let s and t be two fixed integers with $1 \le s < t-1 \le c$ and $P = x_1 \cdots x_m$ be a path with $m \ge c$. We denote Q_m' the (c+1)-partite tournament consisting of the vertex set $\{x_1, \ldots, x_m\}$ and the

arc set $A(P) \cup \{x_i x_j : i - j > 1 \text{ and } i \not\equiv j \pmod{c} \text{ where } i, j \in [m] \}$. Deleting arcs of Q'_m between $\{x_i|\ i\equiv s\ (\mathrm{mod}\ (c+1))\}\ \mathrm{and}\ \{x_j|\ j\equiv t\ (\mathrm{mod}\ (c+1))\}\ \mathrm{and}\ \mathrm{and}\ \mathrm{merging}\ V_i\ \mathrm{and}\ V_j\ \mathrm{into}\ \mathrm{a}\ \mathrm{partite}\ \mathrm{set},$ we obtain a c-partite tournament Q_m^1 . Let $\mathcal{Q}_m = \mathcal{Q}_m^1 \cup \mathcal{Q}_m^2$, where \mathcal{Q}_m^1 and \mathcal{Q}_m^2 are defined as follows.

- (\mathcal{Q}_m^1) The set of all c-partite tournaments obtained from \mathcal{Q}_m^1 by substituting x_i with a vertex set A_i is denoted by \mathcal{Q}_m^1 for
 - (1) $i \in \{1, 2, m 1, m\}$; or
 - (2) i = t when s = 1 and t = 3 or 4; or
 - (3) i = m 2 when $\{m, m 2\} \equiv \{s, t\} \pmod{(c+1)}$, or i = m 3 when $\{m, m 3\} \equiv \{s, t\}$ (mod (c+1)).
- (Q_m^2) Q_m^2 is the set of all c-partite tournaments obtained from a member of Q_m^1 by reversing some arcs satisfying

$$\begin{cases} & (A_2,A_3)\text{-arcs}, & \text{when } t=3,s=1; \\ & (A_1,A_2)\text{-arcs}, & \text{when } t=c+1,s=2; \\ & (A_{m-2},A_{m-1})\text{-arcs}, & \text{when } \{m-2,m\} \equiv \{s,t\} \ (\text{mod } (c+1)); \\ & (A_{m-1},A_m)\text{-arcs}, & \text{when } \{m-1,m-c\} \equiv \{s,t\} \ (\text{mod } (c+1)). \end{cases}$$

Note that, in our main theorem, the parameter c is at least 8. This condition may be not sharp. The characterization of rich c-partite tournaments with $5 \le c \le 7$ needs more techniques. We conclude this section by giving the following organization. In the second section, we set up notation and some helpful lemmas. The proof of the main theorem is presented in the third section.

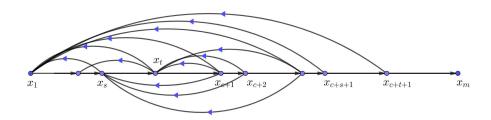


Fig. 2: An example of Q_m^1 . Here, all other possible arcs are of the same direction as the path.

2 Notation and useful lemmas

2.1 Notation

For terminology and notation not defined here, we refer to Bang-Jensen and Gutin (2001). Let D be a digraph. For the vertex $x \in V(D)$, the set of out-neighbours of x is denoted by $N_D^+(x) = \{y \in V(D), x \in V(D), x \in V(D)\}$ $V(D)|\ xy\in A(D)\}$ and the set of in-neighbours of x is denoted by $N_D^-(x)=\{y\in V(D)|\ yx\in A(D)\}$, respectively. For a vertex set $X\subseteq V(D)$, we define $N^+(X)=N_D^+(X)=\cup_{x\in X}N_D^+(x)\setminus X$ and $N^-(X)=N_D^-(X)=\cup_{x\in X}N_D^-(x)\setminus X$. When X is a subdigraph of D, we write $N^+(X)$ instead of $N^+(V(X))$. We define D[X] as the subdigraph of D induced by X, and let $D-X=D[V(D)\setminus X]$. Define $[t]=\{1,\ldots,t\}$ for simplicity.

Let C be a cycle (or path). For a vertex v of C, the successor and the predecessor of v on C are denoted by v^+ and v^- , respectively. We write the i-th successor and the i-th predecessor of v on C as v^{i+} and v^{i-} , respectively. The notation $v_i C v_j$ means the subpath of C from v_i to v_j along the orientation of C. The length of C is the number of arcs of C. We say a vertex x outside C can be inserted into C if there is an in-neighbour of x on C, say v, such that v^+ is an out-neighbour of x.

If xy is an arc in A(D), then we write $x \to y$ and say that x dominates y. If X and Y are two disjoint vertex sets of D, we use $X \to Y$ to denote that every vertex of X dominates every vertex of Y, and define $A \Rightarrow B$ that there is no arc from a vertex in B to a vertex in A. If X or Y consists of a single vertex, we omit the braces in all following notation. Correspondingly, $x \nrightarrow y$ expresses that $xy \notin A(D)$.

A path $P = x \cdots y$ is *minimal* if no proper subset of V(P) induces a subdigraph of D which contains a path from x to y. For two vertices x and y in D, the distance from x to y in D, denoted by dist(x,y), is the length of a shortest path from x to y in D. The diameter of D, denoted by diam(D), is the maximum distance between all pairs of its vertices.

2.2 Useful lemmas.

We give the following results that are frequently used in the proof of Theorem 1.3.

Theorem 2.1 Guo and Volkmann (1996) Let D be a strong c-partite tournament. If D has a k-cycle containing vertices from exactly l partite sets with l < c, then D has a t-cycle for all t satisfying $k \le t \le c + (k - l)$.

Remark 2.2 Let C be a k-cycle in a digraph D. If D contains no (k+1)-cycle, then no vertex can be inserted into C.

Lemma 2.3 Let D be a multipartite tournament in \mathcal{D} and C a (c+1)-cycle of D. Suppose that D has no (c+2)-cycle and $D-C\subseteq N^+(C)\cap N^-(C)$. Then for every $y\in D-C$, there exists a vertex $x\in C$ such that x and y belong to the same partite set of D and have the same in-neighbours and out-neighbours in C.

Proof: Let $C = x_1 x_2 \cdots x_i y_1 x_{i+1} \cdots x_c x_1$ a (c+1)-cycle of D, where $x_j \in V_j$ for $j \in [c]$ and $y_1 \in V_1$. Clearly, there exists a vertex $x \in C$ such that x and y belong to the same partite set of D. Assume that $x = x_j$, as the case $x = x_1$ and the case $x = y_1$ are similar.

Suppose that j=1. First suppose that $yx_1^- \in A(D)$. Then $y\Rightarrow x_{i+1}Cx_c$. Obviously, if $y\Rightarrow x_i$ then $y\Rightarrow x_2Cx_i$. Since $y\in N^+(C)\cap N^-(C)$, we have $x_i\Rightarrow y$. Thus $x_i\Rightarrow y\Rightarrow x_{i+1}$ and y and y_1 have the same in-neighbours and out-neighbours in C. Second suppose that $x_1^+y\in A(D)$. Similarly, we obtain that y and y_1 have the same in-neighbours and out-neighbours in C again. Hence $x_1^-y,yx_1^+\in A(D)$ and y and x_1 have the same in-neighbours and out-neighbours in C.

Set $j \in [c] \setminus \{1\}$. If $yx_j^- \in A(D)$, then $y \Rightarrow C$ because D has no (c+2)-cycle, which contradicts the assumption. Thus $x_j^- y \in A(D)$ for $j \in \{2, \ldots, c\}$. Similarly, we obtain that $y \to x_j^+$ for $j \in \{2, \ldots, c\}$. Thus y and x_j have the same in-neighbours and out-neighbours in C.

Lemma 2.4 Let D be a c-partite tournament in \mathcal{D} and \mathcal{C} the family of all (c+1)-cycles of D. Suppose that D has no (c+2)-cycle. If $D-C\subseteq N^+(C)\cap N^-(C)$ for every $C\in\mathcal{C}$, then $D\in\mathcal{H}$.

Proof: Since D has no (c+2)-cycle, it follows by Theorem 2.1 that each (c+1)-cycle of D meets all partite sets of D. This implies that each (c+1)-cycle contains exactly two vertices from one partite set and one vertex from other each partite set. Let $C \in \mathcal{C}$ and assume that it contains two vertices of V_1 , that is, $C = x_1x_2 \cdots x_iy_1x_{i+1} \cdots x_cx_1$, where $x_j \in V_j$ for $j \in [c]$ and $y_1 \in V_1$.

Since every partite set of D has at least two vertices, there exist c-1 vertices y_2,\ldots,y_c such that $y_j\in V_j$ for $j\in \{2,\ldots,c\}$. By Lemma 2.3, we have $x_j^-\to y_j\to x_j^+$ for $j\in [c]\setminus \{1\}$. Note that x_i and y_i have the same in-neighbours and out-neighbours in C. Since y_i is any vertex that is distinct with x_i in V_i , each vertex in V_i has the same in- or out-neighbors with x_i for $i=2,\ldots,c$. In the following, we often use this property to determine the direction of the arcs in A(D). We get $x_1\to V_2\to\cdots\to V_i\to y_1\to V_{i+1}\to\cdots\to V_c\to x_1$. Let C' be the (c+1)-cycle $x_1y_2\cdots y_iy_1y_{i+1}\cdots y_cx_1$.

Claim 2.1 The following statements hold.

(1)
$$\{V_2, \dots, V_{j-1}\} \to V_j \to \{V_{j+1}, \dots, V_i\}$$
 for $2 \le j \le i-1$;
(2) $\{V_{i+1}, \dots, V_{j-1}\} \to V_j \to \{V_{j+1}, \dots, V_c\}$ for $i+1 \le j \le c-1$.

By Claim 2.1, we can obtain a (c+2)-cycle from a cycle with larger length. Now consider the arcs between V_i and V_{i+1} . If $x_ix_{i+1}, x_{i+1}y_i \in A(D)$, then D has a (c+3)-cycle $x_ix_{i+1}y_iy_1y_{i+1}x_{i+2}Cx_i$. If $x_{i+1}x_i, x_iy_{i+1} \in A(D)$, then there is a (c+3)-cycle $x_{i+1}x_iy_{i+1}C'y_1x_{i+1}$. By $c \geq 8$ and Claim 2.1, we can obtain a (c+2)-cycle from such a (c+3)-cycle. Thus $V_i \to V_{i+1}$ or $V_{i+1} \to V_i$.

Suppose that $V_i \to V_{i+1}$. We show that $\{V_2, \dots, V_{i-1}\} \to \{V_i, \dots, V_c\}$. If $x_{i+2}x_i \in A(D)$, then $x_{i+2}x_ix_{i+1}y_{i+2}C'y_{i+1}x_{i+2}$ is a (c+4)-cycle. We can obtain a (c+2)-cycle because of $c \geq 8$ and Claim 2.1. Based on this, considering arcs between x_{i+3}, \dots, x_c and x_i in order, we get $x_i \to \{x_{i+3}, \dots, x_c\}$. Similarly, it is immediate that $\{x_2, \dots, x_{i-1}\} \to \{x_{i+1}, \dots, x_c\}$. Thus $\{V_2, \dots, V_i\} \to \{V_{i+1}, \dots, V_c\}$.

For $V_{i+1} \to V_i$, we will get $\{V_{i+1}, \dots, V_c\} \to \{V_2, \dots, V_i\}$ in the same way. Note that the structures obtained in two cases are isomorphic, so we only consider the first structure in the following.

We declare that

$$\{V_2, \dots, V_i\} \to y_1 \to \{V_{i+1}, \dots, V_c\}.$$
 (1)

If $y_1x_j \in A(D)$ for some $j \in [i-1]$, then D contains a (c+2)-cycle $x_1C'y_1x_jx_{i+1}Cx_1$, a contradiction. If $x_iy_1 \in A(D)$ for some $j \in \{i+2,\ldots,c\}$, then (c+2)-cycle $x_jy_1C'x_1Cx_ix_j$ is in D. Thus (1) holds.

If $|V_1|=2$, D is a member of \mathcal{H} , which proves this lemma. Thus assume that $|V_1|\geq 3$. We show that every vertex in $V_1\setminus\{x_1,y_1\}$ have the same in-neighbours and out-neighbours in C as y_1 . To see this, let z_1 be a vertex in $V_1\setminus\{x_1,y_1\}$. Suppose that $z_1\to x_i$. It is easy to see $z_1\to x_2,\ldots,x_{i-1}$. If $x_j\to z_1$ for some $j\in\{i+1,\ldots,c\}$, then $x_t\to z_1$ for all $t\in\{j+1,\ldots,c\}$. Recall that $V(D-C)\subseteq N^+(C)\cap N^-(C)$, we have $x_c\to z_1$. Observe that there is a 6-cycle $z_1x_2y_cx_1y_2x_cz_1$ which meets 3 partite sets of D, a contradiction by Theorem 2.1 again. Thus $x_i\to z_1$. Obviously, $\{x_2,\ldots,x_{i-1}\}\to z_1$. Otherwise there exists a (j+4)-cycle $x_iz_1x_jx_cx_1C'y_jx_i$ which meets j+2 partite sets of D for $j\in\{2,\ldots,i-1\}$, a contradiction. On the other hand, it is easy to see $z_1\to\{x_{i+1}\ldots,x_j\}$ if $z_1x_{j+1}\in A(D)$ for some $j\in\{i+1,\ldots,c\}$. Since $z_1\in N^+(C)\cap N^-(C)$, we have $z_1x_{i+1}\in A(D)$. Thus $x_1x_2\cdots x_iz_1x_{i+1}\cdots x_cx_1$ is also a (c+1)-cycle. This implies that z_1 and y_1 have the same in-neighbours and out-neighbours in C. Hence D is a member of \mathcal{H} . We are done.

3 Proof of Theorem 1.3

Now we are ready to prove our main theorem. It is easy to see that every element of \mathcal{H} and \mathcal{Q}_m has no (c+2)-cycle. Hence, it suffices to show the converse is true as well.

Suppose that D is a c-partite tournament in \mathcal{D} such that D has no (c+2)-cycle and is not isomorphic to any element of \mathcal{H} and \mathcal{Q}_m . Let V_1, \ldots, V_c be partite sets of D. By Theorem 1.1, we know that Dcontains a (c+1)-cycle. It follows by Theorem 2.1 that each (c+1)-cycle of D visits exactly one partite set twice and each other partite sets once. Let C be the set of all (c+1)-cycles of D. Lemma 2.4 gives that for every $C \in \mathcal{C}$, if all vertices of D-C are contained in $N^+(C) \cap N^-(C)$, then $D \in \mathcal{H}$. Thus there exists at least one cycle C in C such that D-C contains a vertex outside $N^+(C) \cap N^-(C)$. Denote $C = x_1 x_2 \cdots x_{c+1} x_1$, where $x_j \in V_j$ for $j \in [i-1]$, $x_j \in V_{j-1}$ for $j \in \{i+1, \dots, c+1\}$ and $x_i \in V_1$. Without loss of generality, assume that there exists a vertex $z \notin N^-(C)$. Because D is strong, there is a path from z to C. Let $P=z_1z_2\cdots z_p$ be such a minimal path with $z_1=z$ and assume that $z_p=x_t$. It is clear that, $p \geq 3$ and $z_2, \ldots, z_{p-2} \notin N^-(C)$, particularly, $z_{p-2} \nrightarrow x_{t-2}$ and $x_{t-1} \to z_{p-2}$. Since D has no (c+2)-cycle, we see that $x_{t-2}z_{p-2} \notin A(D)$. This implies that there is no arc between z_{p-2} and x_{t-2} , that is x_{t-2} and z_{p-2} must belong to the same partite set of D. It is not hard to get $z_{p-1} \nrightarrow x_{t-1}$. Since, otherwise, x_{t-3} and z_{p-2} must belong to the same partite set of D, which is impossible. Together with $x_{t-1} \nrightarrow z_{p-1}$ we obtain that x_{t-1} and z_{p-1} belong to the same partite set of D. Further, vertices x_{t-i} and z_{p-i} belong to the same partite set for $1 \le i \le p-1$. It is obvious that $x_{t-2} \to z_{p-1}$ due to $V(C) \setminus x_t \Rightarrow z_{p-1}$.

We may assume that x_t is on the path $x_{i+1}Cx_1$. If C has a path from x_t to x_{t-2} with at most c-1 vertices, then together with the path $x_{t-2}x_{t-1}z_{p-2}z_{p-1}x_t$, we can form the path into a cycle of length at most c+2, which contains the vertices x_{t-1} and z_{p-1} , and x_{t-2} , z_{p-2} in the same partite sets respectively. We deduce that D has a (c+2)-cycle from Theorem 2.1, a contradiction. This gives $x_{j+2}Cx_{t-2} \Rightarrow x_j \Rightarrow x_tCx_{j-2}$ for $x_j \in x_tCx_{t-2}$. For the same reason, $x_{t-1} \Rightarrow x_iCx_{t-3}$ when t > i+2; and $x_{t-1} \Rightarrow x_1Cx_{t-3}$ when t = i+1 or i+2.

Claim 3.1 diam(D) > c + 2.

Proof: Since $x_{t-2}z_{p-1} \in A(D)$, it is not hard to obtain that x_{t-1} dominates each vertex of $x_{t+1}Cx_{i-1}$ when $t \geq i+2$. Otherwise, there exists a vertex x_j in $x_{t+1}Cx_{i-1}$ with $x_{t-1} \to x_{j+1}Cx_i$ and $x_j \to x_{t-1}$. Observe that $x_jx_{t-1}x_{j+1}Cx_{t-2}z_{p-1}x_tCx_j$ is a (c+2)-cycle, a contradiction. Therefore, D[C]

is isomorphic to \mathcal{Q}_{c+1} with the initial vertex x_t and the terminal vertex x_{t-1} . This implies that every minimal path from z to C must end at x_t and $dist(z, x_{t-1}) \geq c+2$. Thus $diam(D) \geq c+2$ when $t \geq i+2$. If there is a vertex $x \notin V(C) \cup N^-(C)$ such that the minimal path from x to C which ends at the path $x_{i+2}Cx_1$, we complete the proof. Then all such minimal paths from x to C end at x_{i+1} , that is $x_t = x_{i+1}$. The following proof is divided into two cases.

Case 1 $i \geq 5$; or i = 4 and $x_i \rightarrow x_{c+1}$.

If $x_j \to x_i$ for $i+3 \le j \le c$ and $i \ge 4$, or $x_{c+1} \to x_i$ when $i \ge 5$, observe that $x_j x_i x_2 x_3 x_1 z_{p-2} z_{p-1} x_{i+1} C x_j$ is a cycle of length at most c+2 which visits V_1 three times, a contradiction. Thus in this case we have $x_i \to x_j$ for $i+3 \le j \le c+1$. Hence $dist(z,x_{t-1}) \ge c+2$, which implies that $diam(D) \ge c+2$.

Case 2 i = 4 and $x_{c+1} \to x_i$; or i = 3.

Recall that every partite set of D has at least two vertices. Hence there is at least one vertex y in $V_c \setminus \{x_{c+1}\}$. If $y \in N^+(C) \setminus N^-(C)$, then we choose y as the vertex z, that is y = z. It is easy to check that $dist(y, x_1) \geq c + 2$. If $y \in N^-(C) \setminus N^+(C)$, by considering the digraph D' obtained by reversing all arcs of D, we get $diam(D') \geq c + 2$, that is $diam(D) \geq c + 2$. If $y \in N^+(C) \cap N^-(C)$, then $x_c \to y \to x_1$ by Lemma 2.3. This implies that D contains a (c+2)-cycle $x_c y x_1 C x_{i-1} x_{c+1} z_{p-1} x_{i+1} C x_c$, a contradiction. Thus $diam(D) \geq c + 2$ when $i \in \{3, 4\}$. This completes the proof of the claim.

Let $P=x_1x_2\cdots x_m$ be a path of D with $dist(x_1,x_m)=diam(D)=m-1\geq c+2$. As D contains no (c+2)-cycle, vertices x_i and x_{c+i+1} must belong to the same partite set. If there exists vertex set $\{x_{i_1},x_{j_1},x_{i_2},x_{j_2}\}\subset V(D)$ with $\max\{i_1,j_1,i_2,j_2\}-\min\{i_1,j_1,i_2,j_2\}\leq c$ such that x_{i_1} and x_{j_1} belong to the same partite set and x_{i_2} and x_{j_2} belong to the same another partite set, then D contains a (c+2)-cycle by applying Theorem 2.1, a contradiction. Thus x_1Px_{c+1} meets all partite sets of D and contains two vertices of exactly one partite set. Therefore, D[P] is isomorphic to Q_m with the initial vertex x_1 and the terminal vertex x_m . If |V(D)|=m, we are done. So |V(D)|>m. Assume that $x_j\in V_j$ for $j\in [i-1], x_j\in V_{j-1}$ for $j\in \{i+1,\ldots,c\}$ and $x_i\in V_t$ for some $t\in [i-1]$. Let x be a vertex of D-P. Suppose that $x\in N^+(P)\cap N^-(P)$, we now consider the arcs between x and y and y belongs to.

Claim 3.2 Suppose that $x \in N^+(P) \cap N^-(P)$. If there exist two vertices x_p and x_q on P with p < q such that $x_p \to x \to x_q$, then x belongs to one of $\{V_1, V_2, V_{m-1}, V_m\}$. Moreover, x has the same in-neighbours and out-neighbours on P as $x_l \in \{x_2, x_3, x_4, x_{m-3}, x_{m-2}, x_{m-1}\}$, where

$$\begin{cases} x_3 \in V_1, & \textit{when } l = 3; \\ x_4 \in V_1, & \textit{when } l = 4; \\ x_{m-2} \in V_m, & \textit{when } l = m-2; \\ x_{m-3} \in V_m, & \textit{when } l = m-3. \end{cases}$$

Proof: Since D has no (c+2)-cycle, there is an integer l such that $x_{l-1} \to x \to x_{l+1}$ and x is in the same partite set with x_l . If $3 \le l \le m-2$, it easy to check that $D[P \cup \{x\}]$ contains a (c+2)-cycle C as follows:

- (1) when $m \ge c + l 1$,
 - $C = x_{l-1}xx_{l+1}Px_{c+l-2}x_{l}x_{l-2}x_{l-1}$, or

- $C=x_{l-1}xx_{l+1}Px_{c+l-3}x_{l}x_{l-3}x_{l-2}x_{l-1}$ unless $x_3\in V_1$ and l=3, or
- $C = x_{l-1}xx_{l+1}Px_{c+l-4}x_lx_{l-4}Px_{l-1}$ unless l = 3, or $x_4 \in V_1$ and l = 4;

or (2) when $l \geq c - 1$,

- $C = x_{l-1}xx_{l+1}x_{l+2}x_{l}x_{l+2-c}Px_{l-1}$, or
- $C = x_{l-1}xx_{l+1}x_{l+2}x_{l+3}x_{l}x_{l+3-c}Px_{l-1}$ unless $x_{m-2} \in V_m$ and l = m-2, or
- $C = x_{l-1}xx_{l+1}Px_{l+4}x_{l}x_{l+4-c}Px_{l-1}$ unless l = m-2, or $x_{m-3} \in V_m$ and l = m-3.

Then we consider m < c + l - 1 and l < c - 1. Recall that $m \ge c + 3$. This implies that $l \ge 4$. Clearly, if $x_l x_1, x_{c+1} x_l \in A(D)$, there is a (c+2)-cycle $x_1 P x_{l-1} x x_{l+1} P x_{c+1} x_{l}$. If $x_l, x_{c+1} \in V_l$, then $D[P \cup \{x\}]$ contains a (c+2)-cycle $x_2 P x_{l-1} x x_{l+1} P x_{c+2} x_{l} x_2$. Thus $x_l \in V_1$. Observe that $D[P \cup \{x\}]$ contains $x_3 P x_{l-1} x x_{l+1} P x_{c+3} x_l x_3$ which is a (c+2)-cycle unless l=4. Thus x belongs to V_1, V_2, V_{m-1} or V_m . We also get $l \in \{2, 3, 4, m-3, m-2, m-1\}$ and $x_3 \in V_1$ when l=3; $x_4 \in V_1$ when l=4; $x_{m-2} \in V_m$ when l=m-2; and $x_{m-3} \in V_m$ when l=m-3.

In all cases, it is easy to check that x and x_l have the same in-neighbours and out-neighbours on P, otherwise $D[P \cup \{x\}]$ contains a cycle of length at most (c+2) and two pairs of vertices which belong to the same partite set. By Theorem 2.1, we get a contradiction.

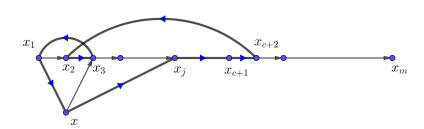


Fig. 3: A cycle of length at most (c+2) in $D[P \cup \{x\}]$ which contains two vertices in V_1 and two vertices in V_2 .

Claim 3.3 Suppose that $x \in N^+(P) \cap N^-(P)$. If all vertices x_p and x_q with $x_p \to x \to x_q$ satisfy p > q, then

- (i) x has the same in-neighbours and out-neighbours on P as x_1 or x_m ; or
- (ii) $D[P \cup \{x\}]$ has four specific structures as described in Fig. 4; or
- (iii) $x \to x_1$, $x_2 P x_m \Rightarrow x$ and $x \in V_{c+1}$; or
- (iv) $x_m \to x$, $x \Rightarrow x_1 P x_{m-1}$ and $x \in V_{m-c}$.

Proof: Note that if some vertex $x_q \to x$ (or $x \to x_q$) then $x_q P x_m \Rightarrow x$ (or $x \Rightarrow x_1 P x_q$). Let q be the maximum integer such that $x \to x_q$.

First we suppose that x has at least two in-neighbours and two out-neighbours on P. Let x_{q_2} be the previous out-neighbour of x before x_q on P and let x_{p_1}, x_{p_2} be two in-neighbours of x on P which is

nearest to x_q . Clearly, we have $p_2-q_2<5$. Otherwise, $xx_{q_2}Px_{p_2}x$ is a 7-cycle meeting five partite sets of D, a contradiction by Theorem 2.1. Hence there exists at most one vertex in $x_{q_2}Px_{p_2}$ such that it is non-adjacent to x. Let x_l be such vertex if it exists. According to the position of x_l , there are four possible sequences of $x_{q_2}Px_{p_2}$: (1) $x_{q_2}x_qx_{p_1}x_{p_2}$, (2) $x_{q_2}x_lx_qx_{p_1}x_{p_2}$, (3) $x_{q_2}x_qx_lx_{p_1}x_{p_2}$ and (4) $x_{q_2}x_qx_{p_1}x_lx_{p_2}$.

If $m \geq c+q$, then there is the (c+2)-cycle $xx_qPx_{c+q}x$ for sequences (1), (3) and (4) and the (c+2)-cycle $xx_{q2}Px_{q2+c}x$ for sequence (2) unless x and x_{q+c-2} are in the same partite set. Observe that for sequence (2) if $q \geq 5$, then there still exists a (c+2)-cycle $xx_{q-5}Px_{q-5+c}x$ via $x_{q+c-2}x \notin A(D)$. If $q \geq c$, then $xx_{q-c+1}Px_{p_1}x$ is a (c+2)-cycle for sequences (1)-(3) and $xx_{q-c+3}Px_{p_2}x$ is a (c+2)-cycle for sequence (4) unless x and x_{q-c+3} belong to the same partite set. We also note that for sequence (4) if $q \geq m-5$ and $q \geq c$, then there still exists a (c+2)-cycle $xx_{q-c+5}Px_{q+5}x$. Hence, m < c+q and q < c or P meets the partite sets of D along two special orders as described in Fig. 4. Moreover, x has exactly two in-neighbours or two out-neighbours and $x \in \{V_1, V_2, V_m, V_{m-1}\}$.

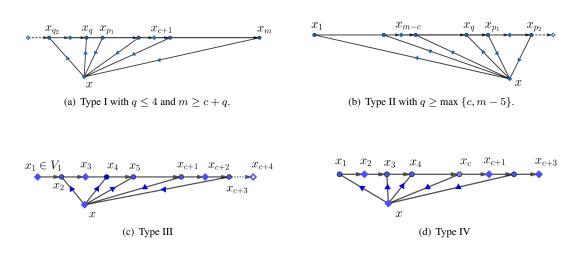


Fig. 4: The structure of $D[P \cup \{x\}]$ of Type I – Type IV in Claim 3.3.

Clearly, $x \in V_1$ or $x \in V_{c+1}$. Otherwise, there is a (c+2)-cycle $xx_1Px_{c+1}x$ in D. Suppose that $x \in V_1$. If $x_3 \notin V_1$, then $xx_3Px_{c+3}x$ is an (c+2)-cycle because x has at least two out-neighbours. Then $x_3 \in V_1$ and $x \to x_4$. Note that x_4 and x_{c+5} belong to the same partite set. If $x_5 \to x$, then q=4, $m \geq c+4$ and P is isomorphic to Type I in Fig. 4. If $x \to x_5$, we obtain a (c+2)-cycle $xx_5Px_{c+5}x$ when $m \geq c+5$. Hence m=c+3 or m=c+4 and $P=x_1x_2\cdots x_{c+3}(x_{c+4})$ where $x_3, x_{c+2}(x_{c+4}) \in V_1$. Next, suppose that $x \in V_{c+1}$. Obviously, there is a (c+2)-cycle $xx_2Px_{c+2}x$ when $x \notin V_2$. Similarly, when $x_4 \to x$ we have q=3 and m=c+2, which is impossible. Then $x \to x_4$. We obtain a (c+2)-cycle $xx_4Px_{c+4}x$ when $m \geq c+4$. Hence m=n+3 and $P=x_1x_2\cdots x_{c+3}$ where $x_{c+1} \in V_2$. In a word, when x has at least two in-neighbours and two out-neighbours, x has four specific structures as described in Fig. 4 based on the partite set which x belongs to.

Second, we suppose that x has either one in-neighbour or one out-neighbour. Then (i) $x \in V_1$ or $x \in V_m$ and x has the same in-neighbours and out-neighbours on P as x_1 or x_m ; or (ii) $x \to x_1$, $x_2Px_m \Rightarrow x_m$

and
$$x \in V_{c+1}$$
; or (iii) $x_m \to x$, $x \Rightarrow x_1 P x_{m-1}$ and $x \in V_{m-c}$.

By Claim 3.2 and Claim 3.3, we get the following.

Proposition 3.4 If $x \in N^+(P) \cap N^-(P)$, then x and P satisfy one of the following statements.

- (i) x and one of $\{x_1, x_2, x_{m-1}, x_m\}$ belong to the same partite set and their in-neighbours and outneighbours on P are same;
- (ii) x and $x_l \in \{x_3, x_4, x_{m-3}, x_{m-2}\}$ belong to the same partite set and their in-neighbours and outneighbours on P are same, where $x_3 \in V_1$ when l=3; $x_4 \in V_1$ when l=4; $x_{m-2} \in V_m$ when l=m-2; and $x_{m-3} \in V_m$ when l=m-2;
 - (iii) $D[P \cup x]$ has four specific structures Type(I–IV) which are shown in Fig. 4;
 - (iv) $x \to x_1$, $x_2Px_m \Rightarrow x$ and $x \in V_{c+1}$ or $x_m \to x$, $x \Rightarrow x_1Px_{m-1}$ and $x \in V_{m-c}$.

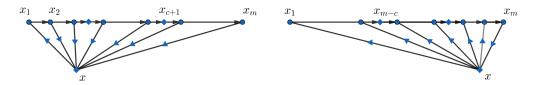


Fig. 5: The structure of $D[P \cup \{x\}]$ of Proposition 3.4(iv).

Next, suppose that x has only in-neighbours in V(P), i.e., $P\Rightarrow x$. Since D is strong, there is a path from x to P. Let $P'=x\cdots x'x''$ be a shortest path such that $x''\in N^-(P)$. If $N^-(x'')\cap V(P)=\emptyset$, then there is an integer $j\leq 4$ such that $x_{j+c-1}\to x'$ and further D contains a (c+2)-cycle $x'x''x_jPx_{j+c-1}x'$, a contradiction. Then $x''\in N^-(P)\cap N^+(P)$. By Proposition 3.4, there are several all possible structures of $D[x''\cup P]$. In each case, we obtain a (c+2)-cycle or $diam(D)\geq m$, which contradicts the initial assumption or Claim 3.1.

- Case 1: x'' satisfies Proposition 3.4 (i) and $x_l \in \{x_2, x_{m-1}, x_m\}$. It is easy to check that D contains a (c+2)-cycle.
- Case 2: x'' satisfies Proposition 3.4 (ii). There exists a (c+2)-cycle $x_2x_3x'x''$ $x_4Px_cx_2$ when l=3 (or $x_2x_3x_4x'x''x_5Px_cx_2$ when l=4, resp.). For the case l=m-2 and l=m-3, we can obtain a (c+2)-cycle similarly.
- Case 3: x'' satisfies Proposition 3.4 (iii). $D[P \cup \{x', x''\}]$ contains a (c+2)-cycle $x_1x_2x'x''x_3Px_cx_1$ (or $x_1x_2x_3x'x''x_4Px_cx_1$) for Type I, III. For Type II, there is a (c+2)-cycle $x_{m-1}x'x''x_{m-c}Px_{m-1}$ unless there is no arc between x'' and x_{m-c} . Moreover D contains $x_mx'x''x_{m-c+1}Px_m$ unless there is no arc between x' and x_m . Then $x_{m-2}x'x''x_{m-c-1}Px_{m-2}$ is a cycle when $x'' \nrightarrow x_{m-c}$ and $x_m \nrightarrow x'$.
- Case 4: x'' satisfies Proposition 3.4 (iv) and $x_m \to x''$. There is a (c+2)-cycle $x_{m-c}x'x''x_{m-1}x_{m-c}$ unless x_{m-c} , x'' and x_{m-1} belong to the same partite set. Then $D[P \cup \{x', x''\}]$ contains a (c+2)-cycle $x_{m-c}x'x''x_{m-c+2}Px_mx_{m-c}$.

Case 5: x'' satisfies Proposition 3.4 (i) $x_l \in \{x_2, x_{m-1}, x_m\}$ or (iv) $x'' \to x_1$. According to the analysis of Cases 1 – 4, we get $dist(x', x_m) \ge m$, a contradiction.

Hence, it is impossible that x has only in-neighbours on P. Analogously, we can show that D-P does not have any vertex which only has out-neighbours on P.

Since each partite set of D has at least two vertices, P is not of Type III or and Type IV $m \geq 2c+1$. In the following, we show that no vertex out of P satisfies (iv). Assume that there is a vertex x satisfying (iv) and $x \to x_1, x_2 P x_m \Rightarrow x$. If there is a vertex y out of P such that $x \to y$, it is easy to obtain that y and x_1 have the same in-neighbours and out-neighbours on P; or y satisfies (iv) and $x_m \to y, y \Rightarrow x_1 P x_{m-1}$; or $D[P \cup y]$ is of Type II. Thus $x_c xy x_{c+1} x_2 P x_c$ is a (c+2)-cycle unless $x_{c+1} \to x_2$. However, we obtain that D contains $x_3 xy x_4 P x_c x_1 x_2 x_3$ or $x_3 xy x_5 P x_{c+1} x_1 x_2 x_3$. Thus y and x_1 have the same adjacency to P. This implies that $dist(x, x_m) = m$, a contradiction. Analogously, if there is a vertex x satisfying (iv) and $x_m \to x, x \Rightarrow x_1 P x_{m-1}$, we will get $dist(x_1, x) = m$, a contradiction. Hence no vertex out of P satisfies (iv). Finally, if there exist vertices x of Type I and y of Type II such that $x \to y$, then D contains a (c+2)-cycle $x_1 P x_4 xy x_5 P x_c x_1$ or $x_1 P x_4 x_6 P x_{c+1} x_1$, a contradiction. Thus for any vertex x of Type I and any vertex y of Type II, there is no arc between x and y or $y \to x$. Observe that D is isomorphic to a member of Q_m . This proves Theorem 1.3. \square

4 Data Availability Statement

No data were generated or used during the study.

References

- R. Balakrishnan and P. Paulraja. Note on the existence of directed (k+1)-cycles in diconnected complete k-partite digraphs. *J. Graph Theory*, 8(3):423–426, 1984.
- J. Bang-Jensen and G. Gutin. Generalizations of tournaments: a survey. *J. Graph Theory*, 28(4):171–202, 1998.
- J. Bang-Jensen and G. Gutin. *Digraphs: Theory, algorithms and applications*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2001.
- J. Bang-Jensen and G. Gutin. Classes of directed graphs. Springer Monographs in Mathematics. Springer, Cham, 2018.
- L. W. Beineke and K. B. Reid. *Tournaments, Selected topics in graph theory*. Academic Press, Inc., London-New York, 1978.
- J. A. Bondy. Diconnected orientations and a conjecture of Las Vergnas. *J. London Math. Soc.* (2), 14(2): 277–282, 1976.
- Y. Guo and L. Volkmann. A complete solution of a problem of Bondy concerning multipartite tournaments. *J. Combin. Theory Ser. B*, 66(1):140–145, 1996.
- G. Gutin. On cycles in complete *n*-partite digraphs. *Depon. in VINITI, No. 2473, Gomel Politechnic Institute*, 1982.

- G. Gutin. Cycles in strong n-partite tournaments. $Vests\bar{\imath}$ Akad. Navuk BSSR Ser. $F\bar{\imath}z.-Mat.$ Navuk, (5): 105–106, 1984.
- L. Volkmann. Multipartite tournaments: a survey. Discrete Math., 307(24):3097–3129, 2007.