

# On Mixed Cages

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revisions 10<sup>th</sup> Mar. 2023, 29<sup>th</sup> June 2023; accepted 13<sup>th</sup> Aug. 2023.

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Mixed graphs have both directed and undirected edges. We consider mixed graphs that are regular in both directed and undirected degree and with given girth. Araujo-Pardo, Hernández-Cruz, and Montellano-Ballesteros recently established a lower bound for regular mixed graphs with directed degree one that generalizes the Moore bound. In this paper we show that, in at least one non-trivial case, their bound can be achieved. For a few other specific values of direct degree, undirected degree and girth we are able to identify graphs (mixed cages) of minimum possible order. We also obtain some fairly tight bounds for a few other cases.

**Keywords:** cage,mixed graph,girth,voltage graph

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## 1 Introduction

The cage problem, which asks for the identification of smallest regular graphs of given girth, has been extensively studied for graphs and for digraphs. A series of survey papers have tracked progress on the problem for graphs Wong (1982); Biggs (1998); Exoo and Jajcay (2013). The digraph version of the problem was first considered in 1970 by Behzad, Chartrand and Wall Behzad et al. (1970). Recently, Araujo-Pardo, Hernández-Cruz, and Montellano-Ballesteros Araujo-Pardo et al. (2019) considered the problem for mixed graphs and raised a number of interesting questions.

Finding graphs that give reasonable upper bounds and proving lower bounds both seem to be difficult problems for the undirected case. The situation appears to be different for digraphs. It is generally believed that the extremal digraphs were correctly identified by Behzad, Chartrand and Wall Behzad et al. (1970), who proposed a fundamental conjecture discussed below. For mixed graphs, as suggested in Araujo-Pardo et al. (2019), the level of difficulty in finding good candidate graphs may depend on the relative sizes of the directed and undirected degrees.

In this paper, we address and resolve some of the questions raised in Araujo-Pardo et al. (2019). In particular, it is shown below that the Moore-like bound they established for directed degree 1 can be achieved. In the next section, we review basic notation and terminology. Then we review the basic bounds of Moore, of Behzad, Chartrand and Wall, and of Araujo-Pardo, Hernández-Cruz, and Montellano-Ballesteros. Finally, we present a number of new results.

## 2 Notation and Terminology

A mixed graph is a graph with both directed and undirected edges. We refer to directed edges as *arcs* and undirected edges as *edges*. The sets of vertices, edges, and arcs for a mixed graph  $G$  are denoted  $V(G)$ ,  $E(G)$ , and  $A(G)$ , respectively. The *degree* of a vertex  $v$  in a mixed graph  $G$  is the number of edges incident with the vertex and is denoted  $deg(v, G)$  or simply  $deg(v)$  if  $G$  is clear from the context. Similarly, the

*in-degree* and *out-degree* of  $v$ , denoted  $ideg(v, G)$  and  $odeg(v, G)$ , are the numbers of arcs incident to and from  $v$ . A mixed graph  $G$  is regular if the degree and out-degree are constant as  $v$  ranges over the vertices of  $G$ . If, in addition, the in-degree is constant then the graph is *totally regular*.

A *cycle* in a mixed graph is a sequence of vertices  $v_0, v_1, \dots, v_k$  such that  $v_0 = v_k$ , and each pair of consecutive vertices  $(v_i, v_{i+1})$  is either joined by an edge or an arc (directed from  $v_i$  to  $v_{i+1}$ ), and there are no repeated edges or arcs. The *girth* of a mixed graph is the length of a shortest cycle. Note that the definition considers the possibility of 1-cycles (loops) and 2-cycles. As usual, we use  $C_k$  to denote an undirected  $k$ -cycle, and  $\vec{C}_k$  to denote a directed  $k$ -cycle.

An  $(r, z, g)$ -graph<sup>(i)</sup> is a regular mixed graph with degree  $r$ , out-degree  $z$ , and girth  $g$ . We are interested in cases where  $r, z > 0$  and  $g \geq 5$ . An  $(r, z, g)$ -cage is an  $(r, z, g)$ -graph of minimum possible order. We denote this minimum order by  $f(r, z, g)$ .

For graphs or digraphs  $G$  and  $H$ , the *graph composition*  $G[H]$  is the graph with vertex set  $V(G) \times V(H)$  where  $(g_1, h_1)$  is adjacent to  $(g_2, h_2)$  if  $g_1$  is adjacent to  $g_2$  or if  $g_1 = g_2$  and  $h_1$  is adjacent to  $h_2$  Harary (1969). This is also known as the *lexicographic product*.

Finally, we note that modular arithmetic on subscripts will frequently be employed when specifying constructions. In obvious cases, we will try to avoid cluttering the presentation with frequent reminders. In one case we use expressions of the form  $a + b \bmod c$  and remind the reader that the modulus operator has the precedence of multiplication.

### 3 Basic Bounds

The following conjecture is fundamental to the cage problem for directed graphs.

**Conjecture 1** (Behzad-Chartrand-Wall, Behzad et al. (1970)). *The order of a smallest  $r$  regular digraph of girth  $g$  is  $n = r(g - 1) + 1$ .*

Behzad, Chartrand and Wall also identified the digraphs, for each  $r$  and  $g$ , that they believed to be extremal. These digraphs, which we denoted  $BCW(r, g)$ , have orders  $r(g - 1) + 1$  with vertices  $v_i$  for  $0 \leq i \leq r(g - 1)$  and arcs  $v_i v_{i+j}$  for  $1 \leq j \leq r$ . A small example is shown in Figure 1.

In Araujo-Pardo et al. (2019) Araujo-Pardo, Hernández-Cruz, and Montellano-Ballesteros considered the problem of finding mixed cages. They focused on the case  $z = 1$  and found a lower bound for  $f(r, 1, g)$  based on the well known Moore bound for undirected graphs. Their idea is to attach undirected Moore trees to each vertex of a directed path of order  $g$ , choosing trees whose depth is as large as possible while still guaranteeing that all tree vertices are distinct.

Recall that the Moore bound for an  $r$ -regular graph of diameter  $d$  is given by:

$$n(r, d) \geq \frac{r(r-1)^d - 2}{r-2} \quad (1)$$

Let  $v_0, v_1, \dots, v_{g-1}$  be the vertices of a path of order  $g$ . Attach a Moore tree of depth  $i$  to  $v_i$  and  $v_{g-1-i}$  (if distinct), for  $0 \leq i \leq \lfloor g/2 \rfloor$ . This gives the following bound.

**Theorem 1** (The AHM Bound).

$$f(r, 1, g) \geq \sum_{i=0}^{g-1} n(r, \min(i, g-i-1))$$

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<sup>(i)</sup> This notation is compatible with most of the degree/diameter literature. In Araujo-Pardo et al. (2019) a notation is used that reverses the order of  $r$  and  $z$ .

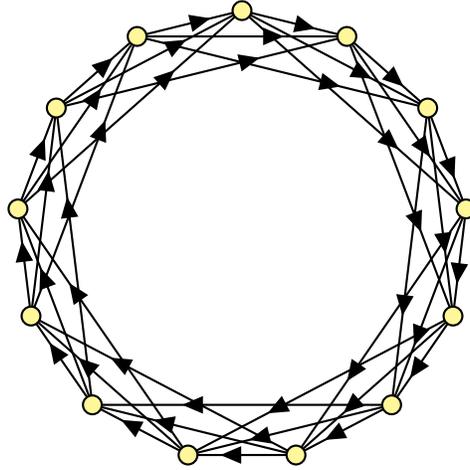


Fig. 1: The Behzad-Chartrand-Wall Graph for degree 3 and girth 5 has order 13.

A small example AHM tree is shown in Figure 2. We will see that this bound can, at least occasionally, be achieved. A central question for mixed cages is to determine how often it is achieved.

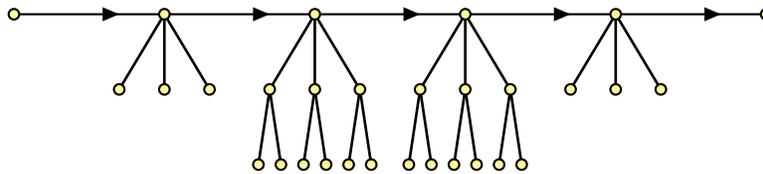


Fig. 2: The AHM Tree for  $r = 3, z = 1, g = 6$ .

Upper bounds on  $f(r, 1, g)$  can be obtained from known cages (or candidates) of degree  $r + 2$  and girth  $g$  whenever the graph is Hamiltonian (or at least has a 2-factor with no short cycles). We shall refer to the upper bound obtained this way as the *Cage Bound*. In general, the Cage Bound is probably weak, but in at least one case it may be useful. The Hoffman-Singleton graph is the  $(7, 5)$ -cage, and is Hamiltonian. If any Hamiltonian cycle is cyclically oriented, a  $(5, 1, 5)$  mixed graph of order 50 is obtained.

It appears that most cages are Hamiltonian. A theorem of Singer (1938) on collineations implies that girth 6 cages constructed from field planes are Hamiltonian. More recently, Lazebnik, Mellinger and Vega Lazebnik et al. (2013) showed that the incidence graphs of all finite projective planes are Hamiltonian. This author has tested all the remaining cages and cage candidates listed in Table 6 of the Dynamic Cage Survey Exoo and Jajcay (2013) that are relevant to the results in this paper and found them all to be Hamiltonian. The cage bound may at least serve as a useful benchmark. One might view finding  $(r, 1, g)$ -graphs whose orders are less than the orders of the  $(r+2, g)$ -cages (or candidates) as significant first steps.

## 4 The Case $r + z \leq 3$

In this section, mixed cages for  $r, z \leq 2$  are determined. The first theorem uses what could be viewed as a degenerate case of the AHM bound, as it employs Moore trees of degree one, which are single edges.

**Theorem 2.**

$$f(1, 1, g) = 2g - 2$$

**Proof:** The lower bound follows from the AHM bound. The AHM bound is achieved by the Möbius ladder, where the outer cycle (see Figure 3) is cyclically oriented. ■

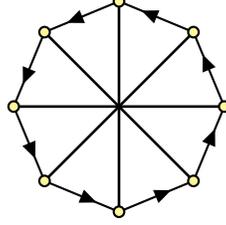


Fig. 3: The Möbius ladder of order 8: the  $(1, 1, 5)$ -cage. An AHM tree can be found by using any directed path of length four and the three edges incident with the non-endvertices of the path.

In Araujo-Pardo et al. (2019) the authors consider and resolve the case,  $r = 2, z = 1$ .

**Theorem 3** (AHM Araujo-Pardo et al. (2019)).

$$f(2, 1, g) = \begin{cases} \frac{g^2 + 1}{2} & , g \text{ odd} \\ \frac{g^2}{2} & , g \text{ even} \end{cases}$$

The extremal graphs in Araujo-Pardo et al. (2019) are derived from the Behzad, Chartrand and Wall digraphs by replacing one directed spanning cycle by an undirected cycle. The lower bound follows from the AHM bound, giving the first examples where the bound is achieved.

The value of  $f(1, 2, g)$  can be determined using the  $r = 2$  case of Behzad-Chartrand-Wall Conjecture, proved by Behzad, and by the graph  $\vec{C}_g[K_2]$ .

**Theorem 4.**

$$f(1, 2, g) = 2g$$

**Proof:** Let  $G$  be a  $(1, 2, g)$ -graph. Consider the directed subgraph. The result of Behzad Behzad (1973) on the case  $r = 2$  of the Behzad-Chartrand-Wall Conjecture implies that  $G$  must have at least  $2g - 1$  vertices. But since  $r$  is odd, the order of  $G$  must be even. So  $f(1, 2, g) \geq 2g$ . The extremal graph is  $\vec{C}_g[K_2]$ . The graph  $\vec{C}_7[K_2]$  is shown in Figure 4. ■

## 5 The Case $r = 2$

In this section we present three theorems that provide upper bounds for  $f(2, z, g)$ . When  $r = 2$ , a simple upper bound can be obtained using subgraphs of  $\vec{C}_g[C_g]$ . Behzad et al. (1970); Araujo-Pardo et al. (2019).

**Theorem 5.**

$$f(2, z, g) \leq g^2, \text{ for } z \leq g.$$

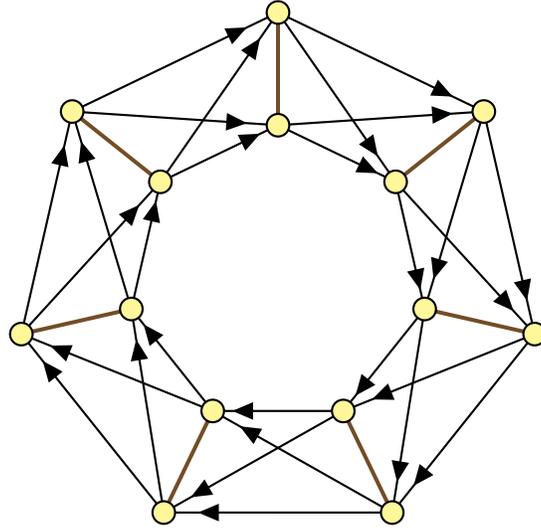


Fig. 4: The graph  $\vec{C}_7[K_2]$  of order 14 with  $(r, z, g) = (1, 2, 7)$ .

**Proof:** The required graphs are all subgraphs of  $\vec{C}_g[C_g]$ . Construct a  $(2, z, g)$ -graph  $G$  as follows. The vertices of  $G$  are  $\{v_{i,j} \mid 0 \leq i, j < g\}$ , with edges  $v_{i,j}v_{i,j+1}$ , and arcs  $v_{i,j}v_{i+1,j+k}$  for  $0 \leq k < z$  (subscript arithmetic is modulo  $g$ ). ■

For a restricted range of  $z$ , we can remove one of the  $g$ -cycles and still maintain the girth.

**Theorem 6.**

$$f(2, z, g) = \begin{cases} g^2 - g, & g \text{ even and } z \leq g/2 \\ g^2 - 1, & g \text{ odd and } z \leq (g+1)/2 \end{cases}$$

**Proof:** If  $g$  is even, let  $g = 2h$ , and then  $z \leq h$ . Construct a  $(2, z, g)$ -graph  $G$  of order  $g^2 - g$  as follows. Let  $V(G) = \{v_{i,j} \mid 0 \leq i < g - 1 \text{ and } 0 \leq j < g\}$ ,  $E(G) = \{v_{i,j}v_{i,j+1} \mid 0 \leq j < g - 1 \text{ and } 0 \leq j < g\}$ . The arc set is a bit more complex:

$$\begin{aligned} A(G) = & \{v_{i,j}v_{i+1,(j+k) \bmod h} \mid 0 \leq i < g-2, 0 \leq j < h, 0 \leq k < z\} \cup \\ & \{v_{i,h+j}v_{i+1,h+(j+k) \bmod h} \mid 0 \leq i < g-2, 0 \leq j < h, 0 \leq k < z\} \cup \\ & \{v_{g-2,j}v_{0,h+(j+k) \bmod h} \mid 0 \leq j < h, 0 \leq k < z\} \cup \\ & \{v_{g-2,j+h}v_{0,(j+k) \bmod h} \mid 0 \leq j < h, 0 \leq k < z\}. \end{aligned}$$

A small example of the construction in Theorem 6 is shown in Figure 5. ■

Finally, when  $r = z = 2$  we reduce the coefficient of  $g^2$  in the upper bound to  $3/4$ .

**Theorem 7.**

$$f(2, 2, g) \leq \left\lfloor \frac{g}{2} \right\rfloor \left\lfloor \frac{3g}{2} \right\rfloor = \begin{cases} \frac{3g^2}{4} & g \text{ even,} \\ \frac{3g^2+2g-1}{4} & g \text{ odd} \end{cases}$$

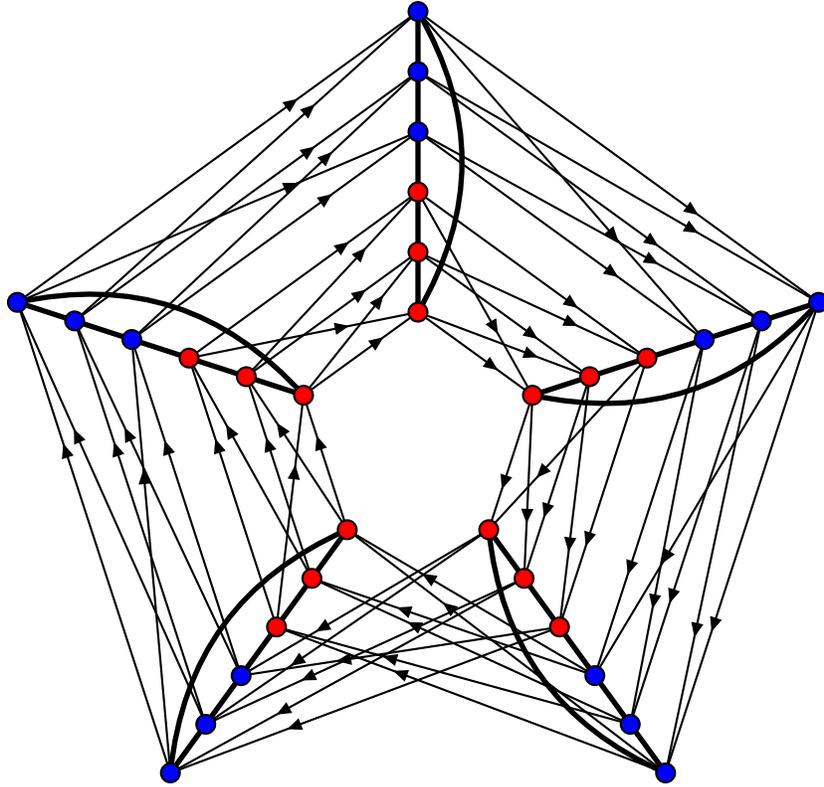


Fig. 5: A  $(2, 2, 6)$ -graph with 5 undirected 6-cycles.

**Proof:** Let  $s = \lceil \frac{g}{2} \rceil$ , and  $t = \lfloor \frac{3g}{2} \rfloor$ . We describe a graph  $G$  of order  $st$  with  $r = z = 2$ , and girth  $g$ . Let

$$V(G) = \{v_{i,j} \mid 0 \leq i < s, 0 \leq j < t\}.$$

The undirected subgraph of  $G$  consists of  $s$  disjoint  $t$ -cycles, where the first subscript of a vertex indicates which cycle the vertex is on, and the second subscript indicates its position on the cycle. We imagine the cycles are numbered according to the first coordinate their vertices, and refer to cycle 0, cycle 1, on up to cycle  $s - 1$ , with the vertices of each cycle numbered in the natural order.

Next we specify the arcs so that all arcs from vertices on cycle  $i$  go to vertices on cycle  $i + 1$ . So  $v_{i_1, j_1} v_{i_2, j_2} \in A(G)$  if  $i_1 + 1 = i_2 < s$  and either  $j_1 = j_2$  or  $j_1 + 1 = j_2$ .

Finally the arcs from cycle  $s-1$  back to cycle 0:  $v_{s-1, j_1} v_{0, j_2}$  is an arc if  $j_1 + \lfloor \frac{g}{2} \rfloor = j_2$  or  $j_1 + \lfloor \frac{g}{2} \rfloor + 1 = j_2$ .

Note that the function that maps  $v_{i,j}$  to  $v_{i,j+1}$  is a graph automorphism.

Any possible cycle of length less than  $g$  must contain arcs, since the undirected subgraph consists of disjoint  $t$ -cycles. But any cycle containing an arc contains at least  $t$  arcs and at least one arc joining each pair of consecutive  $t$ -cycles. Hence it also contains at least one vertex from each of the undirected cycles. So by symmetry, if there is a cycle of length less than  $g$  there is such a cycle containing vertex 0. But any such cycle must contain exactly  $r$  arcs and fewer than  $g - r = \lceil \frac{g}{2} \rceil$  edges. By construction, this is impossible. ■

Note that if this construction were modified for  $f(2, 3, g)$  we would need undirected cycles of length  $2g$  and would no longer have an improvement over the upper bound in Theorem 5.

For the case  $(2, 2, 5)$ , a graph of order 19 was found by Claudia De La Cruz and Miguel Pizaña (personal communication). Their graph can viewed as a mixed orientation of the cubic residue graph. The graph in Figure 6 is a smallest  $(2, 2, 6)$ -graph.

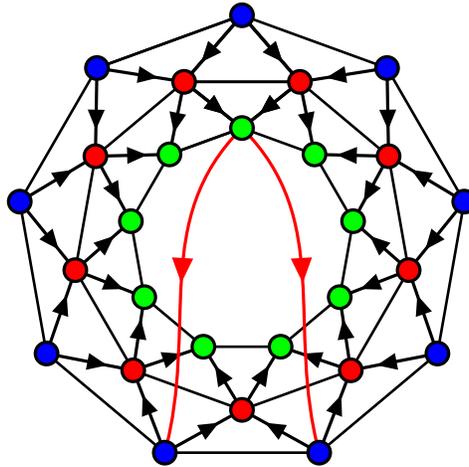


Fig. 6: A  $(2, 2, 6)$ -graph of order 27. Only one pair of arcs from the green cycle to the blue cycle are shown. The others are obtained by rotation.

## 6 The Case $z = 1$

As noted in Araujo-Pardo et al. (2019), the cases where  $z = 1$  may result in graphs most similar to undirected cages. In these cases the directed subgraph consists of a set of disjoint cycles whose lengths partition the order of the graph.

The case  $(3, 1, 5)$  was considered in Araujo-Pardo et al. (2019) where they established the bounds  $20 \leq f(3, 1, 5) \leq 28$ . We settle this case below.

**Theorem 8.**  $f(3, 1, 5) = 24$

**Proof:** The AHM bound in this case is 20, so one needs to show that  $(3, 1, 5)$ -graphs of orders 20 and 22 do not exist. This was done by exhaustive computer searches, which made heavy use of the *nauty and Traces* McKay and Piperno (2014) package. The searches were completed by two different programs.

The first program breaks the task into cases based on the structure of the (maximal) directed subgraph of a putative  $(3, 1, 5)$ -graph. This subgraph must be a union of cycles. The lengths of these cycles give a partition of 20 (respectively 22) with no part less than 5. There are 13 (resp. 18) such partitions. For each case, the program then does an exhaustive search for an undirected subgraph.

The other program reverses the procedure, and divides the search into cases according to the structure of the undirected subgraph. This subgraph must be a cubic graph whose girth is at least 5. There are 5784 such graphs of order 20 and 90940 of order 22 McKay and Piperno (2014). While this leaves more cases to consider than the first program, the exhaustive search for each case is much faster.

After completing the searches for  $n = 20$  and  $n = 22$ , and finding no graphs, the first of these methods was applied to the case  $n = 24$ , and 23 different graphs were generated. The most symmetric of these graphs is depicted in Figure 7. ■

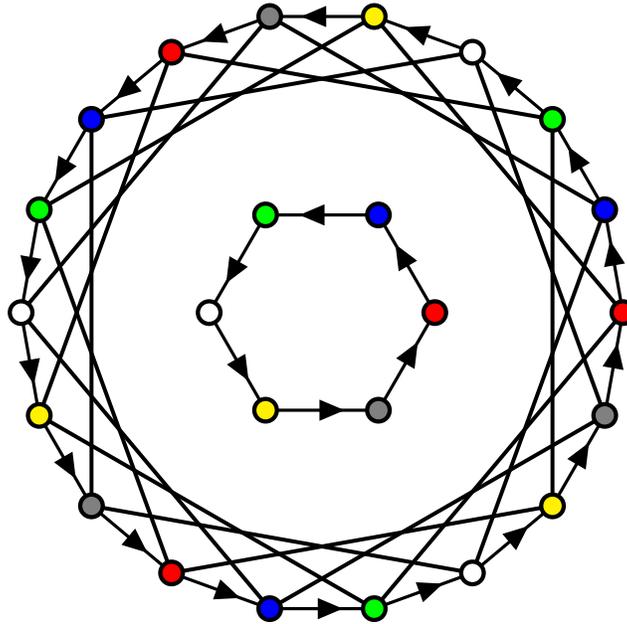


Fig. 7: A graph of order 24 with  $(r, z, g) = (3, 1, 5)$ . There are edges joining inner vertices to all outer vertices of the same color.

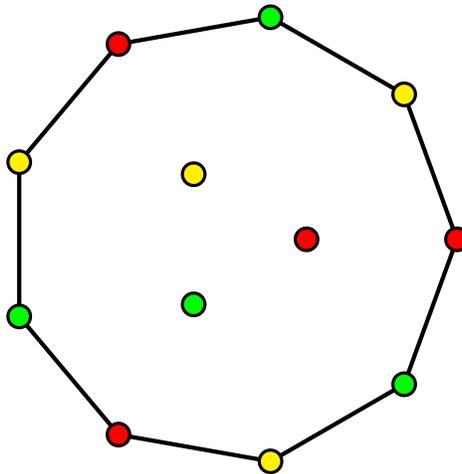


Fig. 8: One of the two cubic graphs of order 12 with girth 5. Each of the three inner vertices is adjacent to the three vertices on the 9-cycle having the matching color.

The undirected subgraph in Figure 7 is isomorphic to two copies of this graph.

The undirected subgraph of the graph in Figure 7 is isomorphic to two copies of the cubic graph of girth

5 and order 12 shown in Figure 8. An alternate view of the  $(3, 1, 5)$ -graph is given in Figure 9. In this graph we fix an ordering of the colors *red, blue, yellow, white, green, gray*, and refer to *red* as color 0, *blue* as color 1, etc. The directed edges of the graph are given as follows. For each of the three central vertices (in both subgraphs), we add an arc from the vertex in color  $i$  to the vertex (in the other subgraph) in color  $i + 1 \pmod{6}$ . This gives the directed 6-cycle from Figure 7.

Next pick any red (color 0) vertex on the outer 9-cycle of the left subgraph, call it  $v_0$ . Then pick any blue (color 1) vertex on the outer 9-cycle of the right subgraph, call it  $v_1$ . Add an arc from  $v_0$  to  $v_1$ . Next let  $v_2$  be the yellow (color 2) vertex in the left subgraph adjacent to  $v_0$ . Add an arc from  $v_1$  to  $v_2$ . Then  $v_3$  is the white (color 3) vertex in the right subgraph adjacent to  $v_1$ , and add an arc from  $v_2$  to  $v_3$ . Continue in this manner to complete the directed 18 cycle as in Figure 7.

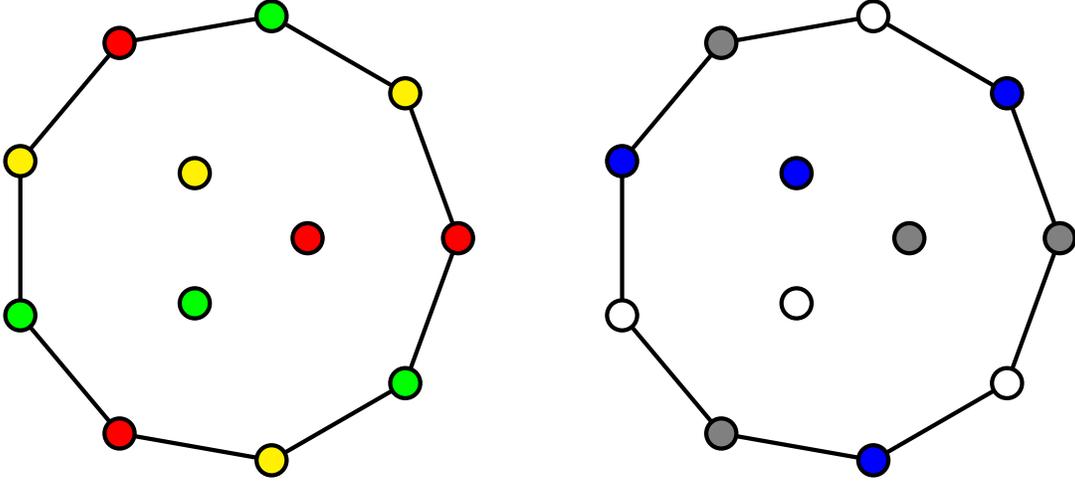


Fig. 9: An alternate view of the  $(3, 1, 5)$ -graph.

The next construction was presented in Exoo (2022) and shows that the AHM bound can be achieved in at least one case where  $r > 2$ .

**Theorem 9.**  $f(3, 1, 6) = 30$

**Proof:** In this case the AHM bound is 30. To construct a  $(3, 1, 6)$ -graph  $G$  of order 30, define the vertex set  $V(G) = X \cup Y \cup Z$ , where

$$X = \{x_i \mid 0 \leq i < 10\}$$

$$Y = \{y_i \mid 0 \leq i < 10\}$$

$$Z = \{z_i \mid 0 \leq i < 10\}$$

The set of arcs  $A(G)$  is

$$\{x_i x_{i+1} \mid 0 \leq i < 10\} \cup \{y_i y_{i+1} \mid 0 \leq i < 10\} \cup \{z_i z_{i+1} \mid 0 \leq i < 10\}$$

The set of edges  $E(G)$  is

$$\{x_i y_{i \pm 2} \mid 0 \leq i < 10\} \cup \{z_i z_{i+5} \mid 0 \leq i < 10\}.$$

In all cases, addition of subscripts is done modulo 10. ■

The graph described in the above proof is depicted in Figure 10.

Next we present a few constructions where we were able to come relatively close to the AHM bound.

**Theorem 10.**

- a)  $52 \leq f(3, 1, 7) \leq 60$
- b)  $74 \leq f(3, 1, 8) \leq 76$
- c)  $29 \leq f(4, 1, 5) \leq 34$
- d)  $46 \leq f(4, 1, 6) \leq 48$
- e)  $40 \leq f(5, 1, 5) \leq 50$
- f)  $66 \leq f(5, 1, 6) \leq 72$

All lower bounds are derived from the AHM bound. The upper bound constructions are handled separately.

(a)  $f(3, 1, 7) \leq 60$

The directed subgraph for the  $(3, 1, 7)$ -graph consists of 6 disjoint directed 10-cycles. Each of these cycles is represented by a node in Figure 11. It will be useful to imagine the 60 vertices of the graph labeled  $v_{i,j}$  for  $0 \leq i < 6$  and  $0 \leq j < 10$ . In this way, node  $i$  in the figure represents all vertices with first subscript  $i$ . Each these sets of 10 vertices induces a directed 10-cycle labeled in the natural order.

Various pairs of 10-cycles are connected by undirected matchings. The arcs in the figure do not represent arcs in the graph, but are there to specify the matching between pairs of 10-cycles. For example, the arc from node 1 to node 2 is labeled with a 4. This means there is a matching connecting 10-cycle number 1 to 10-cycle number 2 such that vertex  $i$  in the first 10-cycle is adjacent to vertex  $i + 4$  in the second 10-cycle, i.e.,  $v_{1,i}$  is adjacent to  $v_{2,i+4}$ , and again subscript addition is modulo 10.

The reader familiar with voltage graphs can think of the arc labels as voltage assignments from the cyclic group of order 10.

(b)  $f(3, 1, 8) \leq 76$

Construct a  $(3, 1, 8)$ -graph  $G$  on vertices  $v_{i,j}$  for  $0 \leq i < 2$  and  $0 \leq j < 38$ . For fixed  $i$ , the 38 vertices  $v_{i,j}$  form a directed cycle, labeled in the natural order. The edge set of  $G$  is comprised of edges of one of the following forms:

$$\begin{aligned} &v_{0,j}v_{0,j+7} \\ &v_{0,j}v_{0,j-7} \\ &v_{1,j}v_{1,j+11} \\ &v_{1,j}v_{1,j-11} \\ &v_{0,j}v_{1,j} \end{aligned}$$

The automorphism group of  $G$  has order 38, with each directed 38-cycle comprising an orbit.

(c)  $f(4, 1, 5) \leq 34$

An example  $(4, 1, 5)$ -graph of order 34 is shown in Figure 12. The directed subgraph consists of two 17-cycles, as shown in the right half of the figure. The undirected subgraph is shown in the left half of the figure. We identify the vertices in the left figure with the corresponding vertices in the right figure. Label the vertices  $v_{i,j}$  for  $0 \leq i < 2$  and  $0 \leq j < 17$ , where we imagine the vertices with first subscript 0 are those on the outer circle. Then there are arcs from  $v_{0,i}$  to  $v_{0,i+7}$  and from  $v_{1,i}$  to  $v_{1,i+6}$ , and edges  $v_{0,j}v_{1,j-2}$  and  $v_{0,j}v_{1,j+2}$ .

(d)  $f(4, 1, 6) \leq 48$

For this case, refer to Figure 13, which is similar to Figure 11 used for part (a). Here each node in the figure represents a directed 8-cycle, and various of these 8-cycles are joined by undirected matchings, as in

part (a). Once again, the reader familiar with voltage graphs can think of the labels as voltage assignments from the cyclic group of order 8.

**(e)**  $f(5, 1, 5) \leq 50$

A  $(5, 1, 5)$ -graph can be easily obtained from the Hoffman-Singleton graph by cyclically orienting a directed Hamiltonian cycle, of which there are many.

**(f)**  $f(5, 1, 6) \leq 72$

The graph is depicted in Figure 14. This time we replace vertices in the base graph by 12-cycles.

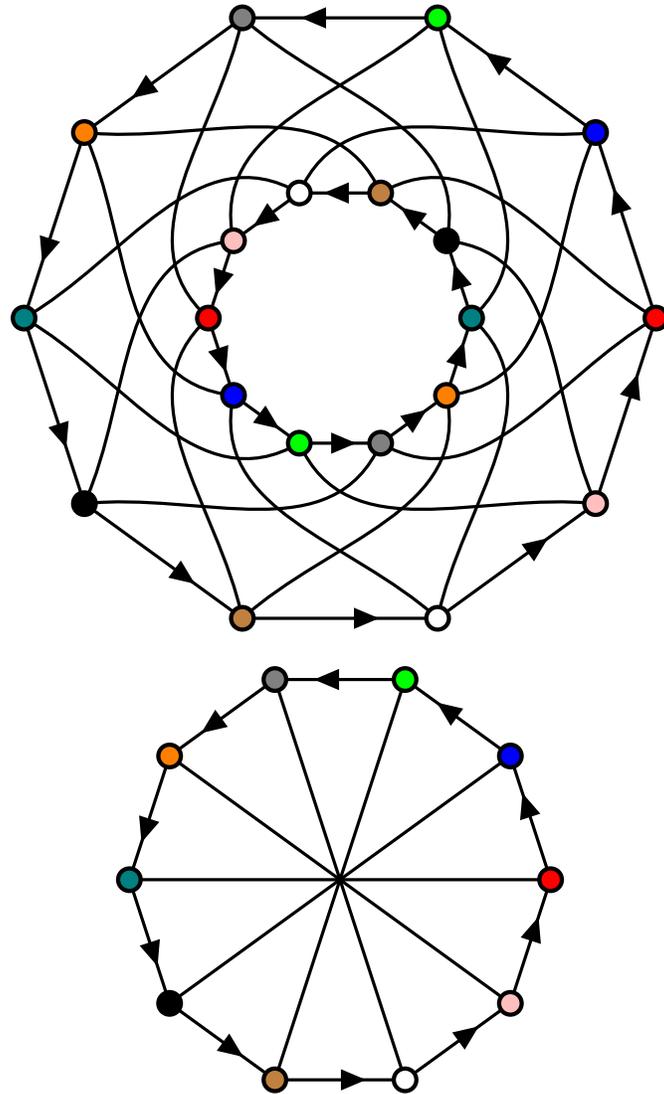


Fig. 10: The unique smallest  $(3, 1, 6)$ -graph of order 30. Vertices in the lower figure are adjacent to vertices in the upper figure that have the same color.

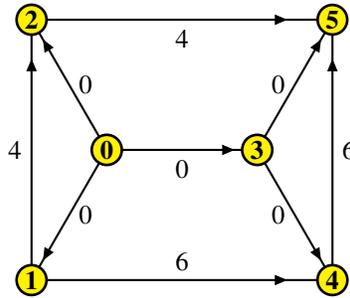


Fig. 11: Structure of an  $(3, 1, 7)$ -graph of order 60. Each node represents a directed 10-cycle.

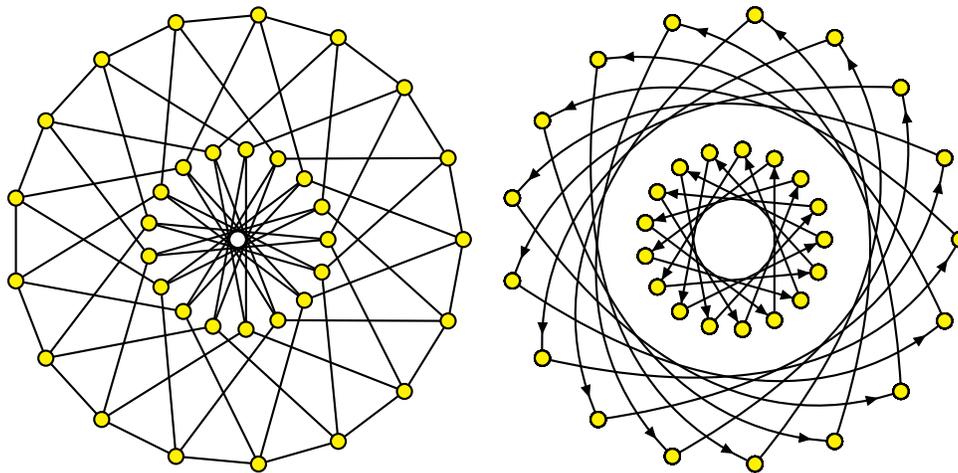


Fig. 12: A  $(4, 1, 5)$ -graph of order 34. The figure on the left shows the edges and the figure on the right shows the arcs.

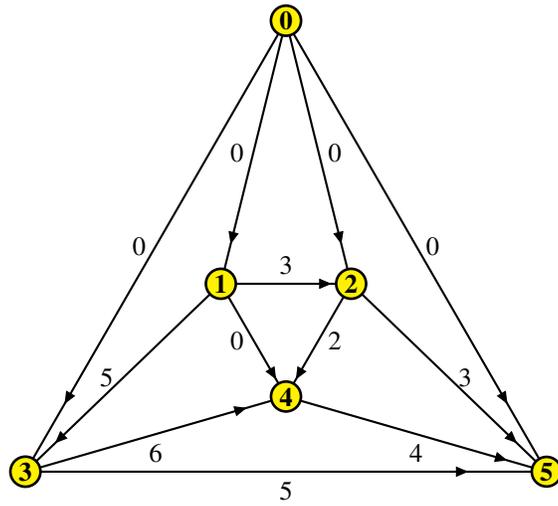


Fig. 13: Structure of an  $(4, 1, 6)$ -graph of order 48. Each node represents a directed 8-cycle.

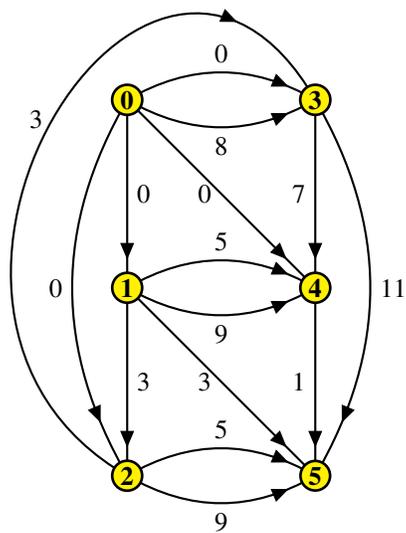


Fig. 14: Structure of a  $(5, 1, 6)$ -graph of order 72. Each node represents a directed 12-cycle.

## 7 Further Work

There are at least four areas for further work on this topic.

1. Table 1 lists cases where we were able to either determine an exact value for  $f(r, z, g)$  or come reasonably close. Note that for the case  $(r, z, g) = (3, 1, 8)$  there are only two remaining possibilities, either there is a graph achieving the AHM bound of 74, or else the graph presented here is a cage. Perhaps techniques from Linear Algebra could be used to eliminate 74 and settle the issue.
2. Recall that the directed subgraph of an  $(r, 1, g)$ -graph consists of a set of directed cycles whose lengths partition the order of the graph. For most of the graph constructions presented here, these partitions contained equal parts. At first, our computer searches were not restricted to partition with equal parts, but it turned out that best constructions we found for smaller cases used such partitions. So for cases where the search space was deemed too large for an exhaustive search, we restricted the search to partitions with equal parts. In each of the cases listed in the table, there is no smaller graph with an equal partition. Given enough computer time, one could probably complete exhaustive searches through all partitions and settle the unresolved cases listed here.
3. In the section on  $r = 2$ , three theorems were given providing upper bounds for  $f(2, z, g)$ . Now the undirected subgraph of a  $(2, z, g)$ -graph consists of disjoint cycles, i.e., (undirected) cages of degree 2. The reader will notice that these theorems could each be generalized for  $r > 2$ , replacing the cycles by  $r$ -cages.
4. As noted above, several of the constructions in this paper can be viewed as voltage graphs over cyclic groups. But they were not found by a search for voltage graphs, so they were not presented as such. Voltage graphs over more interesting groups might be a good place to look for  $(r, z, g)$ -cages.

r	z	g	Lower	Exact	Upper
2	2	5		19	
2	2	6		27	
3	1	5		24	
3	1	6		30	
3	1	7	52		60
3	1	8	74		76
4	1	5	29		34
4	1	6	46		48
5	1	5	40		50
5	1	6	66		72

Tab. 1: Bounds for small  $f(r, z, g)$ . For each of the values of  $r$ ,  $z$ , and  $g$ , either the exact value, or our best lower and upper bounds are given. The result for  $(2, 2, 5)$  is due to Claudia De La Cruz and Miguel Pizaña (unpublished).

## References

- G. Araujo-Pardo, C. Hernández-Cruz, and J. J. Montellano-Ballesteros. Mixed cages. *Graphs and Combinatorics*, 35:989–999, 2019.
- M. Behzad. Minimal 2-regular digraphs with given girth. *Journal of the Mathematical Society of Japan*, 25: 1–6, 1973.
- M. Behzad, G. Chartrand, and C. Wall. On minimal regular digraphs with given girth. *Fund. Math.*, 69: 227–231, 1970.
- J.-C. Bermond. 1-graphes réguliers minimaux de girth donné. *Cahiers du Centre d’Études de Recherche Opérationnelle*, 17:125–135, 1975.
- N. Biggs. Constructions for cubic graphs of large girth. *Electron. J. Combin.*, 5, 1998. URL <https://www.combinatorics.org/ojs/index.php/eljc/article/view/v5i1a1>.
- G. Exoo. A mixed graph achieving a moore-like bound, 2022.
- G. Exoo and R. Jajcay. Dynamic cage survey. *Electron. J. Combin.*, 2013. URL <https://www.combinatorics.org/ojs/index.php/eljc/article/view/DS16>.
- F. Harary. *Graph Theory*. Addison-Wesley, 1969.
- F. Lazebnik, K. E. Mellinger, and O. Vega. Embedding cycles in finite planes. *Electron. J. Combin.*, 20: 1–17, 2013.
- B. D. McKay and A. Piperno. Practical graph isomorphism ii. *J. Symbolic Comput.*, 60:94–112, 2014. URL <https://arxiv.org/abs/1301.1493>.
- M. Miller and J. Sirán. Moore graphs and beyond: A survey of the degree/diameter problem. *Electron. J. Combin.*, 2013. URL <https://www.combinatorics.org/ojs/index.php/eljc/article/view/DS14>.
- J. Singer. A theorem in finite projective geometry and some applications to number theory. *Trans. Amer. Math. Soc.*, 43:377–385, 1938.
- J. Tuite and G. Erskine. On total regularity of mixed graphs with order close to the moore bound. *Graphs and Combinatorics*, 35:1253–1272, 2019.
- J. Tuite and G. Erskine. On networks with order close to the moore bound. *Graphs and Combinatorics*, 38, 2022.
- P. Wong. Cages - a survey. *Journal of Graph Theory*, 6:1–22, 1982.