Corrigendum to "On the monophonic rank of a graph" [Discrete Math. Theor. Comput. Sci. 24:2 (2022) #3]

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In this corrigendum, we give a counterexample to Theorem 5.2 in "On the monophonic rank of a graph" [Discrete Math. Theor. Comput. Sci. 24:2 (2022) #3]. We also present a polynomial-time algorithm for computing the monophonic rank of a starlike graph.

Keywords: monophonic convexity, rank of a graph, starlike graph

1 The counterexample

In the paper "On the monophonic rank of a graph", which appeared in *Discrete Math. Theor. Comput. Sci.* 24:2 (2022) #3, we claimed in Theorem 5.2 the NP-completeness of MONOPHONIC RANK for 2-starlike graphs. This result was used in Corollary 5.1 to prove the NP-completeness of MONOPHONIC RANK for *k*-starlike graphs for any fixed $k \ge 2$. However, the reduction given in Theorem 5.2 is not correct. We show in the sequel that the example given in Figure 1 illustrates a counterexample. Consequently, Corollary 5.1 does not hold as well.

We need some definitions. Consider a graph G. The open and the closed neighborhoods of a vertex v are denoted by N(v) and N[v], respectively. A vertex is *simplicial* if its closed neighborhood induces a complete graph. It is clear that a simplicial vertex is not an internal vertex of an induced path. We say that S is *monophonically convex* if the vertices of every induced path joining two vertices of S are contained in S. The *monophonic convex hull* of S, $\langle S \rangle$, is the smallest monophonically convex set containing S. A set S is *monophonic convexly independent* if v is not in $\langle S - \{v\} \rangle$ for $v \in S$. The *monophonic rank* of G, r(G), is the size of a largest monophonic convexly independent set of G.

Recall that a *starlike graph* G admits a partition of V(G) into cliques (V_0, V_1, \ldots, V_t) such that V_0 is a maximal clique and for $i \in \{1, \ldots, t\}$ and $u, v \in V_i$, it holds $N[u] - V_i = N[v] - V_i \subset V_0$. If $|V_\ell| \le k$ for $\ell \in \{1, \ldots, t\}$, then G is k-starlike (Gustedt (1993); Cerioli and Szwarcfiter (2006)). The 1-starlike

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Fig. 1: A counterexample to the proof of Theorem 5.2 in (Dourado et al. (2022)).

G'

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graphs are the *split graphs*. Denote by Z the set of simplicial vertices of G. For $i \in \{0, 1, ..., t\}$, we denote by X_i the maximal clique containing V_i , $C_i = X_i \cap Z$ and $C'_i = X_i - C_i$. Note that $V_0 = X_0$, that $V_i = C_i$ for $i \in \{1, ..., t\}$, and that C_0 can be an empty set.

Lemma 1.1 If G is a starlike graph and $S \subseteq V(G)$, then every vertex $v \in \langle S \rangle - S$ belongs to C'_0 and is an internal vertex of an induced (u, u')-path such that $u \in S \cap Z \cap N(v)$ and $u' \in \langle S \rangle$.

Proof: Let $v \in \langle S \rangle - S$. It is clear that v is an internal vertex of an induced (w, w')-path P for $w, w' \in \langle S \rangle$. Since Z contains only simplicial vertices, we have that $v \in C'_0$. Since C'_0 is a clique, we have that at least one of w and w', say w, belongs to Z. Suppose first that w' also belongs to Z. Note that P has 3 or 4 vertices. In both cases, v is adjacent to at least one of w and w'. Suppose then that w' belongs to C'_0 . In this case, P has 3 vertices, which means that v is adjacent to w. In all cases, v is adjacent to some vertex of Z belonging to $\langle S \rangle$. Since every vertex of Z is simplicial, we have that such vertex belongs to S, completing the proof.

Figure 1 shows an input graph G with n = 7 vertices and the resulting 2-starlike graph G' according to the reduction given in Theorem 5.2 in (Dourado et al. (2022)), whose vertex set can be partitioned into sets U and W where U is a clique with 70 vertices and G[W] has maximum clique of size 2. In such proof, we claimed that G has an independent set with $\lceil \frac{n+1}{2} \rceil = 4$ vertices if and only if G' has a monophonic convexly independent set with p vertices, where $p = n + (4n - 1) \lceil \frac{n+1}{2} \rceil = 7 + 27 \times 4 = 115$. It is easy to see that G has no independent set of size $\lceil \frac{n+1}{2} \rceil = 4$. In order to see that the set $S \subset V(G')$ with 115 vertices formed by the 56 vertices of U_1, U_2, W_1, W_2 , the 56 vertices of U_4, U_5, U_6, U_7 and the 3 vertices u_5^{15}, u_6^{15} and u_7^{15} is m-convexly independent we use the notation given above where $V_0 = C'_0 = U$ and Z = W. Since the vertices of $S \cap C'_0$ have no neighbors in $S \cap Z$, Lemma 1.1 implies that S is indeed an m-convexly independent set of G'.

2 Monophonic rank is polynomial for starlike graphs

In Theorem 5.1 of (Dourado et al. (2022)), we presented a polynomial-time algorithm for computing the monophonic rank of 1-starlike graphs. Here, in Corollary 2.1, we extend such algorithm so that it works for starlike graphs.

Given a graph G and a set $T \subseteq V(G)$, we write $N(T) = \bigcup_{v \in T} N(v)$. If T is an independent set, we define the *difference* of T as d(T) = |T| - |N(T)|, and the *critical independence difference* of G as $d_c(G) = \max\{d(T) : T \text{ is an independent set of } G\}$. If $d(T) = d_c(G)$, then we say that T is a critical independent set of G.

Using the notation given previously for a starlike graph G with partition (V_0, V_1, \ldots, V_t) of V(G)into cliques, denote by Y_i the union of the sets C_j for $j \in \{0, 1, \ldots, t\} - \{i\}$ such that $C'_j \subseteq C'_i$. If $|N(Y_i) - Y_i| = |C'_i| - 1$ and $|Y_i| \ge |C'_i| - 1$, then we can write $\{x\} = (N(Y_i) - Y_i) - C'_i$ and define $G_i = G[(C'_i - \{x\}) \cup Y_i] - E(G[Y_i])$, otherwise define $G_i = G[C'_i \cup Y_i] - E(G[Y_i])$. Note that G_i is a split graph where Y_i is an independent set which can be empty. If this is the case, G_i can be the empty graph. See an example in Figure 2.

Lemma 2.1 If G is a starlike graph, then for $i \in \{0, 1, ..., t\}$, G_i has a critical independent set T_i contained in Y_i .



Fig. 2: Graphs G_0, G_1 and G_3 constructed from the starlike graph G. Ellipses formed by continuous lines represent cliques, while the ones formed by dashed lines represent independent sets.

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Proof: Note that G_i is a split graph with bipartition (C, Y_i) where $C = C'_i$ or $C = C'_i - \{x\}$ for $\{x\} = C'_i - N(Y_i)$. Let T be an independent set of G_i having a vertex $v \in C$. We show that there is an independent set T' such that $T' \cap C \subset T \cap C$ such that $d(T') \ge d(T)$, which proves the result.

If v has a neighbor $y \in Y_i$, then $d((T - \{v\}) \cup \{y\}) \ge d(T)$ because $N[y] \subseteq N[v]$. Then, we can assume from now on that v has no neighbors in Y_i . By the construction of G_i , we have that $C = C'_i$. If $|N(Y_i) - Y_i| < |C'_i| - 1$, then $d(T - \{v\}) \ge d(T)$. It remains to consider $|N(Y_i) - Y_i| = |C'_i| - 1$. In this case, we have that $|Y_i| < |C'_i| - 1$. Finally, note that $d(T) \le 0$ and that $d(\emptyset) \ge d(T)$. \Box

Theorem 2.1 If G is a starlike graph, then $r(G) = \max_{i \in \{0,1,...,t\}} \{|X_i| + d_c(G_i)\}.$

Proof: We begin by showing that for $i \in \{0, 1, ..., t\}$, G has an m-convexly independent set of size $|X_i| + d_c(G_i)$. Let T_i be a critical independent set of G_i . By Lemma 2.1, we can assume that T_i contains only vertices of Y_i . Define $S_i = (C'_i - N(T_i)) \cup T_i \cup C_i$. Observe that $|S_i| = |X_i| + d_c(G_i)$. Suppose by contradiction that $v \in \langle S_i - \{v\} \rangle$ for some $v \in S_i$. By Lemma 1.1, v is an internal vertex of an induced (u, u')-path P such that $u \in (S_i - \{v\}) \cap Z \cap N(v)$ and $u' \in \langle S_i - \{v\} \rangle$, which implies that $v \in C'_i$ and $u \in C_i$. Lemma 1.1 also implies that $\langle S_i - \{v\} \rangle - (S_i - \{v\}) \subseteq C'_i$. Since u is adjacent to all vertices of C'_i , we conclude that $u' \in C_j$ for some $j \neq i$. Since u' is a simplicial vertex and $u' \in \langle S_i - \{v\} \rangle$, we have that $u' \in (S_i - \{v\})$. Therefore, P has exactly 3 vertices and $vu' \in E(G)$, which is a contradiction, because v has no neighbors belonging to $C_j \cap S_i$ for $j \neq i$.

Now, let S be a maximum m-convexly independent set of G. We will show that $|S| \leq |S_i|$ for some $i \in \{0, 1, \ldots, t\}$. Write $S_Z = S \cap Z$ and $S_{\overline{Z}} = S - S_Z = S \cap C'_0$. First, consider that there is no edge uv such that $u \in S_{\overline{Z}}$ and $v \in S_Z$. Hence, either $S_{\overline{Z}} = \emptyset$ or $S_Z \cap C_0 = \emptyset$. In the former case, since $Y_0 \cup C_0 = Z$, we have that $|S| \leq |Y_0| + |C_0|$. Since $|Y_0| \leq |Y_0| - |N(Y_0) \cap C'_0| + |C'_0|$, we have that $|S| \leq |Y_0| - |N(Y_0) \cap C'_0| + |C'_0|$ we have that $|S| \leq |Y_0| - |N(Y_0) \cap C'_0| + |C'_0| + |C_0| \leq d_c(G_0) + |X_0| = |S_0|$. In the latter case, we have that $|S_{\overline{Z}}| + |N(S_Z) \cap C'_0| \leq |C'_0|$. Since S_Z is an independent set of G_0 , it holds that $|S_Z| - |N(S_Z) \cap C'_0| \leq d_c(G_0)$. Therefore, in this case, $|S| = |S_{\overline{Z}}| + |S_Z| \leq |S_{\overline{Z}}| + |N(S_Z) \cap C'_0| + d_c(G_0) \leq |C'_0| + d_c(G_0) = |S_0|$.

Now, consider that there is an edge uv_i such that $u \in S_{\overline{Z}}$ and $v_i \in S_Z \cap C_i$ for some $i \in \{0, 1, \ldots, t\}$. We claim that if there is another edge u'v' such that $u' \in S_{\overline{Z}}$ and $v' \in S_Z \cap C_j$, then i = j. Suppose the contrary. First, consider that u, v_i and v' can be chosen such that $u \in N(v_i) \cap N(v')$. Then, S is not an m-convexly independent set of G because of the induced path $v_i uv'$. Hence, we can assume that there is no $u'' \in S_{\overline{Z}}$ such that $u'' \in N(v_i) \cap N(v')$. Now, note that $v_i uu'v'$ is an induced path of G, which implies that S is not m-convexly independent. Since we reached a contradiction, the claim does hold.

Next, we prove that for $j \neq i$ and $v_j \in S_Z \cap C_j$, it holds that $N(v_j) \cap C'_0 \subseteq C'_i$. Supposing the contrary, there is $v' \in S_Z \cap C_j$ for $j \neq i$ such that there is an edge u'v' where $u' \in C'_0 - C'_i$. By the claim, we know that $v'u \notin E(G)$. Thus, we have that $v'u'uv_i$ is an induced path of G containing u, which is a contradiction. Hence, $N(v) \cap C'_0 \subseteq C'_i$ for every vertex $v \in S_Z$.

If there was a vertex $w \in C'_0 - C'_i$ belonging to S, then G would have the induced path wuv_i and S would not be m-convexly independent. Therefore, from the above, we conclude that $S - C_i$ is contained in $V(G_i)$.

It remains to show that $|S| \leq |S_i|$ in this case as well. Since $S_Z - C_i$ is an independent set of G_i , it holds that $|S_Z - C_i| - |N(S_Z - C_i) \cap C'_i| \leq d_c(G_i)$. Since there is no edge with one extreme in $S_{\overline{Z}}$ and the other in $S_Z - C_i$, we have that $|S_{\overline{Z}}| + |N(S_Z - C_i) \cap C'_i| \leq |C'_i|$. Therefore,

$$|S| = |S_{\overline{Z}}| + |S_{Z}| =$$

$$|S_{\overline{Z}}| + |S_{Z} - C_{i}| + |S_{Z} \cap C_{i}| \leq$$

$$|S_{\overline{Z}}| + |N(S_{Z} - C_{i}) \cap C_{i}'| + d_{c}(G_{i}) + |S_{Z} \cap C_{i}| \leq$$

$$|C_{i}'| + d_{c}(G_{i}) + |S_{Z} \cap C_{i}| \leq$$

$$|C_{i}'| + d_{c}(G_{i}) + |C_{i}| = |X_{i}| + d_{c}(G_{i}) = |S_{i}|.$$

Corollary 2.1 *The* MONOPHONIC RANK *problem restricted to starlike graphs belongs to* P.

Proof: By Theorem 2.1, for computing the monophonic rank of a starlike graph G, it suffices to compute the critical independence difference of a linear number of subgraphs of G. Since such subgraphs can be found in polynomial time (Peng et al. (2000)) and the critical independence difference can be computed in polynomial time for general graphs (Ageev (1994)), the result thus hold.

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