# Corrigendum to "On the monophonic rank of a graph" [Discrete Math. Theor. Comput. Sci. 24:2 (2022) \#3] 

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In this corrigendum, we give a counterexample to Theorem 5.2 in "On the monophonic rank of a graph" [Discrete Math. Theor. Comput. Sci. 24:2 (2022) \#3]. We also present a polynomial-time algorithm for computing the monophonic rank of a starlike graph.

Keywords: monophonic convexity, rank of a graph, starlike graph

## 1 The counterexample

In the paper "On the monophonic rank of a graph", which appeared in Discrete Math. Theor. Comput. Sci. 24:2 (2022) \#3, we claimed in Theorem 5.2 the NP-completeness of MONOPHONIC RANK for 2-starlike graphs. This result was used in Corollary 5.1 to prove the NP-completeness of Monophonic rank for $k$-starlike graphs for any fixed $k \geq 2$. However, the reduction given in Theorem 5.2 is not correct. We show in the sequel that the example given in Figure 1 illustrates a counterexample. Consequently, Corollary 5.1 does not hold as well.

We need some definitions. Consider a graph $G$. The open and the closed neighborhoods of a vertex $v$ are denoted by $N(v)$ and $N[v]$, respectively. A vertex is simplicial if its closed neighborhood induces a complete graph. It is clear that a simplicial vertex is not an internal vertex of an induced path. We say that $S$ is monophonically convex if the vertices of every induced path joining two vertices of $S$ are contained in $S$. The monophonic convex hull of $S,\langle S\rangle$, is the smallest monophonically convex set containing $S$. A set $S$ is monophonic convexly independent if $v$ is not in $\langle S-\{v\}\rangle$ for $v \in S$. The monophonic rank of $G$, $r(G)$, is the size of a largest monophonic convexly independent set of $G$.

Recall that a starlike graph $G$ admits a partition of $V(G)$ into cliques $\left(V_{0}, V_{1}, \ldots, V_{t}\right)$ such that $V_{0}$ is a maximal clique and for $i \in\{1, \ldots, t\}$ and $u, v \in V_{i}$, it holds $N[u]-V_{i}=N[v]-V_{i} \subset V_{0}$. If $\left|V_{\ell}\right| \leq k$ for $\ell \in\{1, \ldots, t\}$, then $G$ is $k$-starlike (Gustedt (1993); Cerioli and Szwarcfiter (2006)). The 1 -starlike

[^0]
$G^{\prime}$


Fig. 1: A counterexample to the proof of Theorem 5.2 in (Dourado et al. (2022)).
graphs are the split graphs. Denote by $Z$ the set of simplicial vertices of $G$. For $i \in\{0,1, \ldots, t\}$, we denote by $X_{i}$ the maximal clique containing $V_{i}, C_{i}=X_{i} \cap Z$ and $C_{i}^{\prime}=X_{i}-C_{i}$. Note that $V_{0}=X_{0}$, that $V_{i}=C_{i}$ for $i \in\{1, \ldots, t\}$, and that $C_{0}$ can be an empty set.

Lemma 1.1 If $G$ is a starlike graph and $S \subseteq V(G)$, then every vertex $v \in\langle S\rangle-S$ belongs to $C_{0}^{\prime}$ and is an internal vertex of an induced $\left(u, u^{\prime}\right)$-path such that $u \in S \cap Z \cap N(v)$ and $u^{\prime} \in\langle S\rangle$.

Proof: Let $v \in\langle S\rangle-S$. It is clear that $v$ is an internal vertex of an induced ( $w, w^{\prime}$ )-path $P$ for $w, w^{\prime} \in\langle S\rangle$. Since $Z$ contains only simplicial vertices, we have that $v \in C_{0}^{\prime}$. Since $C_{0}^{\prime}$ is a clique, we have that at least one of $w$ and $w^{\prime}$, say $w$, belongs to $Z$. Suppose first that $w^{\prime}$ also belongs to $Z$. Note that $P$ has 3 or 4 vertices. In both cases, $v$ is adjacent to at least one of $w$ and $w^{\prime}$. Suppose then that $w^{\prime}$ belongs to $C_{0}^{\prime}$. In this case, $P$ has 3 vertices, which means that $v$ is adjacent to $w$. In all cases, $v$ is adjacent to some vertex of $Z$ belonging to $\langle S\rangle$. Since every vertex of $Z$ is simplicial, we have that such vertex belongs to $S$, completing the proof.

Figure 1 shows an input graph $G$ with $n=7$ vertices and the resulting 2-starlike graph $G^{\prime}$ according to the reduction given in Theorem 5.2 in (Dourado et al. (2022)), whose vertex set can be partitioned into sets $U$ and $W$ where $U$ is a clique with 70 vertices and $G[W]$ has maximum clique of size 2 . In such proof, we claimed that $G$ has an independent set with $\left\lceil\frac{n+1}{2}\right\rceil=4$ vertices if and only if $G^{\prime}$ has a monophonic convexly independent set with $p$ vertices, where $p=n+(4 n-1)\left\lceil\frac{n+1}{2}\right\rceil=7+27 \times 4=115$. It is easy to see that $G$ has no independent set of size $\left\lceil\frac{n+1}{2}\right\rceil=4$. In order to see that the set $S \subset V\left(G^{\prime}\right)$ with 115 vertices formed by the 56 vertices of $U_{1}, U_{2}, W_{1}, W_{2}$, the 56 vertices of $U_{4}, U_{5}, U_{6}, U_{7}$ and the 3 vertices $u_{5}^{15}, u_{6}^{15}$ and $u_{7}^{15}$ is m-convexly independent we use the notation given above where $V_{0}=C_{0}^{\prime}=U$ and $Z=W$. Since the vertices of $S \cap C_{0}^{\prime}$ have no neighbors in $S \cap Z$, Lemma 1.1 implies that $S$ is indeed an m-convexly independent set of $G^{\prime}$.

## 2 Monophonic rank is polynomial for starlike graphs

In Theorem 5.1 of (Dourado et al. (2022)), we presented a polynomial-time algorithm for computing the monophonic rank of 1-starlike graphs. Here, in Corollary 2.1, we extend such algorithm so that it works for starlike graphs.
 define the difference of $T$ as $d(T)=|T|-|N(T)|$, and the critical independence difference of $G$ as $d_{c}(G)=\max \{d(T): T$ is an independent set of $G\}$. If $d(T)=d_{c}(G)$, then we say that $T$ is a critical independent set of $G$.

Using the notation given previously for a starlike graph $G$ with partition $\left(V_{0}, V_{1}, \ldots, V_{t}\right)$ of $V(G)$ into cliques, denote by $Y_{i}$ the union of the sets $C_{j}$ for $j \in\{0,1, \ldots, t\}-\{i\}$ such that $C_{j}^{\prime} \subseteq C_{i}^{\prime}$. If $\left|N\left(Y_{i}\right)-Y_{i}\right|=\left|C_{i}^{\prime}\right|-1$ and $\left|Y_{i}\right| \geq\left|C_{i}^{\prime}\right|-1$, then we can write $\{x\}=\left(N\left(Y_{i}\right)-Y_{i}\right)-C_{i}^{\prime}$ and define $G_{i}=G\left[\left(C_{i}^{\prime}-\{x\}\right) \cup Y_{i}\right]-E\left(G\left[Y_{i}\right]\right)$, otherwise define $G_{i}=G\left[C_{i}^{\prime} \cup Y_{i}\right]-E\left(G\left[Y_{i}\right]\right)$. Note that $G_{i}$ is a split graph where $Y_{i}$ is an independent set which can be empty. If this is the case, $G_{i}$ can be the empty graph. See an example in Figure 2.

Lemma 2.1 If $G$ is a starlike graph, then for $i \in\{0,1, \ldots, t\}, G_{i}$ has a critical independent set $T_{i}$ contained in $Y_{i}$.


Fig. 2: Graphs $G_{0}, G_{1}$ and $G_{3}$ constructed from the starlike graph $G$. Ellipses formed by continuous lines represent cliques, while the ones formed by dashed lines represent independent sets.

Proof: Note that $G_{i}$ is a split graph with bipartition $\left(C, Y_{i}\right)$ where $C=C_{i}^{\prime}$ or $C=C_{i}^{\prime}-\{x\}$ for $\{x\}=C_{i}^{\prime}-N\left(Y_{i}\right)$. Let $T$ be an independent set of $G_{i}$ having a vertex $v \in C$. We show that there is an independent set $T^{\prime}$ such that $T^{\prime} \cap C \subset T \cap C$ such that $d\left(T^{\prime}\right) \geq d(T)$, which proves the result.

If $v$ has a neighbor $y \in Y_{i}$, then $d((T-\{v\}) \cup\{y\}) \geq d(T)$ because $N[y] \subseteq N[v]$. Then, we can assume from now on that $v$ has no neighbors in $Y_{i}$. By the construction of $G_{i}$, we have that $C=C_{i}^{\prime}$. If $\left|N\left(Y_{i}\right)-Y_{i}\right|<\left|C_{i}^{\prime}\right|-1$, then $d(T-\{v\}) \geq d(T)$. It remains to consider $\left|N\left(Y_{i}\right)-Y_{i}\right|=\left|C_{i}^{\prime}\right|-1$. In this case, we have that $\left|Y_{i}\right|<\left|C_{i}^{\prime}\right|-1$. Finally, note that $d(T) \leq 0$ and that $d(\emptyset) \geq d(T)$.

Theorem 2.1 If $G$ is a starlike graph, then $r(G)=\max _{i \in\{0,1, \ldots, t\}}\left\{\left|X_{i}\right|+d_{c}\left(G_{i}\right)\right\}$.

Proof: We begin by showing that for $i \in\{0,1, \ldots, t\}, G$ has an m-convexly independent set of size $\left|X_{i}\right|+d_{c}\left(G_{i}\right)$. Let $T_{i}$ be a critical independent set of $G_{i}$. By Lemma 2.1, we can assume that $T_{i}$ contains only vertices of $Y_{i}$. Define $S_{i}=\left(C_{i}^{\prime}-N\left(T_{i}\right)\right) \cup T_{i} \cup C_{i}$. Observe that $\left|S_{i}\right|=\left|X_{i}\right|+d_{c}\left(G_{i}\right)$. Suppose by contradiction that $v \in\left\langle S_{i}-\{v\}\right\rangle$ for some $v \in S_{i}$. By Lemma 1.1, $v$ is an internal vertex of an induced $\left(u, u^{\prime}\right)$-path $P$ such that $u \in\left(S_{i}-\{v\}\right) \cap Z \cap N(v)$ and $u^{\prime} \in\left\langle S_{i}-\{v\}\right\rangle$, which implies that $v \in C_{i}^{\prime}$ and $u \in C_{i}$. Lemma 1.1 also implies that $\left\langle S_{i}-\{v\}\right\rangle-\left(S_{i}-\{v\}\right) \subseteq C_{i}^{\prime}$. Since $u$ is adjacent to all vertices of $C_{i}^{\prime}$, we conclude that $u^{\prime} \in C_{j}$ for some $j \neq i$. Since $u^{\prime}$ is a simplicial vertex and $u^{\prime} \in\left\langle S_{i}-\{v\}\right\rangle$, we have that $u^{\prime} \in\left(S_{i}-\{v\}\right)$. Therefore, $P$ has exactly 3 vertices and $v u^{\prime} \in E(G)$, which is a contradiction, because $v$ has no neighbors belonging to $C_{j} \cap S_{i}$ for $j \neq i$.

Now, let $S$ be a maximum m-convexly independent set of $G$. We will show that $|S| \leq\left|S_{i}\right|$ for some $i \in\{0,1, \ldots, t\}$. Write $S_{Z}=S \cap Z$ and $S_{\bar{Z}}=S-S_{Z}=S \cap C_{0}^{\prime}$. First, consider that there is no edge $u v$ such that $u \in S_{\bar{Z}}$ and $v \in S_{Z}$. Hence, either $S_{\bar{Z}}=\emptyset$ or $S_{Z} \cap C_{0}=\emptyset$. In the former case, since $Y_{0} \cup C_{0}=Z$, we have that $|S| \leq\left|Y_{0}\right|+\left|C_{0}\right|$. Since $\left|Y_{0}\right| \leq\left|Y_{0}\right|-\left|N\left(Y_{0}\right) \cap C_{0}^{\prime}\right|+\left|C_{0}^{\prime}\right|$, we have that $|S| \leq\left|Y_{0}\right|+\left|C_{0}\right| \leq\left|Y_{0}\right|-\left|N\left(Y_{0}\right) \cap C_{0}^{\prime}\right|+\left|C_{0}^{\prime}\right|+\left|C_{0}\right| \leq d_{c}\left(G_{0}\right)+\left|X_{0}\right|=\left|S_{0}\right|$. In the latter case, we have that $\left|S_{\bar{Z}}\right|+\left|N\left(S_{Z}\right) \cap C_{0}^{\prime}\right| \leq\left|C_{0}^{\prime}\right|$. Since $S_{Z}$ is an independent set of $G_{0}$, it holds that $\left|S_{Z}\right|-\left|N\left(S_{Z}\right) \cap C_{0}^{\prime}\right| \leq d_{c}\left(G_{0}\right)$. Therefore, in this case, $|S|=\left|S_{\bar{Z}}\right|+\left|S_{Z}\right| \leq\left|S_{\bar{Z}}\right|+\left|N\left(S_{Z}\right) \cap C_{0}^{\prime}\right|+$ $d_{c}\left(G_{0}\right) \leq\left|C_{0}^{\prime}\right|+d_{c}\left(G_{0}\right) \leq\left|X_{0}\right|+d_{c}\left(G_{0}\right)=\left|S_{0}\right|$.

Now, consider that there is an edge $u v_{i}$ such that $u \in S_{\bar{Z}}$ and $v_{i} \in S_{Z} \cap C_{i}$ for some $i \in\{0,1, \ldots, t\}$. We claim that if there is another edge $u^{\prime} v^{\prime}$ such that $u^{\prime} \in S_{\bar{Z}}$ and $v^{\prime} \in S_{Z} \cap C_{j}$, then $i=j$. Suppose the contrary. First, consider that $u, v_{i}$ and $v^{\prime}$ can be chosen such that $u \in N\left(v_{i}\right) \cap N\left(v^{\prime}\right)$. Then, $S$ is not an m-convexly independent set of $G$ because of the induced path $v_{i} u v^{\prime}$. Hence, we can assume that there is no $u^{\prime \prime} \in S_{\bar{Z}}$ such that $u^{\prime \prime} \in N\left(v_{i}\right) \cap N\left(v^{\prime}\right)$. Now, note that $v_{i} u u^{\prime} v^{\prime}$ is an induced path of $G$, which implies that $S$ is not m-convexly independent. Since we reached a contradiction, the claim does hold.

Next, we prove that for $j \neq i$ and $v_{j} \in S_{Z} \cap C_{j}$, it holds that $N\left(v_{j}\right) \cap C_{0}^{\prime} \subseteq C_{i}^{\prime}$. Supposing the contrary, there is $v^{\prime} \in S_{Z} \cap C_{j}$ for $j \neq i$ such that there is an edge $u^{\prime} v^{\prime}$ where $u^{\prime} \in \bar{C}_{0}^{\prime}-C_{i}^{\prime}$. By the claim, we know that $v^{\prime} u \notin E(G)$. Thus, we have that $v^{\prime} u^{\prime} u v_{i}$ is an induced path of $G$ containing $u$, which is a contradiction. Hence, $N(v) \cap C_{0}^{\prime} \subseteq C_{i}^{\prime}$ for every vertex $v \in S_{Z}$.

If there was a vertex $w \in C_{0}^{\prime}-C_{i}^{\prime}$ belonging to $S$, then $G$ would have the induced path $w u v_{i}$ and $S$ would not be m-convexly independent. Therefore, from the above, we conclude that $S-C_{i}$ is contained in $V\left(G_{i}\right)$.

It remains to show that $|S| \leq\left|S_{i}\right|$ in this case as well. Since $S_{Z}-C_{i}$ is an independent set of $G_{i}$, it holds that $\left|S_{Z}-C_{i}\right|-\left|N\left(S_{Z}-C_{i}\right) \cap C_{i}^{\prime}\right| \leq d_{c}\left(G_{i}\right)$. Since there is no edge with one extreme in $S_{\bar{Z}}$ and the other in $S_{Z}-C_{i}$, we have that $\left|S_{\bar{Z}}\right|+\left|N\left(S_{Z}-C_{i}\right) \cap C_{i}^{\prime}\right| \leq\left|C_{i}^{\prime}\right|$. Therefore,

$$
\begin{gathered}
|S|=\left|S_{\bar{Z}}\right|+\left|S_{Z}\right|= \\
\left|S_{\bar{Z}}\right|+\left|S_{Z}-C_{i}\right|+\left|S_{Z} \cap C_{i}\right| \leq \\
\left|S_{\bar{Z}}\right|+\left|N\left(S_{Z}-C_{i}\right) \cap C_{i}^{\prime}\right|+d_{c}\left(G_{i}\right)+\left|S_{Z} \cap C_{i}\right| \leq \\
\left|C_{i}^{\prime}\right|+d_{c}\left(G_{i}\right)+\left|S_{Z} \cap C_{i}\right| \leq \\
\left|C_{i}^{\prime}\right|+d_{c}\left(G_{i}\right)+\left|C_{i}\right|=\left|X_{i}\right|+d_{c}\left(G_{i}\right)=\left|S_{i}\right|
\end{gathered}
$$

Corollary 2.1 The MONOPHONIC RANK problem restricted to starlike graphs belongs to P .

Proof: By Theorem 2.1, for computing the monophonic rank of a starlike graph $G$, it suffices to compute the critical independence difference of a linear number of subgraphs of $G$. Since such subgraphs can be found in polynomial time (Peng et al. (2000)) and the critical independence difference can be computed in polynomial time for general graphs (Ageev (1994)), the result thus hold.

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