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On P_4-tidy graphs

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We study the P_4-tidy graphs, a new class defined by Rusu [30] in order to illustrate the notion of P_4-domination in perfect graphs. This class strictly contains the P_4-extendible graphs and the P_4-lite graphs defined by Jamison & Olariu in [19] and [23] and we show that the P_4-tidy graphs and P_4-lite graphs are closely related. Note that the class of P_4-lite graphs is a class of brittle graphs strictly containing the P_4-sparse graphs defined by Hoàng in [14].

McConnel & Spinrad [2] and independently Cournier & Habib [5] have shown that the modular decomposition tree of any graph is computable in linear time. For recognizing in linear time P_4-tidy graphs, we apply a method intoduced by Giakoumakis in [9] and Giakoumakis & Fouquet in [6] using modular decomposition of graphs and we propose linear algorithms for optimization problems on such graphs, as clique number, stability number, chromatic number and scattering number. We show that the Hamiltonian Path Problem is linear for this class of graphs.

Our study unifies and generalizes previous results of Jamison & Olariu ([18], [21], [22]), Hochstättler & Schindler [16], Jung [25] and Hochstättler & Tinhofer [15].

Keywords: graph, modular decomposition, perfection, P_4-structure.

1 Introduction and motivations

H. A. Jung in [25] studied the existence of a Hamiltonian path or a Hamiltonian cycle in a graph G without induced chordless path of four vertices (P_4*-free graph*), by examining the value of the *scattering number* of G.

The class of P_4-free graphs (or *cographs*) has been discovered independently in different areas of Mathematics and Computer Sciences. Corneil *et al.* in [4] proposed a linear (in the number of edges of G) recognition algorithm obtained from a unique tree representation of a cograph G (a *cotree* associated with G).

The numerous structural properties of P_4-free graphs motivated researchers to define classes of graphs obtained as extensions of cographs. In [14] Hoàng introduced the class of P_4*-sparse graphs* as the graphs for which every set of five vertices induces at most one P_4. For any graph $G = (V, E)$ and W a proper subset of V inducing a P_4, let $S(W)$ be the set of vertices of $V \setminus W$ belonging to a P_4 sharing vertices with W. Jamison and Olariu in [19] defined $G = (V, E)$ to be P_4*-extendible* if for every proper subset W of V inducing a P_4, $S(W)$ contains at most one vertex. Previously they have defined in [17] the P_4*-reducible graphs* as the class of graphs such that any vertex belongs to at most one P_4, and it is easy to see that this class is the intersection of classes of the P_4-extendible and P_4-sparse graphs.

By extending the notion of cotree, Jamison and Olariu proposed a unique tree representation for P_4-reducible graphs [18], for P_4-sparse graphs [20] and for P_4-extendible graphs [19]. These trees are used as framework to linear recognition algorithms for these graphs in papers of Jamison & Olariu ([21] and [18]) and in a paper of Hochstättler & Schindler [16]. This tree representation is also the underlying data structure in [15] for studying the Hamiltonicity of P_4-extendible graphs and of *spiders*, a subclass of P_4-sparse graphs.

McConnel & Spinrad [2] and independently Cournier & Habib [5] show that the *modular decomposition tree* of a graph $G = (V, E)$ is computable in linear time $O(|E| + |V|)$. Using the modular decomposition of graphs Giakoumakis in [9] and Giakoumakis & Vanherpe in [11] study two classes of graphs strictly containing, respectively, the class of P_4-sparse graphs and that of P_4-reducible graphs (we shall see later that these classes are also contained in the class of P_4-tidy graphs). Note here, in the same vein, a paper of Fouquet & Giakoumakis [6] concerned by a large class containing the P_4-sparse graphs. In these three papers, that generalize results of Jamison and Olariu ([18], [21] and [22]), linear algorithms for the recognition as well as for classical optimization problems are obtained from the unique up to isomorphism *modular decomposition tree* associated with any graph G.

In this paper, we define the *P_4-tidy graphs*, a new class of graphs strictly containing the previous considered classes (excepted the class defined by Fouquet & Giakoumakis in [6]). We show that the modular decomposition tree $T(G)$ of a graph $G = (V, E)$ can be used to recognize, in $O(|V|)$ time, a P_4-tidy graph. Namely, we design an algorithm able to state precisely if the considered graph is a cograph, is P_4-reducible, is P_4-extendible, is P_4-sparse, is P_4-lite or is a general P_4-tidy graph. We also use the modular decomposition tree of a P_4-tidy graph for finding, in linear time, *the scattering number*, the clique number, the stability number and the chromatic number of such a graph. At last, we use a method introduced in [10] for studying Hamiltonicity or path partition of graphs, by associating the modular decomposition tree of a graph G with its scattering number. We apply this method to the 3-*sun free* P_4-tidy graphs in order to generalize results of Jung [25] and Hochstättler & Tinhofer [15]. Note that, as corollary, we extend to Hamilton-connected P_4-extendible graphs a result of [15].

2 Definitions, notations and general properties

2.1 Generalities

For terms not defined in this paper the reader can be referred to [12]. In this paper we deal only with simple graphs (that is undirected graphs with no loops and no two edges joining the same pair of vertices). For any graph G, $V(G)$ denotes the set of its vertices and $E(G)$ the set of its edges (or V and E if there is no confusion, and we shall denote $G = (V, E)$). For any vertex v in V, the neighbourhood of v is the set $N_G(v) = \{u \in V \mid uv \in E\}$ (or $N(v)$ if there is no confusion). For every set of vertices $W \subseteq V \setminus \{v\}$ we shall say that v *misses* W if $W \cap N(v)$ is empty and that v *dominates* W if $W \subseteq N(v)$. For any set of vertices A of G the subgraph induced by A is denoted by $G[A]$, while the subgraph $G[V \setminus A]$ is simply denoted by $G \setminus A$. For any set of edges F of G the subgraph $(V, E \setminus F)$ is denoted by $G \setminus F$. The *complement* $\overline{G}$ of G is the graph $(V, \overline{E})$ where $\overline{E}$ is the set $\{xy \mid x \in V, y \in V \text{ and } xy \notin E\}$. A connected component of a graph is simply said to be a *component* and the number of components of a graph G is denoted by $c(G)$. A set $S \subseteq V$ is said to be a *cutset* of G if $c(G \setminus S) > 1$. Note that this definition is distinct from the usual one ($c(G \setminus S) > c(G)$). Following our definition, $S = \emptyset$ is a cutset of G if and only if G is disconnected. For any path P, the *length* of P is the number of its edges. An induced path on k vertices shall be denoted by P_k. An induced subgraph of G isomorphic to a P_k is simply said to be *a P_k in G*. A vertex of a path P distinct from an end-vertex is said to be an *internal* vertex. If u and v are vertices of a path P then $P[u, v]$ denotes the subpath of P whose end-vertices are u and v. If $V(P) = \{v_1, \ldots, v_k\}$ and $E(P) = \{v_i v_{i+1} \mid i \in \{1, \ldots, k-1\}\ \}$, P is also denoted by $[v_1, \ldots, v_k]$. In a P_4, $[a, b, c, d]$, the two internal vertices b and c are referred to as *midpoints* while the end-vertices a and d as *endpoints*. A chordless cycle on k vertices is denoted by C_k or by $[v_1, \ldots, v_k, v_1]$ if its vertex set is $\{v_1, \ldots, v_k\}$ and its edge set is $\{v_i v_{i+1} \mid 1 \leq i \leq k-1\} \cup \{v_k v_1\}$.

Let H be a simple graph with vertices $\{v_1, \ldots, v_n\}$ and let $\{G_1, \ldots, G_n\}$ be a family of vertex-disjoint simple graphs. The *join of* $\{G_1, \ldots, G_n\}$ *over* H (or *composition of* $\{G_1, \ldots, G_n\}$ *over* H) is the graph denoted by $J_H(G_1, \ldots, G_n)$ having $V(G_1) \cup \cdots \cup V(G_n)$ as vertices and a pair $\{u, v\}$, with $u \in V(G_i)$ and $v \in V(G_j)$, is an edge of the join if either $i = j$ and $\{u, v\}$ is an edge of G_i, or $i \neq j$ and $\{v_i, v_j\}$ is an edge of H. We shall say that the join $J_H(G_1, \ldots, G_n)$ arises by replacing the vertices of H by the graphs G_i.

Let $\mathcal{Z}$ be a set of graphs. We shall say that a graph G is *$\mathcal{Z}$-free* if no induced subgraph of G is isomorphic to a graph of $\mathcal{Z}$. A set of graphs $\mathcal{F}$ will be *$\mathcal{Z}$-free* if every graph of $\mathcal{F}$ is $\mathcal{Z}$-free. The subset $\mathcal{F}$ of all $\mathcal{Z}$-free graphs of a set of graphs $\mathcal{G}$ is said to be *defined by the forbidden configurations* $\mathcal{Z}$.

Let $k \geq 3$ be an integer. A *k-sun* is a graph obtained from a chordless cycle $[x_1, \ldots, x_k, x_1]$ by adding k new vertices $y_1, \ldots, y_k$ and the edges $x_1 y_1, \ldots, x_k y_k$. In this paper, we shall be interested by the 3-sun-free P_4-tidy graphs.

A *spider* is a graph $G = (V, E)$ such that V is partitioned into sets S, K and R such that

(a) S is a stable, K is a clique and $|S| = |K| \geq 2$.

(b) Every vertex in R is adjacent to all the vertices in K and adjacent to no vertex in S.

(c) There exists a bijection $f : S \to K$ such that either

(c.1) for all vertices $s \in S$, $N_G(s) \cap K = \{f(s)\}$

or else,

(c.2) for all vertices $s \in S$, $N_G(s) \cap K = K \setminus \{f(s)\}$.

If (c.1) holds, we say that G is a spider with *thin legs*, otherwise it has *thick legs*. R is called *head*, K is called *body* and S *feet* of the spider G. We shall denote a spider by (R, K, S) or by (K, S) if R is empty.

Clearly, the complement $\overline{G}$ of a spider G with thin (resp. thick) legs is a spider with thick (resp. thin) legs. If the head R is empty or contains one vertex then a spider with thin legs is called an *urchin* and a spider with thick legs is called a *starfish*. A P_4 and a *bull* are simultaneously urchin and starfish, a 3-sun is an example of urchin, and its complement (the Hajós graph) is an example of starfish (see Figure 1).

2.2 Scattering number and Jung graphs

Let $G = (V, E)$ be a graph distinct from a complete graph. Let Σ be the family of cutsets of G. The *scattering number* of G is the number

$$s(G) = \max\{s \mid \exists S \in \Sigma \text{ and } s = c(G \setminus S) - |S|\}.$$

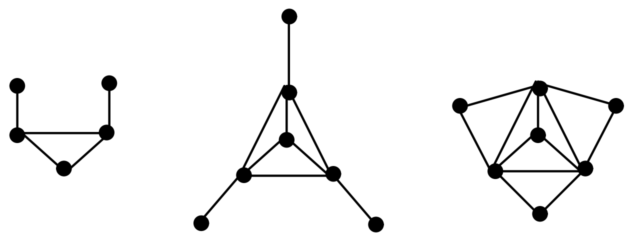

Fig. 1. Bull, urchin and starfish.

A cutset S such that $(c(G \setminus S) - |S|) = s(G)$ is called a *scattering set* of G. By convention, the scattering number of a complete graph $K = (V, E)$ is $-|V|$ and V is the unique scattering set. Remark that for a graph G of order n, $s(G) = -n$ if and only if G is isomorphic to the complete graph K_n, $s(G) = n$ if and only if G is isomorphic to the stable graph $S_n = \overline{K}_n$. Note that if G is not connected then $s(G) \geq c(G)$. The following lemma is implicit in [25].

Lemma 2.1 *Let $G = (V, E)$ be a graph and S be a scattering set of G. Then, for any subset A of V, $s(G \setminus A) \leq s(G) + |A|$. Moreover, if A is a subset of S then $s(G \setminus A) = s(G) + |A|$.*

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \emptyset$. We recall that the *disjoint union* of G_1 and G_2 is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, and the *disjoint sum* is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup \{xy \mid x \in V_1, y \in V_2\}$. We shall denote the disjoint union of G_1 and G_2 by $G_1 ⓪ G_2$ and the disjoint sum by $G_1 ① G_2$.

Lemma 2.2 ([25]) *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \emptyset$. Then*

1. $s(G_1 ⓪ G_2) = \max(1, s(G_1)) + \max(1, s(G_2))$,
2. $s(G_1 ① G_2) = \max(s(G_1) - |V_2|, s(G_2) - |V_1|)$.

The scattering number $s(G)$ of a graph $G = (V, E)$ distinct from a complete graph is closely related to the *toughness* $t(G) = \min\{s \mid \exists S \subset V, c(G \setminus S) > 1 \text{ and } s = |S|/c(G \setminus S)\}$ introduced by Chvátal [1] in order to study Hamiltonicity. More precisely, if S_0 and S_1 are subsets of V such that $c(G \setminus S_0) - |S_0| = s(G)$ (a scattering set) and $c(G \setminus S_1)t(G) = |S_1|$ (a tough cutset) then $c(G \setminus S_1)(1 - t(G)) \leq s(G) \leq c(G \setminus S_0)(1 - t(G))$. Then, we see that $s(G) > 0$ if and only if $t(G) < 1$, and $s(G) = 0$ if and only if $t(G) = 1$. Since for any proper subset S of vertices of a Hamiltonian graph G $c(G \setminus S) \leq |S|$, for such a graph $s(G) \leq 0$ (or equivalently $t(G) \geq 1$). We note that the problem: *'Given a graph G and an integer k, decide whether $s(G) \geq k$'* is *NP*-complete (see [27]).

For a graph G we shall denote by $\rho(G)$ the minimum number of elementary disjoint paths which cover $V(G)$ (i.e. the *minimum path partition number* of G, or simply the *path number* of G). Skupień [32] studied some graphs whose scattering number is $\rho(G)$ and Jung [25] studied relationships between minimum path partition (in particular, Hamiltonicity) and scattering number in P_4-free graphs. Namely he proved the following result:

Theorem 2.1 ([25]) *Let $G = (V, E)$ be a P_4-free graph. Then*

1. $\rho(G) = \max(1, s(G))$,

2. *G is Hamiltonian if and only if $s(G) \leq 0$ and $|V| \geq 3$,*

3. *G is Hamilton-connected if and only if $s(G) < 0$.*

Using Theorem 2.1, Corneil *et al.* [3] proved that the Hamiltonian Decision Problem for cographs is linear, by showing that the scattering number of a P_4-free graph is computable in linear time. We note that,

- for an arbitrary graph G, $\rho(G) \geq \max(1, s(G))$,
- for any Hamiltonian graph G, $s(G) \leq 0$ and
- for any Hamilton-connected graph G, $s(G) < 0$.

Definition 2.1 *A graph G is said to be a* Jung graph *if it verifies the following conditions:*

1. $\rho(G) = \max(1, s(G))$,

2. *if $s(G) = 0$ then G is Hamiltonian,*

3. *if $s(G) < 0$ then G is Hamilton-connected.*

A given class of Jung graphs is said to be a Jung's family.

A 3-sun H is not a Jung graph (because $s(H) = 1$ and $\rho(H) = 2$). By Theorem 2.1, the class of P_4-free graphs is an example of Jung's family. The following result is implicit in [25], [3] and [15].

Proposition 2.1 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two Jung graphs with $V_1 \cap V_2 = \emptyset$. Then the disjoint union $G_1 ⓪ G_2$ and the disjoint sum $G_1 ① G_2$ are Jung graphs.*

2.3 Modular decomposition of graphs

Let $G = (V, E)$ be an arbitrary graph. A set M of vertices is called a *module* if every vertex in $V \setminus M$ is either adjacent to all the vertices in M, or to none of them. Hence, a module M of G is also a module of $\overline{G}$. The empty set, the singletons and V are the *trivial modules* of G. A non-trivial module is called a *homogeneous set*. A module M is called a *strong module* if, for any other module A, the intersection of M and A is empty or one module is contained into the other. A graph having only trivial modules is called *indecomposable*. Any indecomposable graph distinct from K_1, K_2 and S_2 is said to be a *prime graph*. Note that a prime graph is connected and has at least four vertices and that G is a prime graph if and only if $\overline{G}$ is a prime graph. P_4, for $k \geq 5$ P_k, $\overline{P_k}$, C_k, $\overline{C_k}$, bulls, urchins and starfishes are examples of prime graphs.

The *modular decomposition* is a form of decomposition of a graph G that associates with G a unique modular decomposition tree $T(G)$. The leaves of $T(G)$ are the vertices of G and a set of leaves of $T(G)$ having the same least common ancestor in $T(G)$ is a strong module of G. The internal nodes of $T(G)$ are labelled by P, S or N.

More precisely, let r be an internal node of $T(G)$, $M(r)$ be the set of leaves of the subtree of $T(G)$ rooted on r, and $V(r) = \{r_1, \ldots, r_k\}$ be the set of children of r in $T(G)$. If $G[M(r)]$ is disconnected then r is labelled by P (for parallel module) and $G[M(r_1)], \ldots, G[M(r_k)]$ are its components. If $\overline{G}[M(r)]$ is disconnected then r is labelled by S (series module) and $\overline{G}[M(r_1)], \ldots, \overline{G}[M(r_k]$ are its components. Finally, if both graphs $G[M(r)]$ and $\overline{G}[M(r)]$ are connected then r is labelled by N (neighbourhood module) and $M(r_1), \ldots, M(r_k)$ is the unique set of maximal strong submodules of $M(r)$. Then the *representative graph* $G(r)$ of the module $M(r)$ is the graph whose vertex set is $V(r)$ and such that $r_i r_j$ is an edge if and only if there is a vertex of $M(r_i)$ adjacent in G to a vertex of $M(r_j)$. Note that by definition of a module, if a vertex of $M(r_i)$ is adjacent to a vertex of $M(r_j)$, then every vertex of $M(r_i)$ will be adjacent to every vertex of $M(r_j)$. Thus, $G[M(r)]$ is the join $J_{G[r]}(G[M(r_1)], \ldots, G[M(r_k)])$ and $G(r)$ is isomorphic to the graph induced by a subset of $M(r)$ consisting of a single vertex from each maximal strong submodule of $M(r)$ in the modular decomposition of G. It is easy to see that if r is a S-node then $G(r)$ is a complete graph, if r is a P-node then $G(r)$ is a stable set and if r is a N-node then $G(r)$ is a prime graph. Let us denote by $\pi(G)$ the set of prime graphs $\{G(r_1), \ldots, G(r_s)\}$, where $\{r_1, \ldots, r_s\}$ is the set of N-nodes of $T(G)$.

Theorem 2.2 ([9]) *Let Z be is a prime graph then a graph G is Z-free if and only if every graph in $\pi(G)$ is Z-free.*

For more details on modular decomposition, see for instance [7], [28], [26] and [29]. The efficient construction of the modular decomposition tree $T(G)$ had been extensively studied. We recalled previously that McConnel & Spinrad in [2] and independently Cournier & Habib in [5] gave linear algorithms for this purpose ('linear' means here $O(m+n)$ with $m = |E(G)|$ and $n = |V(G)|$).

2.4 p-connectedness

In [24] Jamison & Olariu introduce and investigate the notion of *p-connectedness*. This concept leads them to a general structure for arbitrary graphs and to a unique tree representation extending the modular decomposition.

Let $G = (V, E)$ be a graph. Let $F = \{e \in E \mid e$ belongs to an induced P_4 of $G\}$. Let $G_p = (V, F)$ be the spanning subgraph of G having F as edge-set. Following Jamison & Olariu [24], a connected component of G_p having exactly one vertex is called a *weak* vertex. Remark that any component of G distinct from a weak vertex contains at least four vertices. Such a component is said to be a *p-component* of G. A graph G is said to be *p-connected* if it has only one p-component and no weak vertices. Remark that G is p-connected if and only if for every partition $A \cup B$ of V some P_4 in G contains vertices from both A and B. Then, G is p-connected if and only if $\overline{G}$ is p-connected. Note also that a p-component is a connected subgraph of G and $\overline{G}$, and that the p-connected components of G are the maximal induced subgraphs which are p-connected.

A p-connected graph $G = (V, E)$ is said to be *separable* if there exists a partition $V_1 \cup V_2$ of V such that each P_4 which contains vertices from both V_1 and V_2 has its midpoints in V_1 and its endpoints in V_2.

Proposition 2.2 ([24]) *The partition $V_1 \cup V_2$ of a separable p-connected graph $G = (V, E)$ is unique.*

Jamison & Olariu give the following general structure for arbitrary graphs.

Theorem 2.3 ([24]) *For a graph G, exactly one of the following conditions is satisfied:*

(a) G is disconnected;

(b) $\overline{G}$ is disconnected;

(c) G is p-connected;

(d) *There exists a unique proper separable p-component H of G with partition $W_1 \cup W_2$ (with crossing P_4s between W_1 and W_2 having their midpoints in W_1) such that every vertex in $V(G) \setminus V(H)$ dominates W_1 and misses W_2.*

Jamison & Olariu [24] define the following operation reflecting condition (d) in Theorem 2.3.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \emptyset$ such that G_1 is separable with partition ${V_1}^1$, ${V_1}^2$ (each P_4 which contains vertices from both ${V_1}^1$ and ${V_1}^2$ has its midpoints in ${V_1}^1$). We consider the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup \{xy \mid x \in {V_1}^1, y \in V_2\}$. We shall denote this graph by G_1 ② G_2.

Theorem 2.4 ([24]) *Every graph G is either p-connected or it can be obtained uniquely from its p-components and weak vertices by a finite sequence of disjoint union ⓪, disjoint sum ①, and preceding operation ②.*

3 P_4-tidy graphs

Concerning the recognition problem of a graph G, the key idea of the method introduced in [9] is to transform this problem into that of recognizing a set of prime graphs associated to G. This method uses as basic data structure for studying recognition and classical optimization graph problems, the modular decomposition tree $T(G)$ of G. In [10] we proposed techniques for studying Hamiltonicity of graphs based on this method. We will apply these techniques to the P_4-tidy graphs, a class containing cographs and all classical families with few P_4s.

3.1 Definitions and main properties

Let $G = (V, E)$ be a graph and let A be a P_4 in G. Let $\mathrm{Mid}(A)$ be the midpoints of A and $\mathrm{End}(A)$ its endpoints. Let us define the following sets:

$$\begin{aligned}
T(A) &= \{v \in V(G) \setminus V(A) \mid V(A) \subseteq N(v)\},\\
I(A) &= \{v \in V(G) \setminus V(A) \mid V(A) \cap N(v) = \emptyset\},\\
P(A) &= \{v \in V(G) \setminus V(A) \mid \mathrm{Mid}(A) \subseteq N(v) \text{ and } \ \mathrm{End}(A) \cap N(v) = \emptyset\},\\
R(A) &= V(G) \setminus (V(A) \cup T(A) \cup P(A) \cup I(A)),\\
S(A) &= \{v \in V(G) \setminus V(A) \mid v \text{ belongs to a } P_4 \text{ sharing vertices with } A\}.
\end{aligned}$$

In other words, $T(A)$ is the set of vertices that dominate A, $I(A)$ is the set of vertices that miss A.

Remark 3.1 *A vertex v belongs to $R(A)$ if and only if $G[V(A) \cup \{v\}]$ is isomorphic to one of the seven graphs $Z_1, \ldots, Z_7$ depicted in Figure 2. We shall say that $R(A)$ is the set of* partners *of A. Then, $R(A) = \{v \in V(G) \setminus V(A) \mid V(A) \cup \{v\}$ induces at least two $P_4\}$.*

In order to illustrate a new notion in perfect graphs (P_4-domination), I. Rusu [30] defines *P_4-tidy graphs*: a graph is a P_4-tidy graph if any induced P_4, A, has at most one partner (that is, $|R(A)| \leq 1$). She notes that the C_5-free P_4-tidy graphs are perfect, since they are weakly triangulated graphs (see [13]).

Recall that a graph G is P_4-sparse [14] if every set of five vertices induces at most one P_4. Clearly, by Remark 3.1, a P_4-sparse graph can be defined as a graph such that for any induced P_4, A, $R(A)$ is empty. Thus, every P_4-sparse graph is a P_4-tidy graph.

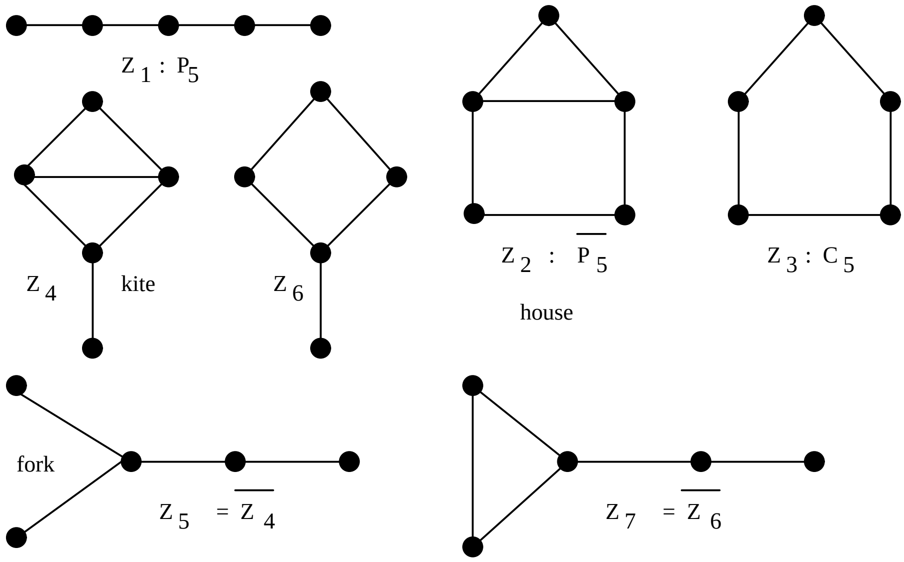

Fig. 2.

Recall also that a graph $G = (V, E)$ is P_4*-extendible* [19] if for every proper subset A of V inducing a P_4, $S(A)$ contains at most one vertex. Since for any P_4, A, $R(A) \subseteq S(A)$, every P_4-extendible graph is a P_4-tidy graph.

Remark 3.2 ([20]) *A graph G is P_4-sparse if and only if G is $(Z_1, \ldots, Z_7)$-free.*

In [23] Jamison & Olariu define P_4*-lite graphs*: a graph G is a P_4-lite graph if every induced subgraph H of G with at most six vertices either contains at most two P_4, or is a 3-sun, or is the Hajós graph (the complement of a 3-sun). They remark that every P_4-sparse graph is a P_4-lite graph and prove that every P_4-lite graph is *brittle* (Chvátal defined a graph G to be brittle if each induced subgraph H of G contains a vertex that is not a midpoint of any P_4 or not an endpoint of any P_4). We shall see now that the family of P_4-tidy graphs and the family of P_4-lite graphs are closely related.

Proposition 3.1 *A graph G is P_4-lite if and only if G is C_5-free P_4-tidy.*

Proof. Let G be a P_4-lite graph. If G is not P_4-tidy then it contains a P_4, A, such that $|R(A)| \geq 2$. Let u and v be two vertices in $R(A)$. Then, $V(A) \cup \{u\}$ induces at least a P_4 containing u (exactly one if $G[V(A) \cup \{v\}]$ is not a C_5, exactly four otherwise). Symetrically, $V(A) \cup \{v\}$ induces at least a P_4 containing v. Thus, $V(A) \cup \{u, v\}$ induces at least three P_4s. Then, $G[V(A) \cup \{u, v\}]$ must be either a 3-sun or its complement. Since $G[V(A) \cup \{u, v\}]$ is an induced subgraph, without loss of generality, either $u \in P(A)$ and $v \in I(A)$, or $u \in P(A)$ and $v \in T(A)$. Thus, we have a contradiction.

Let G be a C_5-free P_4-tidy graph. Let H be an induced subgraph of G with at most six vertices. If $|V(H)| \leq 5$ and H contains an induced P_4 then, since H is not isomorphic to C_5, it induces at most two P_4s. If $|V(H)| = 6$ and H contains a P_4, A, then let u and v be the two vertices in $V(H) \setminus V(A)$.

Case 1: $u \notin R(A)$ and $v \notin R(A)$.

1.1: $uv \notin E(G)$.

If $u \in T(A)$ and $v \in P(A)$ then H is the Hajós graph. Up to symmetries, we see that for the other cases, H induces no P_4 distinct from A.

1.2: $uv \in E(G)$.

If $u \in P(A)$ and $v \in I(A)$ then H is a 3-sun. Up to symmetries, we see as previously that for the other cases, H induces no P_4 distinct from A.

Case 2: $u \in R(A)$ and $v \notin R(A)$ (symmetrically, $u \notin R(A)$ and $v \in R(A)$)

2.1: $uv \notin E(G)$.

We see that, either $G[V(A) \cup \{u\}] \in \{Z_1, Z_2, Z_4, Z_5, Z_6, Z_7\}$ and $v \in I(A)$, or $G[V(A) \cup \{u\}] \in \{Z_5, Z_7\}$ and $v \in P(A)$.

In any other case, $G[V(A) \cup \{u, v\}]$ induces a P_4, B, such that $|R(B)| \geq 2$.

2.2: $uv \in E(G)$.

Then, either $G[V(A) \cup \{u\}] \in \{Z_1, Z_2, Z_4, Z_5, Z_6, Z_7\}$ and $v \in T(A)$, or $G[V(A) \cup \{u\}] \in \{Z_4, Z_6\}$ and $v \in P(A)$.

In any other case, $G[V(A) \cup \{u, v\}]$ induces a P_4, B, such that $|R(B)| \geq 2$.

Thus, either H induces at most two P_4s, or is isomorphic to a 3-sun or its complement, that is, G is a P_4-lite graph. □

Proposition 3.2 *Let $G = (V, E)$ be a P_4-tidy graph. Then every proper subset M of V inducing a subgraph isomorphic to C_5, P_5 or $\overline{P_5}$ is a homogeneous set.*

Proof. Let us suppose that M induces a P_5 and that there exists a vertex v in $V \setminus M$ such that $1 \leq |N(v) \cap M| \leq 4$. It easy to see that in any case we can choose an end-vertex u of $G[M]$ such that $\{u, v\} \subseteq R(M \setminus \{u\})$. Since for any P_4, A, $|R(A)| \leq 1$, we obtain a contradiction. Then, for every v in $V \setminus M$, $N(v) \cap M = \emptyset$ or $M \subseteq N(v)$. Thus, M is a homogeneous set.

The proof is quite analogous if M induces a C_5.

Since a homogeneous set of G is also a homogeneous set of $\overline{G}$, if M induces a $\overline{P_5}$ then M is a homogeneous set of G. □

Let H be any graph isomorphic to one of the four graphs Z_4, Z_5, Z_6 or Z_7 depicted in Figure 2. We shall say that a vertex v in $V(H)$ is an *internal vertex* of H if v is midpoint of a P_4 in H.

Lemma 3.1 *Let G be a P_4-tidy graph. Let H be an induced subgraph of G isomorphic to one of the four graphs Z_4, Z_5, Z_6 or Z_7. Let v be a vertex in $V(G) \setminus V(H)$. Then, either $V(H) \subseteq N(v)$, or $V(H) \cap N(v) = \emptyset$, or v is only adjacent to the internal vertices of H.*

Proof. Let A be a P_4 in H. Let u be the vertex in $V(H \setminus A)$. Clearly, $R(A) = \{u\}$. Then, every vertex $v \in V(G) \setminus V(H)$ is

1. either adjacent to every vertex of A
2. or adjacent to the midpoints and not adjacent to the endpoints of A
3. or adjacent to no vertex of A.

(1) If v is not adjacent to u then we see that $G[V(H) \cup \{v\}]$ contains a P_4, B, containing u such that $|R(B)| \geq 2$. Thus, v is adjacent to every vertex of H.

(2) It is done if H is isomorphic to Z_5 or Z_7. Otherwise, u is midpoint of the P_4, B, of H distinct from A. If v is not adjacent to u then $|R(B)| \geq 2$. Thus, $N(v) \cap V(H)$ is the set of internal vertices of H.

(3) If $N(v) \cap V(H)$ is empty then v is not adacent to u, otherwise we have $|R(B)| \geq 2$ for B, the P_4 in H distinct from A. $\Box$

We are now able to determine the prime P_4-tidy graphs. The following result is due to Jamison and Olariu:

Theorem 3.1 ([20]) *A graph G is a P_4-sparse graph if and only if for every induced subgraph H of G with at least two vertices, exactly one of the following statement is satisfied:*

(a) H is disconnected;

(b) $\overline{H}$ is disconnected;

(c) H is isomorphic to a spider.

Corollary 3.1 *Let G be a prime P_4-sparse graph. Then G is isomorphic to a starfish or to an urchin.*

Theorem 3.2 *Let G be a prime P_4-tidy graph. Then, G is isomorphic to a P_5 or a $\overline{P_5}$ or a C_5 or a starfish or an urchin.*

Proof. Let us suppose that G has an induced subgraph H isomorphic to a P_5 or a $\overline{P_5}$ or a C_5. By Proposition 3.2, either $V(H)$ is a homogeneous set or $G = H$. Since G is prime, $G = H$. Now, suppose that G is distinct from P_5, $\overline{P_5}$ and C_5 and contains an induced subgraph H isomorphic to one of the four graphs Z_4, Z_5, Z_6 or Z_7. By Lemma 3.1, it is easy to see that $V(H)$ contains a homogeneous set of two vertices. This contradiction shows that G is $(Z_1, \ldots, Z_7)$-free, that is, by Remark 3.2, G is a P_4-sparse graph. By Corollary 3.1, G is a starfish or an urchin. $\Box$

Lemma 3.2 *Let G be a P_4-tidy graph and M a neighbourhood module of G. If the representative graph of M is a prime spider H (starfish or urchin) then $G[M]$ is obtained from H by replacing at most one vertex distinct from the head of H by a K_2 or a S_2, and replacing the possible head by the subgraph induced by a module.*

Proof. Let $R = \{r\}$, K and S be respectively the head (if it is not empty), the clique and the stable of the prime spider H. We shall denote by $\{k_1, \ldots, k_l\}$ the vertices of K and by $\{s_1, \ldots, s_l\}$ the vertices of S. For every $i \in \{1, \ldots, l\}$, if H is an urchin then the neighbourhood of s_i is $N(s_i) = \{k_i\}$, and if H is a starfish, $N(s_i) = K \setminus \{k_i, r\}$ ($N(s_i) = K \setminus \{k_i\}$ if R is empty). We know that $G[M]$ is obtained from H by replacing r by a graph A_0, each vertex k_i (respectively s_i) by a graph A_i (resp. B_i) where $V(A_0)$, $V(A_i)$ (resp. $V(B_i)$) are strong modules of G. If $V(A_i)$ (or $V(B_j)$) is a homogeneous set of $G[M] \setminus V(A_0)$ then we can see that there exists a P_4, P, of $G[M] \setminus V(A_0)$ such that $V(P) \cap V(A_i) \neq \emptyset$ (or $V(P) \cap V(B_i) \neq \emptyset$). But this would imply that $|R(P)| \geq |V(A_i)| - 1$ (or $|V(B_i)| - 1$). Thus, $|V(A_i)| \leq 2$ or ($|V(B_i)| \leq 2$). Since every pair of vertices of $H \setminus \{r\}$ belongs to a P_4, at most one vertex in $K \cup S$ can be replaced by K_2 or S_2. $\Box$

Definition 3.1 *A quasi-starfish (resp. quasi-urchin) is a graph obtained from a starfish (resp. urchin) without head by replacing at most one vertex by a K_2 or a S_2. Note that a quasi-starfish (resp. quasi-urchin) is a p-connected graph.*

Remark 3.3 *If the representative graph $G[M]$ of a neighbourhood module of a P_4-tidy graph G is a prime spider then G is isomorphic to the graph arising from a spider (R, K, S) by replacing at most one vertex in $K \cup S$ by a K_2 or a S_2. Then $G[M] \setminus R$ is a quasi-starfish or a quasi-urchin. R will be called the head of $G[M]$. We note that $G[M] = (G[M] \setminus R)$ ② $G[R]$.*

Lemma 3.3 *Let H be a prime spider and G be the graph arising by replacing the possible head of H by a P_4-tidy graph and at most one vertex distinct from the head by a K_2 or a S_2. Then, G is a P_4-tidy graph.*

Proof. By Remark 3.3, G is isomorphic to the graph arising from a spider (R, K, S) by replacing at most one vertex in $K \cup S$ by a K_2 or a S_2. Let

$$K' = \begin{cases} K \cup \{k'\} & \text{if } G \text{ is obtained by replacing } k \text{ in } K \text{ by } \{k, k'\} \\ K & \text{otherwise} \end{cases}$$
$$S' = \begin{cases} S \cup \{s'\} & \text{if } G \text{ is obtained by replacing } s \text{ in } S \text{ by } \{s, s'\} \\ S & \text{otherwise.} \end{cases}$$

Since every vertex of R dominates K' and every vertex of K' dominates R, it is easy to see that there is no P_4 containing vertices from both R and $K' \cup S'$, that every P_4 in R has no partner in K' and that, since every P_4 in $K' \cup S'$ has its midpoints in K' and its endpoints in S', it has no partner in R. Thus, every P_4 in G has at most one partner. □

Lemma 3.4 *Let G be a P_4-tidy graph and M be a neighbourhood module such that the representative graph of M is a prime spider. Let R be the head of $G[M]$. Then, $G[M] \setminus R$ is a p-component of G.*

Proof. By Remark 3.3, $G[M] \setminus R$ is a p-connected induced subgraph of G. By the proof of Lemma 3.3, we know that there is no P_4 containing vertices from both R and $G[M] \setminus R$. By using modular decomposition of G, we see that $G[M] \setminus R$ is a maximal induced p-connected subgraph. □

Proposition 3.3 *A graph G is P_4-tidy if and only if every p-component is isomorphic to either a P_5 or a $\overline{P_5}$ or a C_5 or a quasi-starfish or a quasi-urchin. Quasi-starfishes and quasi-urchins are the separable p-components of G.*

Proof. Suppose that G is P_4-tidy. Let B be a p-component of G. By Proposition 3.1, the set of vertices of a subgraph B isomorphic to a P_5 or a $\overline{P_5}$ or a C_5 is a homogeneous set. Since every P_4 has at most one partner, there no P_4 containing vertices from both $V(B)$ and $V(G) \setminus V(B)$. Thus, B is a p-component.

By Lemma 3.4, the other p-components are quasi-starfishes and quasi-urchins. Then, the 'only if' part is true.

The 'if' part is a consequence of Theorem 2.4. □

Corollary 3.2 ([15]) *A graph G is P_4-extendible if and only if it has no p-component of order greater than 5.*

Proof. Note that the seven graphs given in Figure 2 are the p-connected graphs on five vertices. It is clear that if G has no component of order greater than 5 then G is P_4-extendible. Conversely, if a p-component B of a P_4-extendible graph G is distinct from Z_1, Z_2 and Z_3 then, by Proposition 3.2, B is isomorphic to a quasi-starfish or a quasi-urchin. Since for every P_4 A of a 3-sun (or of its complement) $|S(A)| = 2$, we see that the prime representative graph P of B is a P_4 or a bull. Since a bull is not p-connected, P is a P_4. Then B belongs to $\{Z_4, Z_5, Z_6, Z_7\}$. □

3.2 Linear recognition

Let us consider a graph G. Knowledge of $\pi(G)$ is not sufficient for characterizing G. Let r be a neighbourhood node of the modular decomposition tree $T(G)$, $M(r)$ the corresponding module of G, $G(r)$ the representative prime graph and $V(r) = \{r_1, \ldots, r_k\}$ be the set of children of r in $T(G)$ (vertex set of $G(r)$).

During the construction of a graph $G(r)$ of $\pi(G)$ from $T(G)$, let us mark every vertex r_i whose corresponding module is a homogeneous set. A graph of $\pi(G)$ having no marked vertex is said to be *unmarked*.

Suppose that G is P_4-tidy. By Theorem 3.2, every graph in $\pi(G)$ is isomorphic to a P_5 or a $\overline{P_5}$ or a C_5 or a starfish or an urchin. Then, by Proposition 3.1 and Lemma 3.2, for every neighbourhood node r of $T(G)$, $G(r)$ must verify one of the following statements:

(a) $G(r)$ is isomorphic to a C_5 and $G(r)$ is unmarked.

(b) $G(r)$ is isomorphic to a P_5 and $G(r)$ is unmarked.

(c) $G(r)$ is isomorphic to a $\overline{P_5}$ and $G(r)$ is unmarked.

(d) $G(r)$ is isomorphic to a starfish or an urchin, the head is possibly marked (the corresponding module can be any P_4-tidy graph) and at most one other vertex is marked (the corresponding module contains exactly two vertices).

If $G(r)$ verifies condition (d) then we mark (if necessary) the head by the usual mark and the other marked vertex (if such a vertex exists) by a special mark $*$ and $G(r)$ is said to be *weak-marked*. We shall consider that an unmarked prime spider is weak-marked.

Then we obtain the following characterization.

Theorem 3.3 *A graph G is P_4-tidy if and only if every graph of $\pi(G)$ is either an unmarked P_5 or an unmarked $\overline{P_5}$ or an unmarked C_5 or a weak-marked urchin or a weak-marked starfish.*

Proof. By Proposition 3.1, Theorem 3.2 and Lemma 3.2, the 'only if' part is true. Since P_5, $\overline{P_5}$ and C_5 are P_4-tidy graphs, the 'if' part is obtained by induction on $|V(G)|$ and Lemma 3.3. □

Corollary 3.3 ([11]) *A graph G is P_4-reducible if and only if every graph of $\pi(G)$ is either an unmarked P_4 or a bull having at most one marked vertex: its vertex of degree 2.*

Corollary 3.4 ([11]) *A graph G is extended P_4-reducible if and only if every graph of $\pi(G)$ is either an unmarked C_5 or an unmarked P_4 or a bull having at most one marked vertex: its vertex of degree 2.*

Corollary 3.5 ([9]) *A graph G is P_4-sparse if and only if every graph of $\pi(G)$ is either an unmarked urchin or an unmarked starfish.*

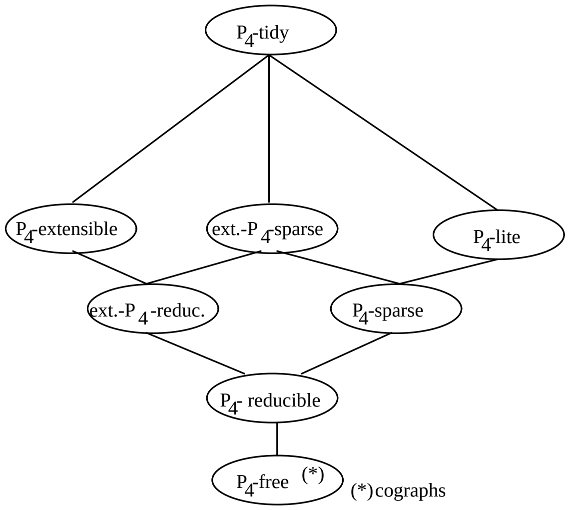

Fig. 3. A Hasse diagram.

Corollary 3.6 ([9]) *A graph G is extended P_4-sparse if and only if every graph of $\pi(G)$ is either an unmarked C_5 or an unmarked urchin or an unmarked starfish.*

Corollary 3.7 *A graph G is P_4-extendible if and only if every graph of $\pi(G)$ is either a weak-marked P_4 or an unmarked P_5 or an unmarked $\overline{P_5}$ or an unmarked C_5 or a weak-marked bull.*

Corollary 3.8 *A graph G is P_4-lite if and only if every graph of $\pi(G)$ is either an unmarked P_5 or an unmarked $\overline{P_5}$ or a weak-marked urchin or a weak-marked starfish.*

The previous classes are partially ordered by inclusion. We sum up the situation by a Hasse diagram given in Figure 3.

Recognition algorithm

Let H be a graph and let us define type(H) in the following way:

$$\text{type}(H) = \begin{cases} 1 & \text{if } H \text{ is isomorphic to a } P_4 \\ 2 & \text{if } H \text{ is isomorphic to a bull} \\ 3 & \text{if } H \text{ is isomorphic to a spider distinct from a } P_4 \text{ and a bull} \\ 4 & \text{if } H \text{ is isomorphic to a } C_5 \\ 5 & \text{if } H \text{ is isomorphic to a } P_5 \\ 6 & \text{if } H \text{ is isomorphic to a } \overline{P_5} \\ 7 & \text{otherwise.} \end{cases}$$

Now, we define a function M on $\{1, \ldots, 6\} \times \pi(G)$ in the following way:

$$M(1, H) = \begin{cases} 0 & \text{if } H \text{ is not isomorphic to a } P_4 \\ 1 & \text{if } H \text{ is isomorphic to an unmarked } P_4 \\ 2 & \text{if } H \text{ is isomorphic to a weak-marked } P_4 \\ 3 & \text{if } H \text{ is isomorphic to a marked } P_4 \text{ (not weak-marked)} \end{cases}$$

$$M(2,H) = \begin{cases} 0 & \text{if } H \text{ is not isomorphic to a bull} \\ 1 & \text{if } H \text{ is isomorphic to a bull with unmarked } P_4 \\ 2 & \text{if } H \text{ is isomorphic to a weak-marked bull} \\ 3 & \text{if } H \text{ is isomorphic to a marked bull (not weak-marked)} \end{cases}$$

$$M(3,H) = \begin{cases} 0 & \text{if } H \text{ is not isomorphic to a spider} \\ 1 & \text{if } H \text{ is isomorphic to an unmarked spider} \\ 2 & \text{if } H \text{ is isomorphic to a weak-marked spider} \\ 3 & \text{if } H \text{ is isomorphic to a marked spider (not weak-marked)} \end{cases}$$

$$M(4,H) = \begin{cases} 0 & \text{if } H \text{ is not isomorphic to a } C_5 \\ 1 & \text{if } H \text{ is isomorphic to an unmarked } C_5 \\ 2 & \text{if } H \text{ is isomorphic to a marked } C_5 \end{cases}$$

$$M(5,H) = \begin{cases} 0 & \text{if } H \text{ is not isomorphic to a } P_5 \\ 1 & \text{if } H \text{ is isomorphic to an unmarked } P_5 \\ 2 & \text{if } H \text{ is isomorphic to marked } P_5 \end{cases}$$

$$M(6,H) = \begin{cases} 0 & \text{if } H \text{ is not isomorphic to a } \overline{P_5} \\ 1 & \text{if } H \text{ is isomorphic to an unmarked } \overline{P_5} \\ 2 & \text{if } H \text{ is isomorphic to a marked } \overline{P_5}. \end{cases}$$

Input: A graph G.

Output: The message 'G is MSG' (with MSG $\in$ {a cograph, P_4-reducible, extended P_4-reducible, P_4-sparse, extended P_4-sparse, P_4-extendible, P_4-lite, P_4-tidy, not P_4-tidy}).

Step 1:

Construct the modular decomposition tree T(G), the set $\pi(G)$ and mark the vertices of every graph in $\pi(G)$ as previously explained. Let $\pi(G) = \{G_1, \dots, G_p\}$.

Step 2:

[1] For each $j \in \{1, \dots, 7\}$ do MASK[j] := 0
[2] For each $G_i \in \pi(G)$ do
[3] if $1 \le$ type(G_i) ≤ 6 then
[4] MASK[type(G_i)] := max{M(type(G_i), G_i), MASK[type(G_i)]}
[5] else MASK[7] := 1
[6] The array MASK is compared with the rows of the table given in Figure 4.

In Figure 4, 'spider' means 'spider distinct from a P_4 and a bull'.

By Theorem 3.3 and Corollaries 3.3 to 3.8, the correctness of the preceding algorithm is easy to prove.

Complexity

Let $n = |V(G)|$ and $m = |E(G)|$. Step 1 can be done in $O(m+n)$ time (see [2], [5] and [33]).

Line [1] of Step 2 is accomplished in constant time. For each G_i in $\pi(G)$ we can sort the vertices of G_i by increasing order of their degrees in $O(|V(G_i)|)$ time by an usual technique of sorting integers having values between 0 and $(|V(G_i)|)$ (by bucketsort for instance). Testing if G_i is either an unmarked P_5 or

P_4	bull	spider	C_5	P_5	$\overline{P_5}$	other	MSG
0	0	0	0	0	0	0	cograph
≤ 1	≤ 1	0	0	0	0	0	P_4-red.
≤ 1	≤ 1	0	1	0	0	0	ext. P_4-red.
≤ 1	≤ 1	≤ 1	0	0	0	0	P_4-sparse
≤ 1	≤ 1	≤ 1	1	0	0	0	ext. P_4-sparse
≤ 2	≤ 2	0	≤ 1	≤ 1	≤ 1	0	P_4-extens.
≤ 2	≤ 2	≤ 2	0	≤ 1	≤ 1	0	P_4-lite
≤ 2	≤ 2	≤ 2	≤ 1	≤ 1	≤ 1	0	P_4-tidy

Fig. 4.

an unmarked $\overline{P_5}$ or an unmarked C_5 is done in constant time. Testing if G_i is a weak-marked urchin or a weak-marked starfish can be decomposed in three steps:

1. Verify if G_i is a split-graph $K_i + S_i$ (where K_i is a clique and S_i is a stable). This can be done in $\mathrm{O}(|V(G_i)|)$ time (see [12]).

2. Check if all vertices of S_i are either of degree 1 or $|K_i| - 1$ or $|K_i| - 2$ (recall that G_i has no homogeneous set).

3. Test if there is at most one vertex in $K_i \cup S_i$ weak-marked.

These three steps are done in $\mathrm{O}(|V(G_i)|)$ time. It is easy to see that $\sum_{i=1}^{p} |V(G_i)| < 2n$. Thus, lines [2] to [5] are accomplished in $\mathrm{O}(n)$ time. Line [6] is accomplished in constant time. It follows that the time complexity of Step 2 is $\mathrm{O}(n)$ as claimed.

3.3 Optimization algorithms

We will show that the modular decomposition tree $T(G)$ of a P_4-tidy graph G and the set $\pi(G)$ can be used to obtain linear time solutions to a number of classical combinatorial optimization problems.

We consider the following parameters:

$\omega(G)$, *the clique number of* G (maximum number of pairwise adjacent vertices),

$\chi(G)$, *the chromatic number of* G (smallest number of stables which cover all the vertices),

$\alpha(G)$, *the stability number of* G (maximum number of pairwise nonadjacent vertices),

$\theta(G)$, *the clique cover number of* G (smallest number of cliques which cover all the vertices).

It is well known that each of the problems of recognizing graphs G and integers k with $\omega(G) \geq k$, $\chi(G) \leq k$, $\alpha(G) \geq k$, and $\theta(G) \leq k$ is *NP*-complete (see [8]). We propose to solve the above optimizations problems for the class of P_4-tidy graphs. More precisely, given the modular decomposition tree of a P_4-tidy graph G, we show how to compute parameters $\omega(G)$, $\chi(G)$, $\alpha(G)$ and $\theta(G)$.

Since $\alpha(G) = \omega(\overline{G})$ and $\theta(G) = \chi(\overline{G})$, and since the complement $\overline{G}$ of any P_4-tidy graph G is a P_4-tidy graph, we shall only compute $\omega(G)$ and $\chi(G)$.

Lemma 3.5 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \emptyset$. Then*

1. $\omega(G_1 \textcircled{0} G_2) = \max\{\omega(G_1), \omega(G_2)\}$,
2. $\omega(G_1 \textcircled{1} G_2) = \omega(G_1) + \omega(G_2)$,
3. $\chi(G_1 \textcircled{0} G_2) = \max\{\chi(G_1), \chi(G_2)\}$,
4. $\chi(G_1 \textcircled{1} G_2) = \chi(G_1) + \chi(G_2)$.

Let G be a P_4-tidy graph and let r be an internal node of the modular decomposition tree $T(G)$, $M(r)$ be the corresponding module of G, $G(r)$ be the representative graph and $V(r) = \{r_1, \dots, r_k\}$ be the set of children of r in $T(G)$.

Case 1: $M(r)$ is a parallel module (r is a P-node).
By Lemma 3.5 (1)(3), we have

$$\begin{aligned} \omega(G[M(r)]) &= \max\{\omega(G[M(r_1)]), \dots, \omega(G[M(r_k)])\} \\ \chi(G[M(r)]) &= \max\{\chi(G[M(r_1)]), \dots, \chi(G[M(r_k)])\} \end{aligned}$$

Case 2: $M(r)$ is a series module (r is an S-node).
By Lemma 3.5 (2)(4), we have

$$\begin{aligned} \omega(G[M(r)]) &= \sum_{i=1}^{k} \omega(G[M(r_i)]) \\ \chi(G[M(r)]) &= \sum_{i=1}^{k} \chi(G[M(r_i)]). \end{aligned}$$

Case 3: $M(r)$ is a neighbourhood module (r is an N-node).
By Theorem 3.5, we shall consider two subcases:

3.1: $G(r)$ is an unmarked P_5 or $\overline{P_5}$ or C_5. ($G[M(r)]$ is isomorphic to P_5 or $\overline{P_5}$ or C_5).
Then, $\omega(P_5) = 2$, $\omega(\overline{P_5}) = 3$, $\omega(C_5) = 2$, $\chi(P_5) = 2$, $\chi(\overline{P_5}) = 3$, $\chi(C_5) = 3$.

3.2: $G(r)$ is a weak-marked urchin or a weak-marked starfish.
By Remark 3.3, $G[M(r)]$ is isomorphic to $H_1 \textcircled{2} H_2$ where H_1 is a quasi-urchin or a quasi-starfish and $V(H_2)$ is empty or is a strong submodule of $M(r)$. By Lemma 3.2, H_1 is a prime spider without head (K, S) or is obtained from a prime spider without head $H = (K, S)$ by replacing exactly one vertex in $K \cup S$ by a S_2 or a K_2. Let

$$\begin{aligned} \epsilon &= \begin{cases} 1 & \text{if } H_1 \text{ is obtained by replacing a vertex of } K \text{ by a } K_2 \\ 0 & \text{otherwise} \end{cases} \\ \epsilon' &= \begin{cases} 1 & \text{if } H \text{ is a starfish and } H_1 \text{ is obtained by replacing a vertex of } S \text{ by a } K_2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Note that a P_4 is a starfish. Then, $\omega(H_1) = \chi(H_1) = |K| + \epsilon + \epsilon'$ and we have

$$\begin{aligned} \omega(G[M(r)]) &= |K| + \epsilon + \max\{\omega(H_2), \epsilon'\} \\ \chi(G[M(r)]) &= |K| + \epsilon + \max\{\chi(H_2), \epsilon'\} \end{aligned}$$

Clique number and Chromatic number

In their paper Corneil *et al.* [3] solved the previous problems for cographs by performing a certain computation on the corresponding cotree, thus reducing the problem to evaluating an expression on the cotree. This method can be generalized to the modular decomposition tree $T(G)$. For a given parameter, clique number or chromatic number, we associate an expression to each type of internal node (given in Cases 1, 2, 3.1 and 3.2). More precisely, we associate to $T(G)$ a new tree $T'(G)$ obtained in the following way:

- P-nodes and S-nodes are unchanged (Cases 1 and 2),
- an N-node corresponding to an unmarked P_5 or $\overline{P_5}$ or C_5 (Case 3.1) becomes a leaf of $T'(G)$ with its attributes ω and χ,
- an N-node r corresponding to a weak-marked urchin or a weak-marked starfish (Case 3.2) has two sons: a leaf H_1 (quasi-urchin or quasi-starfish) with its attributes $|K|$, ϵ and ϵ', and an internal node corresponding to the strong submodule $V(H_2)$ of $M(r)$ previously described.

The parameters $\omega(G)$ and $\chi(G)$ can be evaluated by traversing $T'(G)$ in postorder and performing the prescribed expression at every internal node as described in the previous study of Cases 1, 2 and 3.

Complexity

$T'(G)$ is a subtree of $T(G)$ and is obtained from $T(G)$ in $O(|V(T(G))|)$ time. It is easy to see that $|V(T(G))| < 2n$. Since the postorder traversal of $T'(G)$ is done in $O(|V(T'(G))|)$ time, $\omega(G)$ and $\chi(G)$ are computable in $O(n)$ time.

3.4 Hamiltonicity

As previously, let $s(G)$ be the scattering number of a graph G and $\rho(G)$ be the minimum path partition number of G. We recall here three lemmas that we shall use in this section.

Lemma 3.6 ([15]) *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \emptyset$. Then*

1. $\rho(G_1 \, ⓪ \, G_2) = \rho(G_1) + \rho(G_2)$,
2. $\rho(G_1 \, ① \, G_2) = \max\{\rho(G_2) - |V_1|, \rho(G_1) - |V_2|, 1\}$.

Lemma 3.7 ([15]) *Let $G = (R, K, S)$ be a spider with head R, clique K and stable S. If R is not empty or if G has thin legs then $s(G) = \max\{s(R), 1\}$, else $s(G) = 0$. If G has thin legs then*

$$\rho(G) = \rho(R) + \left\lceil \max\left\{0, \frac{|K| - 2\rho(R)}{2}\right\} \right\rceil.$$

If G has thick legs then, if R is not empty, we have $\rho(G) = \rho(R)$, otherwise G is Hamiltonian.

Lemma 3.8 ([25]) *Let $G = (V, E)$ be a graph and S be a scattering set of G. Then, for any subset A of V, $s(G \setminus A) \leq s(G) + |A|$. Moreover, if A is a subset of S then $s(G \setminus A) = s(G) + |A|$.*

Let G be a P_4-tidy graph and $T(G)$ be the modular decomposition tree of G. In the following, we shall construct a boolean function B defined on the set of internal nodes of $T(G)$ and verifying the following property:

For every internal node r of $T(G)$, $(B(r) = \text{TRUE}) \Rightarrow G[M(r)]$ is a Jung graph.

(Since it is not always obvious to show that a graph is a Jung graph, we shall only consider such a boolean function.)

Let r be an internal node of the modular decomposition tree $T(G)$. As previously, we denote by $M(r)$ the corresponding module of G, by $G(r)$ the representative graph and by $V(r) = \{r_1, \dots, r_k\}$ the set of children of r in $T(G)$. Let $n(r)$ (resp. $n(r_i)$) be the number of vertices of $G[M(r)]$ (resp. $G[M(r_i)]$).

Case 1: $M(r)$ is a parallel module.

By Lemma 2.2 (2), Lemma 3.6 (2) and Proposition 2.1, we have

$$\begin{aligned} s(G[M(r)]) &= \max_{i\in\{1,\dots,k\}} \{s(G[M(r_i)]) - \sum_{j\neq i} n(r_j)\} \\ \rho(G[M(r)]) &= \max_{i\in\{1,\dots,k\}} \{\rho(G[M(r_i)]) - \sum_{j\neq i} n(r_j), 1\} \end{aligned}$$

Let $B(r) = B(r_1) \wedge \cdots \wedge B(r_k)$.

Case 2: $M(r)$ is a series module.

By Lemma 2.2 (1), Lemma 3.6 (1) and Proposition 2.1, we have

$$\begin{aligned} s(G[M(r)]) &= \sum_{i=1}^{k} s(G[M(r_i)]) \\ \rho(G[M(r)]) &= \sum_{i=1}^{k} \rho(G[M(r_i)]). \end{aligned}$$

Let $B(r) = B(r_1) \wedge \cdots \wedge B(r_k)$.

Case 3: $M(r)$ is a neighbourhood module.

3.1: $G[M(r)]$ is isomorphic to P_5 or $\overline{P_5}$ or C_5.

We know that, $s(P_5) = 1$, $s(\overline{P_5}) = 2$, $s(C_5) = 2$, $\rho(P_5) = 1$, $\rho(\overline{P_5}) = 1$, $\rho(C_5) = 1$ and we set $B(r) = \text{TRUE}$.

3.2: $G(r)$ is a weak-marked spider.

We know that $G[M(r)]$ is isomorphic to $H_1 ② H_2$ where H_1 is a quasi-urchin or a quasi-starfish and $V(H_2)$ is empty or is a strong submodule of $M(r)$. We recall that H_1 is a prime spider without head (K, S) or is obtained from a prime spider without head (K, S) by replacing exactly one vertex in $K \cup S$ by a S_2 or a K_2.

Set $K = \{k_1, \dots, k_l\}$, $S = \{s_1, \dots, s_l\}$. We have $H_1 = G[K' \cup S']$ with $K' \cup S'$ obtained from $K \cup S$ by replacing at most one vertex in $K \cup S$ by a S_2 or a K_2. Without loss of generality, we can set

$$K' = \begin{cases} K \cup \{k_1'\} & \text{if } H_1 \text{ is obtained by replacing } k_1 \text{ in } K \text{ by } \{k_1, k_1'\} \\ K & \text{otherwise.} \end{cases}$$

$$S' = \begin{cases} S \cup \{s'_1\} & \text{if } H_1 \text{ is obtained by replacing } s_1 \text{ in } S \text{ by } \{s_1, s'_1\} \\ S & \text{otherwise.} \end{cases}$$

In order to simplify notations, set $H = G[M(r)]$. In the following, if $H_2 = \emptyset$ then we set $\rho(H_2) = 0$. Let $p = \rho(H_2)$ and $\mathcal{Q} = \{Q_1, \dots, Q_p\}$ be a minimum path partition of $V(H_2)$. For every $i \in \{1, \dots, p\}$, let q_i and q'_i be the end-vertices of Q_i (if $|V(Q_i)| = 1$ then $q_i = q'_i$).

If $H_2 \neq \emptyset$, let

$$A_2 = \begin{cases} \emptyset & \text{if } s(H_2) \leq 0 \\ \text{a scattering set of } H_2 & \text{if } s(H_2) \geq 1. \end{cases}$$

3.2.1: $G(r)$ is a weak-marked urchin.

3.2.1.1: $S' = S$ and $K' = K$.

If $H_2 \neq \emptyset$ (resp. $H_2 = \emptyset$) then, by Lemma 3.7, $s(H) = \max(1, s(H_2))$ (resp. $s(H) = 1$) and

$$\rho(H) = \rho(H_2) + \left\lceil \max\left(0, \frac{|K| - 2\rho(H_2)}{2}\right) \right\rceil.$$

If H_2 is a non-empty Jung graph (resp. $H_2 = \emptyset$) then $s(H) = \rho(H_2)$ (resp. $s(H) = 1$). Moreover, if $|K| \leq 2\rho(H_2)$ (resp. $|K| = 2$) then $\rho(H) = \rho(H_2)$ (resp. $\rho(H) = 1$). Thus, $\rho(H) = s(H)$ and H is a Jung graph.

Then, if H_2 is a non-empty Jung graph (resp. $H_2 = \emptyset$) and $|K| \leq 2\rho(H_2)$ (resp. $|K| = 2$), we set $B(r) = \text{TRUE}$, otherwise we set $B(r) = \text{FALSE}$.

3.2.1.2: $S' = S \cup \{s'_1\}$ and $s_1 s'_1 \notin E(H)$.

If $H_2 \neq \emptyset$ then we can see that $K \cup A_2$ is a scattering set of H. Then, by Lemma 3.8, $s(H) = s(H \setminus K) - |K|$. By Lemma 2.2, $s(H) = \max(1, s(H_2)) + \max(1, s(S')) - |K|$. Thus, $s(H) = \max(1, s(H_2)) + |S'| - |K|$. That is, $s(H) = \max(1, s(H_2)) + 1$.

If $H_2 = \emptyset$ then we can see that $\{k_1\}$ is a scattering set of H and $s(H) = 2$.

If $|K| = 2$ then $\{[s_1, k_1, s'_1], Q_1 \cup [q_1, k_2, s_2], Q_2, \dots, Q_p\}$ (or $\{[s_1, k_1, s'_1], [k_2, s_2]\}$ if $H_2 = \emptyset$) is a minimum path partition of $V(H)$. Then, $\rho(H) = \rho(H_2) + 1$ (or $\rho(H) = 2$ if $H_2 = \emptyset$).

If $|K| > 2$ then let $H' = H_2 \textcircled{2} (H_1 \setminus \{s_1, s'_1, k_1\})$. Clearly, H' verifies the hypotheses of Case 3.2.1.1, then

$$\rho(H') = \rho(H_2) + \left\lceil \max\left(0, \frac{|K| - 1 - 2\rho(H_2)}{2}\right) \right\rceil.$$

Since k_1 is the unique neighbour of s_1 and s'_1 in H and $s_1 s'_1 \notin E(H)$, adding the path $[s_1, k_1, s'_1]$ to a minimum path partition of $V(H')$ gives a minimum path partition of $V(H)$. Then, $\rho(H) = \rho(H') + 1$. Thus

$$\rho(H) = \rho(H_2) + \left\lceil \max\left(0, \frac{|K| - 1 - 2\rho(H_2)}{2}\right) \right\rceil + 1.$$

If H_2 is a non-empty Jung graph (resp. $H_2 = \emptyset$) and $|K| \leq 2\rho(H_2) + 1$ (resp. $|K| \leq 3$), by similar arguments to those of Case 3.2.1.1, we can set $B(r) = \text{TRUE}$, otherwise we set $B(r) = \text{FALSE}$.

3.2.1.3: $S' = S \cup \{s'_1\}$ and $s_1 s'_1 \in E(H)$.

If $H_2 \neq \emptyset$ (resp. $H_2 = \emptyset$) then we can see that $K \cup A_2$ (resp. $\{k_2\}$) is a scattering set of H. Then, by Lemmas 3.8 and 2.2, it is easy to prove, as previously, that $s(H) = \max(1, s(H_2))$ (resp. $s(H) = 1$).

Let $H' = H_2 \textcircled{2} (H_1 \setminus \{s'_1\})$. Clearly, H' verifies the hypotheses of Case 3.2.1.1. Then

$$\rho(H') = \rho(H_2) + \left\lceil \max\left(0, \frac{|K| - 2\rho(H_2)}{2}\right) \right\rceil.$$

Let $\mathcal{P}$ a minimum path partition of $V(H')$. Since the degree of s_1 in H' is one, s_1 is necessary an end-vertex of a path P_0 in $\mathcal{P}$. Thus, $(\mathcal{P} \setminus P_0) \cup \{P_0 \cup [s_1, s'_1]\}$ is a minimum path partition of $V(H)$. Then

$$\rho(H) = \rho(H') = \rho(H_2) + \left\lceil \max\left(0, \frac{|K| - 2\rho(H_2)}{2}\right) \right\rceil.$$

As in Case 3.2.1.1, if H_2 is a non-empty Jung graph (resp. $H_2 = \emptyset$) and $|K| \leq 2\rho(H_2)$ (resp. $|K| = 2$), we set $B(r) = \text{TRUE}$, otherwise we set $B(r) = \text{FALSE}$.

3.2.1.4: $K' = K \cup \{k'_1\}$.

If $H_2 \neq \emptyset$ and $s(H_2) > 1$ (resp. $H_2 = \emptyset$ or $s(H_2) = 1$) then $K \cup A_2$ (resp. $\{k_2\}$) is a scattering set. Then, by Lemmas 3.8 and 2.2, $s(H) = \max(1, s(H_2)) - 1$ (resp. $s(H) = 1$).

Claim 3.1 *If* $|K| \geq 2\rho(H_2)$ *then* $\rho(H) = \lfloor \frac{|K|}{2} \rfloor$ *else* $\rho(H) = \rho(H_2) - 1$.

Proof.

a: $H_2 = \emptyset$.

If $|K| = 2$ then $[s_2, k_2, k_1, s_1, k'_1]$ is a Hamiltonian path of H.

If $|K|$ is odd (resp. even ≥ 4) let $|K| = 2t - 1$ (resp. $|K| = 2t$). Then,

$$\mathcal{P} = \{[s_2, k_2, k_1, s_1, k'_1, k_3, s_3], [s_4, k_4, k_5, s_5], \ldots, [s_{2t-2}, k_{2t-2}, k_{2t-1}, s_{2t-1}]\}$$

(resp. $\mathcal{P} \cup \{[s_{2t}, k_{2t}]\}$) is a minimum path partition of $V(H)$.

b: $H_2 \neq \emptyset$ and $|K| \geq 2\rho(H_2)$.

If $|K|$ is even (resp. odd), then let $2t$ (resp. $2t + 1$) be the value of $|K|$.

Then, if $p = |\mathcal{Q}| \geq 2$,

$$\begin{aligned}\mathcal{P} \quad = \quad & \{[s_2, k_2, q_1] \cup Q_1 \cup [q'_1, k_1, s_1, k'_1, q_2] \cup Q_2 \cup [q'_2, k_3, s_3], [s_4, k_4, q_3] \cup Q_3 \\ & \cup [q'_3, k_5, s_5], \ldots, [s_{2p-2}, k_{2p-2}, q_p] \cup Q_p \\ & \cup [q'_p, k_{2p-1}, s_{2p-1}], [s_{2p}, k_{2p}, k_{2p+1}, s_{2p+1}], \ldots, [s_{2t-2}, k_{2t-2}, k_{2t-1}, s_{2t-1}], [s_{2t}, k_{2t}]\}\end{aligned}$$

(resp. $(\mathcal{P} \setminus \{[s_{2t}, k_{2t}]\}) \cup \{[s_{2t}, k_{2t}, k_{2t+1}, s_{2t+1}]\}$) is a minimum path partition of $V(H)$.

If $|\mathcal{Q}| = 1$ then

$$\begin{aligned}\mathcal{P} \quad = \quad & \{[s_2, k_2, q_1] \cup Q_1 \\ & \cup [q'_1, k_1, s_1, k'_1, k_3, s_3], [s_4, k_4, k_5, s_5], \ldots, [s_{2t-2}, k_{2t-2}, k_{2t-1}, s_{2t-1}], [s_{2t}, k_{2t}]\}\end{aligned}$$

(resp. $(\mathcal{P} \setminus \{[s_{2t}, k_{2t}]\}) \cup \{[s_{2t}, k_{2t}, k_{2t+1}, s_{2t+1}]\}$) is a minimum path partition of $V(H)$.

c: $H_2 \neq \emptyset$ and $|K| < 2\rho(H_2)$.

If $|K| = 2$ then $\{[s_2, k_2, q_1] \cup Q_1 \cup [q_1', k_1, s_1, k_1', q_2] \cup Q_2, Q_3, \ldots, Q_p\}$ is a minimum path partition of H.

If $|K|$ is odd (resp. even ≥ 4) then let $2t - 1$ (resp. $2t$) be the value of $|K|$. Then

$$\begin{aligned}\mathcal{P} \quad = \quad & \{[s_2, k_2, q_1] \cup Q_1 \cup [q_1', k_1, s_1, k_1', q_2] \cup Q_2 \cup [q_2', k_3, s_3], [s_4, k_4, q_3] \cup Q_3 \\ & \cup [q_3', k_5, s_5], \ldots, [s_{2t-2}, k_{2t-2}, q_t] \cup Q_t \cup [q_t', k_{2t-1}, s_{2t-1}], Q_{t+1}, \ldots, Q_p\}\end{aligned}$$

(resp. $(\mathcal{P} \setminus \{Q_{t+1}\}) \cup \{[s_{2t}, k_{2t}, q_{t+1}] \cup Q_{t+1}\}$) is a minimum path partition of $V(H)$.

The reader can verify that in each case we obtain the announced formulas. $\square$

By Claim 3.1, if H_2 is a non-empty Jung graph (resp. $H_2 = \emptyset$) and $|K| < 2\rho(H_2)$ (resp. $|K| \leq 3$), we set $B(r) = \text{TRUE}$, otherwise we set $B(r) = \text{FALSE}$.

3.2.2: $G(r)$ is a weak-marked starfish.

If $|K| = 2$ then we are in Case 3.2.1, so we suppose that $|K| \geq 3$. Let us remark that every scattering set contains necessarily $|K| - 1$ vertices of K. Let $t = \lceil \frac{|K|}{2} \rceil$.

We describe two particular Hamiltonian paths of $G[K \cup S]$ that shall be used in the following:

If $|K| > 6$ and K is odd (resp. even) then let

$$P = [k_l, s_1, k_{l-1}, s_2, \ldots, k_t, s_{t+1}, k_{t-1}, s_t, k_{t-2}, s_{t+2}, \ldots, k_2, s_l, k_1, s_{l-1}]$$

(resp. $P = [k_l, s_1, k_{l-1}, s_2, \ldots, k_{t+1}, s_t, k_{t-1}, s_{t+1}, k_t, s_{t+2}, k_{t-2}, s_{t+3}, \ldots, k_2, s_l, k_1, s_{l-1}]$).

Then, P is a Hamiltonian path of $G[K \cup S]$ joining k_l and s_{l-1}. For $|K| \leq 6$, it is easy to construct such a Hamiltonian path P.

By permuting $\{k_1, s_1\}$ and $\{k_{l-1}, s_{l-1}\}$, we obtain from P a Hamiltonian path P' of $G[k \cup S]$ joining k_l and s_1.

3.2.2.1: $S' = S$ and $K' = K$.

If $H_2 \neq \emptyset$ (resp. $H_2 = \emptyset$) then $K \cup A_2$ (resp. K) is a scattering set. By Lemma 3.7, $s(H) = \max(1, s(H_2))$ (resp. $s(H) = 0$), $\rho(H) = \rho(H_2)$ (resp. $\rho(H) = 1$).

If $H_2 = \emptyset$ then $P \cup [k_l, s_{l-1}]$ is a Hamiltonian cycle of H.

If $H_2 \neq \emptyset$ and H_2 is a Jung graph then, since $\rho(H) = \rho(H_2)$, $\rho(H) = s(H)$.

Then, if $H_2 = \emptyset$ or H_2 is a non-empty Jung graph, we set $B(r) = \text{TRUE}$; otherwise, we set $B(r) = \text{FALSE}$.

3.2.2.2: $S' = S \cup \{s_1'\}$ and $s_1 s_1' \notin E(H)$.

If $H_2 \neq \emptyset$ (resp. $H_2 = \emptyset$) then we can see that $K \cup A_2$ (resp. K) is a scattering set of H. Then, by Lemmas 3.8 and 2.2, it is easy to prove that $s(H) = \max(1, s(H_2)) + 1$ (resp. $s(H) = 1$).

Moreover, $\{P \cup [k_l, s_1'], Q_1, Q_2, \ldots, Q_p\}$ is a minimum path partition of $V(H)$ (resp. $P \cup [k_l, s_1']$ is a Hamiltonian path of H). Then, $\rho(H) = \rho(H_2) + 1$ (resp. $\rho(H) = 1$).

If H_2 is a non-empty Jung graph (resp. $H_2 = \emptyset$), then $s(H) = \rho(H_2) = \rho(H)$ (resp. $s(H) = \rho(H) = 1$), that is, H is a Jung graph. Thus, if $H_2 = \emptyset$ or H_2 is a Jung graph then we set $B(r) = \text{TRUE}$; otherwise we set $B(r) = \text{FALSE}$.

3.2.2.3: $S' = S \cup \{s'_1\}$ and $s_1 s'_1 \in E(H)$.

If $H_2 \neq \emptyset$ (resp. $H_2 = \emptyset$) then $K \cup A_2$ (resp. K) is a scattering set of H. Then, by Lemmas 3.8 and 2.2, $s(H) = \max(1, s(H_2))$ (resp. $s(H) = 0$).

Moreover, $\{[s'_1, s_1] \cup P' \cup [k_l, q_1] \cup Q_1, Q_2, \ldots, Q_p\}$ is a minimum path partition of $V(H)$ (resp. $P' \cup [k_l, s'_1, s_1]$ is a Hamiltonian cycle of H). Then, $\rho(H) = \rho(H_2)$ (resp. $\rho(H) = 1$).

It is clear that, if H_2 is a non-empty Jung graph or $H_2 = \emptyset$, H is a Jung graph. Thus, if $H_2 = \emptyset$ or H_2 is a Jung graph then we set $B(r) = \text{TRUE}$; otherwise we set $B(r) = \text{FALSE}$.

3.2.2.4: $K' = K \cup \{k'_1\}$.

If $H_2 \neq \emptyset$ (resp. $H_2 = \emptyset$) then $K \cup A_2$ (resp. K if $|K| > 3$, $\{k_2, k_3\}$ if $|K| = 3$) is a scattering set of H. Then, by Lemmas 3.8 and 2.2, $s(H) = \max(1, s(H_2)) - 1$ (resp. $s(H) = -1$ if $|K| > 3$, $s(H) = 0$ if $|K| = 3$).

If $H_2 = \emptyset$ then $C' = P \cup [s_{l-1}, k'_1, k_l]$ is a Hamiltonian cycle of H.

If $H_2 \neq \emptyset$ and $p = |\mathcal{Q}| > 1$ (resp. $p = 1$) then $\{Q_1 \cup [q_1, k_l] \cup P \cup [s_{l-1}, k'_1, q_2] \cup Q_2, Q_3, \ldots, Q_p\}$ is a minimum path partition (resp. $C = [k'_1, q_1] \cup Q_1 \cup [q'_1, k_l] \cup P \cup [s_{l-1}, k'_1]$ is a Hamiltonian cycle of H).

Then, if $H_2 \neq \emptyset$ (resp. $H_2 = \emptyset$), $\rho(H) = \max(1, \rho(H_2) - 1)$ (resp. $\rho(H) = 1$).

Claim 3.2 *If H_2 is a non-empty Jung graph or $H_2 = \emptyset$ then H is a Jung graph.*

Proof. For any Hamiltonian cycle D of H, and any vertices u and v, 'v is a neighbour of u in D' means that uv is an edge of D.

a: H_2 is a non-empty Jung graph.

Clearly, $s(H) = \rho(H_2) - 1$. If $\rho(H_2) > 1$ then $\rho(H) = \rho(H_2) - 1 = s(H)$. If $\rho(H_2) = 1$ then $s(H) = 0$ and we know that H has a Hamiltonian cycle C. Thus, H is a Jung graph.

b: $H_2 = \emptyset$ and $|K| = 3$.

Then, $S(H) = 0$ and we know that H has a Hamiltonian cycle C'. Thus, H is a Jung graph.

c: $H_2 = \emptyset$ and $|K| > 3$.

Then, $s(H) = -1$. Let x and y be two arbitrary vertices of H.

c.1: x or y is in $\{k_1, k'_1\}$.

Without loss of generality, we suppose that $x = k'_1$. $H' = H \setminus \{x\}$ verifies the hypotheses of Case 3.2.2.1. Then, there exists a Hamiltonian cycle D of H'. Let a and b be the two neighbours of y in D. Note that, if $y \in K$ (resp. $y \in S$) then $\{a, b\} \subseteq S$ (resp. $\{a, b\} \subseteq K$). Since x is adjacent to all the vertices of $H' \setminus \{s_1, k_1\}$, x is adjacent to a or b. We suppose that x is adjacent to a. Then, $(D \setminus ay) \cup [a, x]$ is a Hamiltonian path of H joining x and y.

c.2: x or y is in K.

Without loss of generality, we suppose that $x = k_2$. $H' = H \setminus \{x, s_2, k'_1\}$ verifies the hypotheses of Case 3.2.2.1. Then, there exists a Hamiltonian cycle D of H'. Remark that x is adjacent to all the vertices of H'.

If $y \in \{k_3, \ldots, k_l\}$ then let a and b be the two neighbours of y in D. Note that $\{a, b\} \subseteq (S \setminus s_2)$. Then $[x, b] \cup (D \setminus \{ay, by\}) \cup [a, k'_1, s_2, y]$ is a Hamiltonian path of H joining x and y.

If $y = s_2$ then let a be a neighbour of k_3 in D. Hence, $[x, a] \cup (D \setminus ak_3) \cup [k_3, k_1', y]$ is a Hamiltonian path of H joining x and y.

If $y = s_1$ then let a and b be the two neighbours of y in D and c be a neighbour of b in D. Note that $\{a, b\} \subseteq (K \setminus \{k_1, k_2\})$ and $c \in (S \setminus \{s_1, s_2\})$. Then $[x, a] \cup (D \setminus \{ay, bc\}) \cup [b, s_2, k_1', c]$ induces a Hamiltonian path of H joining x and y.

If $y \in \{s_3, \ldots, s_l\}$ then let a and b be the two neighbours of y in D. Note that $\{a, b\} \subseteq (K \setminus \{k_2\})$. Then k_1' is adjacent to a or b. We suppose that k_1' is adjacent to a. Then $[x, b] \cup (D \setminus ay) \cup [a, k_1', s_2, y]$ is a a Hamiltonian path of H joining x and y.

c.3: x and y are in S.

Without loss of generality, we suppose that $x = s_2$. $H' = H \setminus \{x, k_2, k_1'\}$ verifies the hypotheses of Case 3.2.2.1. Then, there exists a Hamiltonian cycle D of H'. Remark that k_2 is adjacent to all the vertices of H'. Let a be a neighbour of y in D. Then, $[x, k_1', k_2, a] \cup (D \setminus ay)$ is a Hamiltonian path of H joining x and y.

Since in every case there exists a Hamiltonian path of H joining x and y, H is a Jung graph. □

Thus, by Claim 3.2, if $H_2 = \emptyset$ or H_2 is a Jung graph then we set $B(r) =$ TRUE; otherwise we set $B(r) =$ FALSE.

Concluding remarks

For every internal node r of the modular decomposition tree $T(G)$, we compute the values of $s(G[M(r)])$, $\rho(G[M(r)])$ and $B(r)$. Since these values are computable in $O(|V(r)|)$ time for the P-nodes and S-nodes and in constant time for the N-nodes, the scattering number $s(G)$, the minimum path partition number $\rho(G)$ and $B(G)$ are computable in $O(n)$ time. Thus, for P_4-tidy graphs, the Hamiltonian Path Decision Problem is linear and we can easily deduce, from the proofs, an efficient algorithmic construction of a minimum path partition.

Moreover, for the family $\mathcal{F}$ of P_4-tidy graphs G such that $B(G) =$ TRUE, the Hamiltonian Cycle and the Hamilton-Connected Decision Problems are linear. We also obtain an efficient algorithmic construction of a Hamiltonian cycle. We note, for example, that the family of 3-sun-free P_4-tidy graphs is a subfamily of $\mathcal{F}$.

Problem

Characterize the family of Jung P_4-tidy graphs (determine more precisely B such that G is a Jung graph if and only if $B(G) =$ TRUE).

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