Rigged configurations of type $D_4^{(3)}$ and the filling map

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Abstract. We give a statistic preserving bijection from rigged configurations to a tensor product of Kirillov–Reshetikhin crystals $\bigotimes_{i=1}^{N} B^{1,s_i}$ in type $D_4^{(3)}$ by using virtualization into type $D_4^{(1)}$. We consider a special case of this bijection with $B = B^{1,s}$, and we obtain the so-called Kirillov–Reshetikhin tableaux model for the Kirillov–Reshetikhin crystal.

Résumé. Nous donnons une bijection prservant les statistiques entre les configurations gréées et les produits tensoriels de cristaux de Kirillov–Reshetikhin $\bigotimes_{i=1}^{N} B^{1,s_i}$ de type $D_4^{(3)}$, via une virtualisation en type $D_4^{(1)}$. Nous considérons un cas particulier de cette bijection pour $B = B^{1,s}$ et obtenons ainsi les modèles de tableaux appelés Kirillov–Reshetikhin pour le cristal Kirillov–Reshetikhin.

Keywords: rigged configuration, Kirillov–Reshetikhin crystal, bijection

1 Introduction

Rigged configurations were first introduced by Kerov, Kirillov, and Reshetikhin in [14, 15] as combinatorial objects that index solutions to the Bethe Ansatz for the Heisenberg spin chains. Rigged configurations were shown to be in bijection with semi-standard tableaux and classical highest weight elements of a tensor power of the vector representation in type $A_n^{(1)}$. This bijection was then extended to Littlewood–Richardson tableaux [16], to non-exceptional types [20], and to type $E_6^{(1)}$ [19]. This bijection Φ between rigged configurations and the tensor powers has been further expanded to include classically highest weight elements in a tensor product of certain Kirillov–Reshetikhin (KR) crystals [16, 21, 18, 27, 28].

Rigged configurations have been shown to display remarkable representation theoretic properties. A (classical) crystal structure was first given for simply-laced types [26], which was then extended to all finite types [27] and affine types [24]. While Φ is defined recursively, making it difficult to work with, it preserves certain natural statistics (cocharge and energy), giving a bijective proof of the $X = M$ conjecture of [5, 6]. Furthermore, the combinatorial R-matrix transforms into the identity map on rigged configurations under Φ. Rigged configurations are also well-behaved under virtualization [21, 22, 27], a process of realizing a non-simply-laced type crystal inside of a simply-laced type, and embeddings $B(\lambda) \longrightarrow B(\mu)$ where $\lambda \leq \mu$ component-wise, leading to a model for $B(\infty)$ [24].

[†] Partially supported by NSF grant OCI-1147247

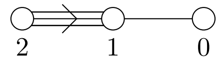

Fig. 2.1: Dynkin diagram of type $D_4^{(3)}$.

KR crystals in non-exceptional types were given a combinatorial model in [4] using Kashiwara–Nakashima tableaux [13]. The bijection Φ has also lead to a new tableaux model for KR crystals, coined Kirillov–Reshetikhin (KR) tableaux, using filled rectangular tableaux [18, 25, 27]. The map between Kashiwara–Nakashima tableaux [13] and the KR tableaux is called the filling map.

The goal of this work is to extend Φ to type $D_4^{(3)}$ and describe the filling map. For this extended abstract, we will be focusing on the KR crystals $B^{1,s}$ and the rigged configurations associated with tensor products of the form $\bigotimes_{i=1}^{N} B^{1,s_i}$. In particular, we show Φ is a classical crystal isomorphism, and we describe the filling map for $B^{1,s}$. We do so by showing the filling map and bijection commute with the virtualization map, proving more special cases of many of the conjectures stated in [27].

This extended abstract is organized as follows. In Section 2, we give background on crystals, virtualization, and rigged configurations. In Section 3, we describe the bijection Φ. In Section 4, we describe the filling map . In Section 5, we describe the virtualization map and our main results. In Section 6, we give possible extensions to $B^{2,s}$ and some open questions. We conclude in Section 7 with some examples using Sage [29].

2 Background

2.1 Crystals

For this extended abstract, let $\mathfrak{g}$ be the Kac–Moody algebra of type $D_4^{(3)}$ with index set $I = \{0, 1, 2\}$, generalized Cartan matrix $A = (A_{ij})_{i,j \in I}$, weight lattice P, root lattice Q, fundamental weights $\{\Lambda_i \mid i \in I\}$, simple roots $\{\alpha_i \mid i \in I\}$, and simple coroots $\{h_i \mid i \in I\}$. There is a canonical pairing $\langle\ ,\ \rangle \colon P^\vee \times P \longrightarrow \mathbb{Z}$ defined by $\langle h_i, \alpha_j \rangle = A_{ij}$, where $P^\vee$ is the dual weight lattice. Let $\mathfrak{g}_0$ denote the classical subalgebra of type G_2 with index set $I_0 = \{1, 2\}$, weight lattice $\overline{P}$, root lattice $\overline{Q}$, fundamental weights $\{\overline{\Lambda}_1, \overline{\Lambda}_2\}$, and simple roots $\{\overline{\alpha}_1, \overline{\alpha}_2\}$.

An *abstract* $U_q(\mathfrak{g})$-*crystal* is a nonempty set $\mathcal{B}$ together with a *weight function* $\mathrm{wt} \colon \mathcal{B} \longrightarrow P$, *crystal operators* $e_a, f_a \colon \mathcal{B} \longrightarrow \mathcal{B} \sqcup \{0\}$, and maps $\varepsilon_a, \varphi_a \colon \mathcal{B} \longrightarrow \mathbb{Z} \sqcup \{-\infty\}$ for $a \in I$, subject to the conditions

1. $\varphi_a(b) = \varepsilon_a(b) + \langle h_a, \mathrm{wt}(b) \rangle$ for all $a \in I$,
2. if $e_a b \in \mathcal{B}$, then $\varepsilon_a(e_a b) = \varepsilon_a(b) - 1$, $\varphi_a(e_a b) = \varphi_a(b) + 1$, and $\mathrm{wt}(e_a b) = \mathrm{wt}(b) + \alpha_a$.
3. if $f_a b \in \mathcal{B}$, then $\varepsilon_a(f_a b) = \varepsilon_a(b) + 1$, $\varphi_a(f_a b) = \varphi_a(b) - 1$, and $\mathrm{wt}(f_a b) = \mathrm{wt}(b) - \alpha_a$.
4. $f_a b = b'$ if and only if $b = e_a b'$ for $b, b' \in \mathcal{B}$ and $a \in I$,
5. if $\varphi_a(b) = -\infty$ for $b \in \mathcal{B}$, then $e_a b = f_a b = 0$.

We define for all $b \in \mathcal{B}$

$$\varepsilon_a(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid e_a^k b \neq 0\}, \qquad \varphi_a(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid f_a^k b \neq 0\}. \tag{2.1}$$

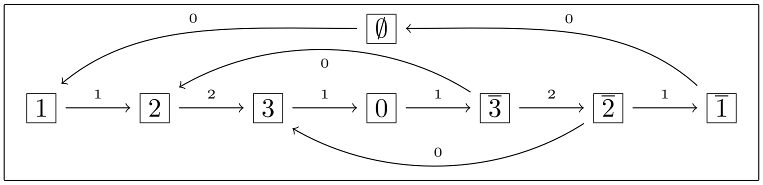

Fig. 2.2: The KR crystal $B^{1,1}$ of type $D_4^{(3)}$ which is isomorphic to $B(\overline{\Lambda}_1) \oplus B(0)$ as $U_q(\mathfrak{g}_0)$-crystals.

An abstract $U_q(\mathfrak{g})$-crystal with ε_a and φ_a defined as above is called a *regular* crystal.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be abstract $U_q(\mathfrak{g})$-crystals. The tensor product of crystals $\mathcal{B}_2 \otimes \mathcal{B}_1$ is defined to be the Cartesian product $\mathcal{B}_2 \times \mathcal{B}_1$ with the crystal structure

$$e_i(b_2 \otimes b_1) = \begin{cases} e_i b_2 \otimes b_1 & \text{if } \varepsilon_i(b_2) > \varphi_i(b_1), \\ b_2 \otimes e_i b_1 & \text{if } \varepsilon_i(b_2) \leq \varphi_i(b_1), \end{cases} \qquad \varepsilon_i(b_2 \otimes b_1) = \max\bigl(\varepsilon_i(b_2), \varepsilon_i(b_1) - \langle h_i, \mathrm{wt}(b_2)\rangle\bigr)$$

$$f_i(b_2 \otimes b_1) = \begin{cases} f_i b_2 \otimes b_1 & \text{if } \varepsilon_i(b_2) \geq \varphi_i(b_1), \\ b_2 \otimes f_i b_1 & \text{if } \varepsilon_i(b_2) < \varphi_i(b_1), \end{cases} \qquad \varphi_i(b_2 \otimes b_1) = \max\bigl(\varphi_i(b_1), \varphi_i(b_2) + \langle h_i, \mathrm{wt}(b_1)\rangle\bigr)$$

$$\mathrm{wt}(b_2 \otimes b_1) = \mathrm{wt}(b_2) + \mathrm{wt}(b_1).$$

Remark 2.1 *Our tensor product convention is the opposite to that given in [12].*

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two abstract $U_q(\mathfrak{g})$-crystals. A *crystal morphism* $\psi \colon \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ is a map $\mathcal{B}_1 \sqcup \{0\} \longrightarrow \mathcal{B}_2 \sqcup \{0\}$ with $\psi(0) = 0$ such that for $b \in \mathcal{B}_1$

1. if $\psi(b) \in \mathcal{B}_2$, then $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$;
2. we have $\psi(e_i b) = e_i \psi(b)$ provided $\psi(e_i b) \neq 0$ and $e_i \psi(b) \neq 0$;
3. we have $\psi(f_i b) = f_i \psi(b)$ provided $\psi(f_i b) \neq 0$ and $f_i \psi(b) \neq 0$.

A crystal embedding or isomorphism is a crystal morphism such that the induced map $\mathcal{B}_1 \sqcup \{0\} \longrightarrow \mathcal{B}_2 \sqcup \{0\}$ is an embedding or bijection respectively. A crystal morphism is *strict* if it commutes with all crystal operators.

If an abstract $U_q(\mathfrak{g})$-crystal $\mathcal{B}$ is isomorphic to the crystal basis of an integrable $U_q(\mathfrak{g})$-module, we simply say $\mathcal{B}$ is a $U_q(\mathfrak{g})$*-crystal*. In particular, an irreducible highest weight $U_q(\mathfrak{g}_0)$-module with highest weight λ admits a crystal basis [11], which we denote by $B(\lambda)$. Moreover there is a unique element $u_\lambda \in B(\lambda)$ such that $\mathrm{wt}(u_\lambda) = \lambda$ and $e_a u_\lambda = 0$ for all $a \in I_0$. For each dominant integral weight $\lambda = k_1\overline{\Lambda}_1 + k_2\overline{\Lambda}_2$, we can associate a partition $(k_1 + k_2, k_2)$. We can realize $B(\lambda)$ as semistandard tableaux of shape λ filled with entries in $B(\overline{\Lambda}_1)$ whose crystal structure is given by embedding into $B(\overline{\Lambda}_1)^{\otimes|\lambda|}$ using the reverse far-eastern reading word. The resulting tableaux were explicitly described by Kang and Misra [9].

2.2 Kirillov–Reshetikhin crystals

An important class of finite dimensional $U_q'(\mathfrak{g})$-representations are Kirillov–Reshetikhin (KR) modules $W^{r,s}$ indexed by $r \in I_0$ and $s \in \mathbb{Z}_{>0}$. KR modules are characterized by their Drinfeld polynomials [2, 3] and correspond to the minimal affinization of $B(s\overline{\Lambda}_r)$ [1]. The KR modules $W^{1,s}$ admit a crystal basis called *Kirillov–Reshetikhin (KR) crystals* and denoted by $B^{1,s}$. As $U_q(\mathfrak{g}_0)$-crystals, we have $B^{1,s} \cong \bigoplus_{k=1}^{s} B(k\overline{\Lambda}_1)$, and $B^{1,s}$ is a perfect crystal [10]. This means we can use a semi-infinite tensor product of $B^{1,s}$ to realize highest weight $U_q(\mathfrak{g})$-crystals, see [7] for details.

There is a statistic called *energy* defined on $B = \bigotimes_{i=1}^{N} B^{1,s_i}$ [5]. First we define the *local energy function* on $B^{1,s} \otimes B^{1,t}$ as follows. The combinatorial R-matrix is the unique $U'_q(\mathfrak{g})$-crystal isomorphism $R\colon B^{1,s} \otimes B^{1,t} \longrightarrow B^{1,t} \otimes B^{1,s}$ [10]. Let $c' \otimes c = R(b \otimes b')$.

$$H\bigl(e_i(b \otimes b')\bigr) = H(b \otimes b') + \begin{cases} -1 & i = 0 \text{ and } e_0(b \otimes b') = b \otimes e_0 b' \text{ and } e_0(c' \otimes c) = c' \otimes e_0 c, \\ 1 & i = 0 \text{ and } e_0(b \otimes b') = e_0 b \otimes b' \text{ and } e_0(c' \otimes c) = e_0 c' \otimes c, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

The local energy function is defined up to an additive constant [8], and so we normalize H by the condition $H(1^s \otimes 1^t) = 0$ where 1^k is row of length k filled with 1. Next we define $D_{B^{1,s}}\colon B^{1,s} \to \mathbb{Z}$ by

$$D_{B^{1,s}}(b) = H(b \otimes b^\sharp) - H(1^s \otimes b^\sharp), \tag{2.3}$$

where $b^\sharp$ is the unique element such that $\varphi(b^\sharp) = s\Lambda_0$. Then we define

$$D(b_N \otimes \cdots \otimes b_1) = \sum_{1 \leq i < j \leq N} H_i R_{i+1} R_{i+2} \cdots R_{j-1} + \sum_{j=1}^{N} D_{B^{1,s_j}} R_1 R_2 \cdots R_{j-1}, \tag{2.4}$$

where R_i and H_i are the combinatorial R-matrix and local energy function, respectively, acting on the i-th and $(i+1)$-th factors and $D_{B^{1,s_j}}$ acts on the rightmost factor. We say the *energy* of an element $b \in B$ is $D(b)$.

2.3 Rigged configurations

Let $\mathcal{H}_0 = I_0 \times \mathbb{Z}_{>0}$. Consider a multiplicity array $L = \bigl(L_i^{(a)} \in \mathbb{Z}_{\geq 0} \mid (a,i) \in \mathcal{H}_0\bigr)$ and a dominant integral weight λ of $\mathfrak{g}_0$. A $(L;\lambda)$-*configuration* is a sequence of partitions $\nu = \{\nu^{(a)} \mid a \in I\}$ such that

$$\sum_{(a,i) \in \mathcal{H}_0} i m_i^{(a)} \overline{\alpha}_a = \sum_{(a,i) \in \mathcal{H}_0} i L_i^{(a)} \overline{\Lambda}_a - \lambda, \tag{2.5}$$

where $m_i^{(a)}$ is the number of parts of length i in the partition $\nu^{(a)}$. We denote the set of (L,λ)-configurations by $C(L,\lambda)$. The *vacancy numbers* of $\nu \in C(L;\lambda)$ are defined as

$$p_i^{(a)} = \sum_{j \geq 1} \min(i,j) L_j^{(a)} - \sum_{(b,j) \in \mathcal{H}_0} A_{ab} \min(i,j) m_j^{(b)}. \tag{2.6}$$

A *rigged configuration* of classical weight λ is a $(L;\lambda)$-configuration ν, along with a sequence of multisets of integers $J = \{J_i^{(a)} \mid (a,i) \in \mathcal{H}_0\}$ such that $|J_i^{(a)}| = m_i^{(a)}$ (the size of $J_i^{(a)}$) and $\max J_i^{(a)} \leq p_i^{(a)}$. (Often each $J_i^{(a)}$ will be sorted in weakly decreasing order.) So to each row of length i, we have an integer $x \in J_i^{(a)}$ and we call the pair (i,x) a *string*. The integers $x \in J_i^{(a)}$ are called *label*, *rigging*, or *quantum number*. The *colabel* of a string (i,x) is defined as $p_i^{(a)} - x$. A rigged configuration is *highest weight* if $\min J_i^{(a)} \geq 0$ for all $(a,i) \in \mathcal{H}_0$ and is *valid* if $\max J_i^{(a)} \leq p_i^{(a)}$. We say a string (i,a) is *singular* if $p_i^{(a)} = x$ and is *quasi-singular* if $p_i^{(a)} = x - 1$ and $\max J_i^{(a)} \neq p_i^{(a)}$.

Example 2.2 *Rigged configurations will be depicted with vacancy numbers on the left and labels on the right. For example,*

$$\begin{array}{rl} 5 & \square\square\square\square\ 3 \\ 1 & \square\ 1 \end{array} \qquad \begin{array}{rl} -2 & \square\square\square\square\ -2 \end{array}$$

is a rigged configuration of weight $2\overline{\Lambda}_1 + \overline{\Lambda}_2$ with L is given by $L_1^{(1)} = L_2^{(1)} = L_1^{(2)} = 1$ with all other $L_i^{(a)} = 0$. See Section 7 on how to construct this example in Sage.

Denote by $\mathrm{RC}^*(L;\lambda)$ the set of valid highest weight rigged configurations. Rigged configurations have an abstract $U_q(\mathfrak{g}_0)$-crystal structure [27]. To obtain the weight, we first note that we can compute the classical weight by

$$\overline{\mathrm{wt}}(\nu, J) = \sum_{(a,i)\in\mathcal{H}_0} i\big(L_i^{(a)}\overline{\Lambda}_a - m_i^{(a)}\overline{\alpha}_a\big). \tag{2.7}$$

We can extend this to $\mathrm{wt}\colon \mathrm{RC}(L;\lambda) \longrightarrow P$ by $\mathrm{wt}(\nu,J) = k_0\Lambda_0 + \overline{\mathrm{wt}}(\nu,J)$, where k_0 is such that $\langle \mathrm{wt}(\nu,J), c\rangle = 0$ with c the canonical central element of $\mathfrak{g}$ (i.e., we make $\mathrm{wt}(\nu,J)$ be level 0). Explicitly, if $\overline{\mathrm{wt}}(\nu,J) = c_1\overline{\Lambda}_1 + c_2\overline{\Lambda}_2$, then we have $k_0 = -2c_1 - 3c_2$. Next we recall the crystal operators.

Definition 2.3 *Let $\mathfrak{g}_0$ be a Lie algebra of finite type and L a multiplicity array. Let (ν,J) be a valid rigged configuration. Fix $a \in I_0$ and let x be the smallest label of $(\nu,J)^{(a)}$, the strings associated to $\nu^{(a)}$.*

1. *If $x \geq 0$, then set $e_a(\nu,J) = 0$. Otherwise, let ℓ be the minimal length of all strings in $(\nu,J)^{(a)}$ which have label x. The rigged configuration $e_a(\nu,J)$ is obtained by replacing the string (ℓ,x) with the string $(\ell-1,x+1)$ and changing all other labels so that all colabels remain fixed.*

2. *If $x > 0$, then add the string $(1,-1)$ to $(\nu,J)^{(a)}$. Otherwise, let ℓ be the maximal length of all strings in $(\nu,J)^{(a)}$ which have label x and replace the string (ℓ,x) by the string $(\ell+1,x-1)$. In both cases, change all other labels so that all colabels remain fixed. If the result is a valid rigged configuration, then it is $f_a(\nu,J)$. Otherwise $f_a(\nu,J) = 0$.*

Remark 2.4 *The condition for highest weight rigged configurations matches with the usual crystal theoretic definition; i.e., that $e_a(\nu,J) = 0$ for all $(\nu,J) \in \mathrm{RC}^*(L;\lambda)$.*

Example 2.5 *Let (ν,J) be the rigged configuration from Example 2.2. Then*

$$e_1(\nu,J) = 0, \qquad e_2(\nu,J) = \begin{array}{rl} 2 & \square\square\square\square\ 0 \\ 1 & \square\ 1 \end{array} \quad \begin{array}{rl} -1 & \square\square\square\ -1 \end{array},$$

$$f_1(\nu,J) = \begin{array}{rl} 3 & \square\square\square\square\ 1 \\ -1 & \square\ -1 \\ -1 & \square\ -1 \end{array} \quad \begin{array}{rl} -1 & \square\square\square\square\ -1 \end{array}, \qquad f_2(\nu,J) = 0.$$

Let $\mathrm{RC}(L;\lambda)$ denote the set generated from $\mathrm{RC}^*(L;\lambda)$ by the crystal operators. Let $\mathrm{RC}(L)$ be the closure under the crystal operators of the set $\mathrm{RC}^*(L) = \bigsqcup_{\lambda\in P^+} \mathrm{RC}^*(L;\lambda)$.

Theorem 2.6 ([27]) *Let $\mathfrak{g}_0$ be a Lie algebra of finite type. For $(\nu, J) \in \mathrm{RC}^*(L; \lambda)$, let $X_{(\nu,J)}$ be the closure of (ν, J) under e_a, f_a for $a \in I_0$. Then $X_{(\nu,J)} \cong B(\lambda)$ as $U_q(\mathfrak{g}_0)$-crystals.*

There is a statistic called *cocharge* on rigged configurations given by

$$\mathrm{cc}(\nu, J) = \frac{1}{2} \sum_{a,b \in I_0} \sum_{i,j \in \mathbb{Z}_{>0}} (\alpha_a | \alpha_b) \min(i, j) m_i^{(a)} m_j^{(b)} + \sum_{(a,i) \in \mathcal{H}_0} \sum_{x \in J_i^{(a)}} x. \tag{2.8}$$

Moreover cocharge is invariant under e_a and f_a for $a \in I_0$ [27].

2.4 Virtual crystals

Let $\widehat{\mathfrak{g}}$ be the Kac–Moody algebra with index set $\widehat{I}$ of type $D_4^{(1)}$ and $\widehat{\mathfrak{g}}_0$ be of type D_4. We consider the diagram folding $\phi \colon \widehat{I} \searrow I$ defined by $\phi(0) = 0$, $\phi(2) = 1$, and $\phi(1) = \phi(3) = \phi(4) = 2$. The folding ϕ restricts to a diagram folding of type $\widehat{\mathfrak{g}}_0 \searrow \mathfrak{g}_0$, and by abuse of notation, we also denote this folding by ϕ.

Remark 2.7 *To simplify our notation, for any object X or $\overline{X}$ of $\mathfrak{g}_0$, we denote the corresponding object of $\widehat{\mathfrak{g}}_0$ by $\widehat{X}$.*

Furthermore, the folding ϕ induces an embedding of weight lattices $\Psi \colon \overline{P} \longrightarrow \widehat{P}$ given by

$$\overline{\Lambda}_a \mapsto \sum_{b \in \phi^{-1}(a)} \widehat{\Lambda}_b, \qquad \overline{\alpha}_a \mapsto \sum_{b \in \phi^{-1}(a)} \widehat{\alpha}_b. \tag{2.9}$$

This gives an embedding of crystals as sets $v \colon B(\lambda) \longrightarrow B\big(\Psi(\lambda)\big)$, and let $V(\lambda)$ denote the image of v. We can define a crystal structure on V which is induced from the crystal $B\big(\Psi(\lambda)\big)$ by

$$\begin{aligned} e^v &:= \prod_{b \in \phi^{-1}(a)} \widehat{e}_b, & f^v &:= \prod_{b \in \phi^{-1}(a)} \widehat{f}_b, \\ \varepsilon_a^v &:= \widehat{\varepsilon}_x, & \varphi_a^v &:= \widehat{\varphi}_x, \\ & & \mathrm{wt} := \Psi^{-1} \circ \widehat{\mathrm{wt}}, \end{aligned} \tag{2.10}$$

where we fix some $x \in \phi^{-1}(a)$. We say the pair $\Big(V(\lambda), B\big(\Psi(\lambda)\big)\Big)$ is a *virtual crystal* and the isomorphism v is the *virtualization map*.

Proposition 2.8 ([27]) *Let $\mathfrak{g}_0$ be of finite type. Then we have $B(\lambda) \cong V(\lambda)$ as $U_q(\mathfrak{g}_0)$-crystals.*

In particular, we can define a virtualization map on rigged configurations by

$$\widehat{\nu}^{(b)} = \nu^{(a)}, \tag{2.11a}$$

$$\widehat{J}_i^{(b)} = J_i^{(a)} \tag{2.11b}$$

for all $b \in \phi^{-1}(a)$ [27].

3 The bijection Φ

Consider a tensor product of KR crystals $B = \bigotimes_{i=1}^{N} B^{r_i, s_i}$. We write $\mathrm{RC}(B)$ for $\mathrm{RC}(L)$ with $L_i^{(a)}$ equal to the number of factors $B^{a,i}$ occurring in B. In this section, we describe the map $\Phi \colon \mathrm{RC}(B) \longrightarrow B$.

3.1 The basic algorithm δ

We begin by describing the basic step $\delta\colon \mathrm{RC}(B^{1,1} \otimes B^*) \longrightarrow \mathrm{RC}(B^*)$, where B^* is some tensor product of KR crystals. Each step δ returns some element $b \in B^{1,1}$, which we use to create B. We note that this is the special case of the algorithm given in [17] for type $D_4^{(3)}$.

Set $\ell_0 = 1$. Do the following process for $a = 1$. Find the minimal integer $i \geq \ell_{a-1}$ such that $\nu^{(a)}$ has a singular string of length i. If no such i exists, then set $b = a$ and $\ell_a = \infty$ and terminate. Otherwise set $\ell_a = i$ and repeat the above process for $a = 2$.

Suppose the process has not terminated. We remove the selected (singular) string of length ℓ_1 from consideration. If there are no singular or quasi-singular strings in $\nu^{(a)}$ larger than ℓ_2 or if $\ell_2 = \ell_1$ and there is only one string of length ℓ_1 in $\nu^{(1)}$, then set $b = 3$ and terminate. Otherwise find the smallest $i \geq \ell_2$ that satisfies one of the following three mutually exclusive conditions:

(S) $J^{(1,i)}$ is singular and $i > 1$;

(P) $J^{(1,i)}$ is singular and $i = 1$;

(Q) $J^{(1,i)}$ is quasi-singular.

If (P) holds, set $b = \emptyset$, and $\ell_3 = i$ and terminate. If (S) holds, set $\ell_3 = i - 1$, $\overline{\ell}_3 = i$, say case (S) holds for $a = n$, and continue. If (Q) holds, find the minimal $j > i$ such that (S) holds. If no such j exists, set $b = 0$ and terminate. Else set $\overline{\ell}_3 = j$ and say case (Q, S) holds and continue.

Suppose the process has not terminiated, and let $a = 2$. If $\ell_a = \overline{\ell}_{a+1}$, then set $\overline{\ell}_a = \ell_a$, afterwards reset $\ell_a = \overline{\ell}_a - 1$, and say case (S2) holds for a. Otherwise find the minimal index $i \geq \overline{\ell}_{a+1}$ such that $\nu^{(a)}$ has a singular string of length i. If no such i exists, set $b = \overline{a+1}$ and terminate. Otherwise set $\overline{\ell}_a = i$ and repeat this for $a = 1$ (there must exists at least two singular strings if $\overline{\ell}_3 = \overline{\ell}_1$ and case (S2) does not hold). If the process has not terminated, set $b = \overline{1}$.

Set all undefined ℓ_a and $\overline{\ell}_a$ for $a = 1, 2, 3$ to ∞.

3.2 Change in the rigged configuration

The rigged configurations change under δ as follows. We first remove a box from $\overline{\ell}_a$ in $\nu^{(a)}$ for $a = 1, 2$, and if case (S2) holds for a, we remove another box from that particular row, otherwise we remove a box from ℓ_a. If case (S) holds, then remove two boxes from $\overline{\ell}_3$ and make the resulting string singular. If case (Q) holds, remove a box from ℓ_3 and make the resulting string singular. If case (Q, S) holds, then we remove both boxes corresponding to ℓ_3 and $\overline{\ell}_3$, but we make the smaller one (i.e. the row corresponding to ℓ_3) singular and the larger one quasi-singular. Also make all the changed strings in $\nu^{(2)}$ singular.

Remark 3.1 *We can determine the inverse algorithm by roughly doing the opposite of the above; in paritcular, selecting largest (quasi)singular strings at most as long as before.*

Example 3.2 *Using the rigged configuration (ν, J) from Example 2.2 and $B = B^{1,1} \otimes B^{1,2} \otimes B^{2,1}$. Applying the map δ, we get $b = 3$ and*

$$\delta(\nu, J) = \quad 3\ \square\square\square\square\ 3 \qquad -2\ \square\square\square\ -2$$

3.3 Extending to arbitrary rectangles

We now extend Φ to $B = \bigotimes_{i=1}^{N} B^{1,s_i}$ by defining the map

$$\mathrm{ls}\colon \mathrm{RC}(B^{1,s} \otimes B^*) \longrightarrow \mathrm{RC}(B^{1,1} \otimes B^{1,s-1} \otimes B^*),$$

which is known as *left-split*. On the rigged configurations, the map ls is the identity (but perhaps increases the vacancy numbers) and a strict crystal embedding. Thus iterating ls with δ, we obtain a map $\Phi\colon \mathrm{RC}(B) \longrightarrow B$.

4 Filling map

We determine the highest weight rigged configurations for $B^{1,s}$ by using the virtual Kleber algorithm [22].

Lemma 4.1 *Consider the KR crystal $B^{1,s}$. We have $\mathrm{RC}(B^{1,s}) = \bigoplus_{k=0}^{s} \mathrm{RC}(B^{1,s}; k\overline{\Lambda}_1)$. Moreover the highest weight rigged configurations in $\mathrm{RC}(B^{1,s}; k\overline{\Lambda}_1)$ are given by $\nu^{(1)} = (s-k, s-k)$ and $\nu^{(2)} = (s-k)$ with all labels 0.*

From Lemma 4.1 and the $U_q(\mathfrak{g}_0)$-crystal decomposition of $B^{1,s}$ is multiplicity free, there exists a natural $U_q(\mathfrak{g}_0)$-crystal isomorphism $\iota\colon \mathrm{RC}(B^{1,s}) \longrightarrow B^{1,s}$. For type $D_4^{(3)}$, we note that $k\overline{\Lambda}_1$ can be considered as the partition (k).

Definition 4.2 *Let $B^{1,s}$ be a KR crystal of type $D_4^{(3)}$ and consider the classical component $B(k\overline{\Lambda}_1) \subseteq B^{1,s}$. The filling map $\mathrm{fill}\colon B^{1,s} \longrightarrow (B^{1,1})^{\otimes s}$ is given by adding $\left\lfloor \frac{s-k}{2} \right\rfloor$ copies of the horizontal domino $\boxed{\overline{1}}\boxed{1}$ and an additional $\boxed{\emptyset}$ if $s-k$ is odd.*

Let $T^{1,s}$ denote the image of $B^{1,s}$ under fill written as a $1 \times s$ rectangle. We note that $T^{1,s}$ inherits a classical crystal structure from $(B^{1,1})^{\otimes s}$.

Example 4.3 *Consider the element*

$$b = \boxed{3}\boxed{0}\boxed{\overline{2}}\boxed{\overline{2}}\boxed{\overline{1}} \in B(5\overline{\Lambda}_1) \subseteq B^{1,9},$$

then we have

$$\mathrm{fill}(b) = \boxed{3}\boxed{0}\boxed{\overline{2}}\boxed{\overline{2}}\boxed{\overline{1}}\boxed{\overline{1}}\boxed{1}\boxed{\overline{1}}\boxed{1}.$$

Now suppose $b \in B^{1,8}$, then we have

$$\mathrm{fill}(b) = \boxed{3}\boxed{0}\boxed{\overline{2}}\boxed{\overline{2}}\boxed{\overline{1}}\boxed{\overline{1}}\boxed{1}\boxed{\emptyset}.$$

We give a $U_q'(\mathfrak{g})$-crystal structure to $T^{1,s}$ by following [10, 30] as the conditions for e_0 and f_0 are preserved under the filling map.

Proposition 4.4 *The filling map $\mathrm{fill}\colon B^{1,s} \to T^{1,s}$ given in Definition 4.2 is a $U_q'(\mathfrak{g})$-crystal isomorphism.*

We also can show the following.

Proposition 4.5 *Let $B = B^{1,s}$. Then $\Phi = \mathrm{fill} \circ \iota$ with fill as in Definition 4.2 on highest weight elements.*

5 Virtualization Map

Lemma 5.1 *The virtualization map $v\colon B^{1,s} \longrightarrow B^{2,s}$ for types $D_4^{(3)} \hookrightarrow D_4^{(1)}$ is given column-by-column by*

$$\boxed{1} \mapsto \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \qquad \boxed{2} \mapsto \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \qquad \boxed{3} \mapsto \begin{array}{|c|}\hline 2\\\hline \overline{3}\\\hline\end{array} \qquad \boxed{0} \mapsto \begin{array}{|c|}\hline 3\\\hline \overline{3}\\\hline\end{array}$$

$$\boxed{\overline{3}} \mapsto \begin{array}{|c|}\hline 3\\\hline \overline{2}\\\hline\end{array} \qquad \boxed{\overline{2}} \mapsto \begin{array}{|c|}\hline \overline{3}\\\hline \overline{1}\\\hline\end{array} \qquad \boxed{\overline{1}} \mapsto \begin{array}{|c|}\hline \overline{2}\\\hline \overline{1}\\\hline\end{array} \qquad \boxed{\emptyset} \mapsto \begin{array}{|c|}\hline 1\\\hline \overline{1}\\\hline\end{array}$$

Using Lemma 5.1 and the analogue of Φ in type $D_4^{(1)}$ [18, 25], we can show the following.

Theorem 5.2 *Consider a tensor product of KR crystals $B = \bigotimes_{i=1}^{N} B^{1,s_i}$ of type $D_4^{(3)}$. The virtualization map v commutes with the map Φ.*

We need to define the *complement rigging map* $\theta\colon \mathrm{RC}(B) \longrightarrow \mathrm{RC}(B_r)$ by sending $(\nu, J) \mapsto (\nu, J')$, where J' is obtained by $x' = p_i^{(a)} - x$ for all labels x and B_r are the factors of B in reverse order. That is to say θ maps each label x to its colabel. We can define $\widetilde{\delta} := \theta \circ \delta \circ \theta$, and using the virtualization map, Proposition 4.5, and the results of [25], we can show the following.

Lemma 5.3 *We have $\delta \circ \widetilde{\delta} = \widetilde{\delta} \circ \delta$.*

Using the results on the combinatorial R-matrix in [30], we can show the following.

Lemma 5.4 *Consider $B = B^{1,s} \otimes B^{1,1}$. We have $\Phi^{-1} \circ R \circ \Phi$ is the identity map on $\mathrm{RC}(B)$.*

Then following [28, Sec. 8], the map $\mathrm{rs} := \theta \circ \mathrm{ls} \circ \theta$ preserves statistics using [30]. From Lemma 5.4, the R-matrix preserves statistics. Thus iterating rs and R-matrices, we preserve statistics to $\bigotimes_{i=1}^{N'} B^{1,1}$. Then we use the results of [23, 25] and Theorem 5.2 to obtain our main result.

Theorem 5.5 *Let $B = \bigotimes_{i=1}^{N} B^{1,s_i}$ of type $D_4^{(3)}$. The map $\Phi\colon \mathrm{RC}(B) \longrightarrow B$ is a $U_q(\mathfrak{g}_0)$-crystal isomorphism and $\Phi \circ \theta$ sends cocharge to energy.*

From Proposition 4.4, Lemma 5.1, Theorem 5.5, and the filling map for type $D_n^{(1)}$ given in [18], we can show the following.

Theorem 5.6 *Let $B = B^{1,s}$. Then $\Phi = \mathrm{fill} \circ \iota$ with* fill *as in Definition 4.2 as $U_q(\mathfrak{g}_0)$-crystal morphisms.*

Thus we can define a $U_q'(\mathfrak{g})$-crystal structure on $\mathrm{RC}(B)$ by extending Φ to be a $U_q'(\mathfrak{g})$-crystal isomorphism. Thus we have a special case in type $D_4^{(3)}$ of the conjectures given in [27].

6 Extensions and questions

The $U_q(\mathfrak{g}_0)$-crystal decomposition of $B^{2,s}$ and the highest weight rigged configurations will appear in the full version of this work. The author hopes to use this to determine the filling map for $B^{2,s}$.

There is a map $\mathrm{lt}\colon \mathrm{RC}(B^{2,1} \otimes B^*) \longrightarrow \mathrm{RC}(B^{1,1} \otimes B^{1,1} \otimes B^*)$ called *left-top* which adds a singular string of length 1 to $\nu^{(1)}$. In the full version, this is used to extend the $U_q(\mathfrak{g}_0)$-crystal isomorphism Φ to tensor products also containing $B^{2,1}$.

Example 6.1 *Continuing from Example 3.2, we obtain*

$$\Phi(\nu, J) = \boxed{3} \otimes \boxed{2\,|\,3} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}.$$

The computations for the Kleber algorithm can be modified to determine the $U_q(\mathfrak{g}_0)$-crystal decomposition of $B^{r,s}$ of type $G_2^{(1)}$. However there is a difficulty with determining what the map δ should be. This would need to be overcome to define the filling map for type $G_2^{(1)}$.

There is a conjecture [22, Conj. 3.7] that we can realize $B^{1,s}$ of type $D_4^{(3)}$ as a virtual crystal in $B^{1,s}$ of type $D_4^{(1)}$. Therefore obtaining a direct description of e_0 and f_0 on rigged configurations could lead to an answer to this conjecture using the results of [18]. The author hopes to have this description and prove this conjecture in this special case in the full version of this work.

7 Examples using Sage

The bijection Φ and the rigged configurations have been implemented by the author in Sage [29]. We begin by setting up the Sage environment to give a more concise printing.

```
sage: RiggedConfigurations.global_options(display="horizontal")
```

We construct our the rigged configuration from Example 2.2 by specifying the partitions and corresponding labels.

```
sage: nu = RC(partition_list=[[4,1], [4]], rigging_list=[[3,1], [-2]]); nu
5[ ][ ][ ][ ]3   -2[ ][ ][ ][ ]-2
1[ ]1
```

We apply the full bijection and print the output using Sage's ASCII art.

```
sage: ascii_art(nu.to_tensor_product_of_kirillov_reshetikhin_tableaux())
  3 #   2  3 #   1
                -2
```

Acknowledgements

The author would like to thank Masato Okado for a very valuable discussion and pointing out the reference [17]. The author would also like to thank Anne Schilling for reading a draft of this extended abstract and her mentorship.

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