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A kicking basis for the two column Garsia-Haiman modules

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In the early 1990s, Garsia and Haiman conjectured that the dimension of the Garsia-Haiman module R_μ is $n!$, and they showed that the resolution of this conjecture implies the Macdonald Positivity Conjecture. Haiman proved these conjectures in 2001 using algebraic geometry, but the question remains to find an explicit basis for R_μ which would give a simple proof of the dimension. Using the theory of Orbit Harmonics developed by Garsia and Haiman, we present a "kicking basis" for R_μ when μ has two columns.

Keywords: Macdonald polynomials, Garsia-Haiman modules, combinatorial basis

1 Introduction

In 1988, Macdonald [14] found a remarkable new basis of symmetric functions in two parameters which specializes to Schur functions, complete homogeneous, elementary and monomial symmetric functions and Hall-Littlewood functions, among others. With an appropriate analog of the Hall inner product, the transformed Macdonald polynomials $\widetilde{H}_\mu(Z; q, t)$ are uniquely characterized by certain triangularity and orthogonality conditions, from which their symmetry follows. The *Kostka-Macdonald* polynomials, $\widetilde{K}_{\lambda\mu}(q, t)$, are defined by

$$\widetilde{H}_\mu(Z; q, t) = \sum_\lambda \widetilde{K}_{\lambda\mu}(q, t) s_\lambda(Z).$$

The Macdonald Positivity Conjecture states that $\widetilde{K}_{\lambda\mu}(q, t) \in \mathbb{N}[q, t]$.

In 1993, Garsia and Haiman [6] conjectured that the transformed Macdonald polynomials could be realized as the bigraded characters for a diagonal action of S_n on two sets of variables. Moreover, they were able to show that knowing the dimension of this module is enough to determine its character. Therefore the $n!$ Conjecture, which states that the dimension of the Garsia-Haiman module is $n!$, implies the Macdonald Positivity Conjecture.

By analyzing the algebraic geometry of the Hilbert scheme of n points in the plane, Haiman [13] was able to prove the $n!$ Conjecture and consequently establish Macdonald Positivity. However, it remains an important open problem in the theory of Macdonald polynomials to prove the $n!$ Theorem directly by finding an explicit basis for the module. After reviewing Macdonald polynomials and the Garsia-Haiman modules in Section 2, we give an explicit basis for the Garsia-Haiman modules indexed by a partition with at most two columns in Section 3. A new basis for hooks is also given in Section 4.

2 Macdonald polynomials and graded $\mathcal{S}_n$-modules

We assume the definitions and notations from [15] of partitions and the classical bases for symmetric functions. So as to avoid confusions when defining various modules, we use the alphabet $Z = z_1, \ldots, z_n$ for symmetric functions. For example, the *Schur functions* shall be denoted $s_\lambda(Z)$.

2.1 Macdonald positivity

Departing slightly from Macdonald's convention of defining $P_\mu(Z; q, t)$ [14], we instead use the transformed Macdonald polynomials $\widetilde{H}_\mu(Z; q, t)$ as presented in [6].

Definition 2.1 *The transformed Macdonald polynomials $\widetilde{H}_\mu(Z; q, t)$ are the unique functions satisfying the following triangularity and orthogonality conditions:*

(i) $\widetilde{H}_\mu(Z; q, t) \in \mathbb{Q}(q, t)\{s_\lambda[Z/(1-q)] \,:\, \lambda \geq \mu\}$;

(ii) $\widetilde{H}_\mu(Z; q, t) \in \mathbb{Q}(q, t)\{s_\lambda[Z/(1-t)] \,:\, \lambda \geq \mu'\}$;

(iii) $\widetilde{H}_\mu[1; q, t] = 1$.

The square brackets in Definition 2.1 stand for *plethystic substitution*. In short, $s_\lambda[A]$ means s_λ applied as a Λ-ring operator to the expression A, where Λ is the ring of symmetric functions. For a thorough account of plethysm, see [12].

The existence of such a family of functions is a theorem, following in large part from Macdonald's original proof of existence. Once established, the symmetry of $\widetilde{H}_\mu(Z; q, t)$ follows by definition. Of particular importance are the change of basis coefficients from the transformed Macdonald polynomials to the Schur functions, defined by

$$\widetilde{H}_\mu(Z; q, t) = \sum_\lambda \widetilde{K}_{\lambda,\mu}(q, t) s_\lambda(Z). \tag{1}$$

A priori, the $\widetilde{K}_{\lambda,\mu}(q, t)$ are rational functions in q and t with rational coefficients.

Theorem 2.2 ([13]) *We have $\widetilde{K}_{\lambda,\mu}(q, t) \in \mathbb{N}[q, t]$.*

Macdonald originally conjectured Theorem 2.2 when he introduced the polynomials in 1988. The original proof, due to Haiman in 2001, realizes $\widetilde{H}_\mu(Z; q, t)$ as the bigraded character of certain modules for the diagonal action of $\mathcal{S}_n$ on $\mathbb{Q}[X, Y]$; see sections 2.2 and 2.3. From this it follows that the character can be written as a sum of irreducible representations of $\mathcal{S}_n$ with coefficients in $\mathbb{N}[q, t]$. Under the Frobenius image, these coefficients exactly give $\widetilde{K}_{\lambda,\mu}(q, t)$. The aim of this paper is to follow this method of proof until it departs the realm of representation theory for algebraic geometry.

It is worth noting that there are now two additional proofs of Macdonald positivity, both of which utilize an expansion of Macdonald polynomials in terms of LLT polynomials conjectured by Haglund [10] and proved along with Haiman and Loehr [11]. Grojnowski and Haiman [9] have a proof using Kazhdan-Lusztig theory and the first author [3] has a purely combinatorial proof.

2.2 Garsia-Haiman modules

To define the modules mentioned in Section 2.1, we consider the diagonal action of the symmetric group $\mathcal{S}_n$ on the polynomial ring $\mathbb{Q}[X,Y] = \mathbb{Q}[x_1,\dots,x_n; y_1,\dots,y_n]$ permuting the x_i's and y_j's simultaneously and identically. Let the coordinates of the diagram of a partition μ of n be $\{(p_1,q_1),\dots,(p_n,q_n)\}$, where p gives the row coordinate and q the column coordinate indexed from zero; see Figure 1.

(2,0)		
(1,0)	(1,1)	
(0,0)	(0,1)	(0,2)

Fig. 1: The coordinates for each cell of $\mu = (3,2,1)$.

Define the polynomial $\Delta_\mu \in \mathbb{Q}[X,Y]$ by

$$\Delta_\mu(X,Y) = \det \begin{pmatrix} x_1^{p_1}y_1^{q_1} & x_2^{p_1}y_2^{q_1} & \cdots & x_n^{p_1}y_n^{q_1} \\ x_1^{p_2}y_1^{q_2} & x_2^{p_2}y_2^{q_2} & \cdots & x_n^{p_2}y_n^{q_2} \\ \vdots & \vdots & & \vdots \\ x_1^{p_n}y_1^{q_n} & x_2^{p_n}y_2^{q_n} & \cdots & x_n^{p_n}y_n^{q_n} \end{pmatrix}. \tag{2}$$

Since the bi-exponents are all distinct, Δ_μ is a non-zero homogeneous $\mathcal{S}_n$-alternating polynomial with top degree $n(\mu) = \sum_i (i-1)\mu_i$ in X and $n(\mu')$ in Y. Taking $\mu = (1^n)$ or $\mu = (n)$ gives the Vandermonde determinant in X or Y, respectively.

Let $\mathcal{I}_\mu \subset \mathbb{Q}[X,Y]$ be the ideal of polynomials φ such that

$$\varphi(\partial/\partial x_1,\dots,\partial/\partial x_n; \partial/\partial y_1,\dots,\partial/\partial y_n)\Delta_\mu = 0.$$

Clearly this defines an $\mathcal{S}_n$ invariant doubly homogeneous ideal. Define the *Garsia-Haiman module* $\mathcal{H}_\mu$ to be the quotient ring $\mathbb{Q}[X,Y]/\mathcal{I}_\mu$ with its natural structure of a doubly graded $\mathcal{S}_n$-module.

Garsia and Haiman [7] proved that if this module has the correct dimension (the $n!$ Conjecture), then the bi-graded character is given by the transformed Macdonald polynomial.

Theorem 2.3 ([7]) *If $\mathcal{H}_\mu$ affords the regular representation of $\mathcal{S}_n$, then the bi-graded Frobenius character, given by*

$$\mathrm{Frob}_{\mathcal{H}_\mu}(Z;q,t) = \sum_{i,j} t^i q^j \psi\left((\mathcal{H}_\mu)_{i,j}\right),$$

where ψ is the usual Frobenius map sending the Specht module S^λ to the Schur function s_λ, is equal to the transformed Macdonald polynomials $\widetilde{H}_\mu(Z;q,t)$. In particular, $\widetilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$.

The following theorem is the famed $n!$ Conjecture of Garsia and Haiman [6], proved by Haiman [13] in 2001.

Theorem 2.4 ([13]) *The dimension of $\mathcal{H}_\mu$ is $n!$.*

By Theorem 2.3, Haiman's proof of the $n!$ Conjecture provided the first proof of the Macdonald positivity conjecture. Haiman's proof analyzes the isospectral Hilbert scheme of n points in a plane, ultimately showing that it is Cohen-Macaulay (and Gorenstein). As this proof uses difficult machinery in algebraic geometry, it remains an important open problem to prove Theorem 2.4 directly by finding an explicit basis for the module $\mathcal{H}_\mu$.

2.3 Orbit Harmonics

Let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be sequences of distinct rational numbers. Let $(p_1, q_1), \ldots, (p_n, q_n)$ be the coordinates of the cells of μ taken in some order, recorded by the standard filling S of μ given by placing the entry i in the cell (p_i, q_i). To each S, associate the *orbit point of* S, denoted p_S, defined by

$$p_S = (\alpha_{p_1+1}, \ldots, \alpha_{p_n+1}; \beta_{q_1+1}, \ldots, \beta_{q_n+1}). \tag{3}$$

Here the shift in indices is a notational convenience. For example,

$$p_{\tiny\begin{array}{|c|c|}\hline 5 & 3 \\ \hline 6 & 1 \\ \hline 2 & 4 \\ \hline\end{array}} = (\alpha_2, \alpha_1, \alpha_3, \alpha_1, \alpha_3, \alpha_2; \beta_2, \beta_1, \beta_2, \beta_2, \beta_1, \beta_1).$$

Let $\mathcal{S}_n$ act on $\mathbb{Q}^{2n}$ by permuting the first n and second n coordinates simultaneously and identically. Let $[p_S]$ denote the ***regular*** orbit of p_S under this action. Regarding $\mathbb{Q}[X, Y]$ as the coordinate ring of $\mathbb{Q}^{2n}$, define $\mathcal{J}_\mu \subset \mathbb{Q}[X, Y]$ to be the ideal of polynomials vanishing on $[p_S]$. Define the module R_μ to be the coordinate ring of $[p_S]$, i.e. $\mathbb{Q}[X, Y]/\mathcal{J}_\mu$, with its natural $\mathcal{S}_n$ action.

Since R_μ clearly affords the regular representation, the aim is to relate this module to $\mathcal{H}_\mu$. To do this, construct the associated graded module $\mathrm{gr}R_\mu = \mathbb{Q}[X, Y]/\mathrm{gr}\mathcal{J}_\mu$. Garsia and Haiman showed that if $\mathcal{H}_\mu$ and $\mathrm{gr}R_\mu$ have the same Hilbert series, then $\mathcal{H}_\mu = \mathrm{gr}R_\mu$. While this would demonstrate the $n!$ Conjecture, the obvious problem is that one needs first to know the Hilbert series of $\mathcal{H}_\mu$, in which case the dimension can be directly calculated. The way around this problem lies in the theory of Orbit Harmonics developed by Garsia and Haiman. The main result is the following.

Theorem 2.5 ([5]) *Let Φ_μ be a basis for R_μ. Let $F_\mu(q, t) = \sum_{\varphi \in \Phi_\mu} \widehat{\varphi}(t, \ldots, t; q, \ldots, q)$, where $\widehat{\varphi}$ is the leading term of φ. If F_μ is symmetric in the following sense,*

$$\left[t^i q^j\right] \; F_\mu(q, t) \; = \; \left[t^{n(\mu)-i} q^{n(\mu')-j}\right] \; F_\mu(q, t), \tag{4}$$

then $\widehat{\Phi}_\mu = \{\widehat{\varphi} \mid \varphi \in \Phi_\mu\}$ is a basis for $\mathrm{gr}R_\mu$. Moreover, $\mathrm{gr}R_\mu \cong \mathcal{H}_\mu$ as doubly-graded $\mathcal{S}_n$ modules. In particular, $\dim \mathcal{H}_\mu = n!$.

Theorem 2.5 suggests the following strategy for constructing a basis for the Garsia-Haiman module $\mathcal{H}_\mu$. To each filling S of μ, define a polynomial $\varphi_S \in \mathbb{Q}[X, Y]$ so that the evaluation matrix $(\varphi_S(p_T))$ of polynomials on orbit points is nonsingular and the corresponding degree polynomial $F_\mu(q, t)$ is symmetric in the sense of equation (4). The remainder of this paper is devoted to carrying out this strategy in the cases when μ is a two column shape (Section 3).

3 Two columns

Throughout this section, we restrict our attention to partitions with at most two columns. Following the procedure laid out in Section 2.3, we will construct a basis for R_μ such that the degree polynomial is symmetric. Following the idea of the kicking basis for the Garsia-Procesi modules described in [8], we will construct the basis together with a linear order on fillings of μ so that the evaluation matrix has nice triangularity properties. While the Garsia-Procesi case results in an upper triangular matrix with nonzero diagonal entries, our matrix will only be block triangular with respect to the largest entry.

3.1 The kicking tree

The *kicking tree of* μ provides a nice visualization of the recursive construction of the proposed basis. Though proving that the resulting collection is a basis with symmetric Hilbert series is better done from the recursive definition, the construction is better motivated from this viewpoint.

To construct the kicking tree, entries will be added to an empty shape one at a time in all possible ways in some specified order, ultimately resulting in a total ordering for the fillings. We begin by recalling the Garsia-Procesi ordering for row-increasing tableaux [8].

Let S be a partial filling of μ with distinct entries. Define a total ordering on the rows of S containing at least one empty cell, called the *row preference order*, as follows: empty rows of length 2 from top to bottom followed by (empty) rows of length 1 from top to bottom followed by rows of length 2 with a single occupant beginning with the largest occupant. Given two rows i and j of a (partial) filling S, say that *k prefers row j over row i*, denoted $j \succ_k i$, if j occurs before i in the row ordering on the filling obtained by removing entries less than $k+1$ from S. For example, Figure 2 shows the ranking of the rows (on the left) for two partial fillings of $(2,2,2,1,1)$.

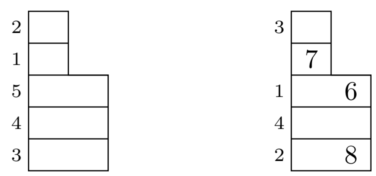

Fig. 2: The row preference order for partial fillings.

The row preference order is enough to define a total order on fillings with unsorted rows. The basic construction of the tree is to fill entries into unsorted rows one at a time according to row preference, where a row of length 2 is sorted, increasing then decreasing, as soon as it is fully occupied. The real power of the kicking tree lies in the weights assigned at each stage which we now describe.

Let S be a partial, partially sorted filling of μ with entries $n > n-1 > \cdots > k+1$. That is, each entry is assigned a row of μ, and an entry is assigned a specific column if and only if the row is fully occupied. Below S with arrows going down, place k into a row, ordered from left to right by row preference with respect to k. Label the arrow going down from S to the filling with k by

$$\prod_{j \succ_k \mathrm{row}(k)} (x_k - \alpha_j)\,.$$

If k completed a row of length 2, say with $m > k$ already in the row, then below this with arrows going down make two partial fillings: the left one having k before m and the right having m before k. Label the left branch put 1, and label the right branch

$$(y_k - \beta_1)\,.$$

If ignoring entries larger than m does not form a rectangle, then move the label from the arrow going down from S to the left-hand arrow just added, and add to the right-hand arrow

$$\prod_{\mathrm{row}(k) \succ_k i} (x_k - \alpha_i)\,.$$

The tree so constructed beginning with the empty shape μ is called the *kicking tree for* μ. For example, the kicking tree for $(2, 1)$ is constructed in Figure 3. For this example, we omit vertical lines to indicate an unsorted row.

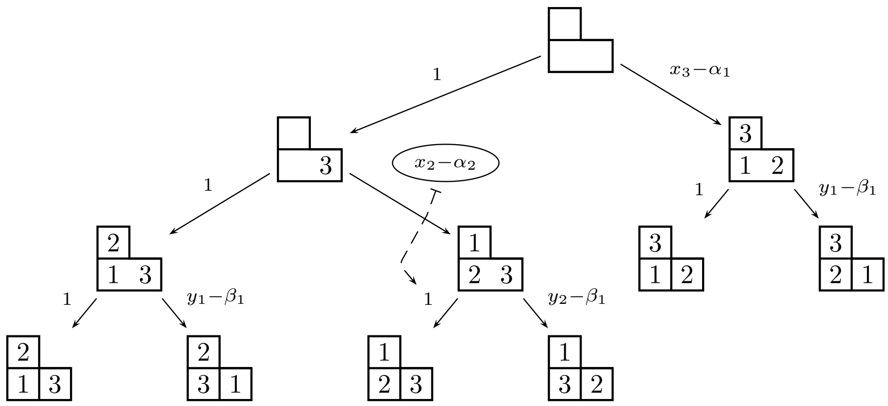

Fig. 3: The kicking tree for (2,1). Here the circled term $x_2-\alpha_2$ indicates that this term is pushed to the leftmost branch below. From left to right, the corresponding polynomials are 1 , $y_1-\beta_1$, $x_2-\alpha_2$, $y_2-\beta_1$, $x_3-\alpha_1, (x_3-\alpha_1)(y_1-\beta_1)$.

From the construction of the kicking tree, the product of the branch labels from a leaf S back to the empty shape μ is clearly a polynomial. The collection of polynomials for each filling of μ forms the proposed kicking basis for R_μ.

3.2 A recursive construction

In order to give an alternative recursive description of the kicking basis, we first need a bit more terminology.

For S a standard filling of size n, define $S \setminus n$ to be the standard filling of size $n-1$ obtained by removing the cell containing n and *straightening* the shape as follows. If n lies in a row of length 2, then move the remaining cell in the same row as n above rows of length 2 and below rows of length 1 and

$$\begin{array}{|c|c|}\hline 5 & 3 \\ \hline 6 & 1 \\ \hline 2 & 4 \\ \hline \end{array} \xrightarrow{S\backslash 6} \begin{array}{|c|c}\hline 1 & \multicolumn{1}{c}{} \\ \hline 5 & \multicolumn{1}{c|}{3} \\ \hline 2 & \multicolumn{1}{c|}{4} \\ \hline \end{array} \xrightarrow{S\backslash 5} \begin{array}{|c|c}\hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{4} \\ \hline \end{array} \xrightarrow{S\backslash 4} \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 2 \\ \hline \end{array} \xrightarrow{S\backslash 3} \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \xrightarrow{S\backslash 2} \begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

Fig. 4: An illustration of straightening after removing the largest entry.

push it to the left if necessary. Otherwise slide the cells down, preserving their order, to close the gap; see Figure 4. Notice that row dominance order commutes with straightening.

In Definition 3.1, when the largest entry of a tableau is removed and the remaining shape is straightened, the orbit point of the resulting tableau is defined using the original labelling of the rows and columns. That is, the orbit point of $S \setminus n$ is the orbit point of S with the nth and $2n$th coordinates removed. For example, in Figure 4, the orbit point of the filling of shape $(2, 1, 1)$ will be $(\alpha_2, \alpha_1, \alpha_3, \alpha_1; \beta_2, \beta_1, \beta_2, \beta_2)$.

Definition 3.1 *Define $\varphi_{\boxed{1}} = 1$. For S a standard filling of μ, $|\mu| > 1$, define φ_S recursively by*

$$\varphi_S \;=\; \varphi_{S\backslash n} \cdot \prod_{j \succ_n \mathrm{row}(n)} (x_n - \alpha_j) \cdot \begin{cases} 1 & \textit{if } n \textit{ at the end of row}(n) \\ (y_k - \beta_1) & \textit{if } \mu = (2^b) \textit{ and col}(n) = 1 \\ \dfrac{(y_k - \beta_1) \displaystyle\prod_{\mathrm{row}(k) \succ_k i} (x_k - \alpha_i)}{\displaystyle\prod_{j \succ_k \mathrm{row}(k)} (x_k - \alpha_j)} & \textit{otherwise} \end{cases}$$

where k is such that row(k)=row(n).

Using the example in Figure 4, we compute

$$\varphi_{\tiny\begin{array}{|c|c|}\hline 5 & 3 \\ \hline 6 & 1 \\ \hline 2 & 4 \\ \hline \end{array}} = \overbrace{(x_6 - \alpha_3)(y_1 - \beta_1)}^{\boxed{6}} \cdot \overbrace{(y_3 - \beta_1) \left(\frac{x_3 - \alpha_1}{x_3 - \alpha_2} \right)}^{\boxed{5}} \cdot \overbrace{1}^{\boxed{4}} \cdot \overbrace{(x_3 - \alpha_2)}^{\boxed{3}} \cdot \overbrace{(x_2 - \alpha_2)}^{\boxed{2}} \cdot 1,$$

where each step in the recursion is indicated by the cell removed to obtain the given terms.

The above formula associates to each standard filling S of μ the same polynomial as the kicking tree from Section 3.1. Notice that the denominator in the last case is precisely the label which is 'pushed down' when constructing the kicking tree. Analyzing this statement in terms of the recursive definition yields the following result.

Proposition 3.2 *For S a standard filling of μ, φ_S is a polynomial.*

Proof: The result for $\mu = (1)$ is clear, so we proceed by induction on $n = |\mu|$. It suffices to assume k, n reside in the same length 2 row with n in column 1. We must show that each term in the denominator occurs in the numerator of $\varphi_{S\backslash n}$. The only terms that ever appear in any denominator are $x_i - \alpha_j$ where

i lies in the second column and the entry to its left is greater. In particular, if $x_k - \alpha_j$ ever occurs in a numerator in the construction of φ, it remains there through $\varphi_{S \setminus n}$. Now notice that the product outside of the brace (for k, not n) is precisely the denominator in question. $\square$

To show that these polynomials form a basis for R_μ, we show that the evaluation matrix of polynomials on orbit points is nonsingular. The argument uses a nested induction to show that the matrix is almost block triangular.

Theorem 3.3 *The $n! \times n!$ matrix $(\varphi_S(p_T))$, where S, T range over all fillings of μ, is nonsingular. In particular, the set $\{\varphi_S\}$ of polynomials associated to fillings of μ forms a basis for R_μ.*

Proof: We proceed by induction on $n = |\mu|$, the case $n = 1$ being trivial. The row preference order with respect to n makes $(\varphi_S(p_T))$ block triangular with respect to the row of n. Therefore we must show that each block, corresponding to n in a particular row, is nonsingular. If n lies in a row of length 1, this is immediate by induction, so assume n lies in a row of length 2.

For $k < n$, let T_k be a partial, partially sorted filling of μ with entries $k+1, k+2, \ldots, n$ (here n must lie in its designated row of length 2). By partially sorted, we mean that the row of each entry is determined, but the column is determined if and only if the row is fully occupied; see Figure 5 for an example. Let $\mathcal{T}_k$ be the set of standard fillings of μ which restrict to T_k on $\{k+1, \ldots, n\}$, where here again the restriction allows the column of an entry to be undetermined exactly when the other occupant of the same row is at most k; again, see Figure 5. We will show that the evaluation matrix for $\mathcal{T}_k$ is nonsingular by induction on k. As usual, the base case, $k = 1$, is trivial.

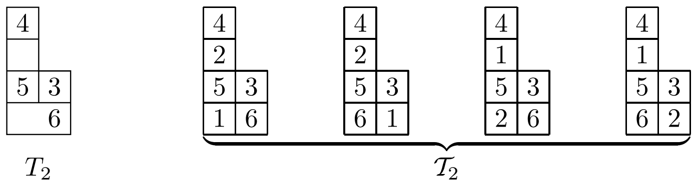

Fig. 5: An illustration of T_k and $\mathcal{T}_k$.

Restricting our attention to the set of polynomials and orbit points associated to standard fillings $S \in \mathcal{T}_k$, we put the following block ordering based on the position of k: k is the largest entry in a row of length 2 from highest row to lowest row; k lies in a row of length 1 from highest row to lowest; k lies to the left of a larger entry from largest entry to smallest; and k lies to the right of a larger entry again from smallest entry to largest. Note that the order for the first three blocks comes from the kicking tree, but the order of the fourth block is the reverse of the kicking order. By the definition of φ_S, each of the four blocks is triangular with respect to the row of k, therefore by induction each block is nonsingular since each is a fixed polynomial times the polynomials associated with $\mathcal{T}_{k-1}$ for a fixed partial filling T_{k-1}.

Also from the definition of φ_S, the first three blocks are triangular with respect to one another in the given order, and the third and fourth blocks are triangular with respect to each other as well. Moreover, for S in one of the first three cases, the monomial $(y_m - \beta_i)$ does not divide φ_S for $i = 1, 2$ and any $m \geq k$ that appears by itself in a row of length two in $\mathcal{T}_k$. Therefore the block structure of the evaluation matrix

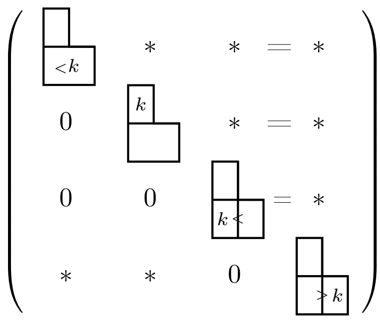

Fig. 6: Block structure of the evaluation matrix of $\mathcal{T}_k$.

is as depicted in Figure 6. Since the first two blocks are nonsingular, we may perform row reductions to eliminate the nonzero elements in the bottom block-row of the matrix. These reductions will change the bottom block-row 0 into some matrix, say M, and so by previous remarks the reductions will alter the fourth block by M as well. Hence using the row reductions from the third block to restore the 0 will also restore the fourth block. Hence the matrix can be made block triangular. Since the determinant of the matrix is the product of the determinants of the blocks, the full matrix is nonsingular. □

3.3 Symmetry of the Hilbert series

Now that we have a basis for R_μ, we must show that the associated degree polynomial, denoted $F_\mu(q,t)$, is symmetric. Recall that $F_\mu(q,t)$ is given by

$$F_\mu(q,t) = \sum_{S:\mu \xrightarrow{\sim} [n]} \widehat{\varphi}_S(t,\ldots,t;q,\ldots,q), \tag{5}$$

where $\widehat{\varphi}_S$ is the highest degree term of φ_S. That is, $F_\mu(q,t)$ is the polynomial in q and t obtained by adding leading terms of the kicking basis and recording the total x degree with t and the total y degree with q.

Our aim is to show that $F_\mu(q,t)$ is symmetric, i.e.

$$F_\mu(q,t) \;=\; t^{n(\mu)}\, q^{n(\mu')}\, F_\mu(1/q, 1/t). \tag{6}$$

For example, from Figure 3 we see that $F_{(2,1)}(q,t) = 1 + 2q + 2t + qt$, which indeed exhibits the desired symmetry.

In order to establish symmetry, we will exploit a recurrence relation that follows naturally from the recursive definition of φ_S. To do this, we must first define a more general degree polynomial, denoted $J^m_{a,b}$, by

$$J^m_{a,b}(q,t) \;=\; \frac{1}{q^m} \sum_{\substack{S:\mu \xrightarrow{\sim} [n] \text{s.t.} \\ \text{for } j=0,\ldots,m-1 \\ \text{row}(n-j)=b-j, \\ \text{col}(n-j)=1}} \widehat{\varphi}_S(t,\ldots,t;q,\ldots,q), \tag{7}$$

where $a \geq b \geq m \geq 0$. Note that $J^m_{a,b}$ is a polynomial with maximum q and t exponents given by $b-m$ and $\binom{a-m}{2} + \binom{b}{2}$, respectively. Pictorially, $J^m_{a,b}$ is the degree polynomial of fillings of $(2^b, 1^{a-b})$ with the top m cells on the left-hand side of the rectangle (2^b) deleted. In particular, we have

$$J^0_{a,b}(q,t) = F_\mu(q,t). \tag{8}$$

Therefore it is enough to show that $J^m_{a,b}$ is symmetric.

Proposition 3.4 *The degree polynomials $J^m_{a,b}$ satisfy the following recurrence relations*

$$J^m_{a,b} \;=\; [m]_t J^{m-1}_{a-1,b-1} + t^{b-m}[a-b]_t J^m_{a-1,b} + t^m[b-m]_t J^m_{a,b-1} + q[b-m]_t J^{m+1}_{a,b}, \tag{9}$$
$$J^m_{a,b} \;=\; t^{b-m}[m]_t J^{m-1}_{a-1,b-1} + [a-b]_t J^m_{a-1,b} + q[b-m]_t J^m_{a,b-1} + t^{a-b}[b-m]_t J^{m+1}_{a,b}, \tag{10}$$

with initial conditions

$$J^b_{a,b} = \binom{a}{b}\,[b]_t!\;[a-b]_t! \qquad \textit{and} \qquad J^m_{b,b} = J^0_{b,b-m},$$

where $J^m_{a,b} = 0$ unless $a \geq b \geq m \geq 0$.

The above recurrence relations follow from the recursive description in Definition 3.1. Expanding $J^m_{a,b}$ twice using both recurrence relations in Proposition 3.4 taken in one order followed by the other establishes the desired symmetry.

Theorem 3.5 *For $a \geq b \geq m \geq 0$, we have*

$$J^m_{a,b}(q,t) \;=\; t^{\binom{a-m}{2}+\binom{b}{2}}\, q^{b-m}\; J^m_{a,b}(1/q, 1/t).$$

In particular, by equation (8), Theorem 3.5 shows that the degree polynomial for the two column kicking basis is indeed symmetric. Therefore by Theorem 2.5, we have the following consequence.

Corollary 3.6 *For μ a two column partition, $\{\widehat{\varphi}_S \mid S : \mu \xrightarrow{\sim} [n]\}$ is a basis for $\mathrm{gr}R_\mu$ and so too for $\mathcal{H}_\mu$. In particular, $\dim(\mathcal{H}_\mu) = n!$ and $\widetilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$.*

4 Hooks

We next treat the case of hooks, i.e. partitions $\mu = (n-m, 1^m)$. Though there exist several known bases for Garsia-Haiman modules indexed by hooks, the first in [7] and several more in [16, 4, 2, 1]. we present this new construction because it is compatible with our two column case, i.e. the definitions of φ_S will agree on shapes of the form $(2, 1^{n-2})$, and thus suggests how to extend this approach to arbitrary shapes.

As with the two column case, we will construct a basis for R_μ such that the degree polynomial is symmetric following the idea of the kicking basis for the Garsia-Procesi modules [8]. In this case, the linear order on fillings of μ will have the property that the evaluation matrix is upper triangular with nonzero diagonal entries with respect to this basis. In the interest of brevity, we omit the direct description of the kicking tree in favor of the recursive description.

Definition 4.1 *Define* $\varphi_{\boxed{1}} = 1$. *For* S *a standard filling of* μ, $|\mu| = n$, *define* φ_S *by*

$$\varphi_S = \varphi_{S\setminus n} \prod_{j \succ_n \mathrm{row}(n)} (x_n - \alpha_j) \cdot \begin{cases} \displaystyle\prod_{\mathrm{col}(n) < i \leq \mu_1} (y_n - \beta_i) & \text{if } \mu \text{ is a single row;} \\ \displaystyle\prod_{\mathrm{col}(n) < \mathrm{col}(k)} (y_k - \beta_{l_k}) & \text{if } \mathrm{row}(n) > 1 \text{ or } \mathrm{col}(n) > 1, \\ \dfrac{\displaystyle\prod_{\mathrm{col}(n) < \mathrm{col}(k)} (y_k - \beta_{l_k})}{\displaystyle\prod_{j \succ_K 1} (x_K - \alpha_j) \prod_{\substack{1 \succ_j \mathrm{row}(j) \\ j > k \ \forall k \in \mathrm{row}(n) \setminus n}} (x_j - \alpha_1)} & \text{otherwise,} \end{cases}$$

where l_k *is the maximum column index of all entries in row* 1 *larger than and to the left of* k, *and* K *is the entry in the second column of the bottom row.*

For example, we compute

$$\varphi_{\begin{smallmatrix} \boxed{1} & & & \\ \boxed{4} & & & \\ \boxed{5} & \boxed{3} & \boxed{6} & \boxed{2} \end{smallmatrix}} = \overbrace{(y_2 - \beta_3)}^{\boxed{6}} \cdot \overbrace{\frac{(y_3 - \beta_1)(y_2 - \beta_2)}{(x_4 - \alpha_1)}}^{\boxed{5}} \cdot \overbrace{(x_4 - \alpha_1)(x_4 - \alpha_3)}^{\boxed{4}} \cdot \overbrace{\frac{(y_2 - \beta_1)}{(x_2 - \alpha_2)}}^{\boxed{3}} \cdot \overbrace{(x_2 - \alpha_2)}^{\boxed{2}} \cdot 1,$$

where each step in the recursion is indicated by the cell removed to obtain the given terms.

Both Proposition 4.2 and Theorem 4.3 are evident from the kicking tree description and are straightforward from the recursive definition.

Proposition 4.2 *For* S *a standard filling of a hook* μ, φ_S *is a polynomial.*

Theorem 4.3 *The* $n! \times n!$ *evaluation matrix* $(\varphi_S(p_T))$, *where* S, T *range over all fillings of* μ, *is upper triangular with nonzero diagonal entries. In particular, the set* $\{\varphi_S\}$ *forms a basis for* R_μ.

As before, let Φ_μ denote the kicking basis and define

$$F_\mu(q,t) = \sum_{S:\mu \xrightarrow{\sim} [n]} \widehat{\varphi}_S(t, \ldots, t; q, \ldots, q),$$

where $\widehat{\varphi}_S$ is the leading term of φ_S. Note that for a hook $\mu = (n-m, 1^m)$, the largest powers of q and t are $(n-m)(n-m-1)/2$ and $m(m+1)/2$, which again agree with $n(\mu')$ and $n(\mu)$, respectively.

Similar to the two column case, we can show the desired symmetry for $F_\mu(q,t)$ by defining a more general function $J_\mu(q,t)$. By deriving suitable recurrence relations for F and J in order to establish the following theorem.

Theorem 4.4 *For* μ *a hook partition, both* $F_\mu(q,t)$ *and* $J_\mu(q,t)$ *exhibit the desired symmetry. In particular, we have a basis for* $\mathcal{H}_\mu$ *of size* $n!$, *and so* $\widetilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$.

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