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# *Minimal transitive factorizations of a permutation of type* $(p, q)$

Jang Soo Kim[1] Seunghyun Seo[2] Heesung Shin[3]

[1] *School of Mathematics, University of Minnesota, Minneapolis, MN, USA*
[2] *Department of Math Education, Kangwon National University, Chuncheon, South Korea*
[3] *Department of Mathematics, Inha University, Incheon, South Korea*

**Abstract.** We give a combinatorial proof of Goulden and Jackson's formula for the number of minimal transitive factorizations of a permutation when the permutation has two cycles. We use the recent result of Goulden, Nica, and Oancea on the number of maximal chains of annular noncrossing partitions of type $B$.

**Résumé.** Nous donnons une preuve combinatoire de formule de Goulden et Jackson pour le nombre de factorisations transitives minimales d'une permutation lorsque la permutation a deux cycles. Nous utilisons le résultat récent de Goulden, Nica, et Oancea sur le nombre de chaînes maximales des partitions non-croisées annulaires de type $B$.

**Keywords:** minimal transitive factorizations, annular noncrossing partitions, bijective proof

## 1 Introduction

Given an integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of $n$, denote by $\alpha_\lambda$ the permutation

$$(1 \ldots \lambda_1)(\lambda_1 + 1 \ldots \lambda_1 + \lambda_2) \ldots (n - \lambda_\ell + 1 \ldots n)$$

of the set $\{1, 2, \ldots, n\}$ in the cycle notation. Let $\mathcal{F}_\lambda$ be the set of all $(n + \ell - 2)$-tuples $(\eta_1, \ldots, \eta_{n+\ell-2})$ of transpositions such that

(1) $\eta_1 \cdots \eta_{n+\ell-2} = \alpha_\lambda$ and

(2) $\{\eta_1, \ldots, \eta_{n+\ell-2}\}$ generates the symmetric group $\mathcal{S}_n$.

Such tuples are called *minimal transitive factorizations* of the permutation $\alpha_\lambda$ of type $\lambda$, which are related to the branched covers of the sphere suggested by Hurwitz [Hur91, Str96].

In 1997, using algebraic methods Goulden and Jackson [GJ97] proved that

$$|\mathcal{F}_\lambda| = (n + \ell - 2)!\, n^{\ell-3} \prod_{i=1}^{\ell} \frac{\lambda_i^{\lambda_i}}{(\lambda_i - 1)!}. \tag{1}$$

[0] Emails: kimjs@math.umn.edu, shyunseo@kangwon.ac.kr, shin@inha.ac.kr

Bousquet-Mélou and Schaeffer [BMS00] proved a more general formula than (1) and obtained (1) using the principle of inclusion and exclusion. Irving [Irv09] studied the enumeration of minimal transitive factorizations into cycles instead of transpositions.

If $\lambda = (n)$, the formula (1) yields

$$\left|\mathcal{F}_{(n)}\right| = n^{n-2}, \tag{2}$$

and there are several combinatorial proofs of (2) [Bia02, GY02, Mos89].

If $\lambda = (p, q)$, the formula (1) yields

$$\left|\mathcal{F}_{(p,q)}\right| = \frac{pq}{p+q}\binom{p+q}{q}p^p q^q. \tag{3}$$

A few special cases of (3) have bijective proofs: by Kim and Seo [KS03] for the case $(p, q) = (1, n-1)$, and by Rattan [Rat06] for the cases $(p, q) = (2, n-2)$ and $(p, q) = (3, n-3)$. There are no simple combinatorial proofs for other $(p, q)$.

Recently, Goulden et al. [GNO11] showed that the number of maximal chains in the poset $NC^{(B)}(p, q)$ of annular noncrossing partitions of type $B$ is

$$\binom{p+q}{q}p^p q^q + \sum_{c\geq 1} 2c\binom{p+q}{p-c}p^{p-c}q^{q+c}. \tag{4}$$

Interestingly it turns out that half the sum in (4) is equal to the number in (3):

$$\sum_{c\geq 1} c\binom{p+q}{p-c}p^{p-c}q^{q+c} = \frac{pq}{p+q}\binom{p+q}{q}p^p q^q.$$

In this paper we will give a combinatorial proof of (3) using the results in [GNO11]. The rest of this paper is organized as follows. In Section 2 we recall the poset $\mathcal{S}^B_{\mathrm{nc}}(p, q)$ of annular noncrossing permutations of type $B$ which is isomorphic to the poset $NC^{(B)}(p, q)$ of annular noncrossing partitions of type $B$, and show that the number of connected maximal chains in $\mathcal{S}^B_{\mathrm{nc}}(p, q)$ is equal to $\frac{2pq}{p+q}\binom{p+q}{q}p^p q^q$. In Section 3 we prove that there is a 2-1 map from the set of connected maximal chains in $\mathcal{S}^B_{\mathrm{nc}}(p, q)$ to $\mathcal{F}_{(p,q)}$, thus completing a combinatorial proof of (3).

We note that the present paper is part of [KSS12]. In the full version [KSS12] we give another combinatorial proof of (3) by introducing marked annular noncrossing permutations of type $A$.

## 2 Connected maximal chains

A *signed permutation* is a permutation $\sigma$ on $\{\pm 1, \ldots, \pm n\}$ satisfying $\sigma(-i) = -\sigma(i)$ for all $i \in \{1, \ldots, n\}$. We denote by $B_n$ the set of signed permutations on $\{\pm 1, \ldots, \pm n\}$.

We will use the two notations

$$[a_1\ a_2 \ldots a_k] = (a_1\ a_2 \ldots a_k\ -a_1\ -a_2 \ldots -a_k),$$
$$((a_1\ a_2 \ldots a_k)) = (a_1\ a_2 \ldots a_k)(-a_1\ -a_2 \ldots -a_k),$$

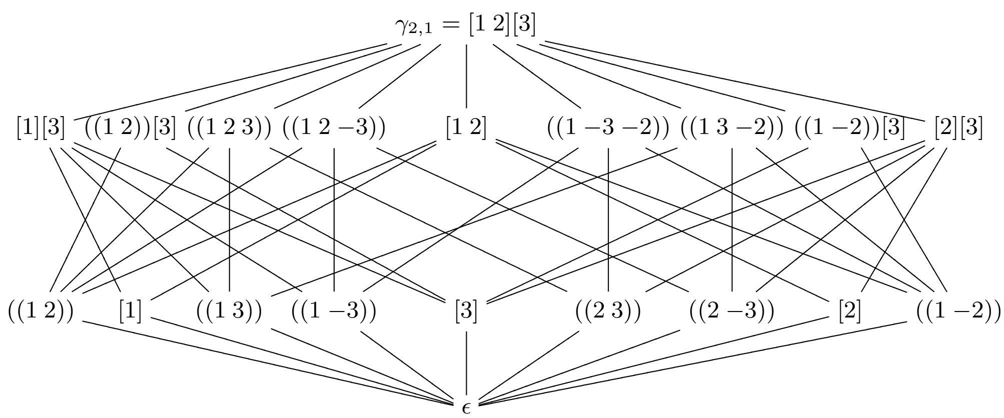


**Fig. 1:** The Hasse diagram for $\mathcal{S}^B_{\mathrm{nc}}(2, 1)$.

and call $[a_1\ a_2 \dots a_k]$ a *zero cycle* and $((a_1\ a_2 \dots a_k))$ a *paired nonzero cycle*. We also call the cycles $\epsilon_i := [i] = (i\ -i)$ and $((i\ j))$ *type $B$ transpositions*, or simply transpositions if there is no possibility of confusion.

For $\pi \in B_n$, the *absolute length* $\ell(\pi)$ is defined to be the smallest integer $k$ such that $\pi$ can be written as a product of $k$ type $B$ transpositions. The *absolute order* on $B_n$ is defined by

$$\pi \le \sigma \quad \Leftrightarrow \quad \ell(\sigma) = \ell(\pi) + \ell(\pi^{-1}\sigma).$$

From now, we fix positive integers $p$ and $q$. The poset $\mathcal{S}^B_{\mathrm{nc}}(p, q)$ of *annular noncrossing permutations of type $B$* is defined by

$$\mathcal{S}^B_{\mathrm{nc}}(p, q) := [\epsilon, \gamma_{p,q}] = \{\sigma \in B_{p+q} : \epsilon \le \sigma \le \gamma_{p,q}\} \subseteq B_{p+q},$$

where $\epsilon$ is the identity in $B_{p+q}$ and $\gamma_{p,q} = [1 \dots p][p+1 \dots p+q]$. Figure 1 shows the Hasse diagram for $\mathcal{S}^B_{\mathrm{nc}}(2, 1)$. Then $\mathcal{S}^B_{\mathrm{nc}}(p, q)$ is a graded poset with rank function

$$\mathrm{rank}(\sigma) = (p + q) - (\#\text{ of paired nonzero cycles of } \sigma). \tag{5}$$

Nica and Oancea [NO09] showed that $\sigma \in \mathcal{S}^B_{\mathrm{nc}}(p, q)$ if and only if $\sigma$ can be drawn without crossing arrows inside an annulus in which the outer circle has integers $1, 2, \dots, p, -1, -2, \dots, -p$ in clockwise order and the inner circle has integers $p+1, p+2, \dots, p+q, -p-1, -p-2, \dots, -p-q$ in counterclockwise order, see Figure 2. They also showed that $\mathcal{S}^B_{\mathrm{nc}}(p, q)$ is isomorphic to the poset $NC^{(B)}(p, q)$ of annular noncrossing partitions of type $B$.

A paired nonzero cycle $((a_1\ a_2 \dots a_k))$ is called *connected* if the set $\{a_1, \dots, a_k\}$ intersects with both $\{\pm 1, \dots, \pm p\}$ and $\{\pm(p + 1), \dots, \pm(p + q)\}$, and *disconnected* otherwise. A zero cycle is always considered to be disconnected. For $\sigma \in \mathcal{S}^B_{\mathrm{nc}}(p, q)$, the *connectivity* of $\sigma$ is the number of connected paired nonzero cycles of $\sigma$.

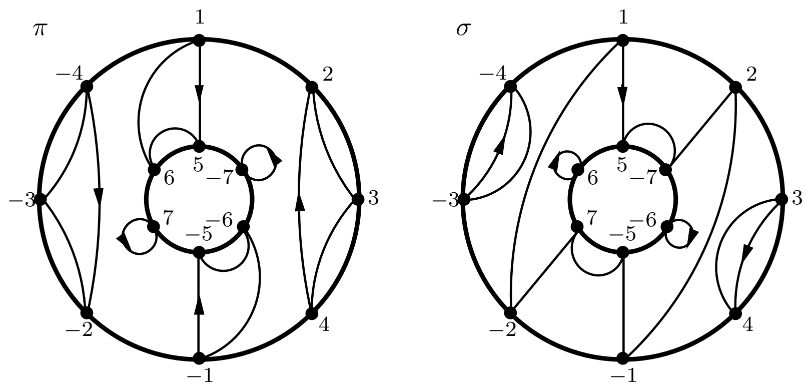


**Fig. 2:** $\pi = ((1\ 5\ 6))((2\ 3\ 4))$ and $\sigma = [1\ 5\ {-7}\ 2]((3\ 4))$ in $\mathcal{S}^B_{\mathrm{nc}}(4,3)$

We say that a maximal chain $C = \{\epsilon = \pi_0 < \pi_1 < \cdots < \pi_{p+q} = \gamma_{p,q}\}$ of $\mathcal{S}^B_{\mathrm{nc}}(p,q)$ is *disconnected* if the connectivity of each $\pi_i$ is zero. Otherwise, $C$ is called *connected*. Denote by $\mathcal{CM}(\mathcal{S}^B_{\mathrm{nc}}(p,q))$ the set of connected maximal chains of $\mathcal{S}^B_{\mathrm{nc}}(p,q)$.

For a maximal chain $C = \{\pi_0 < \pi_1 < \cdots < \pi_n\}$ of the interval $[\pi_0, \pi_n]$, we define $\varphi(C) = (\tau_1, \tau_2, \ldots, \tau_n)$, where $\tau_i = \pi_i^{-1}\pi_{i+1}$. Note that each $\tau_i$ is a type $B$ transposition and $\pi_i = \tau_1\tau_2\cdots\tau_i$ for all $i = 1, 2, \ldots, n$.

**Lemma 1** *If $C$ is a connected maximal chain of $\mathcal{S}^B_{\mathrm{nc}}(p,q)$, then $\varphi(C)$ has no transpositions of the form $\epsilon_i = [i]$ and has at least one connected transposition. If $C$ is a disconnected maximal chain of $\mathcal{S}^B_{\mathrm{nc}}(p,q)$, then $\varphi(C)$ has only disconnected transpositions.*

**Proof:** By (5), $\sigma$ covers $\pi$ in $\mathcal{S}^B_{\mathrm{nc}}(p,q)$ if and only if one of the following conditions holds, see [NO09, Proposition 2.2]:

(a) $\pi^{-1}\sigma = \epsilon_i$ and the cycle containing $i$ in $\pi$ is nonzero, i.e., $\pi$ has $((i\cdots))$ and $\sigma$ has $[i\cdots]$.

(b) $\pi^{-1}\sigma = ((i\ j))$ and no two of $i$, $-i$, $j$, $-j$ belong to the same cycle in $\pi$ with $|i| \neq |j|$, i.e., $\pi$ has $((i\cdots))((j\cdots))$ and $\sigma$ has $((i\cdots j\cdots))$.

(c) $\pi^{-1}\sigma = ((i\ j))$ and the cycle containing $i$ in $\pi$ is nonzero and the cycle containing $j$ in $\pi$ is zero with $|i| \neq |j|$, i.e., $\pi$ has $((i\cdots))[j\cdots]$ and $\sigma$ has $[i\cdots j\cdots]$.

(d) $\pi^{-1}\sigma = ((i\ j))$ and $i$ and $-j$ belong to the same nonzero cycle in $\pi$ with $|i| \neq |j|$, i.e., $\pi$ has $((i\cdots -j\cdots))$ and $\sigma$ has $[i\cdots][-j\cdots]$.

If $\sigma$ covers $\pi$ in $\mathcal{S}^B_{\mathrm{nc}}(p,q)$, we have $\mathrm{zc}(\sigma) \geq \mathrm{zc}(\pi)$, where $\mathrm{zc}(\sigma)$ is the the number of zero cycles in $\sigma$. More precisely we have

$$\mathrm{zc}(\sigma) - \mathrm{zc}(\pi) = \begin{cases} 0 & \text{if type (b) or (c),} \\ 1 & \text{if type (a),} \\ 2 & \text{if type (d).} \end{cases}$$

Since $\gamma_{p,q}$ has two zero cycles, each $\pi \in \mathcal{S}_{\mathrm{nc}}^B(p,q)$ has at most two zero cycles. Moreover, if $\pi$ has two zero cycles, then one of them belongs to $\{\pm, 1, \dots, \pm p\}$ and the other belongs to $\{\pm(p+1), \dots, \pm(p+q)\}$. Consider a maximal chain $C$ in $\mathcal{S}_{\mathrm{nc}}^B(p,q)$.

- If $C$ has a permutation $\pi$ with $\mathrm{zc}(\pi) = 1$, there are two cover relations of type (a) and no cover relations of type (d) in $C$. For each cover relation $\pi < \sigma$ of type (a), (b), or (c), $\sigma$ is obtained by merging cycles in $\pi$. Since $\gamma_{p,q}$ has only disconnected cycles, all permutations in $C$ are disconnected, which implies that $C$ is disconnected.

- Otherwise, there is a cover relation $\pi < \sigma$ of type (d) in $C$. Then $\sigma$ has two zero cycles $[i \cdots]$ and $[-j \cdots]$, one of which is contained in $\{\pm, 1, \dots, \pm p\}$ and the other is contained in $\{\pm(p+1), \dots, \pm(p+q)\}$. Thus $\pi$ has a connected nonzero cycle $((i \cdots -j \cdots))$, and $C$ is connected. Since $C$ has no cover relations of type (a), $\varphi(C)$ has no transposition of the form $\epsilon_i$.

Therefore, if $C$ is a disconnected maximal chain of $\mathcal{S}_{\mathrm{nc}}^B(p,q)$, then $\varphi(C)$ has two transpositions of the form $\epsilon_i$. So all transpositions of $\varphi(C)$ are disconnected. Also, if $C$ is a connected maximal chain of $\mathcal{S}_{\mathrm{nc}}^B(p,q)$, then $\varphi(C)$ has no transposition of the form $\epsilon_i$ and has at least one connected transposition. $\square$

The following proposition is a refinement of (4).

**Proposition 2** *The number of disconnected maximal chains of* $\mathcal{S}_{\mathrm{nc}}^B(p,q)$ *is equal to*

$$\binom{p+q}{q} p^p q^q \tag{6}$$

*and the number of connected maximal chains of* $\mathcal{S}_{\mathrm{nc}}^B(p,q)$ *is equal to*

$$\sum_{c \geq 1} 2c \binom{p+q}{p-c} p^{p-c} q^{q+c}. \tag{7}$$

We now prove the following identity that appears in the introduction. The proof is due to Krattenthaler [Kra].

**Lemma 3** *We have*

$$\sum_{c \geq 1} c \binom{p+q}{p-c} p^{p-c} q^{q+c} = \frac{pq}{p+q} \binom{p+q}{q} p^p q^q. \tag{8}$$

**Proof:** Since $c = p \cdot \frac{q+c}{p+q} - q \cdot \frac{p-c}{p+q}$, we have

$$\begin{aligned}
\sum_{c=0}^{p} c \binom{p+q}{p-c} p^{p-c} q^{q+c} &= \sum_{c=0}^{p} \left( p \cdot \frac{q+c}{p+q} - q \cdot \frac{p-c}{p+q} \right) \binom{p+q}{p-c} p^{p-c} q^{q+c} \\
&= \sum_{c=0}^{p} \left( \binom{p+q-1}{p-c} p^{p-c+1} q^{q+c} - \binom{p+q-1}{p-c-1} p^{p-c} q^{q+c+1} \right) \\
&= \binom{p+q-1}{p} p^{p+1} q^q = \frac{pq}{p+q} \binom{p+q}{p} p^p q^q.
\end{aligned}$$

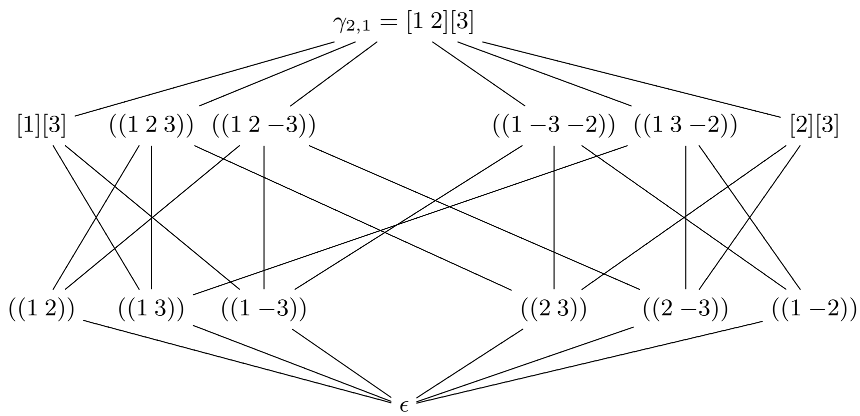


**Fig. 3:** Connected maximal chains in $\mathcal{S}_{\rm nc}^B(2,1)$.

□

By Proposition 2 and Lemma 3, we get the following.

**Corollary 4** *The number of connected maximal chains of* $\mathcal{S}_{\rm nc}^B(p,q)$ *is equal to*

$$\frac{2pq}{p+q}\binom{p+q}{q}p^p q^q. \tag{9}$$

For example, Figure 3 illustrates $16 = \frac{4}{3}\binom{3}{1}2^2$ connected maximal chains of $\mathcal{S}_{\rm nc}^B(2,1)$.

By Corollary 4, in order to prove (3) combinatorially it is sufficient to find a 2-1 map from $\mathcal{CM}(\mathcal{S}_{\rm nc}^B(p,q))$ to $\mathcal{F}_{(p,q)}$. We will find such a map in the next section.

*Remark* 1. One can check that the factorizations $\varphi(C)$ coming from connected maximal chains $C$ in $\mathcal{S}_{\rm nc}^B(p,q)$ are precisely the minimal factorizations of $\gamma_{p,q}$ in the Weyl group $D_{p+q}$. Thus Corollary 4 can be restated as follows: the number of minimal factorizations of $\gamma_{p,q}$ in $D_{p+q}$ is equal to $\frac{2pq}{p+q}\binom{p+q}{q}p^p q^q$. Goupil [Gou95, Theorem 3.1] also proved this result by finding a recurrence relation.

*Remark* 2. Since the proof of Lemma 3 is a simple manipulation, it is easy and straightforward to construct a combinatorial proof for the identity in Lemma 3. Together with the result in Section 3 we get a combinatorial proof of (9). It would be interesting to find a direct bijective proof of (9) without using Lemma 3.

# 3 A 2-1 map from $\mathcal{CM}(\mathcal{S}_{\rm nc}^B(p,q))$ to $\mathcal{F}_{(p,q)}$

Recall that a minimal transitive factorization of $\alpha_{p,q} = (1\dots p)(p+1\dots p+q)$ is a sequence $(\eta_1,\dots,\eta_{p+q})$ of transpositions in $\mathcal{S}_{p+q}$ such that

(1) $\eta_1 \cdots \eta_{p+q} = \alpha_{p,q}$ and

(2) $\{\eta_1,\dots,\eta_{p+q}\}$ generates $\mathcal{S}_{p+q}$,

and $\mathcal{F}_{(p,q)}$ is the set of minimal transitive factorizations of $\alpha_{p,q}$.
In this section we will prove the following theorem.

**Theorem 5** *There is a 2-1 map from the set of connected maximal chains in $\mathcal{S}^B_{\mathrm{nc}}(p,q)$ to the set $\mathcal{F}_{(p,q)}$ of minimal transitive factorizations of $\alpha_{p,q}$.*

In order to prove Theorem 5 we need some definitions.

**Definition 6 (Two maps $(\cdot)^+$ and $|\cdot|$)** *We introduce the following two maps.*

*(1) The map $(\cdot)^+ : B_n \to B_n$ is defined by*

$$\sigma^+(i) = \begin{cases} |\sigma(i)| & \text{if } i > 0,\\ -|\sigma(i)| & \text{if } i < 0.\end{cases}$$

*(2) The map $|\cdot| : B_n \to \mathcal{S}_n$ is defined by $|\sigma|(i) = |\sigma(i)|$ for all $i \in \{1,\dots,n\}$.*

**Definition 7** *A $(p+q)$-tuple $(\tau_1,\dots,\tau_{p+q})$ of transpositions in $B_{p+q}$ is called a* minimal transitive factorization of type $B$ *of $\gamma_{p,q} = [1\dots p][p+1\dots p+q]$ if it satisfies*

*(1) $\tau_1\dots\tau_{p+q} = \gamma_{p,q}$,*

*(2) $\{|\tau_1|,\dots,|\tau_{p+q}|\}$ generates $\mathcal{S}_{p+q}$.*

*Denote by $\mathcal{F}^{(B)}_{(p,q)}$ the set of minimal transitive factorizations of type $B$ of $\gamma_{p,q}$.*

**Definition 8** *A $(p+q)$-tuple $(\sigma_1,\dots,\sigma_{p+q})$ of transpositions in $B_{p+q}$ is called a* positive *minimal transitive factorization of type $B$ of $\beta_{p,q} = ((1\dots p))((p+1\dots p+q))$ if it satisfies*

*(1) $\sigma_1\dots\sigma_{p+q} = \beta_{p,q}$,*

*(2) $\{|\sigma_1|,\dots,|\sigma_{p+q}|\}$ generates $\mathcal{S}_{p+q}$,*

*(3) $\sigma_i = \sigma_i^+$ for all $i = 1,\dots,p+q$.*

*Denote by $\mathcal{F}^+_{(p,q)}$ the set of positive minimal transitive factorizations of type $B$ of $\beta_{p,q}$.*

For the rest of this section we will prove the following:

1. The map $\varphi : \mathcal{CM}(\mathcal{S}^B_{\mathrm{nc}}(p,q)) \to \mathcal{F}^{(B)}_{(p,q)}$ is a bijection. (Lemma 9)
2. There is a 2-1 map $(\cdot)^+ : \mathcal{F}^{(B)}_{(p,q)} \to \mathcal{F}^+_{(p,q)}$. (Lemma 11)
3. There is a bijection $|\cdot| : \mathcal{F}^+_{(p,q)} \to \mathcal{F}_{(p,q)}$. (Lemma 10)

By the above three statements the composition $|\varphi^+| := |\cdot| \circ (\cdot)^+ \circ \varphi$ is a 2-1 map from $\mathcal{CM}(\mathcal{S}^B_{\mathrm{nc}}(p,q))$ to $\mathcal{F}_{(p,q)}$, which completes the proof of Theorem 5. Since the proofs of the first and the third statements are simpler, we will present these first.

**Lemma 9** *The map* $\varphi : \mathcal{CM}(\mathcal{S}_{\mathrm{nc}}^{B}(p,q)) \to \mathcal{F}_{(p,q)}^{(B)}$ *is a bijection.*

**Proof:** Given a connected maximal chain $C = \{\epsilon = \pi_0 < \pi_1 < \cdots < \pi_{p+q} = \gamma_{p,q}\}$ in $\mathcal{S}_{\mathrm{nc}}^{B}(p,q)$, the elements in the sequence $\varphi(C) = (\tau_1, \ldots, \tau_{p+q})$ are transpositions with $\tau_1 \cdots \tau_{p+q} = \gamma_{p+q}$. By Lemma 1, at least one of $\tau_i$'s is connected. Thus $\{|\tau_1|, \ldots, |\tau_{p+q}|\}$ generates $\mathcal{S}_{p+q}$, and $\varphi(C) \in \mathcal{F}_{(p,q)}^{(B)}$. Conversely, if $\tau = (\tau_1, \ldots, \tau_{p+q}) \in \mathcal{F}_{(p,q)}^{(B)}$, then $\varphi^{-1}(\tau) = \{\epsilon = \pi_0 < \pi_1 < \cdots < \pi_{p+q} = \gamma_{p,q}\}$, where $\pi_i = \tau_1 \cdots \tau_i$, is a connected maximal chain in $\mathcal{S}_{\mathrm{nc}}^{B}(p,q)$ because $\{|\tau_1|, \ldots, |\tau_{p+q}|\}$ generates $\mathcal{S}_{p+q}$. □

**Lemma 10** *There is a bijection* $|\cdot| : \mathcal{F}_{(p,q)}^{+} \to \mathcal{F}_{(p,q)}$.

**Proof:** Let $(\sigma_1, \ldots, \sigma_{p+q}) \in \mathcal{F}_{(p,q)}^{+}$. Each $\sigma_i$ can be written as $\sigma_i = ((j\ k))$ for some positive integers $j$ and $k$. In this case we let $\eta_i = |\sigma_i| = (j\ k) \in S_{p+q}$. Then the map $|\cdot| : \mathcal{F}_{(p,q)}^{+} \to \mathcal{F}_{(p,q)}$ sending $(\sigma_1, \ldots, \sigma_{p+q})$ to $(\eta_1, \ldots, \eta_{p+q})$ is a bijection. □

Recall $\epsilon_i = [i] = (i\ \ -i)$. We write $\overline{((i\ j))} := ((i\ -j))$. It is easy to see that for $i, j \in \{\pm 1, \ldots, \pm(p+q)\}$, we have

$$[i\ j] = \epsilon_i((i\ j)) = ((i\ j))\epsilon_j = \overline{((i\ j))}\epsilon_i = \epsilon_j\overline{((i\ j))}. \tag{10}$$

**Lemma 11** *There is a 2-1 map* $(\cdot)^{+} : \mathcal{F}_{(p,q)}^{(B)} \to \mathcal{F}_{(p,q)}^{+}$.

Here we only describe the map $(\cdot)^{+}$: For $(\tau_1, \tau_2, \ldots, \tau_{p+q}) \in \mathcal{F}_{(p,q)}^{(B)}$, we define $(\tau_1, \tau_2, \ldots, \tau_{p+q})^{+} = (\tau_1^{+}, \tau_2^{+}, \ldots, \tau_{p+q}^{+})$. Since $\tau_1^{+} \ldots \tau_{p+q}^{+} = \gamma_{p,q}^{+} = \beta_{p,q}$, we have $(\tau_1, \tau_2, \ldots, \tau_{p+q})^{+} \in \mathcal{F}_{(p,q)}^{+}$. Let us fix $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{p+q}) \in \mathcal{F}_{(p,q)}^{+}$. If $\tau = (\tau_1, \ldots, \tau_{p+q}) \in \mathcal{F}_{(p,q)}^{(B)}$ satisfies $\tau^{+} = \sigma$, then $\tau' = (\tau_1' \ldots \tau_{p+q}') \in \mathcal{F}_{(p,q)}^{(B)}$ defined by

$$\tau_i' = \begin{cases} \tau_i & \text{if } \tau_i \text{ is disconnected} \\ \overline{\tau_i} & \text{if } \tau_i \text{ is connected,} \end{cases} \tag{11}$$

also satisfies $(\tau')^{+} = \sigma$. Thus the map $(\cdot)^{+}$ is two-to-one. For the detailed proof, see [KSS12].

For example, let $\sigma = (\ ((1\ 2)), ((2\ 5)), ((2\ 3)), ((4\ 5)), ((3\ 4))\ ) \in \mathcal{F}_{(3,2)}^{+}$ be the following factorization

$$\beta_{3,2} = ((1\ 2\ 3))((4\ 5)) = ((1\ 2))\ ((2\ 5))\ ((2\ 3))\ ((4\ 5))\ ((3\ 4)).$$

Since $\gamma_{3,2} = \epsilon_4\ \epsilon_1\ \beta_{3,2}$, we can obtain a factorization of $\gamma_{3,2}$ from $\sigma$ as follows:

$$\begin{aligned}
\gamma_{3,2} = [1\ 2\ 3][4\ 5] &= \epsilon_4\ \epsilon_1\ ((1\ 2))\ ((2\ 5))\ ((2\ 3))\ ((4\ 5))\ ((3\ 4)) \\
&= \epsilon_4\ \epsilon_2\ \overline{((1\ 2))}\ ((2\ 5))\ ((2\ 3))\ ((4\ 5))\ ((3\ 4)) \\
&= \epsilon_4\ \epsilon_3\ ((1\ 2))\ \overline{((2\ 5))}\ \overline{((2\ 3))}\ ((4\ 5))\ ((3\ 4)) \\
&= \epsilon_4\ \epsilon_4\ ((1\ 2))\ \overline{((2\ 5))}\ ((2\ 3))\ \overline{((4\ 5))}\ \overline{((3\ 4))} \\
&= ((1\ 2))\ \overline{((2\ 5))}\ ((2\ 3))\ \overline{((4\ 5))}\ \overline{((3\ 4))}.
\end{aligned}$$

Thus $\tau = \Big( ((1\,2)), \overline{((2\,5))}, ((2\,3)), \overline{((4\,5))}, \overline{((3\,4))} \Big) \in \mathcal{F}^{(B)}_{(3,2)}$ satisfies $\tau^+ = \sigma$. The factorization $\tau' = \Big( ((1\,2)), ((2\,5)), ((2\,3)), \overline{((4\,5))}, ((3\,4)) \Big)$ obtained by toggling the connected transpositions of $\tau$ also satisfies $(\tau')^+ = \sigma$.

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