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Circular Separation Dimension of a Subclass of Planar Graphs

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A pair of non-adjacent edges is said to be *separated* in a circular ordering of vertices, if the endpoints of the two edges do not alternate in the ordering. The *circular separation dimension* of a graph G, denoted by $\pi^\circ(G)$, is the minimum number of circular orderings of the vertices of G such that every pair of non-adjacent edges is separated in at least one of the circular orderings. This notion is introduced by Loeb and West in their recent paper. In this article, we consider two subclasses of planar graphs, namely 2-outerplanar graphs and series-parallel graphs. A 2-outerplanar graph has a planar embedding such that the subgraph obtained by removal of the vertices of the exterior face is outerplanar. We prove that if G is 2-outerplanar then $\pi^\circ(G) = 2$. We also prove that if G is a series-parallel graph then $\pi^\circ(G) \leq 2$.

Keywords: Circular separation dimension, planar graph, 2-outerplanar graph, series-parallel graph

1 Introduction

Basavaraju et al. (2016) introduced the notion of separation dimension of a graph. *Separation dimension* $\pi(G)$ of a graph $G = (V, E)$, denotes the minimum number of linear orderings of the vertices in V required to "separate" all non-adjacent edges in E. Here, a pair (e, f) of edges is separated in an ordering if both vertices of e appear after/before both vertices of f.

Very recently, Loeb and West (2016) introduced a similar terminology circular separation dimension. The *circular separation dimension* of a graph $G = (V, E)$, denoted by $\pi^\circ(G)$, is the minimum number of circular orderings of V needed to separate all non-adjacent edges in E. Here, a pair of edges is separated in an ordering if the endpoints of the two edges do not alternate in that ordering. As a pair of edges separated in a linear ordering are also separated in the corresponding circular ordering, we know that $\pi^\circ(G) \leq \pi(G)$. Another variant of the dimensional problem that has been recently studied is the *induced separation dimension* by Ziedan et al. (2016).

Addition of edges in a graph does not decrease both the parameters π and π°; this property is referred to as *monotonicity*. Thus, for a graph with n vertices, these parameters achieve the maximum value when the graph is a complete graph K_n. Basavaraju et al. (2016) showed that in general for any graph with n vertices, $\log_2(\lfloor \frac{1}{2}\omega(G) \rfloor) \leq \pi(G) \leq 4 \log_{\frac{3}{2}} n$. Loeb and West (2016) proved that $\pi^\circ(G) > \log_2 \log_3(\omega(G) - 1)$. Here $\omega(G) = \max\{t | K_t \subseteq G\}$.

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The ranges of both the parameters were studied for special graph classes as well. Alon et al. (2015) showed that $\pi(G) \leq 2^{9log^{\star}d}d$ for a graph G whose degree is bounded by d. They also proved that for almost all d regular graphs, $\pi(G)$ is at least $\lceil d/2 \rceil$. For a complete bipartite graph $K_{m,n}$, $\pi(K_{m,n}) \geq \log_2 \min\{m, n\}$ Basavaraju et al. (2016), and $\pi^{\circ}(K_{m,n}) = 2$ Loeb and West (2016).

If G is a planar graph, $\pi(G) \leq 3$ Basavaraju et al. (2016). Clearly from the previous observation *i.e.* $\pi^{\circ}(G) \leq \pi(G)$, we have that $\pi^{\circ}(G) \leq 3$ when G is planar. On the other hand, it is interesting to note that $\pi^{\circ}(G) = 1$ if and only if G is outerplanar. This result is again by Loeb and West (2016). Thus, it follows easily for planar graphs that are not outerplanar, that $\pi^{\circ}(G)$ is either 2 or 3. In this context, we conjecture the following.

Conjecture. *The circular separation dimension of a planar graph is at most two.*

Note that even for K_4, the smallest 2-outerplanar graph, we have $\pi^{\circ}(K_4) > 1$. On the other hand, from the result of Basavaraju et al. (2016), we know that $\pi(K_4) = 3$. But two circular permutations are enough to separate the edges of K_4 (if $\{a, b, c, d\} = V(K_4)$ then the permutations (a, b, c, d) and (a, c, b, d) separate all pairs of non-adjacent edges).

A planar graph is said to be k-outerplanar, $k \geq 2$, if it has a planar embedding such that by removing the vertices on the unbounded face we obtain a $(k-1)$-outerplanar graph. It is a well known fact that every planar graph is k-outerplanar for some integer k (typically, much smaller than n) Bienstock and Monma (1990). Thus, it is natural to investigate the circular separation dimension of 2-outerplanar graphs, and to see whether one can generalize the result for k-outerplanar graphs. In Section 2, we prove that the circular separation dimension of a 2-outerplanar graph is two.

It is interesting to note that for series-parallel graphs, which is k-outerplanar for some positive integer k, two circular permutations are sufficient to separate all non-adjacent pairs of edges. We discuss this result in Section 3.

1.1 Preliminaries

A graph is *outerplanar* if it has a planar embedding such that all vertices are on the outer face. A graph G is 2-*outerplanar* if G has a planar embedding such that the subgraph obtained by removal of the vertices of the exterior face is outerplanar. In this paper, we consider G with its 2-outerplanar embedding.

A graph is a *series-parallel* graph if it can be turned into a K_2 by a sequence of the following operations.

> Replacement of a pair of parallel edges with a single edge that connects their common endpoints. This operations is called a *parallel operation*.
>
> Replacement of a pair of edges incident to a vertex of degree 2 other than two distinguished vertices, called source and sink vertex, with a single edge. This operation is called a *series operation*.

A *cut vertex* of a connected graph is a vertex whose deletion disconnects the graph. A connected graph is *biconnected* if it requires deletion of at least two vertices to disconnect the graph. We consider a *block* of a graph to be the vertex set of a maximal biconnected subgraph. Other terminologies, which are not defined here, can be found in Diestel (2006).

1.2 Definitions and Notations

Let $G = (V, E)$ be a graph with vertex set V and edge set E. Let $\sigma : V \rightarrow \{1, 2, \ldots, |V|\}$ be a permutation of elements of the vertex set V. A permutation $\sigma = (v_1, v_2, \ldots, v_k)$ of V implies that

$\sigma(v_i) < \sigma(v_j),\ 1 \le i < j \le k$. A *sub-permutation* σ' of σ restricted to $V' \subseteq V$ is defined as $\sigma' : V' \to \{1, 2, \ldots, |V'|\}$ such that $\sigma'(v_i) < \sigma'(v_j)$ if and only if $\sigma(v_i) < \sigma(v_j)$ for all $v_i, v_j \in V'$. We define $reversal(\sigma) = (v_k, v_{k-1}, \ldots, v_2, v_1)$. If $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_m)$ are two permutations, then the permutation (α, β) is defined as $(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)$ where $(\alpha, \beta)(a_i) = \alpha(a_i)$ and $(\alpha, \beta)(b_i) = \beta(b_i) + n$. A pair of non-adjacent edges in G is *separated* by a circular ordering or circular permutation of V if the endpoints of the two edges do not alternate. If two non-adjacent edges are not separated, then they are said to *cross* each other in the permutation. If a pair of edges cross each other in the first permutation and are separated in the second, then the second permutation is said to *resolve* these pair of edges. A family σ of circular permutations of V is called *pairwise suitable* for G if, for every pair of non-adjacent edges in G, there exists a permutation in σ in which the edges are separated. The *circular separation dimension* of a graph G, denoted by $\pi^\circ(G)$, is the minimum cardinality of such a family. We assume that every graph under consideration is connected, as $\pi^\circ(G) = \max\{\pi^\circ(H) \mid H \text{ is a component of } G\}$. We use $N(v)$ to denote the set of all neighbours of v.

2 2-Outerplanar graphs

In this section, we prove the following main theorem.

Theorem. *The circular separation dimension of a* 2*-outerplanar graph is exactly two.*

We use the following result on outerplanar graphs which was proved by Loeb and West.

Lemma 2.1 (Loeb and West (2016))**.** *If G is a maximal outerplanar graph, then any circular ordering of vertices following the outer face produces a circular permutation that separates all pairs of non-adjacent edges of G.*

Let $G = (V, E)$ be a maximal 2-outerplanar graph in the sense that every face is a triangle except the outer face. We fix a planar embedding of G and our proofs are based on such an embedding. Let V_2 be the set of vertices appearing on the exterior face of G, and $V_1 = V \setminus V_2$. Let $n_i = |V_i|$ for $i \in \{1, 2\}$. Let E_i be the set of edges in $[V_i]$. Clearly, each $[V_i]$ is a 1-outerplanar graph. Let $E_{12} = E(G) \setminus (E_1 \cup E_2)$ (this is exactly the set of edges with one end point in V_1 and the other end point in V_2). Thus, we have a partition of the edge set of G as $E(G) = E_1 \cup E_2 \cup E_{12}$. To verify if a family of circular permutations is pairwise suitable, we check if there can be any *(i)* $E_1 - E_1$ crossing *(ii)* $E_2 - E_2$ crossing *(iii)* $E_1 - E_2$ crossing *(iv)* $E_1 - E_{12}$ crossing *(v)* $E_{12} - E_2$ crossing *(vi)* $E_{12} - E_{12}$ crossing in both permutations. Let a vertex of V_i be denoted as s^i.

Note that if $[V_2]$ contains a cut vertex, then by adding edges on the outer face between neighbours of the cut vertex (retaining planarity), one can make $[V_2]$ biconnected (see Figure 1). The addition of edges does not decrease the value of π° (because of its monotonicity property). Thus, we only consider the case when the graph $[V_2]$ is biconnected.

We first introduce a technique which is often used to resolve $E_1 - E_{12}$ crossing called the "arc-removal" technique.

Let G be a maximal 2-outerplanar graph such that $[V_1]$ is biconnected. Let σ' be a permutation of V_1 according to Lemma 2.1. Similarly, let $(s^2_1, s^2_2, \ldots, s^2_{n_2})$ be a permutation of V_2. Let $\sigma_1 = (s^2_1, \sigma', s^2_2, s^2_3, \ldots, s^2_{n_2})$. Suppose $e = (x, y) \in E_1$ where $\sigma'(x) < \sigma'(y)$. Let $f \in E_{12}$ cross e in σ_1. Observe that each $v \in V_1$ such that $\sigma'(x) < \sigma'(v) < \sigma'(y)$ is a "candidate" for an endpoint of f. To

(a) $[V_2]$ is not biconnected.

(b) $[V_2]$ is biconnected.

Fig. 1: Suitable addition of edges when $[V_2]$ contains a cut vertex.

separate e and f, we construct a permutation σ'' where each v appears either before or after both x and y. We need to do this for all of the edges in E_1. Here we use the following fact: As all the non-adjacent pairs of edges in E_1 are separated in σ', all of the edges in E_1 form a "well-parenthesis" structure in the ordering σ' where each parenthesis represents an edge (see Figure 2(a)). So, we can "unfold" the "parenthesis" one after another (see Figure 2(b)).

The arc-removal technique takes as input the permutation σ' and a parameter $p \in \{r, l\}$ which decides whether we need to maintain the position of the right most vertex or the left most vertex of σ'. The arc-removal technique fixing the right end is explained as an algorithm below.

Input: Edge separating permutation $\sigma' = (v_1, v_2, \ldots, v_{n_1})$ of vertices in $[V_1]$ and a parameter $p \in \{l, r\}$.
Output: A permutation σ'' of $\{v_1, v_2, \ldots, v_{n_1}\}$.
If $p = r$, then we perform the following.
Initialization: Set $i = n_1$ and set $k = n_1$. Mark every vertex as `not selected`. Let σ'' be an empty ordering.

1. If v_i is `selected` go to step 4. If v_i is `not selected` check if v_i has already appeared in σ''. If not, set $\sigma''(v_i) = k$. Set $k = k - 1$.

2. Choose minimum j for all $v_j \in N(v_i)$ where v_j does not appear in σ''. Set $\sigma''(v_j) = k$ and $k = k - 1$.

(a) Before arc-removal technique.

(b) After arc-removal technique fixing the right end.

Fig. 2: Resolving $E_1 - E_{12}$ crossing.

3. Repeat step 2 till all the vertices in $N(v_i)$ appear in σ''. Mark v_i as `selected`.

4. Set $i = i - 1$. Go to step 1 if $k > 0$.

If $p = l$, then we perform a similar technique as described above. The only differences are: *(i)* In the initialization, we set both $i, k = 1$ in order to fix the left end; *(ii)* Instead of decrementing, we increment these variables in steps 1, 2 and 4; *(iii)* In step 2, we choose the maximum j among neighbours `not selected`; *(iv)* In step 4, we go to step 1 if $k < n_1$.

Claim 1. *Let σ' be a permutation of V_1 that separates all non-adjacent pairs of edges in $[V_1]$, and $\sigma_1 = (s_1^2, \sigma', s_2^2, s_3^2, \ldots, s_{n_2}^2)$. Then the permutation $\sigma'' = (arc\text{-}removal(\sigma', p), s_1^2, s_2^2, ..., s_{n_2}^2)$ resolves every pair of crossing edges e, f, where $e \in E_{12}, f \in E_1$ and e, f are crossing in σ_1, and $p \in \{l, r\}$.*

Proof: Following our notation, suppose $e = (s_i^1, s_j^2)$ and $f = (s_l^1, s_m^1)$ where without loss of generality, we can assume that $1 \leq l < i < m \leq n_1, 1 \leq j \leq n_2$. Suppose $p = r$. Step 2 of the algorithm ensures that s_i^1 appears before s_l^1 and s_m^1. Thus in σ'', s_l^1 and s_m^1 appear in between s_i^1 and s_j^2 and therefore e, f are separated. It should be noted that s_i^1 cannot be listed in σ'' before s_m^1 is listed, as this would mean there is an edge $g = (s_i^1, s_h^1)$ such that $m < h$, a contradiction as f, g would cross in σ'. When $p = l$, s_l^1 and s_m^1 appear before s_i^1 and s_j^2 in σ''. □

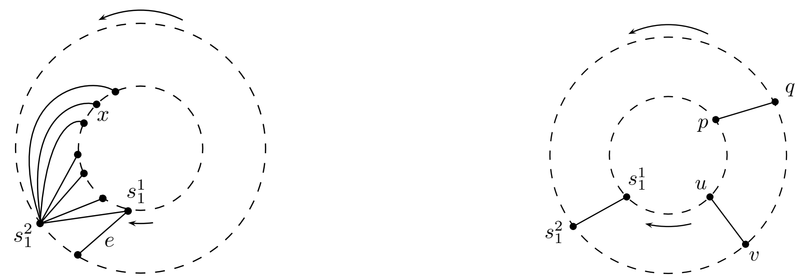
Claim 2. *Let σ' be a permutation of V_1 that separates all non-adjacent pairs of edges in $[V_1]$, and $\sigma_1 = (s_1^2, \sigma', s_2^2, s_3^2, \ldots, s_{n_2}^2)$. Then the permutation $\sigma'' = (reversal(arc\text{-}removal(\sigma', p)),\ s_1^2, s_2^2, ..., s_{n_2}^2)$ resolves every pair of crossing edges e, f, where $e \in E_{12}, f \in E_1$ and e, f are crossing in σ_1, and $p \in \{l, r\}$.*

Proof: Consider edges e, f as in Claim 1. Suppose $p = r$. Then $(arc\text{-}removal(\sigma', p)),\ s_1^2, s_2^2, ..., s_{n_2}^2)$ resolves e, f crossing as s_l^1 and s_m^1 appear in between s_i^1 and s_j^2 due to Claim 1. The reversal would now imply that s_l^1 and s_m^1 appear before s_i^1 and s_j^2 in σ''. If $p = l$, then s_l^1 and s_m^1 appear in between s_i^1 and s_j^2 in σ''. □

We now prove two lemmas that help us to prove the main theorem of this section.

Lemma 2.2. *Let G be a maximal 2-outerplanar graph. If the graph induced on V_1 is biconnected, then $\pi^\circ(G) = 2$.*

Proof: Let s_1^2 be a vertex in V_2 that has a neighbour x in V_1 (such a vertex exists, else G is disconnected). We name s_1^2 as the *start vertex* of G.
Choosing s_1^1: Upon moving counter-clockwise from x on the outer boundary of $[V_1]$, let e (see Figure 3(a)) be the first edge of E_{12} that is seen after s_1^2x that does not have an endpoint in s_1^2. Such an edge exists, else all vertices of V_1 are adjacent to s_1^2 and no other vertex of V_2, which contradicts the maximality of G (note that $n_2 \geq 3$ as V_2 are the vertices of the outer face, see Figure 3(a)). Let the endpoint of e in V_1 be s_1^1 (it could very well be possible that $x = s_1^1$). Note that $s_1^1s_1^2$ is an edge due to the maximality of G.
Defining e': Upon moving clockwise from s_1^1 on the outer face of $[V_1]$, let e' be the first edge of E_{12} that is seen after $s_1^1s_1^2$. Then the edge e' either has s_1^2 or s_1^1 as an endpoint due to the maximality of G.



(a) Existence of e and s_1^1.

(b) Edges of B do not cross each other in σ_1.

Fig. 3: The edges between the layers V_1 and V_2.

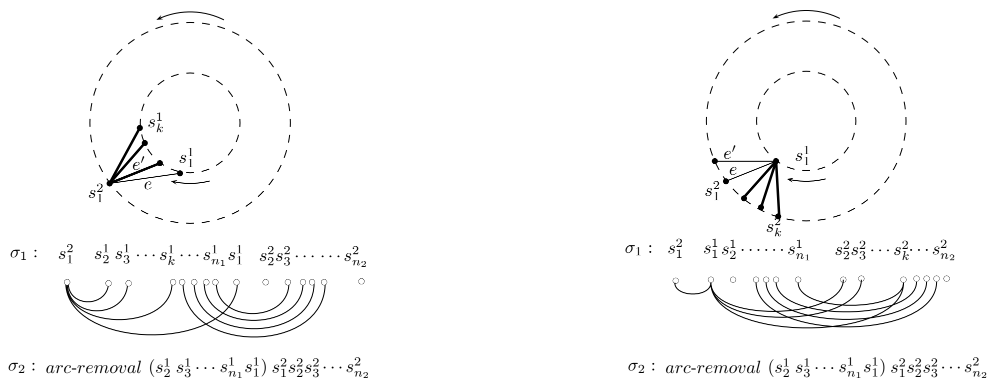
Let $s_1^1, s_2^1, ..., s_{n_1}^1$ be the vertices of $[V_1]$ traversed in a clockwise order on its boundary (it is useful to know that a biconnected outerplanar graph is Hamiltonian Sysło (1979)). Similarly, let $s_1^2, s_2^2, ..., s_{n_2}^2$ be the vertices of $[V_2]$ traversed in an anti-clockwise order on its boundary. Depending on whether e' is incident with s_1^1 or s_1^2, consider the first circular permutation $\sigma_1 = (s_1^2, s_2^1, s_3^1, s_4^1, ..., s_{n_1}^1, s_1^1, s_2^2, ..., s_{n_2}^2)$ or $\sigma_1 = (s_1^2, s_1^1, s_2^1, s_3^1, ..., s_{n_1}^1, s_2^2, ..., s_{n_2}^2)$. We now observe which pair of edges from E_1, E_2, E_{12} can or cannot cross in both the cases of σ_1.

$E_1 - E_1$ crossing, $E_2 - E_2$ crossing, $E_1 - E_2$ crossing: It can be observed that in σ_1 there is no pairwise crossing of edges that are both taken from E_i, since $[V_i]$ is 1-outerplanar and thus applying Lemma 2.1. Clearly, no edge of E_1 crosses any edge of E_2 in σ_1.

Now, we produce two pairwise suitable permutations based on two cases of e' as follows:

Case 1: e' has s_1^2 as an endpoint (see Figure 4(a)).

Let $\alpha = (s_2^1, s_3^1, s_4^1, \ldots, s_{n_1}^1, s_1^1)$. We define $\sigma_1 = (s_1^2, \alpha, s_2^2, s_3^2, \ldots, s_{n_2}^2)$ and $\sigma_2 = (arc\text{-}removal(\alpha, r)$ $, s_1^2\ , s_2^2, \ldots, s_{n_2}^2)$. We now justify that σ_2 is a permutation that resolves crossing edges in σ_1.
$E_1 - E_{12}$ crossing: The crossings between edges of E_1 and E_{12} are resolved by performing the arc removal technique on α (see Claim 1).
$E_{12} - E_2$ crossing: It is not possible that an edge from E_{12} crosses an edge from E_2 in σ_2 except for $s_1^2 s_{n_2}^2$, due to the planarity of G. The crossing of edges with $s_1^2 s_{n_2}^2$ is already resolved in σ_1.
$E_{12} - E_{12}$ crossing: It is not possible for any edge to cross $s_1^1 s_1^2$ as s_1^1 and s_1^2 appear consecutively in both σ_1 and σ_2. Let A denote the set of edges from E_{12} that are incident on s_1^2 except for the edge $s_1^1 s_1^2$. Then there exists an integer k such that the endpoints of the edges of A in V_1 are $A' = \{s_2^1, s_3^1, \ldots, s_{k-2}^1, s_{k-1}^1, s_k^1, \}$ (it might very well be possible that $k = 2$).
It is not possible for an edge from A to cross an edge from $B = E_{12} \setminus (A \cup \{s_1^1 s_1^2\})$ in σ_1 as all end points of edges in A appear together. We observe that no pair of edges from B cross each other in σ_1. This is because of the following: suppose $pq, uv \in B$ and $p, u \in V_1$ and $q, v \in V_2$. Without loss of generality, we can assume $\sigma_1(p) < \sigma_1(u)$ (see Figure 3(b)). This implies that $\sigma_1(v) < \sigma_1(q)$, else pq and uv cross in the planar embedding of G. Since neither q nor v can be s_1^2, the vertices p, u appear before v, q in σ_1.



(a) e' has s_1^2 as an endpoint. (b) e' has s_1^1 as an endpoint.

Fig. 4: Pairwise suitable family when $[V_1]$ is biconnected (edges of set A are marked in bold, $arc\text{-}rem$ and rev denote arc-removal and reversal respectively).

Hence we have $\sigma_1(p) < \sigma_1(u) < \sigma_1(v) < \sigma_1(q)$, thus separating pq and uv in σ_1.

Case 2: e' has s_1^1 as an endpoint (see Figure 4(b)).

For this case, let $\alpha = (s_1^1, s_2^1, s_3^1, \ldots, s_{n_1}^1)$. We define $\sigma_1 = (s_1^2, \alpha, s_2^2, s_3^2, \ldots, s_{n_2}^2)$. Define $\sigma_2 = (reversal\ (arc\text{-}removal(\alpha, l)), s_1^2\ , s_2^2, \ldots, s_{n_2}^2)$. We now justify that σ_2 is a permutation that resolves all of the crossing edges in σ_1.

$E_{12} - E_2$ *crossing:* The same reason as Case 1 holds.

$E_{12} - E_{12}$ *crossing:* On moving counter-clockwise on the boundary of $[V_2]$ starting from s_1^2, let s_k^2 be the first vertex that has a neighbour in V_1 other than s_1^1 (it might very well be possible that $k = 2$). Due to maximality of G, s_1^1 is adjacent to s_k^2. Let A denote the set of edges between s_1^1 and $s_2^2, s_3^2, \ldots, s_k^2$. Just as in Case 1, no pair of edges from $B = E_{12} \setminus (A \cup \{s_1^1 s_1^2\})$ cross each other in σ_1. Thus the only pair of edges from E_{12} that can cross is an edge from A and an edge from B. These crossings are resolved by shifting s_1^1 to just before s_1^2 as given in σ_2. We retain this position of s_1^1 to prevent these edges from crossing again.

$E_1 - E_{12}$ *crossing:* These crossings are resolved by performing the arc removal technique on $\alpha = (s_1^1, s_2^1, \ldots, s_{n_1}^1)$ by fixing the left end and finally reversing this permutation.

□

Lemma 2.3. *Let G be a maximal 2-outerplanar graph. If the graph induced on V_1 is connected, then $\pi^\circ(G) = 2$.*

Proof: We consider that case where $[V_1]$ is not biconnected. We follow a technique of listing the vertices that is almost identical to the one in Lemma 2.2. Let s_1^2 be a vertex in V_2 that has a neighbour x in V_1.

Choosing s_1^1, and defining e': This is done as in Lemma 2.2.

Labelling vertices and blocks: Consider a clockwise walk along the outer face of $[V_2]$ starting from s_1^2. As we move along the outer face, we see one set of endpoints of edges in E_{12}. Consider the order in which

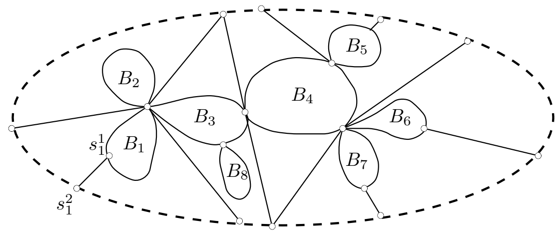


Fig. 5: Edges of B do not cross each other in σ_1.

the corresponding endpoints in V_1 are seen, which would also be clockwise along the outer face of $[V_1]$. We label the vertices of V_1 starting from s_1^1 in this order. We preserve the first occurrence of cut vertices and ignore repeated instances. Hence, in order to obtain the first permutation σ_1, we traverse clockwise along the boundary of the outer face of $[V_1]$ and in a counter-clockwise manner along the boundary of the outer face of $[V_2]$ (as in the previous lemma). We now aim to construct the second permutation through repeated contraction of blocks.

Label each block as B_i seen in the order of the clockwise walk taken on the outer boundary of $[V_1]$ (see Figure 5). We claim that blocks of $[V_1]$ can be contracted in a certain manner to obtain a biconnected outerplanar graph. The block B_i labelled last (*i.e.* the block with the largest label i) can share at most one cut vertex with other blocks. Let G_{i-1} be the graph obtained upon contracting B_i into the cut vertex, (let the vertex remaining after contraction retain the label of the cut vertex). Hence, the inner layer of G_{i-1} induces a maximal outerplanar graph with $i-1$ blocks, where B_{i-1} is a block that shares at most one cut vertex with the remaining blocks. We continue contracting successively (producing graphs G_k with k blocks) till we obtain a maximal 2-outerplanar graph $G_1 = G'$ with B_1 as the vertex set of its inner layer. Thus the inner layer induces a single biconnected component $[B_1]$. Hence, G' satisfies the constraints of Lemma 2.3. Let σ_2' be the second permutation of $V(G')$ according to Lemma 2.3. We now obtain two permutations for $V(G)$: the first from the walk described above and the second from σ_2'.

Constructing σ_1: We define $\sigma_1 = (s_1^2, \textit{permutation of } V_1, s_2^2, s_3^2, \ldots, s_{n_2}^2)$ where the permutation of V_1 is listed according to the walk described above while labelling the vertices. It is to be noted that this walk either begins with s_2^1 or s_1^1 based on whether e' has s_1^2 as en endpoint or s_1^1 as an endpoint respectively (see Lemma 2.2).
Constructing σ_2: Consider the graph G_2. Let s_c^1 be the cut vertex shared by B_2 and B_1. Suppose in σ_1, vertices of B_2 are seen in the relative order $s_c^1, s_b^1, s_{b+1}^1, s_{b+2}^1, \ldots, s_{b+|B_2|-2}^1$, where s_b^1 is the first vertex of B_2 seen after s_c^1. Let $\alpha = (s_c^1, s_b^1, s_{b+1}^1, s_{b+2}^1, \ldots, s_{b+|B_2|-2}^1)$. We construct the second permutation by replacing s_c^1 in σ_2' by $reversal(arc\text{-}removal(\alpha, l))$. Hence, to construct a permutation of $V(G_i)$, we replace the cut vertex in B_i by a permutation of block B_i in a similar manner. We continue inserting such permutations of blocks until we obtain a permutation of $V(G)$, which we fix as the second permutation σ_2.

We prove the lemma by induction on the number of blocks of $[V_1]$. If $[V_1]$ has only one block (and hence no cut vertex), it is easy to see that the permutations of G are as obtained from the previous lemma. Suppose the lemma holds for all maximal 2-outerplanar graphs whose inner layer induces at most $j-1$

(a) Edges e, f crossing in σ_1.

(b) Edges e, f not crossing in σ_2.

Fig. 6: $E_{12} - E_{12}$ crossing

blocks. Let G be a maximal 2-outerplanar graph whose inner layer induces exactly j blocks. Let s_1^2 be a start vertex of G. We construct σ_1 as explained earlier. We find the last block B_j containing only one cut vertex, say s_c^1, and contract B_j to produce the graph G'. Let the vertex set of the inner layer of G' be B'. Since the induction hypothesis can be applied to G', let $\sigma^{'} = \{\sigma_1^{'}, \sigma_2^{'}\}$ be the two permutations separating the edges of G' as described above, with s_1^2 as the start vertex. Let σ_2 be produced from σ_2' after inserting a suitable permutation of B_j (refer construction of σ_2) in the second permutation. Let $\sigma = \{\sigma_1, \sigma_2\}$. We now prove that σ separates the edges of G. It is easy to see that $\sigma = \sigma'$ if we ignore the vertices of $B_j \setminus \{s_c^1\}$ in σ. Hence, edges of G that are separated in σ' are separated in σ.

$E_1 - E_1$ crossing, $E_2 - E_2$ crossing, $E_1 - E_2$ crossing: It can be observed that in σ_1 there is no pairwise crossing of edges that are both taken from E_i, since $[V_i]$ is 1-outerplanar and thus applying Lemma 2.1. Clearly, no edge of E_1 crosses any edge of E_2 in σ_1.

Note that all the vertices of V_2 which are adjacent to a vertex in B_j, are adjacent to s_c^1 in G'. Let s_b^1 be the first vertex of B_j seen after s_c^1 in σ_1. In σ_1, let the vertices of B_j be seen in the order $s_c^1, \ldots, s_b^1, s_{b+1}^1, \ldots, s_{b+|B_j|-2}^1$. Consider $\alpha = (s_c^1, s_b^1, s_{b+1}^1, \ldots, s_{b+|B_j|-2}^1)$. To construct σ_2 we replace s_c^1 in σ_2' by $reversal(arc\text{-}removal(\alpha, l))$. We now discuss the rest of the crossings.

$E_1 - E_{12}$ crossing: The crossings between edges of E_1 and E_{12} are resolved by performing the arc removal technique on α (see Claim 2).
$E_{12} - E_2$ crossing: It is not possible that an edge from E_{12} crosses an edge from E_2 in σ_2 except for $s_1^2 s_{n_2}^2$, due to the planarity of G. The crossing of edges with $s_1^2 s_{n_2}^2$ is already resolved in σ_1.
$E_{12} - E_{12}$ crossing: When we consider the graph G', any crossing involving an edge incident on s_c^1 is resolved in one of the two permutations σ_1' or σ_2'. Hence it is clear that, an edge from E_{12} with one endpoint on $B' \setminus \{s_c^1\}$ and an edge from E_{12} with one endpoint in B_j do not cross in either σ_1 or σ_2. Also, a pair of edges from E_{12} with one end point each in $B_j \setminus \{s_c^1\}$ do not cross each other (as seen similarly in Lemma 2.2). So we only have to consider the case when (see Figure 6) $e \in E_{12}$ such that $e = (s_c^1, s_k^2)$ for some $s_k^2 \in V_2$ and $f = (s_l^1, s_m^2)$ where $s_l^1 \in B_j, s_m^1 \in V_2$. If e and f cross each other in σ_1, then they don't cross each other in σ_2 as s_c^1 appears after all vertices of B_j.

□

Remark 1. *Hence, two permutations are enough to separate the edges of G when $[V_1]$ contains exactly one component. It should be noted that in both permutations, the vertices of V_2 appear relatively according to the permutation described by the outerplanarity of $[V_2]$. It should also be noted that s_1^2 appears first in σ_1 and appears just after all vertices of V_1 in the second permutation.*

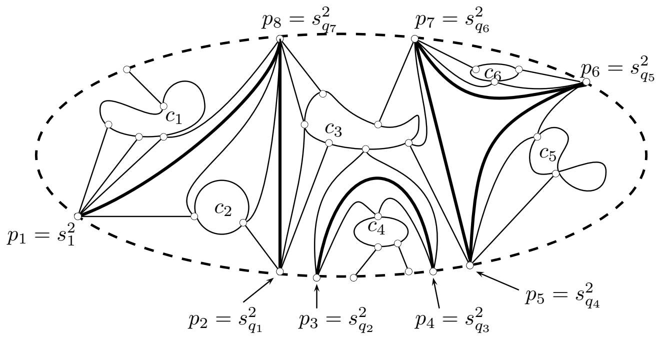


Fig. 7: Example of a 2-outerplanar graph where each C^i is a component of $[V_1]$. Separating chords are marked in bold.

Theorem 2.4. *The circular separation dimension of a maximal* 2*-outerplanar graph is two.*

Proof: As Lemma 2.3 deals with the case where $[V_1]$ is connected, we consider the case where $[V_1]$ is disconnected. Let each component of $[V_1]$ be C^i. We say that an edge $pp' \in E_2$ is a separating chord of G if *(i)* both p, p' have a common neighbour in V_1 and if *(ii)* removal of p and p' disconnects G such that every component of $G - \{p, p'\}$ contains a vertex of V_1. It is easy to see that each C^i is enclosed within the outer boundary of $[V_2]$ and by separating chords of G (see Figure 7). Let R^i be the smallest induced subgraph of G whose vertices are those of C^i and the vertices of the outer layer bounding C^i including the endpoints of separating chords bounding C^i. We label R^i's appropriately later. It is easy to see that each R^i satisfies the conditions of Lemma 2.3. Let p_1 be an endpoint of one separating chord. Let $s^2_1(= p_1), s^2_2, s^2_3, s^2_4, \ldots, s^2_{n_2}$ be the ordering of vertices of $[V_2]$ by traversing its outer boundary in a counter-clockwise manner. Let $P = \{p_j\}$ be the set of endpoints of all separating chords such that p_j is the j^{th} endpoint while traversing the outer boundary of $[V_2]$ in a counter-clockwise manner starting at p_1. It is to be noted that every p_j is some s^2. For any R^i containing some p_j, we claim that p_j can be a start vertex (recall that the start vertex has a neighbour in V_1). This is due to the definition of separating chords and the maximality of G (as G is a maximal 2-outerplanar graph).

Let the start vertex for each R^i be p_j where R^i contains p_j and j is the smallest such integer. Let $\gamma^i = \{\gamma^i_1, \gamma^i_2\}$ be the pairwise suitable family for the graph R^i according to Lemma 2.3. For $k = 1, 2$, let α^i_k be the sub-permutation of γ^i_k restricted to the $V(C^i)$.

Construction of σ: To form the first permutation σ_1 of G, we first list the start vertex $p_1 = s^2_1$. We then consider all the neighbours of p_1 and see the edges between p_1 and its neighbours clockwise from the edge $p_1 s^2_{n_2}$ (see Figure 7). The R^i's containing p_1 are labelled in the order in which they are seen. We list the α^i_1's corresponding to the labelled R^i's in increasing order of i. We then list the vertices of V_2 starting from the vertex succeeding p_1 (which is s^2_2) till we reach p_2. Suppose $p_2 = s^2_{q_1}$. We perform the same procedure, that is we see all neighbours of p_2 in a clockwise manner starting from the edge $p_2 s^2_{q_1-1}$ and label the R^i's seen, ignoring those R^i already labelled. We then list the corresponding α^i_1's. Thus for the example in Figure 7, $\sigma_1 = (p_1 = s^2_1, \alpha^1_1, \alpha^2_1, s^2_2, s^2_3, \ldots, p_2 = s^2_{q_1}, \alpha^3_1, s^2_{q_1+1}, s^2_{q_1+2}, \ldots, p_3 = s^2_{q_2}, \alpha^4_1, s^2_{q_2+1}, s^2_{q_2+2}, \ldots, p_4 = s^2_{q_3}, s^2_{q_3+1}, s^2_{q_3+2}, \ldots, p_5 = s^2_{q_4}, \alpha^5_1, s^2_{q_4+1}, s^2_{q_4+2}, \ldots, p_6 = s^2_{q_5}, \alpha^6_1, s^2_{q_5+1}, s^2_{q_5+2}, \ldots, p_7 = s^2_{q_6}, s^2_{q_6+1}, s^2_{q_6+2}, \ldots, p_8 = s^2_{q_7}, s^2_{q_7+1}, s^2_{q_7+2}, \ldots, s^2_{n_2})$. In order to list the second per-

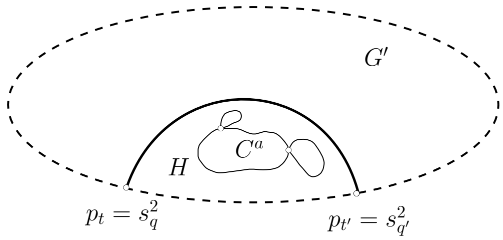


Fig. 8: R^a with exactly one separating chord $p_t p_{t'}$.

mutation, we replace each α^i_1 by α^i_2 in σ_1 and shift each p_j to the right of all the α^i_2's listed immediately after it, where each α^i_2 contains at least one vertex adjacent to p_j. Thus for the example in Figure 7, $\sigma_2 = (\alpha^1_2, \alpha^2_2, p_1 = s^2_1, s^2_2, s^2_3, \ldots, \alpha^3_2, p_2 = s^2_{q_1}, s^2_{q_1+1}, s^2_{q_1+2}, \ldots, \alpha^4_2, p_3 = s^2_{q_2}, s^2_{q_2+1}, s^2_{q_2+2}, \ldots, p_4 = s^2_{q_3}, s^2_{q_3+1}, s^2_{q_3+2}, \ldots, \alpha^5_2, p_5 = s^2_{q_4}, s^2_{q_4+1}, s^2_{q_4+2}, \ldots, \alpha^6_2, p_6 = s^2_{q_5}, s^2_{q_5+1}, s^2_{q_5+2}, \ldots, p_7 = s^2_{q_6}, s^2_{q_6+1}, s^2_{q_6+2}, \ldots, p_8 = s^2_{q_7}, s^2_{q_7+1}, s^2_{q_7+2}, \ldots, s^2_{n_2})$.

We now prove that these two permutations form a suitable family of permutations for G by induction on the number of components of $[V_1]$. The base case of having one component is already dealt with in Lemma 2.3. We assume that the induction hypothesis is true for at most $n-1$ components. Suppose $[V_1]$ has exactly n components.
Let P^i be the set of all p_j's contained in R^i for each i. It is easy to see that there exists a subgraph R^a that is bounded by exactly one separating chord (see Figure 8). In other words, $\exists a \ni P^a = \{p_t, p_{t+1} = p_{t'}\}$ where the addition is $modulo$ $|P|$ (we use $modulo$ $|P|$ to accommodate the case where $t = |P|$ and $t' = 1$). We discuss the case when $p_{t'} \neq p_1$. Suppose $p_t = s^2_q$ and $p_{t'} = s^2_{q'}$. Let H be the graph induced on the vertex set $V(C^a) \cup \{p_t = s^2_q, s^2_{q+1}, s^2_{q+2}, \ldots s^2_{q'-1}, s^2_{q'} = p_{t'}\}$. Let G' be the graph obtained from G by deleting all vertices of H except $p_t, p_{t'}$. Then G' is a maximal 2-outerplanar graph with exactly $n-1$ components.

Let $\sigma' = \{\sigma'_1, \sigma'_2\}$ be the pairwise suitable family for G' according to the above construction with p_1 as the start vertex (observe that all p_i's of G are retained in G'). As defined earlier, let $\gamma^i = \{\gamma^i_1, \gamma^i_2\}$ be the pairwise suitable family for the graph R^i with p_t as the start vertex. For $k = 1, 2$, let α^i_k be the sub-permutation of γ^i_k restricted to $V(C^i)$.
Let $\sigma = \{\sigma_1, \sigma_2\}$ be the pairwise suitable family for G obtained from the method of construction described above, with p_1 as the start vertex. It is easy to see that the same permutation σ_1 can be obtained from σ'_1 by inserting $\alpha^a_1, s^2_{q+1}, s^2_{q+2}, \ldots, s^2_{q'-1}$ immediately after all components following p_t while σ_2 can be obtained from σ'_2 by inserting α^a_2 immediately before p_t and inserting $s^2_{q+1}, s^2_{q+2}, \ldots, s^2_{q'-1}$ immediately after p_t. (If $p_{t'} = p_1$, then α^a_1 appears immediately after p_1 in σ_1, and in σ_2, α^a_2 appears at the very beginning. We observe that $s^2_{q+1}, s^2_{q+2}, \ldots, s^2_{q'-1}$ appears immediately after p_t in both σ_1 and σ_2.)
It is easy to see that the relative ordering of γ^i_k is maintained in σ_k for each i and $k \in \{1, 2\}$. We only need to check whether an edge of H can cross with an edge not contained in H. Let R^b be a subgraph such that $b \neq a$. We now prove that σ is a pairwise suitable family for G.

$E_1 - E_1$, $E_1 - E_2$, $E_2 - E_2$ crossings: Clearly, an edge of C^a will not cross an edge of C^b since all vertices of C^a appear consecutively. Therefore, a pair of edges from E_1 cannot cross each other. Similarly,

an edge in E_1 cannot cross an edge in E_2. Clearly, a pair of edges from E_2 cannot cross each other as vertices of $[V_2]$ are listed according to Lemma 2.1.
$E_1 - E_{12}$ crossings: Suppose $e \in E(C^a)$ and $f \in E_{12} \cap E(R^b)$. Since the vertices of C^a appear consecutively in both permutations, it is not possible for f to cross e. If $e \in E(C^b)$ and $f \in E(R^a)$, the same reason holds as vertices of C^b are written consecutively in both permutations.
$E_2 - E_{12}$ crossings: Suppose $e \in E_2 \setminus E(H)$ and $f \in E_{12} \cap E(R^a)$. Consider only the vertices $V_2 \cup V(H)$ in σ_1. Since all vertices of H appear consecutively, it is not possible for f to cross e. Similar reason holds if $e \in E_2 \cap E(R^a)$ and $f \in E_{12} \setminus E(H)$.
$E_{12} - E_{12}$ crossings: Suppose where $e \in E_{12} \cap E(R^a)$ and $f \in E_{12} \cap E(R^b)$. Note that the only vertices of R^a that f can be incident on are $p_t, p_{t'}$. *Case (i): R^b has a start vertex other than p_t.* Then in σ_1, all vertices of C^b appear either before p_t or after $p_{t'}$ and the vertices of R^a appear consecutively. (if $p_{t'} = p_1$, then then all vertices of R^b appear together). Hence there is no crossing between e and f. *Case (ii): R^b has p_t as the start vertex* (if $p_{t'} = p_1$, then suppose R^b has p_1 as the start vertex). Consider only the vertices $V_2 \cup V(R^a) \cup V(R^b)$ in σ. The respective permutations are $(p_1, ..., p_t = s^2_q, \alpha^b_1, \alpha^a_1, s^2_{q+1}, s^2_{q+2}, \dots, p_{t'} s^2_{q'}, s^2_{q'+1}, s^2_{q'+2}, \dots, s^2_{n_2})$ and $(p_1, ..., \alpha^b_2, \alpha^a_2, p_t = s^2_q, s^2_{q+1}, s^2_{q+2}, \dots, p_{t'} = s^2_{q'}, s^2_{q'+1}, s^2_{q'+2}, \dots, s^2_{n_2})$. In the first permutation, crossing between e and f can occur only if e is incident on p_t and f is incident on $p_{t'}$ or a vertex to its right. It can be seen that this crossing is resolved in the second permutation.

Thus, a pairwise suitable family of two permutations is constructed for a maximal 2-outerplanar graph. □

3 Series-Parallel graphs

Each series-parallel graph can be k-outerplanar for some natural number k. Interestingly, circular separation dimension of series-parallel graphs is at most 2. The proof is as follows.

Theorem 3.1. *If G is a series-parallel graph, then $\pi^\circ(G) \leq 2$.*

Proof: We prove the statement by induction on the number of series or parallel operations. It is easy to see that the statement holds for a series or parallel operation on a single edge. It is easy to see that a parallel operation maintains the separation dimension of any graph. Hence, the theorem has to be proved only for series operations. Suppose the statement holds for $\leq k$ operations and G' is a graph obtained from k operations. Then, $\pi^\circ(G') \leq 2$. Let $\{\sigma'_1, \sigma'_2\}$ be a pairwise suitable family for G' (if $\pi^\circ(G') = 1$, then we can assume $\sigma'_1 = \sigma'_2$). Let G be a graph obtained from G' through a series operation. Let x be the new vertex added where $N(x) = \{a, b\} \subseteq V(G)$. We obtain σ_1 from σ'_1 by replacing a by a, x, and σ_2 from σ'_2 by replacing b by b, x. Thus no edge can cross ax in σ_1, and bx in σ_2 as the vertices appear consecutively in σ_1 and σ_2, respectively. Hence $\{\sigma_1, \sigma_2\}$ is a pairwise suitable family for G. □

4 Conclusion

In this article, we show that the circular separation dimension of a 2-outerplanar graph is exactly two. It is to be noted that if the 2-outerplanar embedding is given as an input, one can construct two pairwise suitable circular permutations in polynomial time. If our conjecture on circular separation dimension of planar graphs is not true, then an obvious question to ask is: *what is the maximum value of k for which the circular separation dimension of a maximal k-outerplanar graph is exactly 2?* We conclude with this open question for future research.

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