Annular and pants thrackles

Grace Misereh

Yuri Nikolayevsky

Department of Mathematics and Statistics, La Trobe University, Melbourne, Australia

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A thrackle is a drawing of a graph in which each pair of edges meets precisely once. Conway's Thrackle Conjecture asserts that a thrackle drawing of a graph on the plane cannot have more edges than vertices. We prove the Conjecture for thrackle drawings all of whose vertices lie on the boundaries of $d \le 3$ connected domains in the complement of the drawing. We also give a detailed description of thrackle drawings corresponding to the cases when d=2 (annular thrackles) and d=3 (pants thrackles).

Keywords: thrackle, Thrackle Conjecture

1 Introduction

Let G be a finite simple graph with n vertices and m edges. A thrackle drawing of G on the plane is a drawing $\mathcal{T}: G \to \mathbb{R}^2$, in which every pair of edges meets precisely once, either at a common vertex or at a point of proper crossing (see [LPS97] for definitions of a drawing of a graph and a proper crossing). The notion of thrackle was introduced in the late sixties by John Conway, in relation with the following conjecture.

Conway's Thrackle Conjecture 1. For a thrackle drawing of a graph on the plane, one has $m \leq n$.

Despite considerable effort [Woo71, LPS97, CN00, CMN04, CN09, PS11, FP11, CN12, CKN15, GX17, FP17, MN18], the conjecture remains open. The best known bound for a thrackleable graph with n vertices is $m \le 1.3984n$ [FP17].

Adding a point at infinity we can consider a thrackle drawing on the plane as a thrackle drawing on the 2-sphere S^2 . The complement of a thrackle drawing on S^2 is the disjoint union of open discs. We say that a drawing belongs to the class T_d , $d \ge 1$, if there exist d open discs D_1, \ldots, D_d whose closures are pairwise disjoint such that all the vertices of the drawing lie on the union of their boundaries (a disk may contain no vertices on its boundary). We say that two thrackle drawings of class T_d are isotopic if they are isotopic as drawings on $S^2 \setminus (\bigcup_{k=1}^d D_k)$. We will also occasionally identify a graph G with its thrackle drawing $\mathcal{T}(G)$ speaking, for example, of the vertices and edges of the drawing.

Thrackles of class T_1 are called *outerplanar*: all their vertices lie on the boundary of a single disc D_1 . Such thrackles are very well understood.

Theorem 1. Suppose a graph G admits an outerplanar thrackle drawing. Then

(a) any cycle in G is odd [CN12, Theorem 1];

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- (b) the number of edges of G does not exceed the number of vertices [PS11, Theorem 2];
- (c) if G is a cycle, then the drawing is Reidemeister equivalent to a standard odd musquash [CN12, Theorem 1].

We say that two thrackle drawings are *Reidemeister equivalent* (or *equivalent up to Reidemeister moves*), if they can be obtained from one another by a finite sequence of Reidemeister moves of the third kind in the complement of vertices (see Section 2.2).

A standard odd musquash is the simplest example of a thrackled cycle: for n odd, distribute n vertices evenly on a circle and then join by an edge every pair of vertices at the maximal distance from each other. This defines a musquash in the sense of Woodall [Woo71]: an n-gonal musquash is a thrackled n-cycle whose successive edges e_0, \ldots, e_{n-1} intersect in the following manner: if the edge e_0 intersects the edges $e_{k_1}, \ldots, e_{k_{n-3}}$ in that order, then for all $j = 1, \ldots, n-1$, the edge e_j intersects the edges $e_{k_1+j}, \ldots, e_{k_{n-3}+j}$ in that order, where the edge subscripts are computed modulo n. A complete classification of musquashes was obtained in [CK99, CK01]: every musquash is either isotopic to a standard n-musquash, or is a thrackled six-cycle.

In this paper, we study thrackle drawings of the next two classes T_d : annular thrackles and pants thrackles.

A thrackle drawing of class T_2 is called *annular*. Up to isotopy, we can assume that the boundaries of D_1 and D_2 are two concentric circles on the plane, and that the thrackle drawing, except for the vertices, entirely lies in the open annulus bounded by these circles. Clearly, any outerplanar drawing can be viewed as an annular drawing. Figure 1 shows an example of an annular thrackle drawing which is not outerplanar. Note however that the underlying graph has some vertices of degree 1 (which must always be the case by Theorem 2(c) below).

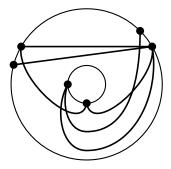


Fig. 1: An annular thrackle drawing.

We show that the three assertions of Theorem 1 also hold for annular drawings.

Theorem 2. Suppose a graph G admits an annular thrackle drawing. Then

- (a) any cycle in G is odd;
- (b) the number of edges of G does not exceed the number of vertices;
- (c) if G is a cycle, then the drawing is, in fact, outerplanar (and as such, is Reidemeister equivalent to a standard odd musquash).

We next proceed to the thrackle drawings of class T_3 . We call such drawings pants thrackle drawings or pants thrackles.

Any annular thrackle drawing is trivially a pants thrackle drawing. The pants thrackle drawing of a six-cycle in Figure 2 is not annular.

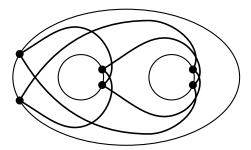


Fig. 2: Pants thrackle drawing of a six-cycle.

We prove the following.

Theorem 3. Suppose a graph G admits a pants thrackle drawing. Then

- (a) any even cycle in G is a six-cycle, and its drawing is Reidemeister equivalent to the one in Figure 2;
- (b) if G is an odd cycle, then the drawing can be obtained from a pants drawing of a three-cycle by a sequence of edge insertions;
- (c) the number of edges of G does not exceed the number of vertices.

The procedure of edge insertion replaces an edge in a thrackle drawing by a three-path such that the resulting drawing is again a thrackle – see Section 2.1 for details.

The ideas of the proofs are roughly as follows. There is a toolbox of operations one can do on a thrackled graph while preserving thrackleability and that have been used in the past literature on thrackles; these include edge insertion, edge removal and vertex splitting. We investigate how these operations interact with the more restrictive annular or pants conditions. One key observation (Lemma 1 below) is that, in order to preserve thrackleability, edge removal hinges on some empty triangle condition which blends well with the annular or the pants structure. This allows the study of irreducible thrackles, which are those for which no edge removal is possible. We prove that irreducible thrackled cycles are either triangles, or, in the case of pants drawing, a six-cycle.

2 Thrackle operations

2.1 Edge insertion and edge removal

The operation of edge insertion was introduced in [Woo71, Figure 14]; given a thrackle drawing, one replaces an edge by a three-path in such a way that the resulting drawing is again a thrackle. All the changes to the drawing are performed in a small neighbourhood of the edge, as shown in Figure 3. Edge insertion on a given edge is not uniquely defined, even up to isotopy and Reidemeister moves, as we can



Fig. 3: The edge insertion operation.

choose one of two different orientations of the crossing of the first and the third edge of the three-path by which we replace the edge. We want to formalise and slightly modify the edge insertion procedure. Given an edge e=uv, on the first step, we remove from it a small segment Q_1Q_2 lying in the interior of e and containing no crossings with other edges. On the second step, we slightly extend the segments uQ_1 and Q_2v so that they cross (with one of two possible orientations), and then further extend each of them to cross other edges in such a way that the resulting drawing is again a thrackle. On the third step, we join the two endpoints of degree 1 of the two edges obtained at the first step so that the resulting drawing is again a thrackle. We make two observations regarding this process of edge insertion. First, it may happen that we change the drawing not only in a small neighbourhood of e, but also "far away" from it. Figure 4 shows two Reidemeister inequivalent thrackled seven-cycles obtained from the standard 5-musquash by edge insertion. Note that the orientations of all the crossings in the two thrackles are the same (we note in passing, that up to isotopy and Reidemeister moves there exist only three thrackled seven-cycles: the two shown in Figure 4 and the standard 7-musquash; we can prove that using the algorithm given in the end of Section 3 of [MN18]).

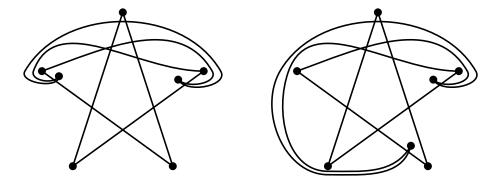


Fig. 4: Two seven-cycles obtained by edge insertions on a five-cycle.

Our second observation is that edge insertion may not always be possible within the same class T_d . For example, in the proof of assertion (a) of Theorem 3 in Section 4, it will be shown that no edge insertion on the pants thrackle drawing of the six-cycle shown in Figure 2 produces a pants thrackle drawing.

The operation of edge removal is inverse to the edge insertion operation. Let $\mathcal{T}(G)$ be a thrackle drawing of a graph G and let $v_1v_2v_3v_4$ be a three-path in G such that $\deg v_2 = \deg v_3 = 2$. Let $Q = \mathcal{T}(v_1v_2) \cap \mathcal{T}(v_3v_4)$. Removing the edge v_2v_3 , together with the segments Qv_2 and Qv_3 we obtain a drawing of a graph with a single edge v_1v_4 in place of the three-path $v_1v_2v_3v_4$ (Figure 5). Edge removal does not necessarily result in a thrackle drawing. Consider the triangular domain \triangle bounded by the arcs

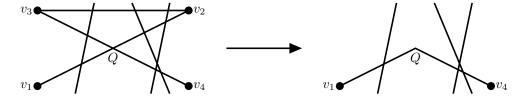


Fig. 5: The edge removal operation.

 v_2v_3 , Qv_2 and v_3Q and not containing the vertices v_1 and v_4 (if we consider the drawing on the plane, \triangle can be unbounded). We have the following lemma.

Lemma 1 ([CN12, Lemma 3]). *Edge removal results in a thrackle drawing if and only if* \triangle *contains no vertices of* $\mathcal{T}(G)$.

Note that for a thrackle drawing of class T_d , the condition of Lemma 1 is satisfied if \triangle contains none of the d circles bounding the discs D_k . Given a thrackle drawing of class T_d of an n-cycle, edge removal, if it is possible, produces a thrackle drawing of the same class T_d of an (n-2)-cycle.

We call a thrackle drawing *irreducible* if it admits no edge removals and *reducible* otherwise.

To a path in a thrackle drawing of class T_d we can associate a word W in the alphabet $X = \{x_1, \ldots, x_d\}$ in such a way that the i-th letter of W is x_k if the i-th vertex of the path lies on the boundary of the disc D_k . For a thrackled cycle, we consider the associated word W to be a cyclic word. For a word w and an integer m we denote w^m the word obtained by m consecutive repetitions of w. We have the following simple observation.

Lemma 2. For a thrackle drawing of a graph G of class T_d ,

- (a) For no two different i, j = 1, ..., d, may a thrackle drawing of class T_d contain two edges with the words x_i^2 and x_j^2 .
- (b) Suppose that for some i = 1, ..., d, a thrackle drawing of class T_d contains a two-path with the word x_i^3 the first two vertices of which have degree 2. Then the drawing is reducible.

Proof: (a) is obvious, as otherwise the thrackle condition will be violated by the corresponding two edges. (b) The complement of the two-path in $S^2 \setminus (\cup_{k=1}^d \overline{D_k})$ is the union of three domains, exactly one of which has the two-path on its boundary. That domain can contain no other vertices of the thrackle inside it or on its boundary, as otherwise the thrackle condition is violated. But then by Lemma 1, edge removal can be performed on the three-path which is the union of the given two-path and the edge of the graph incident to its first vertex.

2.2 Reidemeister moves

A *Reidemeister move* can be performed on a triple of pairwise non-adjacent edges of a thrackle drawing if the open triangular domain bounded by the segments on each of the edges between the crossings with the other two contains no points of the drawing – see Figure 6. We say that two thrackle drawings are *Reidemeister equivalent* if one can be obtained from the other by a finite sequence of Reidemeister moves.

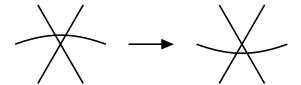


Fig. 6: A Reidemeister move.

Suppose that two thrackles $\mathcal{T}_1(G)$ and $\mathcal{T}_2(G)$ can be obtained from one another by a Reidemeister move on a triple of edges e_i, e_j, e_k . From Lemma 1 it follows that if $\mathcal{T}_1(G)$ admits edge removal on a three-path not containing these three edges, then $\mathcal{T}_2(G)$ also does; moreover, after edge removals the resulting two thrackles can again be obtained from one another by the same Reidemeister move. However, adding an edge to $\mathcal{T}_1(G)$ may result in a thrackle which is not Reidemeister equivalent to any thrackle obtained from $\mathcal{T}_2(G)$ by adding an edge, as the added edge may end at a vertex inside the triangular domain \triangle_{ijk} bounded by e_i, e_j, e_k . The same is true for edge insertion on $\mathcal{T}_1(G)$.

Now suppose that $\mathcal{T}_1(G)$ and $\mathcal{T}_2(G)$ belong to a class T_d . The domains \triangle_{ijk} in both $\mathcal{T}_1(G)$ and $\mathcal{T}_2(G)$ contain no vertices. If we additionally require that they contain no "inessential" discs D_l , those having no vertices on their boundaries, then the edge added to $\mathcal{T}_1(G)$ cannot end in \triangle_{ijk} and so we can add a corresponding edge to $\mathcal{T}_2(G)$ such that the resulting two thrackles are again Reidemeister equivalent.

2.3 Forbidden configurations

A graph having more edges than vertices always contains one of the following subgraphs: a theta-graph (two vertices joined by three disjoint paths), a dumbbell (two disjoint cycles with a path joining a vertex of one cycle to a vertex of another), or a figure-8 graph (two cycles sharing a vertex). To prove Conway's Thrackle Conjecture it is therefore sufficient to show that none of these three graphs admits a thrackle drawing. Repeatedly using the vertex-splitting operation [MN18, Figure 1(a)] one can show that the existence of a counterexample of any of these three types implies the existence of a counterexample of the other two types. However, this may not be true for thrackle drawings of class T_d , as the required vertex-

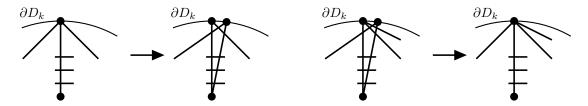


Fig. 7: Splitting a vertex of degree 3 and a vertex of degree 4.

splitting operation on a vertex of degree 3 may not be permitted within the class T_d . The problem is that in order to remain within the class T_d , vertex-splitting on a vertex of degree 3 may only be performed by doubling the "middle" edge (as on the left in Figure 7) and this is too restrictive; for example, starting with a dumbbell, vertex-splitting within the class T_d might only *increase* the length of the dumbbell handle. So one might not be able to reduce a dumbbell to a figure-8 graph. Nevertheless, if we are given a thrackle drawing of class T_d of a figure-8 graph, we can always perform the vertex-splitting operation on the vertex

of degree 4 to obtain a thrackle drawing of the same class T_d of a dumbbell, as on the right in Figure 7. This gives the following lemma.

Lemma 3. To prove Conway's Thrackle Conjecture for thrackle drawings in a class T_d it is sufficient to prove that no dumbbell and no theta-graph admit a thrackle drawing of class T_d . In both cases, the corresponding graph contains an even cycle.

The second assertion is clear for a theta-graph, and for a dumbbell, follows from the fact that a thrack-lable graph contains no two vertex-disjoint odd cycles [LPS97, Lemma 2.1].

3 Annular thrackles

In this section, we prove Theorem 2. We can assume that the thrackle drawing lies in the closed annulus bounded by two concentric circles on the plane, the outer circle A and the inner circle B; the vertices lie in $A \cup B$, and the rest of the drawing, in the open annulus. As in Section 2.1 we can associate to a path within a thrackle a word in the alphabet $\{a,b\}$, where the letter a (respectively b) corresponds to a vertex lying on A (respectively on B). To an annular thrackle drawing of an n-cycle there corresponds a word W defined up to cyclic permutation and reversing.

The following lemma and the fact that edge removal decreases the length of a cycle by 2 imply assertion (a).

Lemma 4. If an n-cycle admits an irreducible annular thrackle drawing, then n=3.

Proof: By Lemma 2(a) we can assume that W contains no two consecutive b's. If W contains no letters b at all, then the thrackle is outerplanar and the assertion of the lemma follows from Theorem 1 (c). Assuming that W contains at least one b we get that W contains a sequence aba. Suppose n>3; then $n\geq 5$, as no 4-cycle admits a thrackle drawing on the plane. Consider the next letter in W. Up to isotopy, there are three possible ways of adding an extra edge. As the reader may verify, two of them produce a reducible thrackle by Lemma 1. The third one is shown in the middle in Figure 8. But then

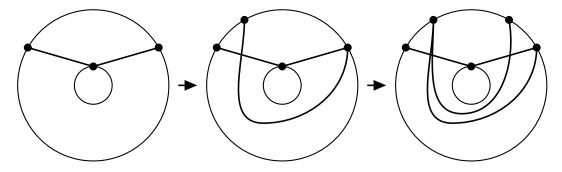


Fig. 8: Adding the third and the fourth edge.

there is only one way to add the next edge, as on the right in Figure 8 and the resulting thrackle drawing is reducible.

By Lemma 3, if in the class of annular thrackles there exists a counterexample to Conway's Thrackle Conjecture, then there exists such a counterexample whose underlying graph contains an even cycle. So assertion (b) follows from assertion (a).

We now prove assertion (c). Suppose a cycle c of an odd length n admits an annular thrackle drawing. We can assume that the corresponding word W contains at least one b and does not contain b^2 .

Lemma 5. Up to cyclic permutation, $W = a^{2p}(ba)^r b$, for some $p \ge 1$, $r \ge 0$.

Proof: As n is odd, W contains a subword a^2 . Let a^k , $k \ge 2$, be a maximal by inclusion string of consecutive a's. If k = n - 1, we are done. Otherwise, up to cyclic permutation, $W = a^k bwb$ for some word w. Consider the edge e defined by the last pair aa in a^k . Let γ be the arc of A joining the endpoints of e such that the domain bounded by $e \cup \gamma$ does not contain B. Every edge of the thrackle not sharing a common vertex with e crosses it, so every second vertex counting from the last a in a^k lies in the interior of γ . It follows that $W = a^k bay_1 ay_2 a \dots y_q ab$, where $y_i \in \{a,b\}$, and so k is necessarily even. By the same reasoning, any maximal sequence of more than one consecutive a's in W is even. But then $y_i = b$, for all $i = 1, \dots, q$, as otherwise W would contain a maximal sequence of consecutive a's of an odd length greater than one.

To prove assertion (c) we show any annular thrackled cycle is alternating; then the claim follows from the fact that alternating thrackles are outerplanar, as was proved in [CN12, Theorem 2]. Recall that a thrackled cycle is called *alternating* if for every edge e and every two-path fg vertex-disjoint from e, the crossings of e by f and g have opposite orientations.

Suppose c is a cycle of the shortest possible length which admits a non-alternating annular thrackle drawing $\mathcal{T}(c)$; the length of c must be at least 7. An easy inspection shows that any edge vertex-disjoint with a two-path aba (or bab) crosses its edges with opposite orientations. The same is true for a two-path a^3 . It remains to show that any edge vertex-disjoint with a two-path aab also crosses the edges of that two-path with opposite orientations. Up to isotopy, the only drawing for which this is not true is the one shown in Figure 9. Note that the edge which violates the alternating condition necessarily joins an

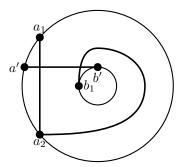


Fig. 9: A non-alternating crossing.

a-vertex and a b-vertex. We claim that such a drawing cannot be a part of $\mathcal{T}(c)$. To see that, we consider possible drawings of the four-path in c which extends the path $a_1a_2b_1$. The vertex following b_1 must be an a-vertex (call it a_3); there are two possible cases: $a_3 = a'$ and $a_3 \neq a'$. In the first case, up to isotopy, we get the drawing on the left in Figure 10, and then there is only one possible way to attach

an edge at a_1 , as shown on the right in Figure 10. But then performing edge removal on a_1a_2 we get a shorter non-alternating annular thrackled cycle, a contradiction. Now suppose $a_3 \neq a'$. We have two

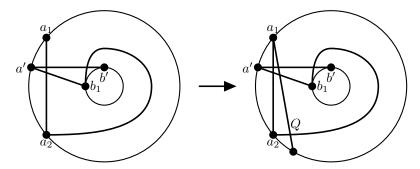


Fig. 10: Path $a_1a_2b_1a_3$, $a_3 = a'$.

cases for adding the edge b_1a_3 , and then by Lemma 5, the letter after a_3 must be a b. In the first case, up to isotopy and a Reidemeister move, we get the drawing on the left in Figure 11, and then we can attach the edge joining a_3 to a b-vertex uniquely, up to isotopy and a Reidemeister move, as on the right in Figure 11. Again, performing edge removal on b_1a_3 we get a shorter non-alternating annular thrackled

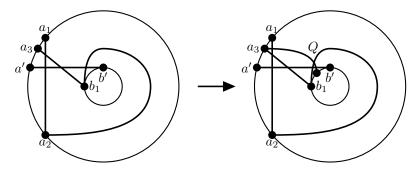


Fig. 11: Path $a_1a_2b_1a_3$, $a_3 \neq a'$, case 1.

cycle. The second possibility of attaching the edge b_1a_3 , $a_3 \neq a'$, to the drawing in Figure 9 is the one shown on the left in Figure 12, up to isotopy. Then the edge joining a_3 to the next b-vertex can be also added uniquely, up to isotopy, as on the right in Figure 12, and yet again, edge removal on b_1a_3 results in a shorter non-alternating annular thrackled cycle. This completes the proof of Theorem 2.

4 Pants thrackles

In this section, we prove Theorem 3. We represent the pair of pants domain P whose closure contains the drawing as the interior of an ellipse, with two disjoint closed discs removed. To a path in a pants thrackle drawing we associate a word in the alphabet $\{a, b, c\}$, where a corresponds to the vertices on the ellipse, and b and c, to the vertices on the circles bounding the discs (e.g., as in Figure 14).

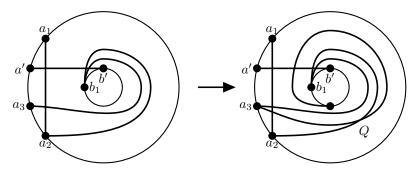


Fig. 12: Path $a_1a_2b_1a_3$, $a_3 \neq a'$, case 2.

We start with the following proposition which implies assertion (b) of Theorem 3 and will also be used in the proof of assertion (a).

Proposition. If a cycle C admits an irreducible pants thrackle drawing, then C is either a three-cycle or a six-cycle, and in the latter case, the drawing is Reidemeister equivalent to the one in Figure 2.

Proof: Let W be the (cyclic) word corresponding to an irreducible pants thrackle drawing of a cycle C. The following lemma can be compared to Lemma 5.

Lemma 6. If W contains a^2 , then one of the two domains of the complement of the corresponding edge in P is a disc, the cycle C is odd, and $W = aay_1ay_2...y_{m-1}ay_m$, where $y_i \in \{b, c\}$ for i = 1,...,m.

Proof: Suppose no domain of the complement of an edge aa is a disc. By Lemma 2(b), neither the letter which precedes a^2 in W, nor the next letter after a^2 is a, and for the corresponding edges to cross, those two letters must be the same, say b. If the corresponding three-path baab is irreducible, it has to be isotopic to the path on the left in Figure 13. But then there is a unique, up to isotopy, way to add to the path the starting segment of the next edge, and it produces a reducible three-path, as on the right in Figure 13. It

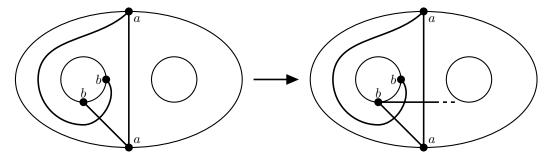


Fig. 13: Adding the fourth edge produces a reducible path.

follows that if W contains a^2 , then one of the two domains of the complement of the corresponding edge in P is a disc. But then by the thrackle condition, every second vertex counting from the second a in aa is again a, so $W = aay_1ay_2 \dots y_{m-1}ay_m$, for some $y_i \in \{a, b, c\}$. In particular, C is an odd cycle and furthermore, none of the y_i can be equal to a by Lemma 2(b).

Lemma 7. Suppose the word W contains no subwords bb and cc. Then it contains no subwords caba or baca.

Proof: Arguing by contradiction (and renaming the letters if necessary) suppose that W contains the subword caba. The only irreducible three-path corresponding to that subword, up to isotopy, is shown on the left in Figure 14. Suppose that the next letter in W is not a. Then the only irreducible fourpath extending caba, up to isotopy, is the one shown on the right in Figure 14. If C is of length five, then

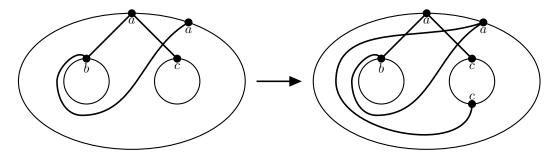


Fig. 14: The irreducible path caba and the next edge ending not in a.

W=cabac which contradicts the fact that the (cyclic) word W does not contain a subword cc. Otherwise, there are only three possible ways, up to isotopy and a Reidemeister move, to add another edge starting at the last added vertex c in such a way that the resulting drawing is a thrackled path. But one of them results in a reducible drawing, and the other two end in c contradicting the fact that W does not contain a subword cc.

It follows that the letter following caba in W must be an a, so we get a subword cabaa. If the length of the cycle C is greater than 5, then by Lemma 6, the letter which precedes c must be a, so W contains the subword acabaa. But then the above argument applied to the subword acab (if we reverse the direction of C and swap b and c) implies that the letter which precedes the starting a is another a, so that W contains the subword acabaa, giving a contradiction with Lemma a.

If C is of length five, then W = cabaa and the resulting drawing is reducible by Lemma 1, as there is just a single b in W, and so the triangular domain corresponding to the three-path caba on the left in Figure 14 contains no other vertices of the thrackle.

Now if W contains the subword aa, then by Lemma 6 and Lemma 7, the word W may contain only one of the letters b or c. Then the drawing is annular, and hence by Lemma 4 is reducible unless C is a three-cycle. Suppose W contains no letter repetitions. Then Lemma 7 applies to any subword xyzy such that $\{x,y,z\}=\{a,b,c\}$. Furthermore, up to renaming the letters we can assume that W starts with ab. If W contains no subword abc, then $W=(ab)^m$, and so the drawing is annular. We can therefore assume that W contains a subword abc. Then the following letter cannot be any of b or c, so it must be an a. Repeating this argument we obtain that $W=(abc)^m$.

We now modify the word W by attaching to every letter a subscript plus (respectively minus) if the tangent vector to the drawing in the direction of the cycle C makes a positive (respectively negative) turn at the corresponding vertex; in other words, the subscript is a plus (respectively a minus) if the path turns

left (respectively right) at the vertex. We will occasionally omit the subscript when it is unknown or unimportant.

Note that if the length of C is greater than 3, then no two consecutive subscripts in the word $W = (abc)^m$ can be the same. Indeed, assume that W contains a subword ab_+c_+a . Then the corresponding irreducible three-path is unique up to isotopy, as shown on the left in Figure 15, and the only possible way to attach an edge ab results in a reducible drawing, as on the right in Figure 15. By reflection, a similar comment applies to subwords of the form ab_-c_-a . It follows that the subscripts in W alternate and in

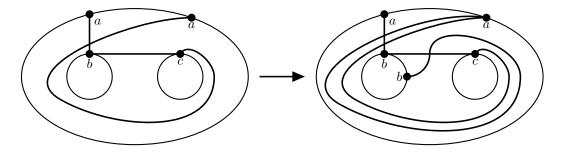


Fig. 15: The path ab_+c_+a extends to a reducible drawing.

particular, the length of C is divisible by 6.

There are two drawings of the three-path ab_+c_-a , both irreducible, as shown in Figure 16. They differ

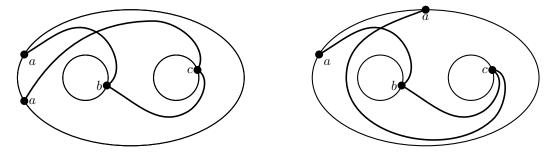


Fig. 16: Two paths ab_+c_-a .

by the orientation of the crossing of the edges ab and ca. If we change the direction on C and swap the letters b and c, the subword ab_+c_-a does not change. By reflection, a similar comment applies to subwords of the form ab_-c_+a . Hence the whole word W is unchanged, with all the subscripts, but the orientation of the crossings of the edges ab and ca are reversed. We therefore lose no generality by assuming that the subword ab_+c_-a is represented by the three-path on the left in Figure 16. We can then uniquely, up to isotopy, add an edge ca to the starting vertex a, as on the left in Figure 17, which produces the four-paths corresponding to the subword $ca_-b_+c_-a$. Furthermore, up to isotopy and a Reidemeister move, we can uniquely add an edge bc to the starting vertex c, as on the right in Figure 17. We get the five-paths corresponding to the subword $bc_+a_-b_+c_-a$. One possibility for completing the cycle would

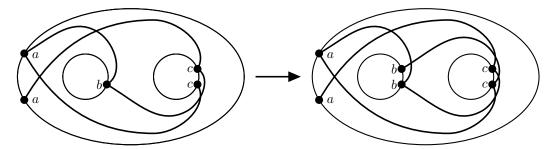


Fig. 17: The four-path paths $ca_-b_+c_-a$ and the five-path $bc_+a_-b_+c_-a$.

be to now join the degree one vertices a and b of the five-path by an edge. This can be done uniquely up to isotopy and produces an irreducible pants thrackle drawing of a six-cycle corresponding to the word $W = b_-c_+a_-b_+c_-a_+$, as in Figure 2. Any other such drawing is equivalent to that up to isotopy and Reidemeister moves (which were possible at the intermediate steps of our construction).

Otherwise, we can extend the five-path to a six-path corresponding to the subword $ab_-c_+a_-b_+c_-a$ by adding an edge ab at the start. The resulting six-path is equivalent, up to isotopy and Reidemeister moves, to the one on the left in Figure 18. But then no edge ca (with the correct orientation at a) can be added at

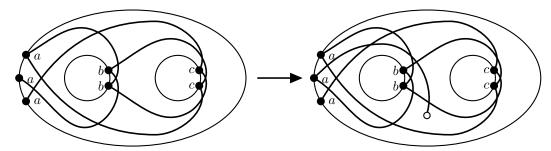


Fig. 18: The six-path $ab_-c_+a_-b_+c_-a$ cannot be extended to a seven-path $ca_+b_-c_+a_-b_+c_-a$.

the start of the six-path: up to isotopy and Reidemeister moves, the only edge we can add does not start at c, as on the right in Figure 18. This completes the proof of the Proposition.

By the Proposition, if an even cycle of length greater than 6 has a pants thrackle drawing, then it must be reducible. Hence, to prove assertion (a) of Theorem 3, it suffices to show that the pants thrackle drawing of the six-cycle in Figure 2 (or Reidemeister equivalent to it) admits no edge insertion such that the resulting thrackle drawing of the eight-cycle is again a pants thrackle drawing. One possible way is to consider all edge insertions following the procedure in Section 2.1. But as the resulting thrackles are sufficiently small, all these cases can be treated by computer. Using the algorithm given in the end of Section 3 of [MN18] we found that up to isotopy and Reidemeister moves, there exist exactly three thrackled eight-cycles; they are shown in Figure 19. Each of them is obtained by edge insertion in a thrackled six-cycle and belongs to class T_4 , but none of them is a pants thrackle. This proves assertion (a) of Theorem 3.

It remains to prove assertion (c). By Lemma 3, it suffices to show that if G is either a theta-graph or a dumbbell, then it admits no pants thrackle drawing. We also know from Lemma 3 that in both cases, G

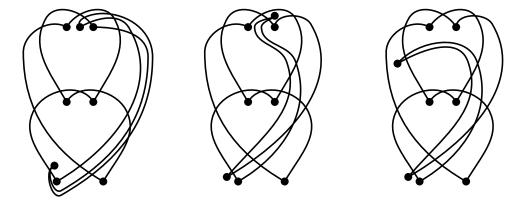


Fig. 19: All thrackled eight-cycles up to Reidemeister equivalency.

contains an even cycle which by assertion (a) must be a six-cycle whose thrackle drawing is Reidemeister equivalent to the one in Figure 2.

The proof goes as follows: we explicitly construct pants thrackle drawings of a six-cycle with certain small trees attached to one of its vertices and first show that in a pants thrackle drawing of a three-path attached to a six-cycle, the drawing of the three-path is reducible. Repeatedly performing edge removals we get a pants thrackle drawing either of a theta-graph obtained from a six-cycle by joining two of its vertices by a path of length at most 2, or of a dumbbell consisting of a six-cycle and some other cycle joined by a path of length at most 2. The resulting theta-graphs are very small, and from [FP11, MN18] we know that they admit no thrackle drawing at all, and in particular, no pants thrackle drawing (the latter fact will also be confirmed in the course of the proof). Every resulting dumbbell contains one of two subgraphs obtained from the six-cycle by attaching a small tree, as in Figure 20. We show that for a pants



Fig. 20: The six-path cycle with a tree attached.

thrackle drawing of each of these two subgraphs, to at least one of the two vertices v_1, v_2 , it is not possible to attach another edge so that the resulting drawing is a pants thrackle drawing.

We start with the pants thrackle drawing of the six-cycle and attach a path to one of its vertices. By cyclic symmetry, we can choose any vertex to attach a path. Moreover, from the arguments in Section 2.2 it follows that Reidemeister moves on the original six-cycle and on the intermediate steps of adding edges will result in a Reidemeister equivalent drawing in the end. So we can attach a path edge-by-edge choosing one of Reidemeister equivalent drawings arbitrarily at each step.

Up to isotopy and Reidemeister moves, there are two ways to attach an edge to a vertex of the drawing

of the six-cycle, as in Figure 21. Note that the second endpoint of this edge is not one of the vertices of the six-cycle (so that no theta-graph obtained by joining two vertices of a six-cycle by an edge admits a pants thrackle drawing) and that in the two cases shown in Figure 21, it lies on different boundary components of P. It follows that renaming b and c and changing the direction on the cycle and the orientation on the plane, we obtain two Reidemeister equivalent drawings. We continue with the one on the left in Figure 21 and attach another edge at the vertex of degree 1. This can be done uniquely up to isotopy and

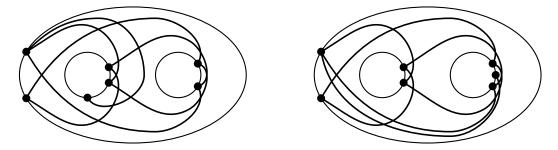


Fig. 21: Six-cycle with an edge attached.

Reidemeister equivalence resulting in the drawing as on the left in Figure 22. Again, the second endpoint of the attached edge cannot be one of the vertices of the six-cycle (so that no theta-graph obtained by joining two vertices of a six-cycle by a two-path admits a pants thrackle drawing). Then we can attach another edge at that vertex. This can be done uniquely up to isotopy and Reidemeister equivalence, as on the right in Figure 22. But if the two vertices other than the endpoints in the so attached three-path have

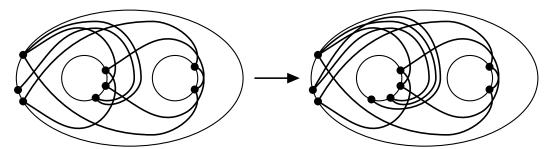


Fig. 22: Six-cycle with two- and three-paths attached.

degree 2 in G, then the three-path is reducible by Lemma 1. Now if G is a theta-graph, then by repeatedly performing edge removals we obtain a pants thrackle drawing of a theta-graph obtained by joining two vertices of a six-cycle by a path of length at most 2, which is impossible, as we have shown above. If G is a dumbbell, then by repeatedly performing edge removals we obtain a pants thrackle drawing of a dumbbell consisting of the six-cycle and a cycle C', with a vertex of the six-cycle joined to the vertex v of C' by either an edge or a two-path. Such a dumbbell contains one of the two subgraphs given in Figure 20. So it remains to deal with these two cases.

The vertex v has degree 3 in G, so we have to attach two edges to it. In the first case, we start with the

drawing on the left in Figure 21 and attach two edges to the vertex v. We obtain a unique drawing, up to isotopy and Reidemeister moves, as on the right in Figure 23. But then no edge can be attached to the vertex a_2 in such a way that the resulting drawing is a pants thrackle drawing.

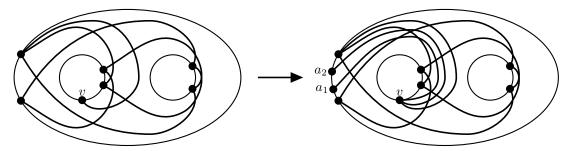


Fig. 23: Pants drawing of the graph on the left in Figure 20.

Similarly, in the second case, we start with the drawing on the left in Figure 22 and attach two edges to the vertex v. We obtain a unique drawing, up to isotopy and Reidemeister moves, as on the right in Figure 24. But then no edge can be attached to the vertex b_1 in such a way that the resulting drawing is a pants thrackle drawing.

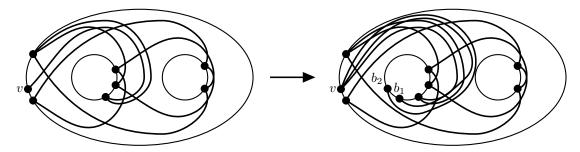


Fig. 24: Pants drawing of the graph on the right in Figure 20.

This completes the proof of Theorem 3.

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