# *Summation formulas for Fox-Wright function*

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By means of inversion techniques and several known hypergeometric series identities, summation formulas for Fox-Wright function are explored. They give some new hypergeometric series identities when the parameters are specified.

**Keywords:** Hypergeometric series, Fox-Wright function, Inversion techniques

## 1 Introduction

For a complex variable $x$ and an nonnegative integer $n$, define the shifted-factorial to be

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{when} \quad n = 1, 2, \cdots .$$

Following Andrews et al. (2000), define the hypergeometric series by

$$ {}_{1+r}F_s \left[ \begin{matrix} a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_s \end{matrix} \Big| \, z \right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_r)_k}{k! (b_1)_k \cdots (b_s)_k} z^k, $$

where $\{a_i\}_{i\geq 0}$ and $\{b_j\}_{j\geq 1}$ are complex parameters such that no zero factors appear in the denominators of the summand on the right hand side. Then Whipple's ${}_3F_2$-series identity (cf. Andrews et al. (2000)[p. 149]) can be stated as

$$ {}_3F_2 \left[ \begin{matrix} a, 1-a, b \\ c, 1+2b-c \end{matrix} \Big| \, 1 \right] = \frac{\Gamma(\frac{c}{2})\Gamma(\frac{1+c}{2})\Gamma(b+\frac{1-c}{2})\Gamma(b+\frac{2-c}{2})}{\Gamma(\frac{a+c}{2})\Gamma(\frac{1-a+c}{2})\Gamma(b+\frac{1+a-c}{2})\Gamma(b+\frac{2-a-c}{2})}, \tag{1} $$

where $Re(b) > 0$ and $\Gamma(x)$ is the well-known gamma function

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad \text{with} \quad Re(x) > 0.$$

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About the research for the generalizations of Watson's ${}_3F_2$-series identity and (1) with integer parameters, the reader may refer to the papers Chu (2012, 2013); Lavoie (1987); Lavoie et al. (1992); Wimp (1983); Zeilberger (1992). The corresponding $q$-analogues can be found in Wei and Wang (2017). Some strange evaluations of hypergeometric series can be seen in the papers Gessel and Stanton (1982); Wang and Wang (2014); Wang (2015); Wang et al. (2018).

Recall Fox-Wright function ${}_p\Psi_q$ (cf. Fox (1928); Wright (1935, 1940); see also Srivastava and Karlsson (1985)[p. 21], which is defined by

$$ {}_p\Psi_q \left[ \begin{matrix} (\alpha_1; A_1), & \cdots, & (\alpha_p; A_p) \\ (\beta_1; B_1), & \cdots, & (\beta_q; B_q) \end{matrix} \Bigg| \, z \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(\alpha_i + A_i k)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j k)} \frac{z^k}{k!} $$

and regarded as a generalization of hypergeometric series, where the coefficients $\{A_i\}_{i\geq 1}$ and $\{B_j\}_{j\geq 1}$ are positive real numbers such that

$$ 1 + \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i \geq 0. $$

Naturally, the domain where $\{A_i\}_{i\geq 1}$ and $\{B_j\}_{j\geq 1}$ take on values can be extended to the complex field and the upper coefficient relation is ignored when the series is terminating. For convenience, we shall frequently use the symbol

$$ (x_1, x_2, \cdots, x_r; A) := \prod_{i=1}^{r} \Gamma(x_i + Ak) $$

in the Fox-Wright function.

The importance of Fox-Wright function lies in that it can be applied to many fields. Miller and Moskowitz (1995) offered the applications of Fox-Wright function to the solution of algebraic trinomial equations and to a problem of information theory. Mainardi and Pagnini (2007) told us that Fox-Wright function plays an important role in finding the fundamental solution of the factional diffusion equation of distributed order in time. More applications of the function can be found in Craven and Csorads (2006); Darus and Ibrahim (2008); Miller (2002); Murugusundaramoorthy and Rosy (2011); Pogány and Srivastava (2009); Srivastava et al. (2005); Srivastava (2007). The following fact should be mentioned. Aomoto and Iguchi (1999) introduced the quasi hypergeometric function, which is exactly multiple Fox-Wright function, and showed that it satisfies a system of difference-differential equations.

One pair of inverse relations implied in the works of Bressoud (1988) and Gasper (1989) (see also Chu (1994)[p. 17]) can be expressed as follows.

**Lemma 1** *Let $x, y$ and $z$ be complex numbers. Then the system of equations*

$$ f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x + zk + k}{(x + zn)_{1+k}} \frac{y - zk + k}{(y - zn)_{1+k}} (z^{-1}(x - y) + k)_n g(k) \tag{2} $$

*is equivalent to the system of equations*

$$ g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x + zk)_n (y - zk)_n \frac{z^{-1}(x - y) + 2k}{(z^{-1}(x - y) + n)_{1+k}} f(k). \tag{3} $$

Although the importance of Fox-Wright function has been realized for many years, there are only a small number of summation formulas for this function to our knowledge. The corresponding results, which can be seen in the papers Chu and Wang (2008); Krattenthaler (1996); Wei et al. (2012), are all from inversion techniques and identities related to Saalschutz's theorem. The reader may refer to Ma (2007, 2011); Warnaar (2002) for more details on inversion techniques.

Inspired by the importance of Fox-Wright function and the lack of summation formulas for this function, we shall derive several summation formulas for Fox-Wright function according to Lemma 1, (1) and some other hypergeometric series identities in Sections 2-3.

# 2 Whipple-type series and summation formulas for Fox-Wright function

**Theorem 2** *Let $a$ and $\lambda$ be complex numbers. Then*

$$\begin{aligned}
&{}_5\Psi_6\left[\begin{matrix}(1,\frac{3}{2};1),(\lambda-a;\lambda),(\frac{1}{2}+\lambda-a;1+\lambda),(1+2a+n;1+2\lambda)\\(1+n;-1),(2+n,\frac{1}{2};1),(1+a;\lambda),(\frac{3}{2}+a;1+\lambda),(1+2\lambda-2a-n;1+2\lambda)\end{matrix}\,\middle|\,-1\right]\\
&=\frac{(-1)^n4^{2a-\lambda}(1+2a-\lambda)_n}{(\lambda-a+\lambda n)(1+2a+2\lambda n+2n)(1)_n}.
\end{aligned}$$

**Proof:** The case $a=-n$ of (1) reads

$$ {}_3F_2\left[\begin{matrix}-n,1+n,b\\c,1+2b-c\end{matrix}\,\middle|\,1\right]=\frac{(\frac{c-n}{2})_n(c-2b)_n}{(\frac{c-n}{2}-b)_n(c)_n}.$$

Perform the replacements $b\to1+2a-\lambda$ and $c\to2+2a+2\lambda n+n$ to obtain

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1+2a+2\lambda k+k+k}{(1+2a+2\lambda n+n)_{1+k}}\frac{2a-2\lambda-2\lambda k-k+k}{(2a-2\lambda-2\lambda n-n)_{1+k}}(1+k)_n\\
&\times\frac{(1)_k(1+2a-\lambda)_k}{(1+2a+2\lambda k+2k)(2a-2\lambda-2\lambda k)}\\
&=\frac{(1)_n}{2a-2\lambda-2\lambda n-n}\frac{(1+a+\lambda n)_n(2\lambda-2a+2\lambda n+n)_n}{(\lambda-a+\lambda n)_n(1+2a+2\lambda n+n)_{1+n}},
\end{aligned}$$

which fits to (2) with

$$\begin{aligned}
&x=1+2a,\quad y=2a-2\lambda,\quad z=1+2\lambda;\\
&g(k)=\frac{(1)_k(1+2a-\lambda)_k}{(1+2a+2\lambda k+2k)(2a-2\lambda-2\lambda k)};\\
&f(n)=\frac{(1)_n}{2a-2\lambda-2\lambda n-n}\frac{(1+a+\lambda n)_n(2\lambda-2a+2\lambda n+n)_n}{(\lambda-a+\lambda n)_n(1+2a+2\lambda n+n)_{1+n}}.
\end{aligned}$$

Then (3) produces the dual relation

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+2a+2\lambda k+k)_n(2a-2\lambda-2\lambda k-k)_n\frac{1+2k}{(1+n)_{1+k}}$$
$$\times\ \frac{(1)_k}{2a-2\lambda-2\lambda k-k}\frac{(1+a+\lambda k)_k(2\lambda-2a+2\lambda k+k)_k}{(\lambda-a+\lambda k)_k(1+2a+2\lambda k+k)_{1+k}}$$
$$=\ \frac{(1)_n(1+2a-\lambda)_n}{(1+2a+2\lambda n+2n)(2a-2\lambda-2\lambda n)}.$$

Rewrite it in accordance with Fox-Wright function, we get Theorem 2. □

When $\lambda=0$, Theorem 2 reduces to the hypergeometric series identity

$${}_5F_4\left[\begin{matrix}1,\frac{3}{2},\frac{1}{2}-a,1+2a+n,-n\\ \frac{1}{2},\frac{3}{2}+a,1-2a-n,2+n\end{matrix}\Big|\,1\right]=\frac{1+2a}{1+2a+2n}\frac{(2)_n}{(2a)_n},$$

which is a special case of Dougall's ${}_5F_4$-series identity (cf. Andrews et al. (2000)[p. 71]):

$${}_5F_4\left[\begin{matrix}a,1+\frac{a}{2},b,c,d\\ \frac{a}{2},1+a-b,1+a-c,1+a-d\end{matrix}\Big|\,1\right]$$
$$=\ \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}\tag{4}$$

provided that $Re(1+a-b-c-d)>0$.

Other two hypergeometric series identities from Theorem 2 can be displayed as follows.

**Example 1 ($\lambda=1$ in Theorem 2)**

$${}_9F_8\left[\begin{matrix}1,\frac{3}{2},1-a,\frac{3-2a}{4},\frac{5-2a}{4},\frac{1+2a+n}{3},\frac{2+2a+n}{3},\frac{3+2a+n}{3},-n\\ \frac{1}{2},1+a,\frac{5+2a}{4},\frac{3+2a}{4},\frac{5-2a-n}{3},\frac{4-2a-n}{3},\frac{3-2a-n}{3},2+n\end{matrix}\Big|\,1\right]$$
$$=\ \frac{2a(1-a)(1+2a)}{(1-a-n)(2a+n)(1+2a+4n)}\frac{(2)_n}{(2a-2)_n}.$$

**Example 2 ($\lambda=2$ in Theorem 2)**

$${}_{13}F_{12}\left[\begin{matrix}1,\frac{3}{2},\frac{2-a}{2},\frac{3-a}{2},\frac{5-2a}{6},\frac{7-2a}{6},\frac{9-2a}{6},\{\frac{i+2a+n}{5}\}_{i=1}^{5},-n\\ \frac{1}{2},\frac{2+a}{2},\frac{1+a}{2},\frac{7+2a}{6},\frac{5+2a}{6},\frac{3+2a}{6},\{\frac{10-i-2a-n}{5}\}_{i=1}^{5},2+n\end{matrix}\Big|\,1\right]$$
$$=\ \frac{(2-a)(1+2a)}{(2-a+2n)(1+2a+6n)}\frac{(2a-1)_n(2)_n}{(2a-4)_n(2a+1)_n}.$$

**Theorem 3** *Let $a$ and $\lambda$ be complex numbers. Then*

$$\sum_{i=0}^{m}(-1)^i\frac{(-m)_i}{i!}\Omega(\lambda,a,i,m,n)=\frac{(-1)^n2^{4a-2\lambda-m-1}(1+2a-\lambda-\frac{m}{2})_n}{(1+2a+2\lambda n+2n)(\lambda-a+\lambda n)(1)_n},$$

*where the expression on the left hand side is*

$$\Omega(\lambda,a,i,m,n) = {}_7\Psi_8\left[\begin{matrix}(1,\frac{3}{2};1),(\frac{m-i}{2}+\lambda-a;\lambda),(\frac{1+m-i}{2}+\lambda-a;1+\lambda),\\(\frac{3+m}{2}-i+\lambda,1+2a+n;1+2\lambda),(1-i+2\lambda;2+4\lambda)\\(1+n;-1),(2+n,\frac{1}{2};1),(\frac{2-i}{2}+a;\lambda),(\frac{3-i}{2}+a;1+\lambda),\\(\frac{1+m}{2}-i+\lambda,1+2\lambda-2a-n;1+2\lambda),(2+m-i+2\lambda;2+4\lambda)\end{matrix}\,\Big|\,-1\right].$$

Remark: In the symbol $\Omega(\lambda,a,i,m,n)$, the parameters on the first two lines are numerators and other ones are denominators. This remark is also applicative to Theorem 5.

**Proof:** Lemma 1 of Chu (2013) gives

$$\begin{aligned}{}_3F_2\left[\begin{matrix}a,1-a,b\\c,1+2b-c+m\end{matrix}\,\Big|\,1\right] &= \frac{(1+2b-c)_m}{(2+2b-2c)_m}\sum_{i=0}^{m}(-1)^i\frac{(-m)_i(\frac{3}{2}+b-c)_i}{i!(\frac{1}{2}+b-c)_i}\\ &\times\frac{(1-c)_i(1+2b-2c)_i}{(1+2b-c)_i(2+2b-2c+m)_i}\,{}_3F_2\left[\begin{matrix}a,1-a,b\\c-i,1+2b-c+i\end{matrix}\,\Big|\,1\right].\end{aligned}$$

Calculating the ${}_3F_2$-series on the right hand side by (1) and then setting $a=-n$ in the resulting identity, we have

$$\begin{aligned}{}_3F_2\left[\begin{matrix}-n,1+n,b\\c,1+2b-c+m\end{matrix}\,\Big|\,1\right] &= \frac{(1+2b-c)_m}{(2+2b-2c)_m}\sum_{i=0}^{m}(-1)^i\frac{(-m)_i(\frac{3}{2}+b-c)_i}{i!(\frac{1}{2}+b-c)_i}\\ &\times\frac{(1-c)_i(1+2b-2c)_i}{(1+2b-c)_i(2+2b-2c+m)_i}\frac{(\frac{c-n-i}{2})_n(c-2b-i)_n}{(\frac{c-n-i}{2}-b)_n(c-i)_n}.\end{aligned}$$

Employ the substitutions $b\to 1+2a-\lambda-\frac{m}{2}$ and $c\to 2+2a+2\lambda n+n$ to gain

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1+2a+2\lambda k+k+k}{(1+2a+2\lambda n+n)_{1+k}}\frac{2a-2\lambda-2\lambda k-k+k}{(2a-2\lambda-2\lambda n-n)_{1+k}}(1+k)_n\\ &\times\frac{(1)_k(1+2a-\lambda-\frac{m}{2})_k}{(1+2a+2\lambda k+2k)(2a-2\lambda-2\lambda k)}\\ &= \frac{(1)_n}{(1+2a+2\lambda n+n)(2a-2\lambda-2\lambda n-n)}\frac{(2\lambda-2a+2\lambda n+n)_m}{(1+2\lambda+4\lambda n+2n)_m}\\ &\times\sum_{i=0}^{m}(-1)^i\frac{(-m)_i(\frac{1-m}{2}-\lambda-2\lambda n-n)_i(-1-2\lambda-4\lambda n-2n-m)_i}{i!(-\frac{1+m}{2}-\lambda-2\lambda n-n)_i(1+2a-2\lambda-2\lambda n-n-m)_i}\\ &\times\frac{(-1-2a-2\lambda n-n)_i}{(-2\lambda-4\lambda n-2n)_i}\frac{(a+\lambda n+\frac{2-i}{2})_n(2\lambda-2a+2\lambda n+n+m-i)_n}{(\lambda-a+\lambda n+\frac{m-i}{2})_n(2+2a+2\lambda n+n-i)_n},\end{aligned}$$

which suits to (2) with

$$\begin{aligned}
&x=1+2a, \quad y=2a-2\lambda, \quad z=1+2\lambda;\\
&g(k)=\frac{(1)_k(1+2a-\lambda-\frac{m}{2})_k}{(1+2a+2\lambda k+2k)(2a-2\lambda-2\lambda k)};\\
&f(n)=\frac{(1)_n}{(1+2a+2\lambda n+n)(2a-2\lambda-2\lambda n-n)}\frac{(2\lambda-2a+2\lambda n+n)_m}{(1+2\lambda+4\lambda n+2n)_m}\\
&\quad\times\sum_{i=0}^{m}(-1)^i\frac{(-m)_i(\frac{1-m}{2}-\lambda-2\lambda n-n)_i(-1-2\lambda-4\lambda n-2n-m)_i}{i!(-\frac{1+m}{2}-\lambda-2\lambda n-n)_i(1+2a-2\lambda-2\lambda n-n-m)_i}\\
&\quad\times\frac{(-1-2a-2\lambda n-n)_i}{(-2\lambda-4\lambda n-2n)_i}\frac{(a+\lambda n+\frac{2-i}{2})_n(2\lambda-2a+2\lambda n+n+m-i)_n}{(\lambda-a+\lambda n+\frac{m-i}{2})_n(2+2a+2\lambda n+n-i)_n}.
\end{aligned}$$

Then (3) offers the dual relation

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+2a+2\lambda k+k)_n(2a-2\lambda-2\lambda k-k)_n\frac{1+2k}{(1+n)_{1+k}}\\
&\times\ \frac{(1)_k}{(1+2a+2\lambda k+k)(2a-2\lambda-2\lambda k-k)}\frac{(2\lambda-2a+2\lambda k+k)_m}{(1+2\lambda+4\lambda k+2k)_m}\\
&\times\ \sum_{i=0}^{m}(-1)^i\frac{(-m)_i(\frac{1-m}{2}-\lambda-2\lambda k-k)_i(-1-2\lambda-4\lambda k-2k-m)_i}{i!(-\frac{1+m}{2}-\lambda-2\lambda k-k)_i(1+2a-2\lambda-2\lambda k-k-m)_i}\\
&\times\ \frac{(-1-2a-2\lambda k-k)_i}{(-2\lambda-4\lambda k-2k)_i}\frac{(a+\lambda k+\frac{2-i}{2})_k(2\lambda-2a+2\lambda k+k+m-i)_k}{(\lambda-a+\lambda k+\frac{m-i}{2})_k(2+2a+2\lambda k+k-i)_k}\\
&=\ \frac{(1)_n(1+2a-\lambda-\frac{m}{2})_n}{(1+2a+2\lambda n+2n)(2a-2\lambda-2\lambda n)}.
\end{aligned}$$

Interchanging the summation order, we achieve Theorem 3 after some simplifications. □

When $m=0$, Theorem 3 reduces to Theorem 2. Performing the the replacements $\lambda\to\lambda-\frac{1}{2}$ and $a\to\frac{\lambda+a-1}{2}$ in the case $m=1$ of Theorem 3, we attain the following reciprocal formula after some reformulations.

**Corollary 4** *Let $a$ and $\lambda$ be complex numbers. Then*

$$A(\lambda;a,n)-A(-\lambda;a,n)=\frac{(-1)^{n+1}2^{2a+1}\,\lambda\,(a)_n}{(a+\lambda+2\lambda n+n)(a-\lambda-2\lambda n+n)(1)_n},$$

*where the symbol on the left hand side stands for*

$$\begin{aligned}
&A(\lambda;a,n)\\
&={}_4\Psi_5\left[\begin{matrix}(1;1),(\frac{1+\lambda-a}{2};\lambda-\frac{1}{2}),(\frac{2+\lambda-a}{2};\lambda+\frac{1}{2}),(\lambda+a+n;2\lambda)\\(1+n;-1),(2+n;1),(\frac{1+\lambda+a}{2};\lambda-\frac{1}{2}),(\frac{2+\lambda+a}{2};\lambda+\frac{1}{2}),(1+\lambda-a-n;2\lambda)\end{matrix}\;\Big|\;-1\right].
\end{aligned}$$

Two hypergeometric series identities from Corollary 4 can be laid out as follows.

**Example 3 ($\lambda = \frac{1}{2}$ in Corollary 4)**

$$\begin{aligned}&\frac{1}{1+2a}\,{}_4F_3\left[\begin{matrix}1, \frac{5-2a}{4}, \frac{1}{2}+a+n, -n\\ \frac{5+2a}{4}, \frac{3}{2}-a-n, 2+n\end{matrix}\,\Big|\, 1\right]\\ &+\frac{1}{1-2a}\,{}_4F_3\left[\begin{matrix}1, \frac{3-2a}{4}, \frac{1}{2}+a+n, -n\\ \frac{3+2a}{4}, \frac{3}{2}-a-n, 2+n\end{matrix}\,\Big|\, 1\right]\\ &=\frac{2}{(1-2a)(1+2a+4n)}\frac{(2)_n(a)_n}{(\frac{1}{2}+a)_n(-\frac{1}{2}+a)_n}.\end{aligned}$$

**Example 4 ($\lambda = \frac{3}{2}$ in Corollary 4)**

$$\begin{aligned}&\frac{1}{3+2a}\,{}_8F_7\left[\begin{matrix}1, \frac{5-2a}{4}, \frac{7-2a}{8}, \frac{11-2a}{8}, \frac{3+2a+2n}{6}, \frac{5+2a+2n}{6}, \frac{7+2a+2n}{6}, -n\\ \frac{5+2a}{4}, \frac{7+2a}{8}, \frac{11+2a}{8}, \frac{9-2a-2n}{6}, \frac{7-2a-2n}{6}, \frac{5-2a-2n}{6}, 2+n\end{matrix}\,\Big|\, 1\right]\\ &+\frac{1}{3-2a}\,{}_8F_7\left[\begin{matrix}1, \frac{3-2a}{4}, \frac{5-2a}{8}, \frac{9-2a}{8}, \frac{3+2a+2n}{6}, \frac{5+2a+2n}{6}, \frac{7+2a+2n}{6}, -n\\ \frac{3+2a}{4}, \frac{5+2a}{8}, \frac{9+2a}{8}, \frac{9-2a-2n}{6}, \frac{7-2a-2n}{6}, \frac{5-2a-2n}{6}, 2+n\end{matrix}\,\Big|\, 1\right]\\ &=\frac{6}{(3-2a+4n)(3+2a+8n)}\frac{(2)_n(a)_n}{(\frac{3}{2}+a)_n(-\frac{3}{2}+a)_n}.\end{aligned}$$

**Theorem 5** *Let $a$ and $\lambda$ be complex numbers. Then*

$$\sum_{i=0}^{m}\frac{(-m)_i}{i!}\Theta(\lambda,a,i,m,n)=\frac{(-1)^n 2^{4a-2\lambda-m-1}(1+2a-\lambda+\frac{m}{2})_n(\lambda-2a-\frac{m}{2})_m}{(1+2a+2\lambda n+2n)(\lambda-a+\lambda n)(1)_n},$$

*where the expression on the left hand side is*

$$\begin{aligned}&\Theta(\lambda,a,i,m,n)\\ &={}_8\Psi_9\left[\begin{matrix}(1,\frac{3}{2};1),(\frac{m-i}{2}+\lambda-a;\lambda),(\frac{1+m-i}{2}+\lambda-a;1+\lambda),\\ (\frac{3+m}{2}-i+\lambda,\frac{2+m}{2}+\lambda,1+2a+n;1+2\lambda),(1-i+2\lambda;2+4\lambda)\\ (1+n;-1),(2+n,\frac{1}{2};1),(\frac{2-i}{2}+a;\lambda),(\frac{3-i}{2}+a;1+\lambda),\\ (\frac{1+m}{2}-i+\lambda,\frac{2-m}{2}+\lambda,1+2\lambda-2a-n;1+2\lambda),(2+m-i+2\lambda;2+4\lambda)\end{matrix}\,\Big|\,-1\right].\end{aligned}$$

**Proof:** Utilizing Lemma 2 of Chu (2013), it is not difficult to obtain

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}a,1-a,b+\frac{m}{2}\\ c,1+2b-c\end{matrix}\,\Big|\,1\right]=\frac{(c-2b)_m(c-b-\frac{m}{2})_m}{(2c-2b-1)_m(1-b-\frac{m}{2})_m}\sum_{i=0}^{m}\frac{(-m)_i(\frac{3-m}{2}+b-c)_i}{i!(\frac{1-m}{2}+b-c)_i}\\ &\times\frac{(1-c)_i(1+2b-2c-m)_i}{(2+2b-2c)_i(1+2b-c-m)_i}{}_3F_2\left[\begin{matrix}a,1-a,b-\frac{m}{2}\\ c-i,1+2b-c-m+i\end{matrix}\,\Big|\,1\right].\end{aligned}$$

Evaluating the ${}_3F_2$-series on the right hand side by (1) and then taking $a=-n$ in the resulting identity, we have

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}-n,1+n,b+\frac{m}{2}\\ c,1+2b-c\end{matrix}\,\Big|\,1\right]=\frac{(c-2b)_m(c-b-\frac{m}{2})_m}{(2c-2b-1)_m(1-b-\frac{m}{2})_m}\sum_{i=0}^{m}\frac{(-m)_i(\frac{3-m}{2}+b-c)_i}{i!(\frac{1-m}{2}+b-c)_i}\\ &\times\frac{(1-c)_i(1+2b-2c-m)_i}{(2+2b-2c)_i(1+2b-c-m)_i}\frac{(\frac{c-n-i}{2})_n(c-2b+m-i)_n}{(\frac{c-n+m-i}{2}-b)_n(c-i)_n}.\end{aligned}$$

Employ the substitutions $b \to 1+2a-\lambda$ and $c \to 2+2a+2\lambda n+n$ to get

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1+2a+2\lambda k+k+k}{(1+2a+2\lambda n+n)_{1+k}}\frac{2a-2\lambda-2\lambda k-k+k}{(2a-2\lambda-2\lambda n-n)_{1+k}}(1+k)_n\\
&\times\frac{(1)_k(1+2a-\lambda+\frac{m}{2})_k}{(1+2a+2\lambda k+2k)(2a-2\lambda-2\lambda k)}\\
&=\frac{(1)_n}{(1+2a+2\lambda n+n)(2a-2\lambda-2\lambda n-n)}\frac{(2\lambda-2a+2\lambda n+n)_m(1+\lambda+2\lambda n+n-\frac{m}{2})_m}{(1+2\lambda+4\lambda n+2n)_m(\lambda-2a-\frac{m}{2})_m}\\
&\times\sum_{i=0}^{m}\frac{(-m)_i(\frac{1-m}{2}-\lambda-2\lambda n-n)_i(-1-2\lambda-4\lambda n-2n-m)_i}{i!(-\frac{1+m}{2}-\lambda-2\lambda n-n)_i(1+2a-2\lambda-2\lambda n-n-m)_i}\\
&\times\frac{(-1-2a-2\lambda n-n)_i}{(-2\lambda-4\lambda n-2n)_i}\frac{(a+\lambda n+\frac{2-i}{2})_n(2\lambda-2a+2\lambda n+n+m-i)_n}{(\lambda-a+\lambda n+\frac{m-i}{2})_n(2+2a+2\lambda n+n-i)_n},
\end{aligned}$$

which satisfies (2) with

$$\begin{aligned}
&x=1+2a,\quad y=2a-2\lambda,\quad z=1+2\lambda;\\
&g(k)=\frac{(1)_k(1+2a-\lambda+\frac{m}{2})_k}{(1+2a+2\lambda k+2k)(2a-2\lambda-2\lambda k)};\\
&f(n)=\frac{(1)_n}{(1+2a+2\lambda n+n)(2a-2\lambda-2\lambda n-n)}\frac{(2\lambda-2a+2\lambda n+n)_m(1+\lambda+2\lambda n+n-\frac{m}{2})_m}{(1+2\lambda+4\lambda n+2n)_m(\lambda-2a-\frac{m}{2})_m}\\
&\quad\times\sum_{i=0}^{m}\frac{(-m)_i(\frac{1-m}{2}-\lambda-2\lambda n-n)_i(-1-2\lambda-4\lambda n-2n-m)_i}{i!(-\frac{1+m}{2}-\lambda-2\lambda n-n)_i(1+2a-2\lambda-2\lambda n-n-m)_i}\\
&\quad\times\frac{(-1-2a-2\lambda n-n)_i}{(-2\lambda-4\lambda n-2n)_i}\frac{(a+\lambda n+\frac{2-i}{2})_n(2\lambda-2a+2\lambda n+n+m-i)_n}{(\lambda-a+\lambda n+\frac{m-i}{2})_n(2+2a+2\lambda n+n-i)_n}.
\end{aligned}$$

Then (3) produces the dual relation

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+2a+2\lambda k+k)_n(2a-2\lambda-2\lambda k-k)_n\frac{1+2k}{(1+n)_{1+k}}\\
&\times\frac{(1)_k}{(1+2a+2\lambda k+k)(2a-2\lambda-2\lambda k-k)}\frac{(2\lambda-2a+2\lambda k+k)_m(1+\lambda+2\lambda k+k-\frac{m}{2})_m}{(1+2\lambda+4\lambda k+2k)_m(\lambda-2a-\frac{m}{2})_m}\\
&\times\sum_{i=0}^{m}\frac{(-m)_i(\frac{1-m}{2}-\lambda-2\lambda k-k)_i(-1-2\lambda-4\lambda k-2k-m)_i}{i!(-\frac{1+m}{2}-\lambda-2\lambda k-k)_i(1+2a-2\lambda-2\lambda k-k-m)_i}\\
&\times\frac{(-1-2a-2\lambda k-k)_i}{(-2\lambda-4\lambda k-2k)_i}\frac{(a+\lambda k+\frac{2-i}{2})_k(2\lambda-2a+2\lambda k+k+m-i)_k}{(\lambda-a+\lambda k+\frac{m-i}{2})_k(2+2a+2\lambda k+k-i)_k}\\
&=\frac{(1)_n(1+2a-\lambda+\frac{m}{2})_n}{(1+2a+2\lambda n+2n)(2a-2\lambda-2\lambda n)}.
\end{aligned}$$

Interchanging the summation order, we gain Theorem 5 after some simplifications. □

When $m=0$, Theorem 5 also reduces to Theorem 2. Performing the the replacements $\lambda \to \lambda-\frac{1}{2}$ and $a \to \frac{\lambda+a-1}{2}$ in the case $m=1$ of Theorem 5, we achieve the following reciprocal formula after some reformulations.

**Corollary 6** *Let $a$ and $\lambda$ be complex numbers. Then*

$$B(\lambda;a,n)+B(-\lambda;a,n)=\frac{(-1)^n4^a(a)_{1+n}}{(a+\lambda+2\lambda n+n)(a-\lambda-2\lambda n+n)(1)_n},$$

*where the symbol on the left hand side stands for*

$$\begin{aligned}&B(\lambda;a,n)\\&={}_5\Psi_6\left[\begin{matrix}(1,\frac{3}{2};1),(\frac{1+\lambda-a}{2};\lambda-\frac{1}{2}),(\frac{2+\lambda-a}{2};\lambda+\frac{1}{2}),(\lambda+a+n;2\lambda)\\(1+n;-1),(2+n,\frac{1}{2};1),(\frac{1+\lambda+a}{2};\lambda-\frac{1}{2}),(\frac{2+\lambda+a}{2};\lambda+\frac{1}{2}),(1+\lambda-a-n;2\lambda)\end{matrix}\,\Big|\,-1\right].\end{aligned}$$

Two hypergeometric series identities from Corollary 6 can be displayed as follows.

**Example 5 ($\lambda=\frac{1}{2}$ in Corollary 6)**

$$\begin{aligned}&\frac{1}{1+2a}\,{}_5F_4\left[\begin{matrix}1,\frac{3}{2},\frac{5-2a}{4},\frac{1}{2}+a+n,-n\\\frac{1}{2},\frac{5+2a}{4},\frac{3}{2}-a-n,2+n\end{matrix}\,\Big|\,1\right]\\&-\frac{1}{1-2a}\,{}_5F_4\left[\begin{matrix}1,\frac{3}{2},\frac{3-2a}{4},\frac{1}{2}+a+n,-n\\\frac{1}{2},\frac{3+2a}{4},\frac{3}{2}-a-n,2+n\end{matrix}\,\Big|\,1\right]\\&=\frac{4}{(2a-1)(1+2a+4n)}\frac{(1)_{1+n}(a)_{1+n}}{(\frac{1}{2}+a)_n(-\frac{1}{2}+a)_n}.\end{aligned}$$

**Example 6 ($\lambda=\frac{3}{2}$ in Corollary 6)**

$$\begin{aligned}&\frac{1}{3+2a}\,{}_9F_8\left[\begin{matrix}1,\frac{3}{2},\frac{5-2a}{4},\frac{7-2a}{8},\frac{11-2a}{8},\frac{3+2a+2n}{6},\frac{5+2a+2n}{6},\frac{7+2a+2n}{6},-n\\\frac{1}{2},\frac{5+2a}{4},\frac{7+2a}{8},\frac{11+2a}{8},\frac{9-2a-2n}{6},\frac{7-2a-2n}{6},\frac{5-2a-2n}{6},2+n\end{matrix}\,\Big|\,1\right]\\&-\frac{1}{3-2a}\,{}_9F_8\left[\begin{matrix}1,\frac{3}{2},\frac{3-2a}{4},\frac{5-2a}{8},\frac{9-2a}{8},\frac{3+2a+2n}{6},\frac{5+2a+2n}{6},\frac{7+2a+2n}{6},-n\\\frac{1}{2},\frac{3+2a}{4},\frac{5+2a}{8},\frac{9+2a}{8},\frac{9-2a-2n}{6},\frac{7-2a-2n}{6},\frac{5-2a-2n}{6},2+n\end{matrix}\,\Big|\,1\right]\\&=\frac{4}{(2a-3-4n)(3+2a+8n)}\frac{(1)_{1+n}(a)_{1+n}}{(\frac{3}{2}+a)_n(-\frac{3}{2}+a)_n}.\end{aligned}$$

# 3 Two different summation formulas for Fox-Wright function

**Theorem 7** *Let $a$, $b$ and $c$ be complex numbers. Then*

$$\begin{aligned}&{}_6\Psi_7\left[\begin{matrix}(1,\frac{3}{2};1),(\frac{1}{2}-a+b,a+n;\frac{1}{2}),(b-a,-\frac{1}{2}+a+n;-\frac{1}{2})\\(1+n;-1),(2+n,\frac{1}{2};1),(1+a-c,\frac{1}{2}-a+b+c;\frac{1}{2}),(\frac{1}{2}+a-c,b+c-a;-\frac{1}{2})\end{matrix}\,\Big|\,-1\right]\\&=\frac{2(\frac{1}{2}+2a-b-c)_n(b)_n(c)_n}{(1)_n(1+2a-2b)_n}\frac{\Gamma(2a-1+n)\Gamma(2b-2a)}{\Gamma(1+2a-2c+n)\Gamma(2b+2c-2a+n)}.\end{aligned}$$

**Proof:** A ${}_7F_6$-series identity due to Chu and Wang (2009)[Corollary 16] can be expressed as

$$\begin{aligned}
&{}_7F_6\left[\begin{matrix} a, 1+\frac{a}{3}, b, c, \frac{1}{2}+a-b-c, 1+n, -n \\ \frac{a}{3}, 1+a-2b, 1+a-2c, 2b+2c-a, \frac{1+a-n}{2}, \frac{2+a+n}{2} \end{matrix} \,\Big|\, 1\right] \\
&= \frac{(1+a)_n(2b-a)_n(\frac{1+a-n}{2}-c)_n(b+c-\frac{a+n}{2})_n}{(1+a-2c)_n(2b+2c-a)_n(\frac{1+a-n}{2})_n(b-\frac{a+n}{2})_n}.
\end{aligned}$$

Employ the substitution $a \to 2a$ to attain

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a+\frac{k}{2}+k}{(a+\frac{n}{2})_{1+k}}\frac{a-\frac{1}{2}-\frac{k}{2}+k}{(a-\frac{1}{2}-\frac{n}{2})_{1+k}}(1+k)_n \\
&\times \frac{(1)_k(2a-1)_k(b)_k(c)_k(\frac{1}{2}+2a-b-c)_k}{(1+2a-2b)_k(1+2a-2c)_k(2b+2c-2a)_k} \\
&= \frac{(1)_n(2b-2a)_n(2a-1)_n}{(1+2a-2c)_n(2b+2c-2a)_n}\frac{(\frac{1-n}{2}+a-c)_n(b+c-a-\frac{n}{2})_n}{(a-\frac{1+n}{2})_n(b-a-\frac{n}{2})_n},
\end{aligned}$$

which fits to (2) with

$$\begin{aligned}
&x=a, \quad y=a-\frac{1}{2}, \quad z=\frac{1}{2}; \\
&g(k)=\frac{(1)_k(2a-1)_k(b)_k(c)_k(\frac{1}{2}+2a-b-c)_k}{(1+2a-2b)_k(1+2a-2c)_k(2b+2c-2a)_k}; \\
&f(n)=\frac{(1)_n(2b-2a)_n(2a-1)_n}{(1+2a-2c)_n(2b+2c-2a)_n}\frac{(\frac{1-n}{2}+a-c)_n(b+c-a-\frac{n}{2})_n}{(a-\frac{1+n}{2})_n(b-a-\frac{n}{2})_n}.
\end{aligned}$$

Then (3) gives the dual relation

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(a+\frac{k}{2}\right)_n\left(a-\frac{1}{2}-\frac{k}{2}\right)_n\frac{1+2k}{(1+n)_{1+k}} \\
&\times \frac{(1)_k(2b-2a)_k(2a-1)_k}{(1+2a-2c)_k(2b+2c-2a)_k}\frac{(\frac{1-k}{2}+a-c)_k(b+c-a-\frac{k}{2})_k}{(a-\frac{1+k}{2})_k(b-a-\frac{k}{2})_k} \\
&= \frac{(1)_n(2a-1)_n(b)_n(c)_n(\frac{1}{2}+2a-b-c)_n}{(1+2a-2b)_n(1+2a-2c)_n(2b+2c-2a)_n}.
\end{aligned}$$

Writing it in terms of Fox-Wright function, we obtain Theorem 7. □

When $b \to \infty$, Theorem 7 offers the following formula.

**Corollary 8** *Let $a$ and $c$ be complex numbers. Then*

$$\begin{aligned}
&{}_4\Psi_5\left[\begin{matrix} (1,\frac{3}{2};1), (a+n;\frac{1}{2}), (-\frac{1}{2}+a+n;-\frac{1}{2}) \\ (1+n;-1), (2+n,\frac{1}{2};1), (1+a-c;\frac{1}{2}), (\frac{1}{2}+a-c;-\frac{1}{2}) \end{matrix} \,\Big|\, -1\right] \\
&= \frac{1}{2^{2c+2n-1}}\frac{(c)_n}{(1)_n}\frac{\Gamma(2a-1+n)}{\Gamma(1+2a-2c+n)}.
\end{aligned}$$

Fixing $c = 1 - m - n$ in Corollary 8, we get the following identity.

**Corollary 9** *Let $a$ be a complex number and $m$ a nonnegative integer. Then*

$$\begin{aligned}&{}_{3+2m}F_{2+2m}\left[\begin{matrix}1,\ \frac{3}{2},\ \{2a+2n+2i-2\}_{i=1}^{m},\ \{3-2a-2n-2i\}_{i=1}^{m},\ -n\\ \frac{1}{2},\ \{2a+2n+2i-1\}_{i=1}^{m},\ \{4-2a-2n-2i\}_{i=1}^{m},\ 2+n\end{matrix}\ \Big|\ 1\right]\\ &= (-1)^n\frac{(2)_n(m)_n}{(2a-1+n)_n(2a-1+2m+2n)_n}.\end{aligned}$$

**Theorem 10** *Let $a$ and $b$ be complex numbers. Then*

$$\begin{aligned}&{}_7\Psi_6\left[\begin{matrix}(1,\frac{3}{2},1-a-b,b-a,a+n;1),(\frac{1-a-b}{2},\frac{b-a}{2};-\frac{1}{2})\\ (1+n;-1),(2+n,2-a-n,\frac{1}{2};1),(\frac{1-a-b}{2},\frac{b-a}{2};\frac{1}{2})\end{matrix}\ \Big|\ \frac{1}{4}\right]\\ &= \frac{(b)_n(1-b)_n}{(a+b)_n(1+a-b)_n}\frac{\Gamma(a+n)\Gamma(1-a-b)\Gamma(b-a)}{2\Gamma(1+n)\Gamma(2-a)}.\end{aligned}$$

**Proof:** Setting $d = 1 - b$ and $e = 1 - c$ in the transformation formula (cf. Andrews et al. (2000)[p. 147])

$$\begin{aligned}&{}_6F_5\left[\begin{matrix}a, & 1+a/2, & b, & c, & d, & e\\ & a/2, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e\end{matrix}\ \Big|\ -1\right]\\ &= \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}{}_3F_2\left[\begin{matrix}1+a-d-e, & b, & c\\ 1+a-d, & 1+a-e\end{matrix}\ \Big|\ 1\right]\end{aligned}$$

and calculating the series on the right hand side by Dixon's ${}_3F_2$-series identity(cf. Andrews et al. (2000)[p. 72]):

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}a,b,c\\ 1+a-b,1+a-c\end{matrix}\ \Big|\ 1\right]\\ &= \frac{\Gamma(1+\frac{a}{2})\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{a}{2}-b-c)}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)\Gamma(1+\frac{a}{2}-c)\Gamma(1+a-b-c)},\end{aligned}$$

where $Re(1 + \frac{a}{2} - b - c) > 0$, we gain

$$\begin{aligned}&{}_6F_5\left[\begin{matrix}a,1+\frac{a}{2},b,1-b,c,1-c\\ \frac{a}{2},1+a-b,a+b,1+a-c,a+c\end{matrix}\ \Big|\ -1\right]\\ &= \frac{\pi\Gamma(a+b)\Gamma(1+a-b)\Gamma(a+c)\Gamma(1+a-c)}{2^{2a-1}\Gamma(a)\Gamma(1+a)\Gamma(\frac{a+b+c}{2})\Gamma(\frac{1+a+b-c}{2})\Gamma(\frac{1+a-b+c}{2})\Gamma(\frac{2+a-b-c}{2})}\end{aligned}$$

provided that $Re(a) > 0$. The case $c = -n$ of it can be manipulated as

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a+k+k}{(a+n)_{1+k}}\frac{a-1-k+k}{(a-1-n)_{1+k}}(1+k)_n\frac{(1)_k(a)_k(b)_k(1-b)_k}{(1+a-b)_k(a+b)_k}(-1)^k\\ &= \frac{(1)_n(a)_n(1-a-b)_n(b-a)_n}{(1+a)_n(-a)_n(2-a)_n}\frac{(-\frac{a+n}{2})_n(\frac{1-a-n}{2})_n}{(\frac{1-a-b-n}{2})_n(\frac{b-a-n}{2})_n},\end{aligned}$$

which suits to (2) with

$$x = a, \quad y = a-1, \quad z = 1;$$
$$g(k) = \frac{(1)_k(a)_k(b)_k(1-b)_k}{(1+a-b)_k(a+b)_k}(-1)^k;$$
$$f(n) = \frac{(1)_n(a)_n(1-a-b)_n(b-a)_n}{(1+a)_n(-a)_n(2-a)_n}\frac{(-\frac{a+n}{2})_n(\frac{1-a-n}{2})_n}{(\frac{1-a-b-n}{2})_n(\frac{b-a-n}{2})_n}.$$

Then (3) produces the dual relation

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(a+k)_n(a-1-k)_n\frac{1+2k}{(1+n)_{1+k}}\frac{(1)_k(a)_k(1-a-b)_k(b-a)_k}{(1+a)_k(-a)_k(2-a)_k}$$
$$\times \frac{(-\frac{a+k}{2})_k(\frac{1-a-k}{2})_k}{(\frac{1-a-b-k}{2})_k(\frac{b-a-k}{2})_k} = \frac{(1)_n(a)_n(b)_n(1-b)_n}{(1+a-b)_n(a+b)_n}(-1)^n.$$

Writing it according to Fox-Wright function, we achieve Theorem 10. □

The case $d \to \infty$ of (4) reads

$${}_4F_3\left[\begin{matrix} a, 1+\frac{a}{2}, b, c \\ \frac{a}{2}, 1+a-b, 1+a-c \end{matrix} \;\Big|\; -1\right] = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}$$

with $Re(1+\frac{a}{2}-b-c) > 0$. When $b \to \infty$, Theorem 10 reduces to the special case of it:

$${}_4F_3\left[\begin{matrix} 1, \frac{3}{2}, a+n, -n \\ \frac{1}{2}, 2-a-n, 2+n \end{matrix} \;\Big|\; -1\right] = (-1)^n\frac{(2)_n}{(a-1)_n}.$$

Taking $b = -2m$ in Theorem 10, we attain the following result.

**Corollary 11** *Let $a$ be a complex number and $m$ a nonnegative integer. Then*

$${}_{5+4m}F_{4+2m}\left[\begin{matrix} 1, \frac{3}{2}, \{1-a+2i\}_{i=0}^{m}, \{a+2i\}_{i=1}^{m}, \{1-a-2i\}_{i=1}^{m}, \{2+a-2i\}_{i=1}^{m}, a+n, -n \\ \frac{1}{2}, \{1+a+2i\}_{i=0}^{m}, \{-a+2i\}_{i=1}^{m}, \{1+a-2i\}_{i=1}^{m}, \{2-a-2i\}_{i=1}^{m}, 2-a-n, 2+n \end{matrix} \;\Big|\; 1\right]$$
$$= (-1)^n\frac{(-2m)_n(1+2m)_n(2)_n}{(a-1)_n(a-2m)_n(1+a+2m)_n}.$$


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