Discrete Mathematics and Theoretical Computer Science DMTCS vol. **22**:4, 2020, #6

Even cycles and perfect matchings in claw-free plane graphs

Shanshan Zhang Xiumei Wang* Jinjiang Yuan†

School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, Henan 450001, China

received 31st Jan. 2020, revised 2nd June 2020, accepted 30th Aug. 2020.

Lovász showed that a matching covered graph G has an ear decomposition starting with an arbitrary edge of G. Let G be a graph which has a perfect matching. We call G cycle-nice if for each even cycle C of G, $G - V(C)$ has a perfect matching. If G is a cycle-nice matching covered graph, then G has ear decompositions starting with an arbitrary even cycle of G. In this paper, we characterize cycle-nice claw-free plane graphs. We show that the only cycle-nice simple 3-connected claw-free plane graphs are K_4, W_5 and $\overline{C}_6$. Furthermore, every cycle-nice 2-connected claw-free plane graph can be obtained from a graph in the family $\mathcal{F}$ by a sequence of three types of operations, where $\mathcal{F}$ consists of even cycles, a diamond, K_4, and $\overline{C}_6$.

Keywords: nice cycle, cycle-nice graph, claw-free graph, plane graph

1 Introduction

In this paper, all graphs are connected and loopless, but perhaps have multiple edges. We follow the notation and terminology in (1) except otherwise stated. We use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. For an edge e of G, if there is another edge whose ends are the same as e, then e is called a *multiple edge*; otherwise, e is called a *single edge*. The *underlying simple graph* of G is the simple spanning subgraph of G obtained from G by first deleting all the edges and then connecting each pair of adjacent vertices by a single edge. A *perfect matching* of G is a set of independent edges covering all vertices of G. For a connected plane bipartite graph G with the minimum vertex degree at least 2, a face f of G is said to be a *forcing face* if $G-V(f)$ has exactly one perfect matching. The concept of forcing face was first introduced in Che and Chen (7), which is a natural generalization of the concept of forcing hexagon of a hexagonal system introduced in Che and Chen (6). For research on forcing faces, see (7, 8). In particular, Che and Chen (7) presented a characterization of plane elementary bipartite graphs whose finite faces are all forcing (by using ear decompositions). Here, a graph is *elementary* if the union of all its perfect matchings forms a connected subgraph. A graph G is called *cycle-forced* if for each even

*Corresponding author. Email address: wangxiumei@zzu.edu.cn. Supported by the National Natural Science Foundation of China (Nos. 11801526, 11971445, and 11571323).

†Supported by the National Natural Science Foundation of China (No. 11671368).

ISSN 1365–8050

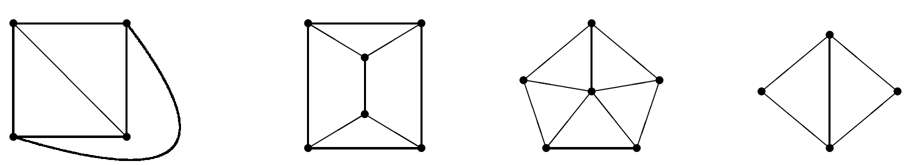
cycle C of G, $G - V(C)$ has exactly one perfect matching. All the cycle-forced Hamiltonian bipartite graphs and bipartite graphs have been characterized, see (10, 21).

A subgraph H of G is called *nice* if $G - V(H)$ has a perfect matching (see (13)). In particular, when H is a cycle, we call H a *nice cycle*. A nice subgraph is also called a conformal subgraph in (5), a well-fitted subgraph in (14), and a central subgraph in (16). A graph G is called *cycle-nice* if each even cycle of G is a nice cycle. Clearly, a cycle-forced graph is also cycle-nice. Given a proper subgraph H of G, an *ear* of G with respect to H is an odd path of G having both ends, but no interior vertices, in H. A graph G has an *ear decomposition* if G can be represented as $G' + P_1 + \cdots + P_r$, where G' is a subgraph of G, P_1 is an ear of G with respect to G', and P_i is an ear of G with respect to $G' + P_1 + \cdots + P_{i-1}$ for $2 \leq i \leq r$. Ear decomposition is a powerful tool in the study of the structure of matchings and the enumeration of matchings (11, 13). The idea of ear decomposition occurred first in Hetyei (9), and was further developed by Lovász, Carvalho, etc. (2–4, 12). A graph G is *matching covered* if each edge of G induces a nice subgraph of G.

Let G be a matching covered graph. Lovász (12) showed that, for a subgraph G' of G, G has an ear decomposition starting with G' if and only if G' is a nice subgraph of G. This implies that, for each edge e of G, there is an ear decomposition starting with e, that is, $G = G' + P_1 + \cdots + P_r$, where G' is induced by the edge e. In this case $G' + P_1$ is an even cycle. If G is a cycle-nice matching covered graph, then G has ear decompositions starting with an arbitrary even cycle.

A graph G is *claw-free* if the underlying simple graph of G contains no induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$. Sumner (17–19) and Las Vergnas (20) studied perfect matchings in claw-free graphs, and independently showed that every connected claw-free graph with an even number of vertices has a perfect matching. A characterization of 2-connected claw-free cubic graphs which have ear decompositions starting with an arbitrary induced even cycle is presented in (15).

In this paper we present a characterization of cycle-nice 2-connected claw-free plane graphs. The paper is organized as follows. In Section 2, we present some basic results. In Section 3, we prove that the only cycle-nice simple 3-connected claw-free plane graphs are K_4, W_5 and $\overline{C}_6$. In Section 4, we show that every cycle-nice 2-connected claw-free plane graph can be obtained from a graph in the family $\mathcal{F}$ by a sequence of three types of operations, where $\mathcal{F}$ consists of even cycles, a diamond, K_4, and $\overline{C}_6$ (see Figure 1).



K_4 $\overline{C_6}$ W_5 a diamond

Fig. 1. The four graphs

2 Some basic results

We begin with some notions and notations. A path P is *odd* or *even*, if its length is odd or even, respectively. A *uv-path* is a path with ends u and v. Let G be a graph. A k-vertex of G is a vertex of degree k.

The set of neighbours of a vertex v in G is denoted by $N_G(v)$. An *even subdivision* of G at an edge e is a graph obtained from G by replacing e by an odd path P_e with length at least three. An *odd expansion* of G at a vertex v of G is a graph obtained from G by the following four operations:
(*i*) splitting v into two vertices v' and v'',
(*ii*) adding an even path P_v with length at least two which connects v' and v'',
(*iii*) distributing the edges of G incident with v among v' and v'' such that v' and v'' each has at least one neighbour in $V(G-v)$, and
(*iv*) adding some edges joining v' and v'' or not.
If the edges which join v' and v'' are added, then the odd expansion of G is called an *odd A-expansion*; otherwise, the odd expansion of G is called an *odd L-expansion*.

A graph G is *outerplanar* if it has a planar embedding in which all vertices lie on the boundary of its outer face. An outerplanar graph equipped with such an embedding is called an *outerplane graph*. Let K be a 2-vertex cut of a graph G, and X the vertex set of a component of $G-K$. The subgraph of G induced by $X \cup K$ is called a *K-component* of G. Modify a K-component by adding a new edge (possible a multiple edge) joining the two vertices of K. We refer to the modified K-components as *marked K-components*.

Let Y be a subset of $V(G)$. We use $\nabla(Y)$ to denote the *edge cut* of G, whose edges have one end in Y and the other in $\overline{Y}$. The graph $G\{Y\}$ and $G\{\overline{Y}\}$ are obtained from G by contracting $\overline{Y}$ and Y to a single vertex, respectively.

The following lemmas will be used to obtain the main results of this paper.

Lemma 2.1. *((1)) In a loopless 3-connected plane graph G, for any vertex v, the neighbours of v lie on a common cycle which is the boundary of the face of $G-v$ where the vertex v is situated.*

Lemma 2.2. *((1)) Every simple planar graph has a vertex of degree at most five.*

From exercise 11.2.7 in (1), we have the following Lemma 2.3.

Lemma 2.3. *Every simple 2-connected outerplanar graph has two nonadjacent vertices of degree two.*

Lemma 2.4. *((1)) Let G be a 2-connected graph, and K a 2-vertex cut of G. Then the marked K-components are also 2-connected.*

Lemma 2.5. *Let e be an edge of a graph G, and G' an even subdivision of G at e. Then G' is cycle-nice if and only if G is cycle-nice.*

Proof: Let u and v be the two ends of e. Since G' is an even subdivision of G at e, G' has an odd uv-path P_e which is used to replace e and whose internal vertices are 2-vertices of G'.

Suppose first that G' is a cycle-nice graph. Let C be an even cycle of G. If $e \in E(C)$, let $C' := (C-e) \cup P_e$. Then C' is an even cycle of G' such that $G - V(C) = G' - V(C')$. Since G' is cycle-nice, $G' - V(C')$ has a perfect matching M. Then M is also a perfect matching of $G - V(C)$. If $e \notin E(C)$, then C is an even cycle of G'. Let M be a perfect matching of $G' - V(C)$. If $M \cap E(P_e)$ is a perfect matching of P_e, let $M' = (M \setminus E(P_e)) \cup \{e\}$; otherwise, let $M' = M \setminus E(P_e)$. Then M' is a perfect matching of $G - V(C)$. Hence, G is cycle-nice.

Conversely, suppose that G is cycle-nice. Let C' be an even cycle of G'. If $E(P_e) \cap E(C') \neq \emptyset$, then P_e is a segment of C'. Let C be the cycle obtained from C' by replacing P_e by the edge e. Then C is an even cycle of G such that $G - V(C) = G' - V(C')$. Let M' be a perfect matching of $G - V(C)$. Then M' is also a perfect matching of $G' - V(C')$. If $E(P_e) \cap E(C) = \emptyset$, then C' is an even cycle of

G. Let M' be a perfect matching of $G - V(C')$, and let M'_1 be a perfect matching of P_e. If $e \in M'$, then $(M' \setminus \{e\}) \cup M'_1$ is a perfect matching of $G' - V(C')$; if $e \notin M'$, then $M' \cup (M'_1 \triangle E(P_e))$ is a perfect matching of $G' - V(C')$. Consequently, G' is cycle-nice. The lemma follows. $\square$

Lemma 2.6. *Let G be a 2-connected claw-free graph with a 2-vertex cut $\{u, v\}$. Then*

(i) $G - \{u, v\}$ has exactly two components G_1 and G_2.

Furthermore, suppose that G is cycle-nice. Let G'_i be the underlying simple graph of the graph which is obtained from the $\{u, v\}$-component of G containing G_i by deleting the possible edges of G connecting u and v. Then

(ii) at least one of G'_1 and G'_2 is a path (suppose that G'_2 is a path in the following statements);

(iii) if G'_2 is an odd path, then the two marked $\{u, v\}$-components of G are cycle-nice;

(iv) if G'_2 is an even path, then $G\{V(G_1)\}$ and $G\{V(G_2)\}$ are cycle-nice. In particular, $uv \notin E(G)$ when neither G'_1 nor G'_2 is a path of length two.

Proof: Suppose that $G_1, G_2, \ldots, G_t$ are the components of $G - \{u, v\}$, where $t \geq 2$. Since G is claw-free, for each vertex in $\{u, v\}$, its neighbours lie in at most two components of $G - \{u, v\}$. If $t \geq 3$, then there exists a component, say G_1, such that only one of u and v, say u, has neighbours in G_1. Then u is a cut vertex of G, a contradiction. So, $t = 2$. (i) follows.

Now, suppose that G is cycle-nice. To show (ii), we first prove the following claim.

Claim. If $|V(G_i)| \geq 2$ and G'_i is not a path, then in G'_i there are two uv-paths which have different parity, $i = 1$ or 2.

For convenience, suppose that $i = 1$. Since G is 2-connected and $|V(G_1)| \geq 2$, there are two nonadjacent edges uu' and vv' with $u', v' \in V(G_1)$. Since G_1 is connected, there is a path P' in G_1 connecting u' and v'. Let $P_1 = uu'P'v'v$. Then P_1 is a uv-path in G'_1 with length at least three. Suppose that $P_1 = x_0x_1 \cdots x_sx_{s+1}$ is a longest such path in G'_1, where $s \geq 2$, $x_0 = u$ and $x_{s+1} = v$. We will find another uv-path P'_1 in G'_1 such that P_1 and P'_1 have different parity.

When $V(G'_1) \neq V(P_1)$, since G_1 is connected, there is a vertex x of $G_1 - V(P_1)$ such that $xx_j \in E(G_1)$, $j \in \{1, 2, \ldots, s\}$. Since G is claw-free, at least one of xx_{j-1}, xx_{j+1} and $x_{j-1}x_{j+1}$ is an edge of G'_1. Since P_1 is a longest uv-path, we have $xx_{j-1}, xx_{j+1} \notin E(G'_1)$, and so, $x_{j-1}x_{j+1} \in E(G'_1)$. Let $P'_1 = x_0P_1x_{j-1}x_{j+1}P_1x_{s+1}$. Then P'_1 is the desired path.

When $V(G'_1) = V(P_1)$, since G'_1 is not a path, there is an edge x_jx_k in $E(G'_1) \setminus E(P_1)$ with $j, k \in \{0, 1, 2, \ldots, s+1\}$, $k > j + 1$ and $\{j, k\} \neq \{0, s+1\}$. Suppose first that for any edge $x_ix_{i'}$ in $E(G'_1) \setminus E(P_1)$, i and i' have different parity. So j and k have different parity and $k > j + 2$. If $j \geq 1$, since $j - 1$, $j + 1$ and k have the same parity, we have $x_{j-1}x_{j+1}, x_{j-1}x_k, x_{j+1}x_k \notin E(G_1)$. Then the subgraph induced by x_j and its three neighbors x_{j-1}, x_{j+1} and x_k contains a claw, a contradiction. If $j = 0$, then $k \geq 3$, $k \neq s + 1$ and k is odd. Then $x_1x_k \notin E(G_1)$. In this case, the subgraph induced by x_0 and its three neighbours x_1, x_k and one in $V(G_2)$ contains a claw, a contradiction. Thus, we may suppose that j and k have the same parity. Let $P'_1 = x_0P_1x_jx_kP_1x_{s+1}$. Then P'_1 is the desired path. The claim follows.

Now we continue the proof of statement (ii). If G_1 or G_2 is trivial, then (ii) holds trivially. Suppose in the following that $|V(G_i)| \geq 2$ for $i = 1, 2$, and neither G'_1 nor G'_2 is a path. Since $|V(G)|$ is even, $|V(G_1)|$ and $|V(G_2)|$ have the same parity. From the above claim, when G_1 and G_2 are odd components, we may suppose that P_i is an odd uv-path of G'_i, $i = 1, 2$; when G_1 and G_2 are even components, we may suppose that P_i is an even uv-path of G'_i, $i = 1, 2$. Let $C = P_1 \cup P_2$. Then C is an even cycle of G.

Since $|V(G_i) \setminus V(C)|$ is odd, $G - V(C)$ has no perfect matching, a contradiction to the assumption that G is cycle-nice. So, at least one of G'_1 and G'_2 is a path. This proves (ii).

(iii) Since G is 2-connected, by Lemma 2.4, the marked $\{u, v\}$-components of G, say H_1 and H_2, are 2-connected. Suppose that $V(H_i) = V(G_i) \cup \{u, v\}$, $i = 1, 2$. Since G'_2 is an odd path, $|V(H_1)|$ and $|V(H_2)|$ are even. So the underlying simple graph of H_2 is an even cycle. Thus H_2 is cycle-nice. Replace the odd path G'_2 of G by an edge, which connects u and v. The resulting graph is H_1. Since G is cycle-nice, Lemma 2.5 implies that H_1 is cycle-nice. (iii) follows.

(iv) Since G'_2 is an even path, G_1 and G_2 are odd components. We first suppose that neither G'_1 nor G'_2 is a path of length two, that is, both G_1 and G_2 are nontrivial. We will show that $uv \notin E(G)$. Recall that G'_2 is an even path. Then G'_2 has at least five vertices. Suppose, to the contrary, that $uv \in E(G)$. If G'_1 is a path, then it also has at least five vertices. This implies that G contains a claw formed by u and its three neighbours (one in G_1, one in G_2, and v), a contradiction. Thus, G'_1 is not a path. Since G is 2-connected and $|V(G_1)| \geq 3$, in G_1 there are a neighbour u_1 of u and a neighbour v_1 of v such that $u_1 \neq v_1$. Since G is claw-free, considering u and its three neighbours u_1, v and one in G'_2, we have $u_1v \in E(G)$, and considering v and its three neighbours v_1, u_1 and one in G'_2, we have $u_1v_1 \in E(G)$. Let C be the cycle uu_1v_1vu. Then G_2 is an odd component of $G - V(C)$, and so, $G - V(C)$ has no perfect matching, a contradiction to the assumption that G is cycle-nice. Hence, $uv \notin E(G)$.

Recall that $G\{V(G_i)\}$ is the graph obtained from G by contracting $\overline{V(G_i)}$ to a vertex x_i, $i = 1, 2$. Since the underlying simple graph of $G\{V(G_2)\}$ is an even cycle, $G\{V(G_2)\}$ is cycle-nice. To show that $G\{V(G_1)\}$ is cycle-nice, let C_1 be an even cycle of $G\{V(G_1)\}$. If $x_1 \in V(C_1)$, let e_1 and e_2 be the two edges incident with x_1 in C_1. If $e_1 \in \nabla(u)$ and $e_2 \in \nabla(v)$, let $C = (C_1 - x_1) \cup \{e_1, e_2\} \cup G'_2$. Then C is an even cycle of G and $G\{V(G_1)\} - V(C_1) = G - V(C)$. Since G is cycle-nice, $G - V(C)$ has a perfect matching M, which is also a perfect matching of $G\{V(G_1)\} - V(C_1)$. If e_1 and e_2 belong to one of $\nabla(u)$ and $\nabla(v)$, then C_1 is an even cycle of G. If $x_1 \notin V(C_1)$, then C_1 is also an even cycle of G. Let M be a perfect matching of $G - V(C_1)$. Since $|V(G_2)|$ is odd, u and v are matched to different components of $G - \{u, v\}$ under any perfect matching of G. Thus, $M \cap E(G\{V(G_1)\})$ is a perfect matching of $G\{V(G_1)\} - V(C_1)$. Consequently, $G\{V(G_1)\}$ is cycle-nice. (iv) follows. $\square$

Lemma 2.7. *Let G be a 2-connected graph and let G' be an even subdivision or an odd expansion of G. If G' is claw-free, then G is claw-free.*

Proof: Suppose first that G' is an even subdivision of G at an edge e, that is, G' is obtained from G by replacing e by an odd path P_e. Let u and v be the two ends of e. If G has a claw, then the only possibility is that the center of the claw is u or v. Since G' is claw-free, $N_G(u) \setminus \{v\}$ and $N_G(v) \setminus \{u\}$ are cliques of G', which are also cliques of G. So G has no claw with center u or v. Thus, G is claw-free.

We next suppose that G' is an odd expansion of G at a vertex u. Let u' and u'' be the two split vertices of G'. Recall that P_u is the $u'u''$-path of G'. Since G' is claw-free, the neighbours of u' in $V(G-u)$ form a clique of G. Otherwise, the subgraph of G' induced by u' and its three neighbours, two nonadjacent neighbours in $V(G - u)$ and one in P_u, contains a claw, a contradiction. Similarly, the subgraph induced by the neighbours of u'' in $V(G - u)$ also contains a clique of G. This implies that G is claw-free. $\square$

Let G be a 2-connected claw-free cycle-nice graph. If G has a 2-vertex cut K, by Lemma 2.6(i), $G - K$ has only two components G_1 and G_2. By Lemma 2.6 and Lemma 2.7, either the two marked K-components of G or both $G\{V(G_1)\}$ and $G\{V(G_2)\}$, which are 2-connected, are claw-free and cycle-nice. If the smaller 2-connected claw-free cycle-nice graph still has a 2-vertex cut, we will repeat this procedure until we obtain a 3-connected claw-free cycle-nice graph, whose underlying simple graph is K_2 or a 3-connected claw-free cycle-nice graph.

3 3-connected graphs

Theorem 3.1. *Let G be a simple 3-connected claw-free plane graph. Then G is cycle-nice if and only if G is K_4, W_5 or $\overline{C}_6$.*

Proof: It is easy to check that K_4, W_5 and $\overline{C}_6$ are cycle-nice. Now, we assume that G is cycle-nice. Lemma 2.1 implies that for any vertex x of G, the neighbours of x lie on a common cycle, denoted by C_x, which is the boundary of the face of $G-x$ in which x is situated. If C_x is an even cycle, then x is an isolated vertex of $G-V(C)$, a contradiction to the assumption that G is cycle-nice. Thus, for any vertex x of G, C_x is an odd cycle. By Lemma 2.2, $\delta(G) \le 5$. Since G is 3-connected, we have $3 \le \delta(G) \le 5$. Let u be a vertex of G such that $d(u) = \delta(G)$. Let $C_u = u_0u_1 \cdots u_su_0$. Then s is even and $s \ge 2$. We first prove the following claim. All subscripts are taken modulo $s+1$ in the following.
Claim 1. $V(G) = V(C_u) \cup \{u\}$.

Suppose, to the contrary, that there is a vertex $v \in V(G) \setminus (V(C_u) \cup \{u\})$. We may further suppose that $vu_i \in E(G)$ for some $i \in \{0, 1, \ldots, s\}$. If $u_{i-1}v$ or $u_{i+1}v$ is an edge of G, let $C' = C_u - u_{i-1}u_i + u_{i-1}vu_i$ or $C' = C_u - u_iu_{i+1} + u_ivu_{i+1}$. Then C' is an even cycle of G such that u is an isolated vertex of $G-V(C')$, a contradiction. Thus $u_{i-1}v, u_{i+1}v \notin E(G)$. Since G is claw-free, we have $u_{i-1}u_{i+1} \in E(G)$. Furthermore, when $s > 2$, $u_{i-1}u_{i+1}$ lies in the exterior of C_u, and v lies in the interior of the cycle $u_{i-1}u_iu_{i+1}u_{i-1}$. If $uu_i \notin E(G)$, since $\delta(G) \ge 3$, we have $s > 2$. Then $G - \{u_{i-1}, u_{i+1}\}$ has at least two components, one contains v and the other contains u, a contradiction to the assumption that G is 3-connected. Thus, $uu_i \in E(G)$. Consider the vertex u_i and its three neighbours u, u_{i-1} and v. Since G is claw-free and vu_{i-1}, $vu \notin E(G)$, we have $uu_{i-1} \in E(G)$. When consider u_i and its three neighbours u, u_{i+1} and v, we have $uu_{i+1} \in E(G)$. Recall that C_{u_i} is the facial cycle of the face of $G-u_i$ which contains all neighbours of u_i. Then we have $u, v, u_{i-1}, u_{i+1} \in V(C_{u_i})$. Let P_1 and P_2 be the two segments of C_{u_i} such that $u_{i-1}P_1vP_2u_{i+1}uu_{i-1} = C_{u_i}$. Since C_{u_i} is an odd cycle, one of P_1 and P_2 is an odd path, and the other is an even path. Suppose, without loss of generality, that P_1 is an odd path. Let $C' = (C_u - u_{i-1}u_i) + u_{i-1}P_1vu_i$. Then C' is an even cycle of G such that u is an isolated vertex of $G - V(C')$, a contradiction. Claim 1 follows. □

By Claim 1, $G-u$ is a 2-connected outerplane graph. We now let u be situated in the outer face of $G-u$. By Lemma 2.3, $G-u$ has a vertex x of degree two. Since $\delta(G) \ge 3$, we have $xu \in E(G)$ and $d_G(x) = 3$. Recall that $d_G(u) = \delta(G)$. We have $d_G(u) = 3$. Let $N_G(u) = \{u_i, u_j, u_l\}$, $0 \le j < l < i \le s$. Then u_i, u_j and u_l effect a partition of C_u into three edge-disjoint paths P_1, P_2 and P_3, which connect u_l and u_i, u_i and u_j, u_j and u_l, respectively.

Since G is claw-free, at least one of u_iu_j, u_ju_l and u_lu_i is an edge of G. Suppose, without loss of generality, that $u_ju_l \in E(G)$. If $P_3 \ne u_ju_l$, from the fact that $G-u$ is an outerplane graph, we find that $\{u_j, u_l\}$ is a 2-vertex cut of G, a contradiction to the assumption that G is 3-connected. Thus, $P_3 = u_ju_l$. So we may suppose, without loss of generality, that $N(u) = \{u_0, u_1, u_i\}$ for some $i \in \{2, 3, \ldots, s\}$. Then $j = 0$ and $l = 1$, and so, P_1 is a u_1u_i-path, P_2 is a u_iu_0-path, and P_1 and P_2 have the same parity. Since $G-u$ is a simple outerplane graph, there are at least two nonadjacent 2-vertices in $G-u$ by Lemma 2.3. Since $\delta(G) \ge 3$, each 2-vertex of $G-u$ is a neighbour of u in G. Thus $G-u$ has at most three 2-vertices (u_0, u_1, or u_i) and $d_{G-u}(u_i) = 2$.

Consider u_i and its three neighbours. For distinct indices i' and i'', if $\{i', i''\} \subseteq \{1, 2, 3, \dots, i-1\}$ or $\{i', i''\} \subseteq \{i+1, i+2, \dots, s, 0\}$ satisfying $|i' - i''| \geq 2$, then $u_{i'}u_{i''} \notin E(G)$. Otherwise, since $G - u$ is a simple outerplane graph, $\{u_{i'}, u_{i''}\}$ is a 2-vertex cut of G, a contradiction to the 3-connectivity of G. Recall that $\delta(G) \geq 3$ and $d_G(u_i) = 3$. If $k \in \{2, 3, \dots, i-1\}$, then $\emptyset \neq N_G(u_k) \setminus \{u_{k-1}, u_{k+1}\} \subseteq V(P_2) \setminus \{u_i\}$; if $k \in \{i+1, i+2, \dots, s\}$, then $\emptyset \neq N_G(u_k) \setminus \{u_{k-1}, u_{k+1}\} \subseteq V(P_1) \setminus \{u_i\}$. We now show the following claim, which implies that the number of vertices of C_u is at most five.

Claim 2. $s \leq 4$.

Suppose, to the contrary, that $s > 4$. Then $s \geq 6$. If $i = 2$ or $i = s$, by symmetry, we need only consider the case where $i = 2$. Since $G - u$ has two nonadjacent 2-vertices, we have $d_{G-u}(u_0) = d_{G-u}(u_2) = 2$. Then, for each $j \in \{3, 4, \dots, s\}$, u_j is adjacent to u_1, which is the only vertex in $V(P_1) \setminus \{u_i\}$. Since $G - u$ is an outerplane graph, u_1 and its three neighbours u_0, u_2 and u_4 form a claw in G, a contradiction. Thus, $i \neq 2$ and $i \neq s$. This implies that the lengths of P_1 and P_2 are at least two.

If one of P_1 and P_2 has length two, say P_1, then $i = 3$ and $P_1 = u_1u_2u_3$. Since P_1 and P_2 have the same parity and $s \geq 6$, P_2 has at least five vertices. If $u_1u_s \in E(G)$, let $C = uu_3u_4 \dots u_su_1u$. Then C is an even cycle of G such that u_0 and u_2 are two isolated vertices of $G - V(C)$. Thus $G - V(C)$ has no perfect matching. This contradiction implies that $u_1u_s \notin E(G)$. Then u_s is adjacent to u_2, the only vertex in $V(P_1) \setminus \{u_i, u_1\}$. Consequently, all internal vertices of P_2 are adjacent to u_2. Then u_2 and its three neighbours u_1, u_3 and u_s form a claw, a contradiction. Hence, P_1 and P_2 have length at least three.

We now consider u_i and its three neighbours u_{i-1}, u_{i+1} and u. Since $uu_{i-1}, uu_{i+1} \notin E(G)$, we have $u_{i-1}u_{i+1} \in E(G)$. We assert that $u_{i-2}u_{i+2} \in E(G)$. In fact, if $u_{i-1}u_{i+2} \in E(G)$, by considering u_{i-1} and its three neighbours u_{i-2}, u_i and u_{i+2}, we have $u_{i-2}u_{i+2} \in E(G)$. Similarly, if $u_{i-2}u_{i+1} \in E(G)$, we have $u_{i-2}u_{i+2} \in E(G)$. For the case where $u_{i-1}u_{i+2} \notin E(G)$ and $u_{i-2}u_{i+1} \notin E(G)$, we have $N_G(u_{i-2}) \setminus \{u_{i-1}, u_{i-3}\} \subseteq \{u_{i+2}, \dots, u_s, u_0\}$ and $N_G(u_{i+2}) \setminus \{u_{i+1}, u_{i+3}\} \subseteq \{u_1, \dots, u_{i-2}\}$. Recall that $\delta(G) \geq 3$. If $u_{i-2}u_{i+2} \notin E(G)$, then there is an index $j \in \{1, 2, \dots, i-3\}$ such that $u_{i+2}u_j \in E(G)$. Recall that $G - u$ is a outerplane graph, whose boundary of outer face is C_u. Then u_{i-2} is a 2-vertex of G, a contradiction. The assertion follows.

If P_1 and P_2 are even, let $C = u_{i+2}u_{i-2}u_{i-1}u_iuu_1u_0u_s \dots u_{i+2}$; if P_1 and P_2 are odd, let $C = u_{i+2}u_{i-2}u_{i-1}u_iuu_0u_s \dots u_{i+2}$. Then C is an even cycle of G such that u_{i+1} is an isolated vertex of $G - V(C)$, a contradiction. Claim 2 follows. □

Recall that $s \geq 2$. If $s = 2$, G is K_4. If $s = 4$, we have $C_u = u_0u_1u_2u_3u_4u_0$. If $i = 2$, then $d_{G-u}(u_3) \geq 3$, $d_{G-u}(u_4) \geq 3$, and so, $u_1u_3, u_1u_4 \in E(G)$. Hence, G is W_5. By symmetry, if $i = 4$, G is also W_5. If $i = 3$, then $d_{G-u}(u_2) \geq 3$, $d_{G-u}(u_4) \geq 3$. If u_4 is adjacent to u_1, let $C = uu_1u_4u_3u$. Then $G - V(C)$ has no perfect matching. So $u_4u_1 \notin E(G)$. By symmetry, we have $u_2u_0 \notin E(G)$. Consequently, $u_2u_4 \in E(G)$. Hence, G is $\overline{C}_6$. The result follows. □

4 2-connected graphs

In this section, we use G_{sim} to denote the underlying simple graph of G. An edge of G is *admissible* if it is contained in a perfect matching of G. A *quasi-diamond* is an odd A-expansion of a cycle of length two (see Fig. 2). Obviously, a diamond is a quasi-diamond. Note that a quasi-diamond is also an even subdivision of a diamond at an edge whose one end vertex is a 2-vertex of the diamond.

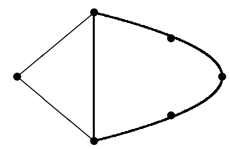


Fig. 2. A quasi-diamond with six vertices

Lemma 4.1. *Let G be a 2-connected claw-free graph, and G' an odd A-expansion of G. If G' is claw-free and cycle-nice, then G_{sim} is an even cycle or K_2, and G'_{sim} is a quasi-diamond.*

Proof: Suppose that G' is an odd A-expansion of G at a vertex v, and v' and v'' are the two split vertices. Then $v'v'' \in E(G')$, and P_v is the even $v'v''$-path. Since G' is cycle-nice, $v'v''$ does not lie in any even cycle of G'. Suppose, to the contrary, that G_{sim} is neither an even cycle nor K_2. Since G is 2-connected and G_{sim} is not K_2, in G the vertex v has at least two neighbours. According to the number of neighbours of v in G, we distinguish the following two cases.

Case 1. $|N_G(v)| = 2$. Let v_1 and v_2 be the two neighbours of v such that $v'v_1, v''v_2 \in E(G')$. Let P' be a shortest v_1v_2-path in $G_{sim} - v$, and C' the union of the two paths $v_1v'v''v_2$ and P'. Then C' is a cycle of G' containing $v'v''$. Thus, C' is an odd cycle, and P' is an even path. Recall that G_{sim} is not an even cycle. Since P' is a shortest v_1v_2-path, C' has no chord. Therefore, there is a vertex $u \in V(G_{sim} - v) \setminus V(P')$ such that u is adjacent to a vertex u' of P'. Since C' has no chord, the two neighbours of u' in C' are not adjacent. Since G' is claw-free, at least one neighbour of u' in P' is adjacent to u, say u''. Then the union of the two paths $C' - u'u''$ and $u'uu''$ is an even cycle of G', which contains $v'v''$, a contradiction.

Case 2. $|N_G(v)| \geq 3$. Since G is claw-free, at least two neighbours, say v_1 and v_2, of v are adjacent in G. If $v_1 \in N_{G'}(v')$ and $v_2 \in N_{G'}(v'')$, then $v'v_1v_2v''v'$ is an even cycle of G' containing $v'v''$, a contradiction. Thus, we may suppose that $\{v_1, v_2\} \subseteq N_{G'}(v')$. Let $v_3 \notin \{v_1, v_2\}$ be a neighbour of v adjacent to v'' in G'. Since G_{sim} is 2-connected, there exists a path Q in $G_{sim} - v$ from v_3 to $\{v_1, v_2\}$. Suppose that Q is a shortest such path and $v_1 \in V(Q)$. Then $v_2 \notin V(Q)$. Thus, one of $v'v_1Qv_3v''v'$ and $v'v_2v_1Qv_3v''v'$ is an even cycle of G' containing $v'v''$, a contradiction.

From the above discussion, G_{sim} is either an even cycle or K_2. Recall that G' is claw-free. If G_{sim} is an even cycle, then P_v is an even path with 3 vertices. Thus, G'_{sim} is a quasi-diamond. If G_{sim} is K_2, then P_v is an even path with at least 3 vertices. Thus, G'_{sim} is a quasi-diamond. □

Lemma 4.2. *Let G be a 2-connected cycle-nice graph, and let G' be an odd L-expansion of G. Then G' is cycle-nice.*

Proof: Suppose that G' is an odd L-expansion of G at a vertex v, and v' and v'' are the two split vertices. Then $v'v'' \notin E(G')$. Let C' be an even cycle of G'. If $|V(C') \cap \{v', v''\}| = 0$ or $|V(C') \cap \{v', v''\}| = 1$, then C' is an even cycle of G. Since G is cycle-nice, $G - V(C')$ has a perfect matching M. If $|V(C') \cap \{v', v''\}| = 0$, then M has an edge e which is incident with v. Suppose that in G' the edge e is incident with v'. If $|V(C') \cap \{v', v''\}| = 1$, then suppose that $v' \in V(C')$. Recall that P_v is an even $v'v''$-path of G'. So $P_v - v'$ has a perfect matching M'. In both cases $M \cup M'$ is a perfect matching of $G' - V(C')$.

If $|V(C')\cap\{v', v''\}| = 2$, then P_v is a segment of C'. Let C be a cycle obtained from C' by contracting P_v to a vertex v. Then C is an even cycle of G, and so, $G - V(C)$ has a perfect matching M. Since $G - V(C) = G' - V(C')$, M is also a perfect matching of $G' - V(C')$.

From the above discussion, we conclude that G' is cycle-nice. □

Lemma 4.3. *If G' is an even subdivision or an odd expansion of W_5 possibly with multiple edges, then G' has a claw.*

Proof: Let G be a graph such that $G_{sim} = W_5$. Suppose that the cycle of G_{sim} of length five is $C = u_1u_2u_3u_4u_5u_1$. Let u be the vertex of G not in $V(C)$. Then $uu_i \in E(G)$, $1 \leq i \leq 5$.

Suppose that G' is an even subdivision of G at an edge e, that is, G' is obtained from G by replacing e by an odd path P_e which has at least four vertices. Note that e has at least one end in C. Suppose that u_1 is one end of e. If the other end of e is u, then $e = uu_1$. Since $u_2u_5 \notin E(G')$, the subgraph of G' induced by u and its three neighbours u_5, u_2, and one in P_e contains a claw. If the other end of e is u_2, then $e = u_1u_2$. If e is a single edge, then the subgraph of G' induced by u and its three neighbours u_1, u_2 and u_4 contains a claw. If e is a multiple edge of G, then u_1 and u_2 are adjacent in G'. Since $u_1u_3 \notin E(G')$, the subgraph of G' induced by u_2 and its three neighbours u_1, u_3, and one in P_e contains a claw. Thus, if G' is obtained from G by an even subdivision of G, then G' has a claw.

We next suppose that G' is an odd expansion of G at a vertex v, and v' and v'' are the two split vertices. Then P_v is the even $v'v''$-path, and v' and v'' are adjacent to at least one vertex in $N_G(v)$, respectively. Note that $v'v''$ is a possible edge of G'. When $v = u$, since u has five neighbours in C, one of v' and v'', say v', has at least three neighbours in C. Then v' has two nonadjacent neighbours in C, say u_1 and u_3. It follows that the subgraph of G' induced by v' and its three neighbours u_1, u_3, and one in P_v contains a claw. When $v \neq u$, we may suppose that $v = u_2$. Since $N_G(u_2) = \{u, u_1, u_3\}$, one of v' and v'' has two neighbours in $N_G(u_2)$. Suppose, without loss of generality, that $\{u, u_1\} \subseteq N(v')$ and $u_3 \in N(v'')$. If u_3 is not adjacent to v' in G', then the subgraph of G' induced by $\{u, v', u_3, u_5\}$ contains a claw. If u_3 is adjacent to v' in G', then the subgraph of G' induced by v' and its three neighbours u_1, u_3, and one in P_v contains a claw. The lemma follows. □

Combining Lemma 2.7 and Lemma 4.3, we see that a 2-connected claw-free graph cannot be obtained from W_5 by a sequence of edge subdivisions and vertex expansions.

Theorem 4.1. *Suppose that G is a 2-connected claw-free plane graph possibly with multiple edges, and G_{sim} is not W_5. Then G is cycle-nice if and only if there exists a sequence $(G_1, G_2, \ldots, G_r)$ of graphs such that (i) G_1 is an even cycle, a diamond, K_4, or $\overline{C}_6$, and $G_r = G$, (ii) G_i is an even subdivision or an odd L-expansion of G_{i-1}, or G_i is obtained from G_{i-1} by replacing some admissible edges of G_{i-1} by some multiple edges.*

Proof: To prove the sufficiency, we show, by induction, that each G_i $(1 \leq i \leq r)$ is a 2-connected cycle-nice graph. Since an even cycle, a diamond, K_4 and $\overline{C}_6$ are 2-connected and cycle-nice, G_1 is a 2-connected cycle-nice graph.

Inductively, suppose that $2 \leq i \leq r$ and G_{i-1} is a 2-connected cycle-nice graph. We consider the following possibilities.

• G_i is an even subdivision or an odd L-expansion of G_{i-1}. By Lemma 2.5 and Lemma 4.2, G_i is cycle-nice. Since an even subdivision and an L-expansion of a 2-connected graph are also 2-connected, G_i is 2-connected.

• G_i is obtained from G_{i-1} by replacing an admissible edge e of G_{i-1} by a set of multiple edges E_e. Then G_i is 2-connected. Let C be an even cycle of G_i. If C contains no edge of E_e, then C is an even cycle of G_{i-1}. Since G_{i-1} is cycle-nice, $G_{i-1} - V(C)$ has a perfect matching, which is also a perfect matching of $G_i - V(C)$. So C is a nice-cycle of G_i. Then we consider the situation that C contains an edge e' of E_e. If the length of C is two, then $V(C) = V(e)$. Since e is an admissible edge of G_{i-1}, $G_i - V(C) = G_{i-1} - V(e)$ has a perfect matching, and so, C is a nice cycle of G_i. If the length of C is at least four, let C' be the cycle obtained from C by replacing the edge e' by the edge e. Then C' is an even cycle of G_{i-1} with $V(C) = V(C')$, and so, $G_i - V(C) = G_{i-1} - V(C')$ has a perfect matching. This implies that C is a nice cycle of G_i. Consequently, G_i is a 2-connected cycle-nice graph, as desired.

Now, we give a proof of necessity by induction on the number of vertices of G. From the assumption, we see that the underlying simple graph G_{sim} of G is a 2-connected claw-free cycle-nice plane graph. If G is 3-connected, then G_{sim} is 3-connected. By Theorem 3.1 and the assumption, G_{sim} is K_4 or $\overline{C}_6$. If G is simple, then $G_1 = G$. If G is not simple, since G is cycle-nice, each cycle of G of length two is a nice cycle. Then each member of multiple edges of G is admissible. So G is obtained from G_{sim} by replacing some admissible edges of G_{sim} by some multiple edges. By setting $G_1 = G_{sim}$ and $G_2 = G$, we are done.

Suppose in the following that G is not 3-connected. Then G has a 2-vertex cut $\{u, v\}$. By Lemma 2.6(i), $G - \{u, v\}$ has exactly two components G_1^* and G_2^*. Let G_i', $i = 1, 2$, be the graph obtained from the $\{u, v\}$-component of G which contains G_i^* by deleting all possible edges of G which connect u and v. Note that $(G_i')_{sim}$ is the underlying simple graph of G_i'. By Lemma 2.6(ii), at least one of $(G_1')_{sim}$ and $(G_2')_{sim}$ is a path, say $(G_2')_{sim}$. Let G' be the graph obtained from G by replacing G_2' by $(G_2')_{sim}$. Since G is cycle-nice, each cycle of G of length two is a nice cycle. So each member of multiple edges of G_2' remained in G' is admissible. Since G is claw-free, G' is claw-free.

If $(G_2')_{sim}$ is an odd path, let H_1 be the marked $\{u, v\}$-component of G which contains G_1^*. Then $(H_1)_{sim}$ is not K_2. Lemma 2.6(iii) implies that H_1 is cycle-nice. Note that G' is an even subdivision of H_1. By Lemma 2.7, H_1 is claw-free, and by Lemma 4.3, $(H_1)_{sim}$ is not W_5. Furthermore, by Lemma 2.4, H_1 is 2-connected. By induction hypothesis, there exists a sequence $(G_1, G_2, \ldots, G_k)$ of graphs such that (i) G_1 is an even cycle, a diamond, K_4, or $\overline{C}_6$, and $G_k = H_1$, (ii) G_i is an even subdivision or an odd L-expansion of G_{i-1}, or G_i is obtained from G_{i-1} by replacing some admissible edges of G_{i-1} by some multiple edges, $i = 2, 3, \ldots, k$. If G_2' is simple, we set $G_{k+1} = G$ and $r = k+1$. If G_2' is not simple, we set $G_{k+1} = G'$, $G_{k+2} = G$ and $r = k+2$. Then the desired sequence of graphs is obtained.

If $(G_2')_{sim}$ is an even path, write $H_1 = G\{V(G_1^*)\}$. Then Lemma 2.6(iv) implies that H_1 is cycle-nice. Note that G' is an odd expansion of H_1. By Lemma 2.7, H_1 is claw-free, and by Lemma 4.3, $(H_1)_{sim}$ is not W_5. Note that H_1 is 2-connected.

If $uv \in E(G)$, then G' is an odd A-expansion of H_1. Recall that G is cycle-nice and claw-free. Then G' is cycle-nice and claw-free. By Lemma 4.1, the underlying simple graph G'_{sim} of G' is a quasi-diamond. Note that a quasi-diamond can be obtained from a diamond by an even subdivision at an edge whose one end vertex is a 2-vertex of the diamond. When G is simple, if G'_{sim} is a diamond, we set $G_1 = G$; otherwise, we set G_1 be a diamond, and $G_2 = G$. When G is not simple, if G'_{sim} is a diamond, we set G_1 be a diamond, $G_2 = G$; Otherwise, we set G_1 be a diamond, $G_2 = G'_{sim}$, and $G_3 = G$. Then the desired sequence of graphs is obtained.

If there is no edge connecting u and v in G, then G' is an odd L-expansion of H_1. Suppose first that $(H_1)_{sim}$ is K_2, then G_{sim} is an even cycle. If G is simple, we set $G_1 = G$; Otherwise, we set $G_1 = G_{sim}$, and $G_2 = G$. Suppose next that $(H_1)_{sim}$ is not K_2. Recall that H_1 is 2-connected, claw-free and cycle-nice, and $(H_1)_{sim}$ is not W_5. By induction hypothesis, there exists a sequence $(G_1, G_2, \ldots, G_k)$ of graphs such that (i) G_1 is an even cycle, a diamond, K_4, or $\overline{C}_6$, and $G_k = H_1$, (ii) G_i is an even subdivision or an odd L-expansion of G_{i-1}, or G_i is obtained from G_{i-1} by replacing some admissible edges of G_{i-1} by some multiple edges, $i = 2, 3, \ldots, k$. Let $G_{k+1} = G'$. Recall that $(G_2')_{sim}$ is an even path. If G_2' is simple, then $G_{k+1} = G' = G$. If G_2' is not simple, we set $G_{k+2} = G$. Again, the desired sequence of graphs is obtained. This completes the proof. $\square$

Acknowledgements

We would like to thank the anonymous referees for their the constructive comments and kind suggestions on improving the representation of the paper.

References

[1] J. Bondy and U. Murty. *Graph Theory*. Springer-Verlag, Berlin, 2008.

[2] M. Carvalho and J. Cheriyan. An *O(VE)* algorithm for ear decompositions of matching-covered graphs. *ACM Trans. Algorithms*, 1:324–337, 2005.

[3] M. Carvalho, C. Lucchesi, and U. Murty. Ear decompositions of matching covered graphs. *Combinatorica*, 19:151–174, 1999.

[4] M. Carvalho, C. Lucchesi, and M. U.S.R. Optimal ear decompositions of matching covered graphs and bases for the matching lattice. *J. Combin. Theory, Ser. B*, 85:59–93, 2002.

[5] M. Carvalho, C. Lucchesi, and U. Murty. On the number of dissimilar pfaffian orientations of graphs. *Theor. Inform. Appl.*, 39:93–113, 2005.

[6] Z. Che and Z. Chen. Forcing hexagons in hexagonal systems. *MATCH Commun. Math. Comput. Chem.*, 56:649–668, 2006.

[7] Z. Che and Z. Chen. Forcing faces in plane bipartite graphs. *Discrete Math*, 308:2427–2439, 2008.

[8] Z. Che and Z. Chen. Forcing faces in plane bipartite graphs (ii). *Discrete Appl. Math.*, 161:71–80, 2013.

[9] G. Hetyei. Rectangular configurations which can be covered by 2×1 rectangles. *Pécsi Tan. Főisk. Közl*, 8:351–367, 1964.

[10] X. Kong, Y. Zhang, and X. Wang. Cycle-forced bipartite graphs. submitted.

[11] Y. Liu and S. Zhang. Ear decomposition of factor-critical graphs and number of maximum matchings. *Bull. Malays. Math. Sci. Soc.*, 38:1537–1549, 2015.

[12] L. Lovász. Ear-decompositions of matching-covered graphs. *Combinatorica*, 3:105–117, 1983.

[13] L. Lovász and M. Plummer. *Mathcing Theory*. Elsevier Science Publishers, B. V. North Holland, 1986.

[14] W. McCuaig. Pólya's permanent problem. *Electron. J. Combin.*, 11:79–83, 2004.

[15] D. Peng and X. Wang. Ear decomposition and induced even cycles. *Discrete Appl. Math.*, 264: 161–166, 2019.

[16] N. Robertson, P. Seymour, and R. Thomas. Permanents, pfaffian orientations and even directed circuits. *Ann. of Math.*, 150:929–975, 1999.

[17] D. Sumner. Graphs with 1-factors. *Proc. Amer. Math. Soc.*, 42:8–12, 1974.

[18] D. Sumner. On tutte's factorization theorem. *in: R. Bari and F. Harary, eds., Graphs and Combinatorics, Lecture Notes in Math.*, 406:350–355, 1974.

[19] D. Sumner. 1-factors and antifactor sets. *J. London Math. Soc. s2*, 13:351–359, 1976.

[20] M. L. Vergnas. A note on matchings in graphs. *Colloque sur la Théorie des Graphs (Paris, 1974). Cahiers Centre Études Rech. Opér.*, 17:257–260, 1975.

[21] X. Wang, Y. Zhang, and J. Zhou. A characterizition of cycle-forced bipartite graphs. *Discrete Math.*, 314:2639–2645, 2018.