Semipaired Domination in Some Subclasses of Chordal Graphs

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A dominating set D of a graph G without isolated vertices is called semipaired dominating set if D can be partitioned into 2-element subsets such that the vertices in each set are at distance at most 2. The semipaired domination number, denoted by $\gamma_{pr2}(G)$ is the minimum cardinality of a semipaired dominating set of G. Given a graph G with no isolated vertices, the MINIMUM SEMIPAIRED DOMINATION problem is to find a semipaired dominating set of G of cardinality $\gamma_{pr2}(G)$. The decision version of the MINIMUM SEMIPAIRED DOMINATION problem is already known to be NP-complete for chordal graphs, an important graph class. In this paper, we show that the decision version of the MINIMUM SEMIPAIRED DOMINATION problem remains NP-complete for split graphs, a subclass of chordal graphs. On the positive side, we propose a linear-time algorithm to compute a minimum cardinality semipaired dominating set of block graphs. In addition, we prove that the MINIMUM SEMIPAIRED DOMINATION problem is APX-complete for graphs with maximum degree 3.

1   Introduction

For a graph $G = (V, E)$, a vertex $v \in V$ is said to *dominate* a vertex $w \in V$ if either $v = w$ or $vw \in E$. A *dominating set* of G is a set $D \subseteq V$ such that every vertex in V is dominated by at least one vertex of D. The minimum cardinality of a dominating set of G is called the *domination number* of G denoted by $\gamma(G)$. Many facility location problems can be modelled using the concept of domination in graphs. Due to vast applications of domination, many variations of dominations have also been introduced in the literature. The domination problem are widely studied from combinatorial as well as algorithmic point of view, see Haynes et al. (1998a,b).

One important variation of domination is paired domination. The concept of paired domination was introduced by Haynes and Slater in Haynes and Slater (1998). For a graph G with no isolated vertices, a dominating set D is called a *paired dominating set*, abbreviated as PD-set, if the graph induced by D has a perfect matching M. Two vertices joined by an edge of M are said to be *paired* and are also called *partners* in D. The MINIMUM PAIRED DOMINATION problem is to find a PD-set of G of minimum cardinality. The cardinality of such a set is known as the *paired domination number* of G, and is denoted by $\gamma_{pr}(G)$. A survey of paired domination can be found in Desormeaux and Henning (2014).

A relaxed version of paired domination, known as *semipaired domination* was introduced by Haynes and Henning Haynes and Henning (2018b), and further studied by others Haynes and Henning (2019); Henning and Kaemawichanurat (2018, 2019); Henning et al. (2020); Zhuang and Hao (2020). For a graph G with no isolated vertex, a *semipaired dominating set*, abbreviated as semi-PD-set, is a dominating set D of G such that the vertices in D can be partitioned into 2-sets such that if $\{u, v\}$ is a 2-set, then $uv \in E(G)$ or the distance between u and v is 2. We say that u and v are *semipaired*, and that u and v are *partners*. The minimum cardinality of a semi-PD-set of G is called the *semipaired domination number* of G, and is denoted by $\gamma_{pr2}(G)$. For a graph G with no isolated vertices, the MINIMUM SEMIPAIRED DOMINATION problem is to find a semi-PD-set of cardinality $\gamma_{pr2}(G)$. For a given graph G and a positive integer k, the SEMIPAIRED DOMINATION DECISION problem is to determine whether G has a semi-PD-set of cardinality at most k or not. Since every PD-set is a semi-PD-set, and since every semi-PD-set is a dominating set, we have the following observation.

Observation 1.1 *(Haynes and Henning (2018b)) For every graph G without isolated vertices,*

$$\gamma(G) \leq \gamma_{pr2}(G) \leq \gamma_{pr}(G).$$

The algorithmic study of the MINIMUM SEMIPAIRED DOMINATION problem was initiated by Henning et al. in Henning et al. (2020). They proved that the SEMIPAIRED DOMINATION DECISION problem is NP-complete even for bipartite graphs and chordal graphs. They also proposed a linear-time algorithm to compute a minimum cardinality semi-PD-set of an interval graph. They proposed a $1 + \ln(2\Delta + 2)$-approximation algorithm for the MINIMUM SEMIPAIRED DOMINATION problem, where Δ denotes the maximum degree of the graph. On the negative side, they proved that MINIMUM SEMIPAIRED DOMINATION problem cannot be approximated within $(1 - \epsilon) \ln |V|$ for any $\epsilon > 0$ unless P=NP. In this paper, we continue the algorithmic study of the MINIMUM SEMIPAIRED DOMINATION problem. The main contributions of the paper are summarized below.

In Section 2, we discuss some definitions and notations. In this section we also observe some graph classes where paired domination and semipaired domination problems differ in complexity. In Section 3, we prove the SEMIPAIRED DOMINATION DECISION problem is NP-complete for split graphs. In Section 4, we propose a linear time algorithm to compute a minimum cardinality semipaired dominating set in block graphs. In Section 5, we show that the MINIMUM SEMIPAIRED DOMINATION problem is APX-hard for graphs with maximum degree 3. Finally, Section 6 concludes the paper.

2   Preliminaries

2.1   Definitions and Notations

Let $G = (V, E)$ be a graph. For a vertex $v \in V$, let $N_G(v) = \{u \in V \mid uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$ denote the *open neighborhood* and the *closed neighborhood* of v, respectively. For a set $S \subseteq V$, the sets $N_G(S) = \bigcup_{u \in S} N_G(u)$ and $N_G[S] = N_G(S) \cup S$ are called *open neighborhood* and the *closed neighborhood* of S, respectively. For a set $S \subset V$, the graph $G \setminus S$ is obtained from G by deleting all vertices in S and all edges incident with vertices in S. If $S = \{v\}$, we write $G \setminus v$ rather than $G \setminus \{v\}$. A *cut vertex* in a connected graph G is a vertex $v \in V$ such that $G \setminus v$ is disconnected. Let n and m denote

the number of vertices and edges of G, respectively. We use standard notation, $[k] = \{1, 2, \ldots, k\}$. In this paper, we only consider connected graphs with at least two vertices.

A set $S \subseteq V$ is called an *independent set* of G if $uv \notin E$ for all $u, v \in S$. A set $K \subseteq V$ is called a *clique* of G if $uv \in E$ for all $u, v \in K$. A graph G is said to be a *chordal graph* if every cycle in G of length at least four has a *chord*, that is, an edge joining two non-consecutive vertices of the cycle. A chordal graph G is a *split graph* if V can be partitioned into two sets C and I such that C is a clique and I is an independent set.

A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. Further, the *grandparent* of v is the vertex at distance 2 from v on the unique (r, v)-path. A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v. A *grandchild* of v in T is a descendant of v at distance 2 from v.

2.2 Complexity difference between paired domination and semipaired domination

In this subsection, we discuss the complexity difference between paired domination and semipaired domination. We show that for the class of GP4 graphs, which we define below, the decision version of the MINIMUM PAIRED DOMINATION problem is NP-complete, but the MINIMUM SEMIPAIRED DOMINATION problem is easily solvable. On the other hand, we introduce a graph class called GP5 graphs, and we show that the SEMIPAIRED DOMINATION DECISION problem is NP-complete for GP5 graphs, but the MINIMUM PAIRED DOMINATION problem is easily solvable for this graph class.

The class of GP4 graphs was introduced by Henning et al. in Henning and Pandey (2019). Below we recall the definition of GP4 graphs.

Definition 2.1 (GP4-graph) A graph $G = (V, E)$ is called a *GP4-graph* if it can be obtained from a general connected graph $H = (V_H, E_H)$ where $V_H = \{v_1, v_2, \ldots, v_{n_H}\}$, by adding a path of length 4 to every vertex of H. Formally, $V = V_H \cup \{w_i, x_i, y_i, z_i \mid i \in [n_H]\}$ and $E = E_H \cup \{v_i w_i, w_i x_i, x_i y_i, y_i z_i \mid i \in [n_H]\}$, where n_H denotes the number of vertices in H.

Theorem 2.1 *If G is a GP4-graph, then $\gamma_{pr2}(G) = \frac{2}{5}|V(G)|$.*

Proof: Let G be a GP4-graph of order $n = |V(G)| = 5|V_H|$ as constructed in Definition 2.1. The set $S = \{w_i, y_i \mid i \in [n_H]\}$ is a semi-PD-set of G, implying that $\gamma_{pr2}(G) \leq \frac{2}{5}|V(G)|$. Every semi-PD-set of G must contain at least two vertices from the set $\{w_i, x_i, y_i, z_i\}$ for each $i \in [n_H]$. Thus, $\gamma_{pr2}(G) \geq \frac{2}{5}|V(G)|$. Consequently, $\gamma_{pr2}(G) = \frac{2}{5}|V(G)|$. □

Lemma 2.1 *If G is a GP4-graph constructed from a graph H as in Definition 2.1, then H has a PD-set of cardinality at most k, $k \leq n_H$, if and only if G has a PD-set of cardinality at most $2n_H + k$.*

Proof: Suppose D is a PD-set of H of cardinality at most k. Then the set $D \cup \{x_i, y_i \mid i \in [n_H]\}$ is a PD-set of G of cardinality at most $2n_H + k$. Conversely, assume that G has a PD-set D' of cardinality at most $2n_H + k$. Now we obtain a PD-set of H of size at most k by updating D'. In order to dominate z_i, the set D' must contain either y_i or z_i. Without loss of generality, we may assume that D' contains y_i and that $x_i \in D'$ with x_i and y_i paired in D'. This implies that D' contains exactly two vertices from the

set $\{x_i, y_i, z_i\}$, where $i \in [n_H]$. Now we consider the set $D = D' \setminus \{x_i, y_i, z_i \mid i \in [n_H]\}$. We note that $|D| \leq k$ and $D \subseteq \{v_i, w_i \mid i \in [n_H]\}$. Further, the set D dominates V_H, and for every vertex u in D, the partner of u is also present in D. Let $G' = G[V(H) \cup \{w_i \mid i \in [n_H]\}]$. Observe that w_i can be paired only with v_i in G' for $i \in [n_H]$. If $w_i \in D$ and $N(v_i) \subseteq D$, then we update the set D as $D \setminus \{w_i, v_i\}$. If there exist a vertex $u \in N(v_i)$ such that $u \notin D$, then we update the set D as $D = (D \setminus \{w_i\}) \cup \{u\}$. We do this update for each $w_i \in D$. Note that the updated set D is a PD-set of H and $|D| \leq k$. Hence, the result follows. $\square$

Since the decision version of the MINIMUM PAIRED DOMINATION problem is known to be NP-complete for general graphs Haynes and Slater (1998), the following theorem follows directly from Lemma 2.1.

Theorem 2.2 *The decision version of the* MINIMUM PAIRED DOMINATION *problem is NP-complete for* GP4-*graphs.*

Next, we define a new graph class and call it GP5-graphs.

Definition 2.2 (GP5-graph) A graph $G = (V, E)$ is called a *GP5-graph* if it can be obtained from a general connected graph $H = (V_H, E_H)$ where $V_H = \{v_1, v_2, \ldots, v_n\}$, by adding a vertex disjoint path P_5 for each vertex v of H and joining v to the central vertex of the path. Formally, $V = V_H \cup \{a_i, b_i, c_i, d_i, e_i \mid i \in [n]\}$ and $E = E_H \cup \{v_i c_i, c_i b_i, c_i d_i, b_i a_i, d_i e_i \mid i \in [n]\}$, where n denotes the number of vertices in H.

For example, when H is a 4-cycle C_4, then a GP5 graph obtained from H is shown in Fig. 1. For a GP5 graph, we show that the SEMIPAIRED DOMINATION DECISION problem is NP-complete, but the MINIMUM PAIRED DOMINATION problem is easily solvable.

Theorem 2.3 *If G is a* GP5-graph*, then $\gamma_{pr}(G) = \frac{2}{3}|V(G)|$.*

Proof: Let G be a GP5-graph of order $n = |V(G)| = 6|V_H|$ as constructed in Definition 2.2. The set $S = \{a_i, b_i, c_i, d_i \mid i \in [n_H]\}$ is a PD-set of G, implying that $\gamma_{pr}(G) \leq \frac{2}{3}|V(G)|$. Also we note that every PD-set of G must contain at least four vertices from the set $\{a_i, b_i, c_i, d_i, e_i\}$ for each $i \in [n_H]$. Hence, $\gamma_{pr}(G) \geq \frac{2}{3}|V(G)|$. Consequently, $\gamma_{pr}(G) = \frac{2}{3}|V(G)|$. $\square$

Lemma 2.2 *If G is a* GP5-graph *constructed from a graph H as in Definition 2.2, then H has a semi-PD-set of cardinality k, $k \leq n_H$ if and only if G has a semi-PD-set of cardinality $2n_H + k$.*

Proof: Suppose D is semi-PD-set of H of cardinality at most k. Then the set $D \cup \{b_i, d_i \mid i \in [n_H]\}$ is a semi-PD-set of G of cardinality at most $2n_H + k$. Conversely, assume that G has a semi-PD-set D' of cardinality at most $2n_H + k$. Now we obtain a PD-set of H of size at most k by updating D'. In order to dominate a_i, the set D' must contain either a_i or b_i. Similarly, in order to dominate e_i, the set D' must contain either d_i or e_i. Without loss of generality, for each $i \in [n_H]$ we may assume that D' contains $\{b_i, d_i\}$ with b_i and d_i semipaired in D'. This implies that D' contains exactly two vertices from the set $\{a_i, b_i, d_i, e_i\}$, where $i \in [n_H]$. Now we consider the set $D = D' \setminus \{a_i, b_i, d_i, e_i \mid i \in [n_H]\}$. We note that $|D| \leq k$ and $D \subseteq \{v_i, c_i \mid i \in [n_H]\}$. Further, the set D dominates V_H, and for every vertex u in
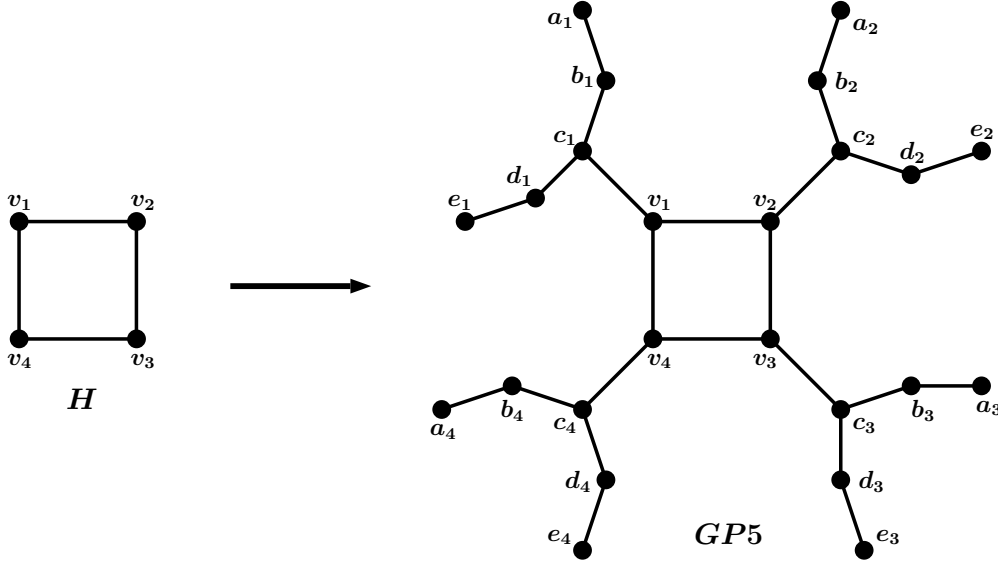
Fig. 1: An illustration of a GP5 graph obtained from a 4-cycle.

D, the semipair of u is also present in D. Let $G' = G[V(H) \cup \{c_i \mid i \in [n_H]\}]$. If $c_i \in D$ and c_i is semipaired with a vertex v_i such that $N(v_i) \subseteq D$, then we update the set D as $D \setminus \{c_i, v_i\}$ and if there exist a vertex $u \in N(v_i)$ such that $u \notin D$, then we update the set D as $D = (D \setminus \{c_i\}) \cup \{u\}$. We do this update for each $c_i \in D$. Note that the updated set D is a semi-PD-set of H and $|D| \leq k$. Hence, the result follows. □

Since the SEMIPAIRED DOMINATION DECISION problem is known to be NP-complete for general graphs Henning et al. (2020), the following theorem follows directly from Lemma 2.2.

Theorem 2.4 *The* SEMIAIRED DOMINATION DECISION *problem is NP-complete for* GP4*-graphs.*

3 NP-Completeness Result for Split Graphs

In this section, we prove that the SEMIPAIRED DOMINATION DECISION problem is NP-complete for split graphs. To prove this NP-completeness result, we use a reduction from the domination problem, which is a well known NP-complete problem Haynes et al. (1998a).

Theorem 3.1 *The* SEMIPAIRED DOMINATION DECISION *problem is NP-complete for split graphs.*

Proof: Clearly, the SEMIPAIRED DOMINATION DECISION problem is in NP. To show the hardness, we give a polynomial time reduction from the DOMINATION DECISION problem for general graphs. Given a non-trivial graph $G = (V, E)$, where $V = \{v_i \mid i \in [n]\}$ and $E = \{e_j \mid j \in [m]\}$, we construct a split graph $G' = (V_{G'}, E_{G'})$ as follows:

Let $V_k = \{v_i^k \mid i \in [n]\}$ and $U_k = \{u_i^k \mid i \in [n]\}$ for $k \in [2]$. Now define $V_{G'} = V_1 \cup V_2 \cup U_1 \cup U_2$, and $E_{G'} = \{uv \mid u, v \in V_1 \cup U_1, u \neq v\} \cup \{v_i^2 v_j^1, u_i^2 u_j^1 \mid i \in [n] \text{ and } v_j \in N_G[v_i]\}$. Note that the set $A = V_1 \cup U_1$ is a clique in G' and the set $B = V_2 \cup U_2$ is an independent set in G'. Since $V_{G'} = A \cup B$, the constructed graph G' is a split graph. Fig. 2 illustrates the construction of G' from G.
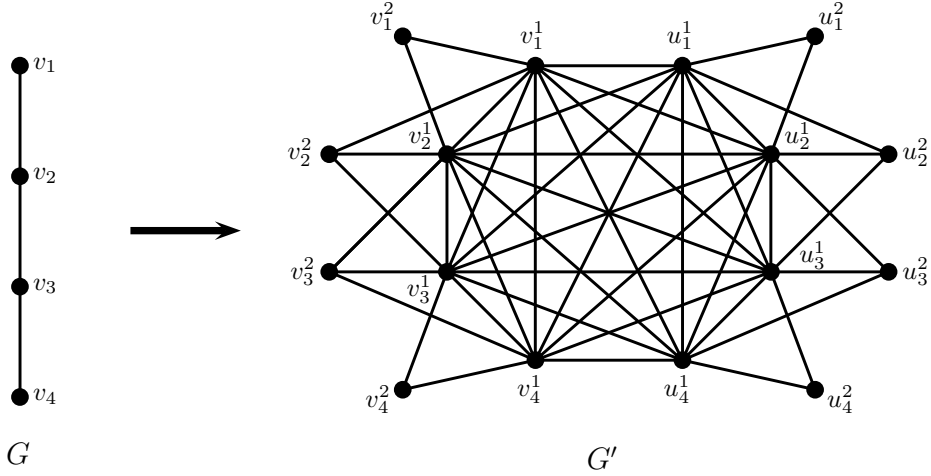
Fig. 2: An illustration to the construction of G' from G in the proof of Theorem 3.1

Now, to complete the proof of the theorem, we only need to prove the following claim.

Claim 3.1 *The graph G has a dominating set of cardinality at most k if and only if G' has a semi-PD-set of cardinality at most $2k$.*

Proof: Let $D = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ be a dominating set of G of cardinality at most k. Then the set

$D' = \{v_{i_1}^1, v_{i_2}^1, \ldots, v_{i_k}^1\} \cup \{u_{i_1}^1, u_{i_2}^1, \ldots, u_{i_k}^1\}$ is a semi-PD-set of G' of cardinality at most $2k$.

Conversely, let D' is a semi-PD-set of G' of cardinality at most $2k$. Now we obtain a dominating set of G of size at most k by updating D'. Note that, either $|D' \cap (V_1 \cup V_2)| \leq k$ or $|D' \cap (U_1 \cup U_2)| \leq k$. Without loss of generality, we assume that $|D' \cap (V_1 \cup V_2)| \leq k$. Let $D = D' \cap (V_1 \cup V_2)$. Note that V_2 is an independent set, hence any vertex $v \in V_2$ is either dominated by itself or by some vertex in V_1. If $v_i^2 \in D$ and none of its neighbors is in D, then update $D = (D \setminus \{v_i^2\}) \cup \{u\}$ where $u \in N(v_i^2)$. We do this update for each vertex $v_i^2 \in V_2$. Now observe that in the updated set D, we have $N(v_i^2) \cap D \neq \emptyset$ for $i \in [n]$. The set $D'' = \{v_i \mid v_i^1 \in D\}$ is a dominating set of G of cardinality at most k. This proves the claim.   □

Hence the result follows.   □

4   Semipaired Domination in Block Graphs

For a graph G, a maximal induced subgraph of G without a cut vertex is called a *block* of G. If B and B' are two blocks of G then $|V(B) \cap V(B')| \le 1$, and a vertex $v \in V(B) \cap V(B')$ if and only if v is a cut vertex. A connected graph whose every block is a complete graph is called a *block graph*. A tree is a block graph in which every block contains exactly two vertices. A block with only one cut vertex is called an *end block*. Every block graph not isomorphic to a complete graph has at least two end blocks.

Lie Chen et al. Chen et al. (2010) have studied an ordering of vertices of block graph, $\alpha = (v_1, v_2, \ldots, v_n)$ such that $v_i v_j \in E$ and $v_i v_k \in E$ implies $v_j v_k \in E$ for $i < j < k \le n$. Such an ordering of vertices of a block graph is called *Block-Elimination-Ordering* (*BEO*). The procedure to get such an ordering is as follows: if G is not isomorphic to a complete graph, then it must have at least two end blocks. Pick an end block, say B, having a cut vertex x. Staring with the index 1, enumerate the vertices in $V(B) \setminus \{x\}$ in any order and remove $V(B) \setminus \{x\}$ from the graph. Let $k = \max\{s \mid v_s \in V(B) \setminus \{x\}\}$, that is, v_k is the vertex in $V(B) \setminus \{x\}$ having highest index. Now if the remaining graph, say G', is a complete graph, then enumerate the remaining vertices starting from index $k + 1$ to n in any order; otherwise, pick an end block in G', say B', having cut vertex x'. Starting with index $k+1$ enumerate the vertices in $V(B') \setminus \{x'\}$ and continue the procedure in $G' \setminus (V(B') \setminus \{x'\})$.

Let $G = (V, E)$ be a block graph, and $\alpha = (v_1, v_2, \ldots, v_n)$ be a BEO of vertices of G. For $i \ne n$, we define $F(v_i) = v_j$, where $j = \max\{k \mid v_i v_k \in E\}$. We also define $F(v_n) = v_n$. Further, we construct a block tree $T(G)$ rooted at v_n such that $V(T(G)) = V(G)$ and $E(T(G)) = \{uv$ if and only if either $F(u) = v$ or $F(v) = u\}$. Fig. 3 illustrates the construction of $T(G)$ from a block graph G. Note that a cut vertex of G is an internal vertex of $T(G)$. Also if B is a block of G with $V(B) = \{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\}$ where u_{i_k} is the highest index vertex in $V(B)$, then $u_{i_1}, u_{i_2}, \ldots, u_{i_{k-1}}$ are called *siblings* in $T(G)$ and each one is a child of u_{i_k}. The following complexity is already known.

Theorem 4.1 *Chen et al. (2010) For a block graph $G = (V, E)$, a BEO can be computed in $O(n + m)$-time. In addition, given a BEO, the corresponding block tree can also be computed in $O(n + m)$-time.*
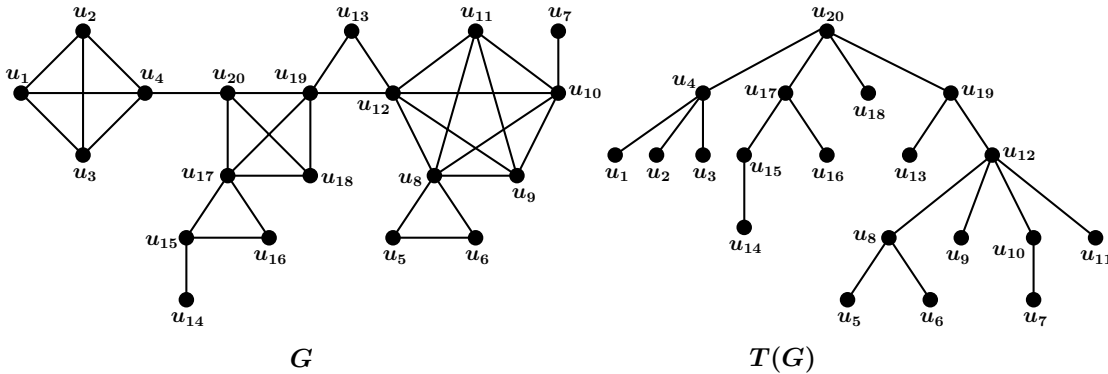

Fig. 3: An illustration of the construction of $T(G)$ from a block graph G.

Observation 4.1 *Let $G = (V(G), E(G))$ be a block graph and let $T(G)$ be a corresponding block tree. If $v \in N_G(u)$, then one of the following holds in $T(G)$,*

    (a) *u is a parent of v.*

    (b) *u is a child of v.*

    (c) *u is a sibling of v.*

Lemma 4.1 *Let G be a block graph with given BEO, $\alpha = (v_1, v_2, \ldots, v_n)$. If B and B' are any two blocks of G such that $v_i \in V(B) \cap V(B')$, then $F(u) = v_i$, for all $u \in V(B) \setminus \{v_i\}$ or for all $u \in V(B') \setminus \{v_i\}$.*

Proof: Let G be a block graph with given BEO, $\alpha = (v_1, v_2, \ldots, v_n)$. Suppose B and B' are blocks of G such that $v_i \in V(B) \cap V(B')$. Clearly, v_i is a cut vertex. By the way the vertices of G are enumerated, either all the vertices in $V(B) \setminus \{v_i\}$ first get enumerated and thereafter the vertices in $V(B')$, or all the vertices in $V(B') \setminus \{v_i\}$ first get enumerated and then the vertices in $V(B)$. Renaming the blocks if necessary, we may assume without loss of generality that the vertices in $V(B) \setminus \{v_i\}$ get enumerated first and thereafter the vertices in $V(B')$. In that case, we note that $i \geq \max\{r \mid v_r \in V(B) \setminus \{v_i\}\}$. Thus, $F(u) = v_i$ for all $u \in V(B) \setminus \{v_i\}$, implying the desired result. $\square$

Using above lemma, we state that if there are exactly k blocks, say $B_1, B_2 \ldots B_k$ of a block graph G such that $v_i \in V(B_1) \cap V(B_2) \cap \cdots \cap V(B_k)$ and $B^* \in \{B_1, B_2 \ldots B_k\}$ is the block whose vertices are enumerated after the enumeration of vertices in $\{V(B_1) \cup V(B_2) \cup \cdots \cup V(B_k)\} \setminus \{V(B^*)\}$ in the BEO α then for every vertex $v_j \in \{V(B_1) \cup V(B_2) \cup \cdots \cup V(B_k)\} \setminus \{V(B^*)\}$, $F(v_j) = v_i$.

Lemma 4.2 *Let G be a block graph and let $T(G)$ be a block tree of G. If $\alpha = (v_n, v_{n-1}, \ldots, v_1)$ is a BFS-ordering of the vertices of $T(G)$ rooted at v_n then the reverse of BFS-ordering $\beta = (v_1, v_2, \ldots, v_n)$ also satisfy BEO in G.*

Proof: Let G be a block graph and $T(G)$ be the corresponding block tree. Let $\beta = (v_1, v_2, \ldots, v_n)$ be the reverse of BFS-ordering $\alpha = (v_n, v_{n-1}, \ldots, v_1)$ of vertices of G as they appear in $T(G)$. For $i < j < k$, let v_i, v_j and v_k satisfy the reverse of BFS-ordering and $v_i v_j, v_i v_k \in E(G)$. To prove the result we need to show $v_j v_k \in E(G)$. By contradiction, suppose $v_j v_k \notin E(G)$. Since $v_j v_k \notin E(G)$, this implies v_j and v_k belongs to different blocks of G, say B and B' respectively, and v_i is a cut vertex. Now, using Lemma 4.1 we have $F(u) = v_i$, for all $u \in V(B) \setminus \{v_i\}$ or for all $u \in V(B') \setminus \{v_i\}$. Therefore in $T(G)$, the vertex v_i is the parent of either v_j or v_k, which is a contradiction as v_i, v_j and v_k satisfy the reverse of BFS-ordering and $i < j < k$. Hence the result follows. $\square$

Let $G = (V(G), E(G))$ be a block graph and $T(G)$ be a corresponding block tree of G. Let $\alpha = (v_n, v_{n-1}, \ldots, v_1)$ be a BFS-ordering of vertices of $T(G)$ rooted at v_n. Recall that the distance between two vertices u and v is the length of shortest path between u and v, denote by $d(u, v)$. For a positive integer $l \geq 0$, we say a vertex x is at level l in a tree T rooted at a vertex y, if $d_T(x, y) = l$. In our algorithm, we will process the vertices of the block graph as they appear in the reverse of BFS-ordering $\beta = (v_1, v_2, \ldots, v_n)$ of the corresponding block tree $T(G)$. We will use the following notation while processing the vertices in the algorithm:

$$D(v_i) = \begin{cases} 0 & \text{if } v_i \text{ is not dominated,} \\ 1 & \text{if } v_i \text{ is dominated.} \end{cases}$$

$$L(v_i) = \begin{cases} 0 & \text{if } v_i \text{ is not selected,} \\ 1 & \text{if } v_i \text{ is selected but not semipaired,} \\ 2 & \text{if } v_i \text{ is selected and semipaired.} \end{cases}$$

$$m(v_i) = \begin{cases} k & \text{if } v_k \text{ need to be semipaired with a vertex in } N_{T(G)}[v_i] \text{ or with some sibling of } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

Also we use $N_i(v_k) = \{v_j \mid v_k v_j \in E(G) \text{ and } j \geq i\}$ and $N_i[v_k] = \{v_j \mid v_k v_j \in E(G) \text{ and } j \geq i\} \cup \{v_k\}$.

Lemma 4.3 *Let G be a block graph, and $T(G)$ be a block tree of G. Let $\beta = (v_1, v_2, \ldots, v_n)$ be the reverse of BFS-ordering of vertices in $T(G)$, then $v_i v_j \in E(G)$ implies $N_i[v_i] \subseteq N_i[v_j]$ where $j \geq i$.*

Proof: If v_j is parent of v_i in $T(G)$, then a neighbor v_k of v_i with $k > i$ is a sibling of v_i in $T(G)$. Hence,

$v_k v_j \in E(G)$ and the result follows. If v_j is a sibling of v_i and $v_i v_j \in E(G)$, then v_i and v_j are in the same block B of G. Now any neighbor v_k of v_i with $k > i$ is either a sibling or parent of v_i in $T(G)$. If v_k is parent of v_i then $v_j v_k \in E(G)$ (as v_j is a sibling of v_i). Next we consider that v_k is a sibling of v_i. Note that either $i < j < k$ or $i < k < j$, and the vertices v_i, v_j, v_k appears in the reverse of BFS-ordering, which is also a BEO (by Lemma 4.2). Since $v_i v_k, v_i v_j \in E(G)$, $v_j v_k \in E(G)$ by using the property of BEO. □

Observation 4.2 *Let G be a block graph, and let $T(G)$ be a block tree of G. If $\beta = (v_1, v_2, \ldots, v_n)$ is the reverse of BFS-ordering of vertices in $T(G)$, then for any descendant v_r of v_j, $N_i(v_r) \subseteq N_i(v_j)$.*

Using the properties of a tree and Observation 4.1 we have the following important observation.

Observation 4.3 *If a vertex v_i is at level $l + 2$, then v_i does not have any neighbor at level l. Also if v_i is semipaired with v_j, then v_j may be one of the following in $T(G)$.*
(i) Parent of v_i that is $F(v_i)$ or a grand parent of v_i that is $F(F(v_i))$.
(ii) Child of v_i or a grand child of v_i.
(iii) Sibling of v_i or sibling of $F(v_i)$.
(iv) Child of some sibling v_s of v_i.

Next, we present the detailed algorithm to compute a minimum cardinality semi-PD-set of a given block

graph.

Algorithm 1: Minimum Semipaired Domination in Block Graphs

Input: A block graph G=(V,E), corresponding block-tree $T(G)$, reverse of BFS ordering of
vertices of $T(G)$: $\beta = (v_1, v_2, \ldots, v_n)$

Output: A minimum cardinality semi-PD-set D_{sp} of G.

for $i = 1$ *to* n **do**

    if $(D(v_i) = 0 \ and \ i \neq n)$ **then**

        $D_{sp} = D_{sp} \cup \{F(v_i)\}$, let $F(v_i) = v_j$;

        $L(v_j) = 1$;

        $D(u) = 1 \ \forall \ u \in N_G[v_j]$;

        Let $C = \{u \in N_G[v_j] \mid m(u) \neq 0\}$;

        if $(C = \emptyset)$ **then**

          $m(F(v_j)) = j$;

        else

          Let $k = \min\{b \mid v_b \in C\}$ and let $m(v_k) = r$;

          $L(v_j) = L(v_r) = 2$; // semipair v_j with v_r

          $m(v_k) = 0$;

    if $(D(v_i) \neq 0 \ and \ m(v_i) = k \neq 0)$ **then**

        if $(L(F(v_i)) = 0)$ **then**

          $D_{sp} = D_{sp} \cup \{F(v_i)\}$;

          $D(u) = 1 \ \forall \ u \in N_G[F(v_i)]$;

          $L(v_k) = L(F(v_i)) = 2$ and $m(v_i) = 0$; // semipair v_k with $F(v_i)$;

        else if $(L(v_i) = 0)$ **then**

          $D_{sp} = D_{sp} \cup \{v_i\}$;

          $L(v_k) = L(v_i) = 2$; // semipair v_k with v_i

          $D(u) = 1 \ \forall \ u \in N_G[v_i]$;

          $m(v_i) = 0$;

        else

          $D_{sp} = D_{sp} \cup \{u\}$ where $u \in N(v_k)$ and $L(u) = 0$;

          $L(v_k) = L(u) = 2$ and $m(v_i) = 0$; // semipair v_k with u;

    if $(D(v_n) = 0)$ **then**

        $L(v_n) = 2$;

        $D(v_n) = 1$;

        $L(u) = 2$ for some $u \in N_G(v_n)$ with $L(u) = 0$; // semipair v_n with u

        $D_{sp} = D_{sp} \cup \{v_n, u\}$;

return D_{sp};

Illustration of the Algorithm with an example

We illustrate the algorithm for computing a minimum cardinality semi-PD-set of the block graph shown
in Fig. 3. Since there are 20 vertices in the graph, the algorithm will terminate in 20-iterations. We
process the vertices as they appear in the reverse of BFS-ordering. For the graph in the Fig. 3, a re-

verse of BFS-ordering of the vertices is given by $\beta = (u_7, u_6, u_5, u_{11}, u_{10}, u_9, u_8, u_{14}, u_{12}, u_{13}, u_{16}, u_{15},$ $u_3, u_2, u_1, u_{19}, u_{18}, u_{17}, u_4, u_{20})$. The iterations of the algorithm are as follows:

INITIALLY
$D(u_i) = L(u_i) = m(u_i) = 0$ for $i \in [20]$ and $D_{sp} = \emptyset$.
ITERATION 1
Since $D(u_7) = 0$, we select $F(u_7) = u_{10}$.
$L(u_{10}) = 1$ and $D(u_7) = D(u_8) = D(u_9) = D(u_{10}) = D(u_{11}) = D(u_{12}) = 1$
As $C = \emptyset$, we have $m(F(u_{10})) = m(u_{12}) = 10$
AFTER ITERATION 1: $D_{sp} = \{u_{10}\}$
ITERATION 2
Since $D(u_6) = 0$, we select $F(u_6) = u_8$.
$L(u_8) = 1$ and $D(u_5) = D(u_6) = 1$
In this iteration note that $C = \{u_{12}\}$, $k = 12$ and $m(u_{12}) = 10$.
$L(u_{10}) = L(u_8) = 2$ and $m(u_{12}) = 0$
AFTER ITERATION 2: $D_{sp} = \{u_8, u_{10}\}$
We do not have any update in ITERATION $3, 4, 5, 6,$ AND 7
ITERATION 8
Since $D(u_{14}) = 0$, we select $F(u_{14}) = u_{15}$.
$L(u_{15}) = 1$ and $D(u_{14}) = D(u_{15}) = D(u_{16}) = D(u_{17}) = 1$
As $C = \emptyset$, hence $m(F(u_{15})) = m(u_{17}) = 15$
AFTER ITERATION 8: $D_{sp} = \{u_8, u_{10}, u_{15}\}$
We do not have any update in ITERATION 9
ITERATION 10
Since $D(u_{13}) = 0$, we select $F(u_{13}) = u_{19}$.
$L(u_{19}) = 1$ and $D(u_{13}) = D(u_{17}) = D(u_{18}) = D(u_{19}) = D(u_{20}) = 1$
As $C = \{u_{17}\}$, $k = 17$ and $m(u_{17}) = 15$
$L(u_{15}) = L(u_{19}) = 2$ and $m(u_{17}) = 0$
AFTER ITERATION 10: $D_{sp} = \{u_8, u_{10}, u_{15}, u_{19}\}$
We do not have any update in ITERATION 11 AND 12
ITERATION 13
Since $D(u_3) = 0$, we select $F(u_3) = u_4$.
$L(u_4) = 1$ and $D(u_1) = D(u_2) = D(u_3) = D(u_4) = 1$
As $C = \emptyset$, hence $m(F(u_4)) = m(u_{20}) = 4$
AFTER ITERATION 13: $D_{sp} = \{u_8, u_{10}, u_{15}, u_{19}, u_4\}$
We do not have any update in ITERATION 14, 15, 16, 17, 18, AND 19
ITERATION 20
In this iteration note that $m(u_{20}) = 4 \neq 0$.
As $L(F(u_{20})) = L(u_{20}) = 0$ hence, $L(u_{20}) = L(u_4) = 2$ and $m(u_{20}) = 0$.
AFTER ITERATION 20: $D_{sp} = \{u_8, u_{10}, u_{15}, u_{19}, u_4, u_{20}\}$
Return D_{sp}

Lemma 4.4 *For $1 \leq i \leq n$, when v_i is the currently considered vertex, the following holds:*

1. $D(v_j) = 1$ *for $j \in [i-1]$.*

2. $m(v_j) = 0$ *for* $j \in [i-1]$.

3. *if* v_i *is at level* l *from the root in* $T(G)$, *then for all vertices* u *at level* $l+2$ *or more, either* $L(u) = 0$ *or* 2.

Proof: Note that in the jth-iteration of the algorithm we dominate v_j if it was not dominated before j^{th}-iteration, that is, if $D(v_j) = 0$ then we include $F(v_j)$ in D_{sp} to dominate v_j and we also update $D(v_j) = 1$. Also, in the jth-iteration if $m(v_j) = r \neq 0$ then v_r is already selected in D_{sp} and we need to semipair v_r with a vertex in $N_{T(G)}[v_j]$ or with some sibling of v_j. In the jth-iteration we semipair v_r with a vertex in $N_{T(G)}[v_j]$ or with some sibling of v_j and update $m(v_j) = 0$. Hence, the first two statements of the lemma follows.

Now, suppose if v_i is at level l from the root in $T(G)$ and there is a vertex v_t at level $l+2$ or more such that $L(v_t) = 1$. Hence v_t is included in D_{sp} but it is not semipaired yet. In this case, there must be a vertex $v_{t'}$ at level $l+1$ or more such that $m(v_{t'}) = t$ and $t' < i$, which is a contradiction to the second statement of the lemma. Hence the result follows.   □

Note that if for any vertex v_r, $m(v_r) = k \neq 0$ then v_k is selected in the semi-PD-set but still need to be semipaired with a vertex in $N_G[v_r] \setminus \{v_k\}$. Now for any vertex $v_t \in N_G[v_r] \setminus \{v_k\}$, if we update $L(v_t) = 1$, that is if we select v_t in the semi-PD-set, according to algorithm, we semipair v_t with v_k and make $m(v_r) = 0$. Hence, in any iteration if for any vertex v_r, $m(v_r) = k \neq 0$ we have the following observation.

Observation 4.4 *In the* i^{th}*-iteration, if for any vertex* v_r, $m(v_r) = k \neq 0$, *then for any* $u \in N_G[v_r] \setminus \{v_k\}$, $L(u) = 0$ *or* 2.

For each $0 \leq i \leq n$, let $D_i = \{v \mid L(v) > 0\}$ when v_i has just been considered. In particular $D_0 = \emptyset$ and $D_n = \{v \mid L(v) > 0\}$, when all the vertices of the graph have been processed. Clearly, D_n is a dominating set. Note that when we are processing the vertex v_n in our algorithm, by Lemma 4.4 for all vertices u at level 2 or more, either $L(u) = 0$ or 2 and $D(v_j) = 1$ for $j \in [n-1]$. Also note that in the nth-iteration, if for a vertex u at level 1, we have $L(u) = 1$, then $m(v_n) \neq 0$ and hence by Observation 4.4 for all vertices w other than u at level 1, we have $L(w) = 0$ or 2. In this iteration, if $L(v_n) = 0$, then we semipair u with v_n; otherwise, we semipair u with one of its neighbors. Also, if $D(v_n) = 0$, this implies no vertex from level 1 is selected in the dominating set. So, according to Algorithm 1, we select v_n in the semi-PD-set and pair it with one of it's children. Therefore, after the nth-iteration for all $u \in V(G)$, we have $L(u) = 0$ or 2 and D_n is a semi-PD-set. So in order to prove the correctness of the algorithm, we need to show that D_n is contained in a minimum semi-PD-set D_{sp}^*.

Lemma 4.5 *For each* $0 \leq i \leq n$, *there is minimum semi-PD-set* D_{sp}' *such that:*

1. $D_i \subseteq D_{sp}'$ *and if* u *and* v *are semipaired in* D_i *then* u *and* v *are also semipaired in* D_{sp}'.

2. *if in the* i^{th}*-iteration we are updating* $m(v_j) = k$ *for some vertex* v_j, *then* v_k *is either semipaired with a vertex in* $N_{T(G)}[v_j]$ *or with a sibling of* v_j *in* D_{sp}'.

Proof: We will prove the result using induction on i. Clearly, when $i = 0$, there is semi-PD-set D_{sp}' such that $D_0 \subseteq D_{sp}'$. Now, suppose that the result holds for any integer less than $i < n$, that is, there is a

minimum semi-PD-set D_{sp}^* such that $D_{i-1} \subseteq D_{sp}^*$ and the second condition of the lemma is satisfied.

Now in the ith-iteration we will have the following cases:

Case 1. $D(v_i) = 0$ and $i \neq n$.

In this case, $D_i = D_{i-1} \cup \{F(v_i)\}$. Let $F(v_i) = v_j$ and assume that v_i is at level $l \geq 1$ from the root v_n. We proceed further with the following claim.

Claim 4.1 *Let $v_j \notin D_{sp}^*$, $v_p \in D_{sp}^*$ be a vertex dominating v_i, and v_q be the vertex which is semipaired with v_p in D_{sp}^*, then either $(D_{sp}^* \setminus \{v_q\}) \cup \{v_j\}$ or $(D_{sp}^* \setminus \{v_p\}) \cup \{v_j\}$ is a minimum semi-PD-set satisfying the first statement of the lemma.*

Proof: Suppose that $v_j \notin D_{sp}^*$. Let v_p be a vertex in D_{sp}^* dominating v_i, and let v_q be the vertex that is semipaired with v_p. Clearly, $j \notin \{p, q\}$ and $v_p \notin D_{i-1}$. If $v_q \in D_{i-1}$ and v_q is at level $l + 2$ or more, then $v_p \in D_{i-1}$ by Lemma 4.4, a contradiction. Hence, if $v_q \in D_{i-1}$, then v_q is at level $l+1$ or less from the root in $T(G)$. Further if $v_q \notin D_{i-1}$ and v_q is at level $l + 2$ or more in $T(G)$, then by Lemma 4.4, the set $D_{sp}' = (D_{sp}^* \setminus \{v_q\}) \cup \{v_j\}$ where v_j is semipaired with v_p is the required minimum semi-PD-set. Hence, we may assume that v_q is at level $l + 1$ or less.

Suppose firstly that the vertex v_p is a child of v_i in $T(G)$. Since v_q is at level $l + 1$ or less and $q \neq j$, we have $v_q \in N_G[v_i]$. Hence, $d(v_j, v_q) \leq 2$. Note that $N_i(v_p) \subseteq N_i(v_j)$. Since $d(v_q, v_j) \leq 2$, the set $D_{sp}' = (D_{sp}^* \setminus \{v_p\}) \cup \{v_j\}$ where v_j is semipaired with v_q is the required minimum semi-PD-set.

Suppose secondly that v_p is a sibling of v_i or $v_p = v_i$ in $T(G)$. In this case, v_p is a child of v_j in $T(G)$. Since v_q is at level $l + 1$ or less hence, either v_q is a child of v_p or a child of some sibling say v_s of v_p or v_q is a sibling of v_p or a sibling of v_j or $v_q = F(v_j)$.

If v_q is a sibling of v_p or a sibling of v_j or $v_q = F(v_j)$, we have $v_q v_j \in E(G)$. If v_q is a child of v_p, then $d(v_j, v_q) = 2$. If v_q is a child of some sibling, say v_s, of v_p, then v_s is a child of v_j. Thus since v_q is a child of v_s in $T(G)$, we again have $d(v_j, v_q) = 2$. In all of the above cases, we have $d(v_j, v_q) \leq 2$. Further, since v_p is a child of v_j, using Lemma 4.4 and the fact that $N_i[v_p] \subseteq N_i[v_j]$, we have that $D_{sp}' = (D_{sp}^* \setminus \{v_p\}) \cup \{v_j\}$ where v_j is semipaired with v_q is the required minimum semi-PD-set. $\square$

By Claim 4.1, we may assume that $v_j \in D_{sp}^*$, for otherwise the desired result holds. We now let $C = \{u \in N_G[v_j] \mid m(u) \neq 0\}$, and consider two subcases.

Case 1.1. $C = \emptyset$.

Here we are updating $m(F(v_j)) = j$, we need to show that v_j is either semipaired with a vertex in $N_{T(G)}[F(v_j)]$ or with some sibling of $F(v_j)$ in D_{sp}^*. Let v_r be the vertex semipaired with v_j in D_{sp}^* and suppose neither $v_r \notin N_{T(G)}[F(v_j)]$ nor v_r is a sibling of $F(v_j)$ in $T(G)$. We note that in this case v_r is at level l or $l + 1$ in $T(G)$. Further, since $C = \emptyset$, $m(u) = 0$ for all $u \in N_G[v_j]$. This implies that for all $u \in N[N[v_j]]$ (that is, for all u such that $d(v_j, u) \leq 2$), $L(u) = 0$ or 2. Thus, $v_r \in D_{i-1}$ implies $v_j \in D_{i-1}$ a contradiction. Hence, $v_r \notin D_{i-1}$. If v_r is at level $l + 1$ or a child of v_j, then using Lemma 4.4 and the fact that $N_i(v_r) \subseteq N_i(v_j)$ we may conclude that $D_{sp}' = (D_{sp}^* \setminus \{v_r\}) \cup \{F(v_j)\}$ with v_j semipaired with $F(v_j)$ in D_{sp}' is the required minimum semi-PD-set. If v_r is at level l and not a child of v_j in $T(G)$, then v_r is a child of some sibling v_s of v_j in $T(G)$ such that $v_s v_j \in E(G)$. In this case $N_i[v_r] \subseteq N_i[v_s]$. Now if $v_s \in D_{sp}^*$ then using Lemma 4.4, $D_{sp}' = (D_{sp}^* \setminus \{v_r\}) \cup \{F(v_j)\}$ where v_j is semipaired with $F(v_j)$ in D_{sp}' is the required minimum semi-PD-set. If $v_s \notin D_{sp}^*$, then using Lemma 4.4,

$D'_{sp} = (D^*_{sp} \setminus \{v_r\}) \cup \{v_s\}$ where v_j is semipaired with v_s in D'_{sp} is the required minimum semi-PD-set.

Case 1.2. $C \neq \emptyset$.

Let $k = \min\{b \mid v_b \in C\}$ and let $m(v_k) = r$. Since in this case we semipair vertices v_r and v_j in D_i, we need to show that they are semipaired in D'_{sp} as well. By Lemma 4.4, $k \geq i$. If v_r is semipaired with v_j in D^*_{sp} then the result follows; otherwise, let v_s and v_t be the vertices semipaired with v_j and v_r, respectively, in D^*_{sp}. Since $m(v_k) = r$ after $(i-1)^{th}$-iteration, v_r still needs to be semipaired with a vertex in $N_{T(G)}[v_k]$ or with some sibling of v_k. Hence, v_r is not semipaired until the $(i-1)^{th}$-iteration in D_{i-1}. Also as $m(v_k) = r$, using the second part of Lemma 4.5, either $v_t \in N_{T(G)}[v_k]$ or v_t is a sibling of v_k such that $v_t v_k \in E(G)$, that is, $v_t \in N_G[v_k]$. Note that if $v_t \in D_{i-1}$ then $L(v_t) = 1$ but as $m(v_k) = r$ by Observation 4.4, for all $u \in N_G[v_k] \setminus \{v_r\}$, $L(u) = 0$ or 2. Hence, $v_t \notin D_{i-1}$.

Suppose that v_k is a child of v_j in $T(G)$. In this case, v_t is a child of either v_j or v_k in $T(G)$. If v_t is a child of v_k then $N_i(v_t) \subseteq N_i[v_j]$ and if v_t is a child of v_j then $N_i[v_t] \subseteq N_i[v_j]$. Now, if $N_G(v_s) \subseteq D^*_{sp}$, then by Lemma 4.4 $D'_{sp} = D^*_{sp} \setminus \{v_s, v_t\}$ is semi-PD-set of smaller size, a contradiction. Hence there exist a vertex $u \in N_G(v_s)$ such that $u \notin D^*_{sp}$ and $D'_{sp} = (D^*_{sp} \setminus \{v_r\}) \cup \{u\}$ with v_j and v_s semipaired with v_r and u, respectively, is the required minimum semi-PD-set. Hence, we may assume that v_k is not a child of v_j in $T(G)$, for otherwise the desired result follows. Also, since v_k is the minimum index vertex such that $v_k \in C$, in further cases we may assume that there is no child $v_{k'}$ of v_j with $m(v_{k'}) \neq 0$. So, v_k can be one of the following: (a) a sibling of v_j, (b) v_j, or (c) $F(v_j)$. We will consider these remaining cases under inclusion or exclusion of v_s in D_{i-1}.

Case 1.2.1 $v_s \in D_{i-1}$.

If $L(v_s) = 2$, then $v_j \in D_{i-1}$, a contradiction. Hence, $L(v_s) = 1$. Thus, if $v_s \in D_{i-1}$, then there exists a vertex v_{s_1} such that $m(v_{s_1}) = s$, where $v_{s_1} = F(v_s)$ and $v_{s_1} \in N_G[v_j]$. If v_s is at level $l+1$ in $T(G)$, then v_{s_1} is a child of v_j, a contradiction noting that $k = \min\{s \mid v_s \in C\}$. If v_s is at level l in $T(G)$, then v_{s_1} is a sibling of v_j in $T(G)$ and $s_1 > k$, implying that $v_k \neq F(v_j)$. If v_s is at level $l-1$, then $v_{s_1} = F(v_j)$ hence, $v_k \neq F(v_j)$. Therefore, we may note that if $v_s \in D_{i-1}$ then $v_k \neq F(v_j)$. So v_k may be a sibling of v_j or $v_k = v_j$.

Let v_k is a sibling of v_j. Now, $v_k v_j \in E(G)$ and either $s_1 = j$ or $v_{s_1} v_j \in E(G)$. If v_t is a child of v_k, then using the fact that $N_i[v_t] \subseteq N_i[v_k]$, the set $D'_{sp} = (D^*_{sp} \setminus \{v_t\}) \cup \{v_k\}$ with v_j semipaired with v_r and with v_s semipaired with v_k is the required minimum semi-PD-set. If v_t is not a child of v_k, then v_t is either a sibling of v_k such that $v_k v_t \in E(G)$ or $v_t = F(v_j)$. We note that if either v_t is a sibling of v_k such that $v_k v_t \in E(G)$ or $v_t = F(v_j)$ then the vertices in the set $\{v_j, v_k, v_{s_1}, v_t\}$ are in the same block of G and $v_s v_{s_1} \in E(G)$. Hence, $d(v_s, v_t) \leq 2$. Therefore, if we exchange the semipairs, that is, if we semipair v_j and v_r, and semipair v_s and v_t in the D^*_{sp}, then D^*_{sp} is the desired minimum semi-PD-set.

Let $v_j = v_k$. If v_t is a child of v_k and $N_G(v_s) \subseteq D^*_{sp}$ then using Lemma 4.4 and the fact that $N_i[v_t] \subseteq N_i[v_k]$, the set $D'_{sp} = D^*_{sp} \setminus \{v_s, v_t\}$ is a semi-PD-set of smaller size, a contradiction. Therefore, if v_t is a child of v_k, then there exists a vertex $u \in N_G(v_s)$ such that $u \notin D^*_{sp}$. In this case, using Lemma 4.4 and the fact that $N_i[v_t] \subseteq N_i[v_k]$, the set $D'_{sp} = (D^*_{sp} \setminus \{v_t\}) \cup \{u\}$ with v_j semipaired with v_r and with v_s semipaired with u is the required minimum semi-PD-set. If v_t is not a child of v_k in $T(G)$, then observe that $d(v_s, v_t) \leq 2$. Hence, if we exchange the semipairs, that is, we semipair v_j and v_r, and semipair v_s and v_t in the D^*_{sp}, then D^*_{sp} is the desired minimum semi-PD-set.

Case 1.2.2. $v_s \notin D_{i-1}$.

If $s \leq i$ or v_s is a child of v_j in $T(G)$ and $N_G(v_t) \subseteq D_{sp}^*$, then $D_{sp}' = D_{sp}^* \setminus \{v_s, v_t\}$ is a semi-PD-set of smaller size, a contradiction noting that $N_i(v_s) \subseteq N_i(v_j)$. Hence, if $s \leq i$ or v_s is a child of v_j in $T(G)$, then there exist a vertex $u \in N_G(v_t)$ such that $u \notin D_{sp}^*$ and in this case $D_{sp}' = (D_{sp}^* \setminus \{v_s\}) \cup \{u\}$ with v_j and v_t semipaired with v_r and u, respectively, is the required minimum semi-PD-set. Now we have, $s > i$ and v_s is not a child of v_j, implying that in $T(G)$, the vertex v_s may be one of the following: (i) a child of some sibling v_s' of v_j in $T(G)$, (ii) v_s is a sibling of v_j, (iii) $v_s = F(v_j)$, (iv) v_s is a sibling of $F(v_j)$ or (v) $v_s = F(F(v_j))$.

Suppose firstly that v_k is a sibling of v_j in $T(G)$. If v_t is a child of v_k, then using the fact that $N_i[v_t] \subseteq N_i[v_k]$, the set $D_{sp}' = (D_{sp}^* \setminus \{v_t\}) \cup \{v_k\}$ with v_j semipaired with v_r, and v_s semipaired with v_k is the required minimum semi-PD-set. If v_t is not a child of v_k, then v_t is either a sibling of v_k or $v_t = F(v_j)$. If v_t is a sibling of v_k or $v_t = F(v_j)$ then vertices v_j, v_k, and v_t belongs to the same block of G and in both cases, we observe that $d(v_s, v_t) \leq 2$. Hence, if we exchange the semipairs, that is, if we semipair v_j and v_r, and semipair v_s and v_t in the D_{sp}^*, then D_{sp}^* is the desired minimum semi-PD-set.

Suppose secondly that $v_j = v_k$. If v_t is a child of v_k and $N[v_s] \subseteq D_{sp}^*$, then using the fact that $N_i[v_t] \subseteq N_i[v_k]$, the set $D_{sp}' = D_{sp}^* \setminus \{v_s, v_t\}$ is semi-PD-set of smaller size, a contradiction. Therefore, if v_t is a child of v_k, then there exists a vertex $u \in N(v_s)$ such that $u \notin D_{sp}^*$. In this case, using Lemma 4.4 and the fact that $N_i[v_t] \subseteq N_i[v_k]$, the set $D_{sp}' = (D_{sp}^* \setminus \{v_t\}) \cup \{u\}$ with v_j semipaired with v_r, and with v_s semipaired with u, is the required minimum semi-PD-set. If v_t is not a child of v_k in $T(G)$, then observe that $d(v_s, v_t) \leq 2$. Hence, if we exchange the semipairs, that is, if we semipair v_j and v_r, and semipair v_s and v_t in the D_{sp}^*, then D_{sp}^* is the desired minimum semi-PD-set.

Suppose next that $v_k = F(v_j)$. If v_s is a child of some sibling v_{s_1} of v_j in $T(G)$ then v_{s_1} is a child of $F(v_j)$. We note that as $v_k = F(v_j)$, this implies $v_t \in N_{T(G)}(F(v_j))$ or v_t is a sibling of $F(v_j)$ and $v_t F(v_j) \in E(G)$. Hence we may observe that $d(v_{s_1}, v_t) \leq 2$. Therefore, if v_s is a child of some sibling v_{s_1} of v_j in $T(G)$, then using Lemma 4.4 and the fact that $N_i[v_s] \subseteq N_i[v_{s_1}]$, the set $D_{sp}' = (D_{sp}^* \setminus \{v_s\}) \cup \{v_{s_1}\}$ with v_j semipaired with v_r, and with v_t semipaired with v_{s_1}, is the required minimum semi-PD-set. If v_s is not a child of some sibling v_{s_1} of v_j in $T(G)$, then we observe that $d(v_s, v_t) \leq 2$. Hence, if we exchange the semipairs, that is, if we semipair v_j and v_r, and semipair v_s and v_t in the D_{sp}^*, then D_{sp}^* is the desired minimum semi-PD-set.

Case 2. $D(v_i) \neq 0$ and $m(v_i) = k \neq 0$.

In this case, we semipair v_k, either with some vertex in $N_{T(G)}[v_i]$ or with a sibling of v_i. Suppose w is the vertex that is semipaired with v_k in D_{i-1}. So we need to show that v_k is semipaired with w in D_{sp}' as well. Using induction, there is minimum semi-PD-set D_{sp}^*, such that $D_{i-1} \subseteq D_{sp}^*$ and v_k is semipaired with a vertex either in $N_{T(G)}[v_i]$ or a sibling of v_i. Using Observation 4.4, we note that for all vertices $u \in N_G[v_i] \setminus \{v_k\}$ either $L(u) = 0$ or 2 and $D(v_j) = 1$ for $j \in [i-1]$ by Lemma 4.4. Let v_k be semipaired with v_p. We note that $v_p \notin D_{i-1}$.

Case 2.1. $L(F(v_i)) = 0$.

If $v_p = F(v_i)$, then the result follows. Let $v_p \neq F(v_i)$, implying that v_p is a child of v_i or a sibling of v_i or $p = i$. If v_p is child of v_i then $N_i(v_p) \subseteq N_i[F(v_i)]$ and if v_p is a sibling of v_i or $p = i$ then we note that $N_i[v_p] \subseteq N_i[F(v_i)]$. Using Lemma 4.4, we can update $D_{sp}' = (D_{sp}^* \setminus \{v_p\}) \cup \{F(v_i)\}$ with v_k semipaired with $F(v_i)$ to get the required minimum semi-PD-set.

Case 2.2. $L(F(v_i)) \neq 0$ and $L(v_i) = 0$.

In this case, $F(v_i) \in D_{i-1}$ and $v_p \neq F(v_i)$. If $v_p = v_i$ in D_{sp}^*, then D_{sp}^* is the desired set. Now suppose $v_p \neq v_i$, implying that v_p is either a child or a sibling of v_i. Similar to the previous case, we note that if v_p is child of v_i then $N_i(v_p) \subseteq N_i[F(v_i)]$ and if v_p is a sibling of v_i or $p = i$ then $N_i[v_p] \subseteq N_i[F(v_i)]$. Using Lemma 4.4 we can update $D_{sp}' = (D_{sp}^* \setminus \{v_p\}) \cup \{v_i\}$ with v_k semipaired with v_i to get the required minimum semi-PD-set.

Case 2.3. $L(F(v_i)) = L(v_i) \neq 0$.

In this case, $F(v_i), v_i \in D_{i-1}$ and neither $F(v_i) = v_p$ nor $v_i = v_p$. Here, v_p is either a child or a sibling of v_i and similar to the previous cases, we note that $N_i[v_p] \subseteq N_i[F(v_i)] \cup N_i[v_i]$. Since $v_p \notin D_{i-1}$, we have $L(v_p) = 0$. Therefore, D_{sp}^* with v_k semipaired with v_p is the desired minimum semi-PD-set.

Case 3. $D(v_n) = 0$.

Since $D(v_n) = 0$, v_n is not dominated by D_{n-1}. Hence, no vertex from $N_G[v_n]$ is selected in D_{n-1} and $L(u) = 0$ for all $u \in N_G[v_n]$. Let v_p be the vertex dominating v_n in D_{sp}^* and v_q be the vertex semipaired with v_p in D_{sp}^*. If $v_q \in D_{n-1}$, then $L(v_q) = 1$. Hence, by Lemma 4.4, v_q is at level 1 or less in $T(G)$, a contradiction as $D(v_n) = 0$. If $v_q \notin D_{n-1}$, then using the fact that $L(u) = 0$ for all $u \in N_G[v_n]$ and by Lemma 4.4, we can state that $D_{sp}' = (D_{sp}^* \setminus \{v_p, v_q\}) \cup \{v_n, u\}$ where $u \in N_G(v_n)$ is the desired minimum semi-PD-set.

This completed the proof of Lemma 4.5. □

Now, we are ready to state the main result of this Section.

Theorem 4.2 *Given a block graph G, a minimum semi-PD-set of G can be computed in $O(n + m)$-time.*

Proof: By Lemma 4.5, we claim that there is a minimum semi-PD-set D_{sp}^* such that $D_n \subseteq D_{sp}^*$ where D_n is the semi-PD-set returned by the Algorithm 1. This proves that the set D_n returned by Algorithm 1 is a minimum semi-PD-set of G. Next, we analyze the complexity of computing D_n for a given block graph G.

By Theorem 4.1, given a block graph $G = (V, E)$, a BEO of vertices of G can be computed in $O(n + m)$-time, and the corresponding block tree can also be constructed in $O(n + m)$-time. Now, given a block tree T, we can find the reverse of BFS ordering of T in $O(n + m)$-time. Also, all the computations in Algorithm 1 can be performed in $O(n + m)$-time. This proves that a minimum semi-PD-set of any block graph can be computed in $O(n + m)$-time. □

5 APX-completeness for Bounded Degree Graphs

In this section, we show that the MINIMUM SEMIPAIRED DOMINAION problem is APX-complete for graphs with maximum degree 3. It is known that the MINIMUM SEMIPAIRED DOMINATION problem for a graph G with maximum degree Δ can be approximated with an approximation ratio of $1 + \ln(2\Delta + 2)$ Henning et al. (2020). Hence the MINIMUM SEMIPAIRED DOMINATION problem is in APX for bounded degree graphs. To show APX-completeness, we use the concept of L-reduction. First, we recall the definition of L-reduction.

Definition 5.1 *Given two NP optimization problems F and G and a polynomial time transformation f from instances of F to instances of G, we say that f is an L-reduction if there are positive constants α and β such that for every instance x of F the following holds.*

1. $opt_G(f(x)) \leq \alpha \cdot opt_F(x)$.

2. *for every feasible solution y of $f(x)$ with objective value $m_G(f(x), y) = c_2$ we can in polynomial time find a solution y' of x with $m_F(x, y') = c_1$ such that $|opt_F(x) - c_1| \leq \beta|opt_G(f(x)) - c_2|$.*

To show the APX-completeness of a problem $\Pi \in$ APX, it suffices to show that there is an L-reduction from some APX-complete problem to Π.

We first show that the MINIMUM SEMIPAIRED DOMINAION problem is APX-complete for graphs with maximum degree 4. To show this result, we prove that the reduction given in the proof of Theorem 2 in Henning et al. (2020) is an L-reduction. So, we show an L-reduction from the MINIMUM VERTEX COVER PROBLEM for graphs with maximum degree 3 Alimonti and Kann (2000). The MINIMUM VERTEX COVER problem is already known to be APX-complete for graphs with maximum degree 3. For a graph $G = (V, E)$, a set $S \subseteq V$ is called a *vertex cover* of G if for every edge $e = uv \in E$, $S \cap \{u, v\} \neq \emptyset$. For a graph G, the MINIMUM VERTEX COVER problem is to find a vertex cover of G of minimum cardinality. Next, we present an L-reduction from the MINIMUM SEMIPAIRED DOMINATION problem for graphs with maximum degree 4 to the MINIMUM SEMIPAIRED DOMINATION problem for graphs with maximum degree 3.

Theorem 5.1 *The* MINIMUM SEMIPAIRED DOMINATION *problem is APX-complete for graphs with maximum degree* 4.

Proof: The MINIMUM SEMIPAIRED DOMINATION problem is in APX for graphs with maximum degree 4. Hence, to prove the APX-completeness, it is sufficient to give an L-reduction f, from the set of instances for the MINIMUM VERTEX COVER PROBLEM for graphs with maximum degree 3, to the set of instances for the MINIMUM SEMIPAIRED DOMINATION problem for graphs with maximum degree 4.

Given a graph $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ with $d_G(v_i) \leq 3$ for each $i \in [n]$ and $E = \{e_1, e_2, \ldots, e_m\}$, we construct a graph $G' = (V', E')$ as follows:

Let $V_k = \{v_i^k \mid i \in [n]\}$, $E_k = \{e_j^k \mid j \in [m]\}$ for $k \in [2]$ and $X = \{w_i, x_i, y_i, z_i \mid i \in [n]\}$. Define $V(G') = V_1 \cup V_2 \cup E_1 \cup E_2 \cup X$, and $E(G') = \{v_i^1 w_i, v_i^2 w_i, w_i x_i, x_i y_i, y_i z_i \mid i \in [n]\} \cup \{v_i^l e_j^l, v_k^l e_j^l \mid l \in [2], j \in [m]$ and $e_j = v_i v_k \in E\}$. Fig. 4 illustrates the construction of G' from G. Note that if the degree of a vertex in G is bounded by 3, then a vertex in G' has degree at most 4.

Next, we show that the above reduction is an L-reduction. The following claim is enough to complete the proof of the theorem.

Claim 5.1 *If V_c^* denotes a minimum vertex cover of G and D_{sp}^* denotes a minimum cardinality semi-PD-set of G', and n denotes the number of vertices of G, then $|D_{sp}^*| = 2|V_c^*| + 2n$. Further, if D_{sp} is an arbitrary semi-PD-set of G', then we can construct a vertex cover V_c of G, such that $|V_c| - |V_c^*| \leq |D_{sp}| - |D_{sp}^*|$.*

Proof: Let V_c^* denotes a minimum vertex cover of G. Then, the set

$$D_{sp} = \{v_i^1, v_i^2 \mid v_i \in V_c^*\} \cup \{w_i, y_i \mid i \in [n]\}$$

is a semi-PD-set of G' which implies that $|D_{sp}| \leq 2|V_c^*| + 2n$. Hence, if D_{sp}^* denotes a semi-PD-set of G' of minimum cardinality, then $|D_{sp}^*| \leq 2|V_c^*| + 2n$.
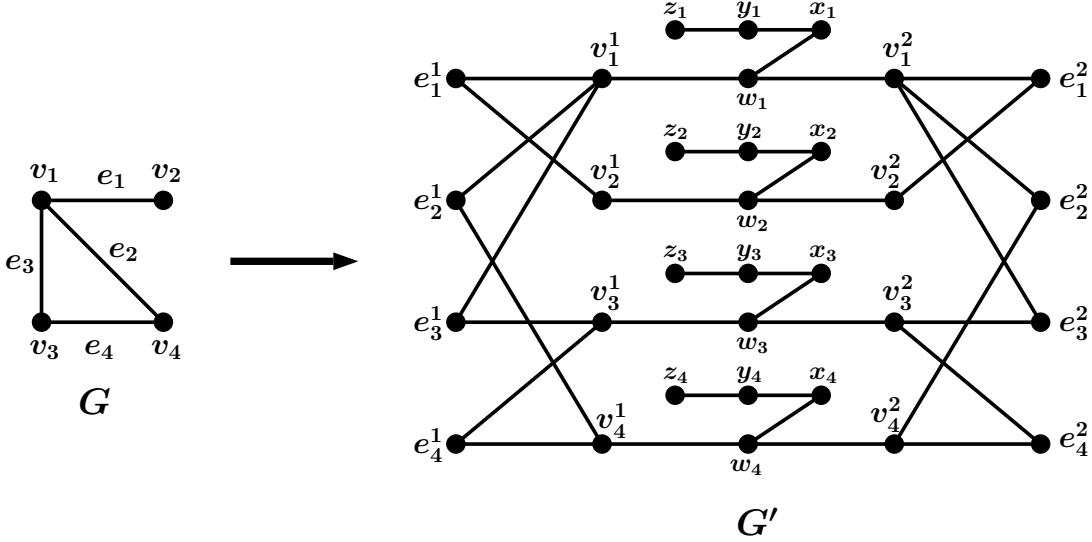
Fig. 4: An illustration of the construction of G' from G in the proof of Theorem 5.1.

Next, suppose that D_{sp} is an arbitrary semi-PD-set of G'. Note that $|D_{sp} \cap \{w_i, x_i, y_i, z_i\}| \geq 2$ for each $i \in [n]$. Hence, without loss of generality, we may assume that $\{w_i, y_i \mid i \in [n]\} \subseteq D_{sp}$ (with w_i and y_i paired in D_{sp}). Let

$$D = D_{sp} \setminus \{w_i, y_i \mid i \in [n]\}.$$

We note that $|D| = |D_{sp}| - 2n$. Let $D_1 = D \cap (V_1 \cup E_1)$ and $D_2 = D \cap (V_2 \cup E_2)$. Renaming the sets if necessary, we may assume that $|D_1| \leq |D_2|$, implying that $|D_1| \leq |D|/2$. In order to dominate a vertex $e_i^1 \in E_1$, either $e_i^1 \in D_{sp}$ or $v_j^1 \in D_{sp}$ where $v_j^1 \in N_{G'}(e_i^1)$. If $N_{G'}(e_i^1) \cap D_{sp} = \emptyset$, then we update D_1 as $D_1 = (D_1 \setminus \{e_i^1\}) \cup \{v_j^1\}$ for some $v_j^1 \in N_{G'}(e_i^1)$. We note that the cardinality of the set D_1 remains unchanged after updating D_1 for all such e_i^1, and so, $|D_1| \leq |D|/2 = (|D_{sp}| - 2n)/2$. Also every vertex in E_1 is now dominated by some vertex in D_1. Therefore the set

$$V_c = \{v_i \mid v_i^1 \in D_1\}$$

is a vertex cover of G and $|V_c| = |D_1| \leq \frac{1}{2}(|D_{sp}| - 2n)$. Hence, $|D_{sp}| \geq 2|V_c| + 2n$. Now, if V_c^* is a minimum vertex cover of G, then $|D_{sp}| \geq 2|V_c^*| + 2n$. This is true for every semi-PD-set, D_{sp}, of G'. In particular, if D_{sp}^* is a minimum semi-PD-set of G', then we have $|D_{sp}^*| \geq 2|V_c^*| + 2n$. As observed earlier, $|D_{sp}^*| \leq 2|V_c^*| + 2n$. Consequently, $|D_{sp}^*| = 2|V_c^*| + 2n$. Further,

$$|V_c| - |V_c^*| \leq \frac{1}{2}(|D_{sp}| - 2n) - \frac{1}{2}(|D_{sp}^*| - 2n) = \frac{1}{2}(|D_{sp}| - |D_{sp}^*|).$$

$\square$

Since the maximum degree of G is 3 and G is connected, $n - 1 \leq m \leq 3|V_c^*|$. Therefore, $|D_{sp}^*| = 2|V_c^*| + 2n \leq 2|V_c^*| + 2(3|V_c^*| + 1) \leq 8|V_c^*| + 2 \leq 10|V_c^*|$. This proves that f is an L-reduction with $\alpha = 10$ and $\beta = \frac{1}{2}$.

$\square$

We observe that the graph G' constructed in Theorem 5.1 is also a bipartite graph. Hence, as a corollary of Theorem 5.1, we have the following result.

Corollary 1 *The* MINIMUM SEMIPAIRED DOMINATION *problem is APX-complete for bipartite graphs with maximum degree* 4.

Theorem 5.2 *The* MINIMUM SEMIPAIRED DOMINATION *problem is APX-complete for graphs with maximum degree* 3.

Proof: To prove the theorem, it is sufficient to describe an L-reduction h, from the MINIMUM SEMI-PAIRED DOMINATION for graphs with maximum degree 4 to the MINIMUM SEMIPAIRED DOMINATION for graphs with maximum degree 3.

Given a graph G with maximum degree 4, we construct a graph G' with maximum degree 3 as follows: for a vertex $v \in V(G)$ with $d_G(v) = 4$, we split and transform v as illustrated in Fig. 5. For a vertex v with $d_G(v) \leq 3$, we do not perform any transformation.
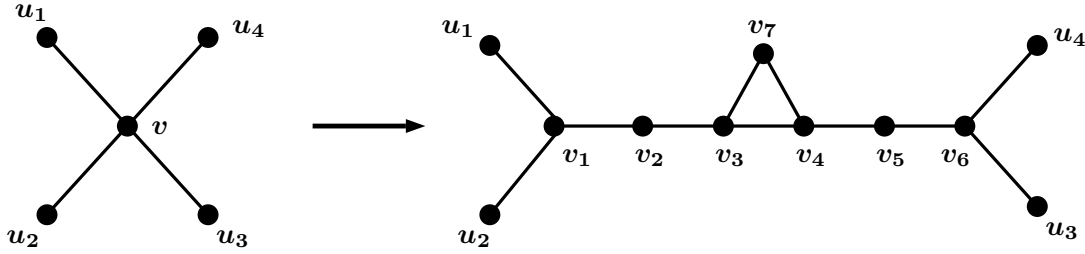
Fig. 5: Transformation of a vertex $v \in V(G)$ with $d_G(v) = 4$.

By construction, for every vertex v in G', we have $d_{G'}(v) \leq 3$. Next, we show that the above reduction is an L-reduction. The following claim is enough to complete the proof of the theorem.

Claim 5.2 *If D_{sp}^* and $D_{sp}'^*$ are minimum semi-PD-sets of G and G', respectively, and k denotes the number of vertices of degree 4 in G, then $|D_{sp}'^*| = |D_{sp}^*| + 2k$. Further, if D_{sp}' is an arbitrary semi-PD-set of G', then we can construct a semi-PD-set D_{sp} of G, such that $|D_{sp}| - |D_{sp}^*| \leq |D_{sp}'| - |D_{sp}'^*|$.*

Proof: Let D_{sp} be a semi-PD-set of G. We construct a semi-PD-set D_{sp}' of G' as follows:

(1) If $d_G(v) \leq 3$, $v \in D_{sp}'$ if and only if $v \in D_{sp}$.

(2) If $d_G(v) = 4$, then we proceed as follows.

   (2.1) If $v \in D_{sp}$ and v is semipaired with a vertex $u \in N[w]$ such that $w \in \{u_1, u_2\}$, take v_1, v_4, v_6 in D_{sp}'.

   (2.2) If $v \in D_{sp}$ and v is semipaired with a vertex $u \in N[w]$ such that $w \in \{u_3, u_4\}$, take v_1, v_3, v_6 in D_{sp}'.

   (2.3) If $v \notin D_{sp}$ and v is dominated by a vertex in the set $\{u_1, u_2\}$, take v_3, v_5 in D_{sp}'.

(2.4) If $v \notin D_{sp}$ and v is dominated by a vertex in the set $\{u_3, u_4\}$, take v_2, v_4 in D'_{sp}.

Let k be the number of vertices of degree 4 in G. We observe that, D'_{sp} is a semi-PD-set of the transformed graph G' and $|D'_{sp}| = |D_{sp}| + 2k$. Thus, $|D'^*_{sp}| \leq |D^*_{sp}| + 2k$, where D'^*_{sp} and D^*_{sp} are minimum semi-PD-sets of G' and G, respectively.

Conversely, let D'_{sp} be a semi-PD-set of G'. Now we construct a semi-PD-set D_{sp} of G from D'_{sp}. If $d_G(v) \leq 3$, then we will include v in D_{sp} if and only if $v \in D'_{sp}$. If $d_G(v) = 4$, then v is transformed as shown in Fig. **??**. For each vertex $v \in V(G)$ of degree 4, let $\phi(v) = |\{v_1, v_2, \ldots, v_7\} \cap D'_{sp}|$. If the degree of v in G is 4, then we include v in D_{sp} if and only if $\phi(v) \geq 3$. Without loss of generality, we may assume that, to dominate v_7 either $v_3 \in D'_{sp}$ or $v_4 \in D'_{sp}$. Also, v_3 and v_4 can be semipaired only with the vertices in the set $\{v_1, v_2, \ldots, v_7\}$. Hence, $\phi(v) \geq 2$.

We note that the vertices in the set $\{v_2, v_3, v_4, v_5, v_7\}$ cannot be dominated by a vertex $w \in D'_{sp} \setminus \{v_1, v_2, \ldots, v_7\}$. Hence, if $\phi(v) = 2$, then without loss of generality we may assume that either $\{v_3, v_5\} \subseteq D'_{sp}$ and v_3 is semipaired with v_5 in D'_{sp} or $\{v_2, v_4\} \subseteq D'_{sp}$ and v_4 is semipaired with v_2 in D'_{sp}. Note that in either case, $V(G') \setminus \{v_1, v_2, \ldots, v_7\}$ is dominated by $D'_{sp} \setminus \{v_1, v_2, \ldots, v_7\}$.

A vertex $u \in D'_{sp} \setminus \{v_1, v_2, \ldots, v_7\}$ can only be semipaired with a vertex in the set $\{v_1, v_2, v_5, v_6\}$ in D'_{sp}. If $\phi(v) = 3$, then only one vertex in the set $\{v_1, v_2, v_5, v_6\}$ is semipaired with a vertex w such that $w \in D'_{sp} \setminus \{v_1, v_2, \ldots, v_7\}$. If $w \in V(G)$, then in D_{sp}, the vertex v will be semipaired with vertex w. If $w \notin V(G)$, that is, if w is obtained by splitting some vertex u of degree 4 in G, then v will be semipaired with vertex u.

Suppose that $\phi(v) \geq 4$ and more than two vertices in the set $\{v_1, v_2, v_5, v_6\}$ are semipaired in $D'_{sp} \setminus \{v_1, v_2, \ldots, v_7\}$. For simplicity, suppose $\{x, y\} \subseteq D'_{sp} \setminus \{v_1, v_2, \ldots, v_7\}$ where x and y are semipaired with some vertices in the set $\{v_1, v_2, v_5, v_6\}$. Note that $V(G') \setminus \{v_1, v_2, \ldots, v_7\}$ is dominated by $(D'_{sp} \setminus \{v_1, v_2, \ldots, v_7\}) \cup \{v\}$. If $\{x, y\} \subseteq D_{sp}$, then semipair x with v in D_{sp}. Now it may be the case that y is not semipaired in D_{sp}. If the vertex $y \in D_{sp}$ has no partner in D_{sp} and $N_G(y) \subseteq D_{sp}$, then remove y from D_{sp}; otherwise, we will include another vertex $u \notin D_{sp}$ which is at distance at most 2 from y.

The resulting set D_{sp} is a semi-PD-set of G. We note that, $|D_{sp}| \leq |D'_{sp}| - 2k$. Thus, $|D^*_{sp}| \leq |D'^*_{sp}| - 2k$ and hence, $|D'^*_{sp}| = |D^*_{sp}| + 2k$. Consequently, we have $|D_{sp}| - |D_{sp}*| \leq |D'_{sp}| - |D'^*_{sp}|$. $\square$

Finally, since G is a graph with maximum degree 4, for any dominating set D of G, we have $|D| \geq |V(G)|/5$. In particular, $|D^*_{sp}| \geq |V(G)|/5$. Since $k \leq |V(G)| \leq 5|D^*_{sp}|$, we have $|D'^*_{sp}| \leq |D^*_{sp}| + 2k \leq 11|D^*_{sp}|$. Hence, we may conclude that h is an L-reduction from the MINIMUM SEMIPAIRED DOMINATION for graphs with maximum degree 4 to the MINIMUM SEMIPAIRED DOMINATION for graphs with maximum degree 3 with $\alpha = 11$ and $\beta = 1$. $\square$

6   Conclusion

The SEMIPAIRED DOMINATION DECISION problem is already known to be NP-complete for chordal graphs. In this paper, we study this problem for two important subclasses of chordal graphs: split graphs and block graphs. We show that the SEMIPAIRED DOMINATION DECISION problem remains NP-complete for split graphs, and we propose a linear-time algorithm to compute a minimum cardinality semi-PD-set of a block graph. We also prove that the MINIMUM SEMIPAIRED DOMINATION is APX-complete for graphs with maximum degree 3. It will be interesting to study the complexity status of the problem for other important subclasses of chordal graphs, for example strongly chordal graphs, doubly chordal graphs, undirected path graphs etc.

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