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The strong chromatic index of 1-planar graphs

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The chromatic index $\chi'(G)$ of a graph G is the smallest k for which G admits an edge k-coloring such that any two adjacent edges have distinct colors. The strong chromatic index $\chi'_s(G)$ of G is the smallest k such that G has an edge k-coloring with the condition that any two edges at distance at most 2 receive distinct colors. A graph is 1-planar if it can be drawn in the plane so that each edge is crossed by at most one other edge.

In this paper, we show that every graph G with maximum average degree $\bar{d}(G)$ has $\chi'_s(G) \leq (2\bar{d}(G) - 1)\chi'(G)$. As a corollary, we prove that every 1-planar graph G with maximum degree Δ has $\chi'_{\mathrm{s}}(G) \leq 14\Delta$, which improves a result, due to Bensmail et al., which says that $\chi'_{\mathrm{s}}(G) \leq 24\Delta$ if $\Delta \geq 56$.

Keywords: Strong edge coloring, strong chromatic index, maximum average degree, 1-planar graph, matching.

1 Introduction

Only simple graphs are considered in this paper unless otherwise stated. Let G be a graph with vertex set $V(G)$, edge set $E(G)$, minimum degree $\delta(G)$, and maximum degree $\Delta(G)$ (for short, Δ), respectively. A vertex v is called a *k-vertex* if the degree $d_G(v)$ of v is k. The *girth* $g(G)$ of a graph G is the length of a shortest cycle in G. The *maximum average degree* $\bar{d}(G)$ of a graph G is defined as follows:

$$\bar{d}(G) = \max\{\frac{2|E(H)|}{|V(H)|} \mid H \subseteq G\}.$$

A *proper edge k-coloring* of a graph G is a mapping $\phi : E(G) \to \{1, 2, \dots, k\}$ such that $\phi(e) \neq \phi(e')$ for any two adjacent edges e and e'. The *chromatic index* $\chi'(G)$ of G is the smallest k such that G has a proper edge k-coloring. The coloring ϕ is called *strong* if any two edges at distance at most two get distinct colors. Equivalently, each color class is an induced matching. The *strong chromatic index*, denoted $\chi'_{\mathrm{s}}(G)$, of G is the smallest integer k such that G has a strong edge k-coloring.

Strong edge coloring of graphs was introduced by Fouquet and Jolivet [12]. It holds trivially that $\chi'_s(G) \geq \chi'(G) \geq \Delta$ for any graph G. In 1985, during a seminar in Prague, Erdős and Nešetřil put forward the following conjecture:

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Conjecture 1 *For a simple graph G,*

$$\chi'_s(G) \le \begin{cases} 1.25\Delta^2, & \text{if } \Delta \text{ is even;} \\ 1.25\Delta^2 - 0.5\Delta + 0.25, & \text{if } \Delta \text{ is odd.} \end{cases}$$

Erdős and Nešetřil provided a construction showing that Conjecture 1 is tight if it were true. Using probabilistic method, Molloy and Reed [23] showed that $\chi'_s(G) \le 1.998\Delta^2$ for any graph G with sufficiently large Δ. This result was gradually improved to that $\chi'_s(G) \le 1.93\Delta^2$ in [6], to that $\chi'_s(G) \le 1.835\Delta^2$ in [4], and to that $\chi'_s(G) \le 1.772\Delta^2$ in [17]. Andersen [1] and independently Horák et al. [14] confirmed Conjecture 1 for graphs with $\Delta = 3$. If $\Delta = 4$, then Conjecture 1 asserts that $\chi'_s(G) \le 20$. However, the currently best known upper bound is 21 for this case, see [15].

A graph G is *d-degenerate* if each subgraph of G contains a vertex of degree at most d. Chang and Narayanan [7] showed that $\chi'_s(G) \le 10\Delta - 10$ for a 2-degenerate graph G. For a general k-degenerate graph G, it was shown that $\chi'_s(G) \le (4k-2)\Delta - k(2k-1) + 1$ in [31], $\chi'_s(G) \le (4k-1)\Delta - k(2k+1) + 1$ in [8], and $\chi'_s(G) \le (4k-2)\Delta - 2k^2 + 1$ in [27].

Suppose that G is a planar graph. Faudree et al. [11] first gave an elegant proof for the result that $\chi'_s(G) \le 4\Delta + 4$, and constructed a class of planar graphs G with $\Delta \ge 2$ such that $\chi'_s(G) = 4\Delta - 4$. For the class of special planar graphs, some better results have been obtained. It was shown in [16] that $\chi'_s(G) \le 3\Delta$ if $g(G) \ge 7$, and in [2] that $\chi'_s(G) \le 3\Delta + 1$ if $g(G) \ge 6$. Kostochka et al. [19] showed that if $\Delta = 3$ then $\chi'_s(G) \le 9$. Hocquard et al. [13] showed that every outerplanar graph G with $\Delta \ge 3$ has $\chi'_s(G) \le 3\Delta - 3$. Wang et al. [29] showed that every K_4-minor-free graph G with $\Delta \ge 3$ has $\chi'_s(G) \le 3\Delta - 2$. Moreover, all upper bounds $9, 3\Delta - 3, 3\Delta - 2$ given in the above results are best possible.

A 1-*planar graph* is a graph that can be drawn in the plane such that each edge crosses at most one other edge. A number of interesting results about structures and parameters of 1-planar graphs have been obtained in recent years. Fabrici and Madaras [10] proved that every 1-planar graph G has $|E(G)| \le 4|V(G)| - 8$, which implies that $\delta(G) \le 7$, and constructed 7-regular 1-planar graphs. Borodin [5] showed that every 1-planar graph is vertex 6-colorable. Wang and Lih [28] proved that the vertex-face total graph of a plane graph, which is a class of special 1-planar graphs, is vertex 7-choosable. Zhang and Wu [33] studied the edge coloring of 1-planar graphs and showed that every 1-planar graph G with $\Delta \ge 10$ satisfies $\chi'(G) = \Delta$.

Bensmail et al. [3] investigated the strong edge coloring of 1-planar graphs and proved that every 1-planar graph G has $\chi'_s(G) \le \max\{18\Delta + 330, 24\Delta - 6\}$. This implies that if $\Delta \ge 56$, then $\chi'_s(G) \le 24\Delta - 6$. In this paper we will improve this result by showing that every 1-planar graph G has $\chi'_s(G) \le 14\Delta$. To obtain this result, we establish a connection between the strong chromatic index and maximum average degree of a graph. More precisely, we will show that $\chi'_s(G) \le (2\bar{d}(G) - 1)(\Delta + 1)$ for any simple graph G.

2 Preliminary

In this section, we summarize some known results, which will be used later.

A *proper k-coloring* of a graph G is a mapping $\phi : V(G) \to \{1, 2, \ldots, k\}$ such that $\phi(u) \ne \phi(v)$ for any two adjacent vertices u and v. The *chromatic number*, denoted $\chi(G)$, of G is the least k such that G has a proper k-coloring.

Using a greedy algorithm, the following conclusion holds automatically.

Lemma 1 *If* G *is a* d*-degenerate graph, then* $\chi(G) \leq d + 1$.

As stated before, Borodin [5] showed the following sharp result:

Theorem 2 ([5]) *Every* 1*-planar graph* G *has* $\chi(G) \leq 6$.

Given a graph G, it is trivial that $\chi'(G) \geq \Delta$. On the other hand, the celebrated Vizing Theorem [26] asserts:

Theorem 3 ([26]) *Every simple graph* G *has* $\chi'(G) \leq \Delta + 1$.

A simple graph G is of *Class I* if $\chi'(G) = \Delta$, and of *Class II* if $\chi'(G) = \Delta + 1$. As early as in 1916, König [18] showed that bipartite graphs are of Class I.

Theorem 4 ([18]) *If* G *is a bipartite graph, then* $\chi'(G) = \Delta$.

Sanders and Zhao [24], and Zhang [32] independently, showed that planar graphs with maximum degree at least seven are of Class I.

Theorem 5 ([24, 32]) *Every planar graph* G *with* $\Delta \geq 7$ *has* $\chi'(G) = \Delta$.

Another result, due to Zhang and Wu [33], claims that 1-planar graphs with maximum degree at least ten are of Class I.

Theorem 6 ([33]) *Every* 1*-planar graph* G *with* $\Delta \geq 10$ *has* $\chi'(G) = \Delta$.

Zhou [34] observed an interesting relation between the degeneracy and chromatic index of a graph.

Theorem 7 ([34]) *If* G *is a* k*-degenerate graph with* $\Delta \geq 2k$*, then* $\chi'(G) = \Delta$.

3 Contracting matchings in a graph

Let G be a simple graph. An edge e of G is said to be *contracted* if it is deleted and its end-vertices are identified. An edge subset M of G is called a *matching* if no two edges in M are adjacent in G. Specifically, a matching M is called *strong* if no two edges in M are adjacent to a common edge. This is equivalent to saying that $G[V(M)] = M$. Thus, a strong matching is also called an *induced matching*. Determining the chromatic index $\chi'(G)$ of a graph G is certainly equivalent to finding the least k such that $E(G)$ can be partitioned into k edge-disjoint matchings, and determining the strong chromatic index $\chi'_s(G)$ of a graph G is equivalent to finding the least k such that $E(G)$ can be partitioned into k edge-disjoint strong matchings. In what follows, an edge k-coloring of G with the color classes $E_1, E_2, \ldots, E_k$ will be denoted by $(E_1, E_2, \ldots, E_k)$.

Given a graph G and a matching M of G, let G_M denote the graph obtained from G by contracting each edge in M. Note that G_M may contain multi-edges, but no loops, even if G is simple.

Let $a \geq 1$ and $b \geq 0$ be integers. A graph G is said to be (a, b)*-graph* if every subgraph G' of G (including itself) has $|E(G')| \leq a|V(G')| - b$.

Theorem 8 *Let* G *be a* (a, b)*-graph with* $a \geq 1$ *and* $b \geq 0$*. Let* M *be a matching of* G*. Then* G_M *is a* $(2a - 1, b)$*-graph.*

Proof: Let H be any subgraph of G_M. Assume that $V(H) = V_1 \cup V_2$, where V_1 is the set of vertices in G_M which are formed by contracting some edges in M, and $V_2 = V(H) \setminus V_1$, say, $V_1 = \{x_1, x_2, \ldots, x_{n_1}\}$, and $V_2 = \{y_1, y_2, \ldots, y_{n_2}\}$. Then $|V(H)| = n_1 + n_2$. Splitting each vertex $x_i \in V_1$ into two vertices u_i and v_i and restoring corresponding incident edges for u_i and v_i in G, we get a subgraph G' of G with

$$V(G') = \{u_1, u_2, \ldots, u_{n_1}; v_1, v_2, \ldots, v_{n_1}; y_1, y_2, \ldots, y_{n_2}\}$$

and

$$E(G') = E(H) \cup M',$$

where

$$M' = \{u_1v_1, u_2v_2, \ldots, u_{n_1}v_{n_1}\} \subseteq M.$$

It is easy to compute that $|V(G')| = 2n_1 + n_2$ and $|E(G')| = |E(H)| + n_1$. By the assumption, $|E(G')| \le a|V(G')| - b$. Since $a \ge 1$, we have $2a - 1 \ge a$. Consequently,

$$\begin{aligned} |E(H)| &= |E(G')| - n_1 \\ &\le a|V(G')| - b - n_1 \\ &= a(2n_1 + n_2) - b - n_1 \\ &= (2a-1)n_1 + an_2 - b \\ &\le (2a-1)(n_1 + n_2) - b \\ &= (2a-1)|V(H)| - b. \end{aligned}$$

This shows that G_M is a $(2a-1, b)$-graph. □

Corollary 9 *Let G be a (a, b)-graph with $a, b \ge 1$. Let M be a matching of G. Then G_M is $(4a-3)$-degenerate.*

Proof: It suffices to verify that $\delta(H) \le 4a - 3$ for any $H \subseteq G_M$. Suppose to the contrary that $\delta(H) \ge 4a - 2$. Since $b \ge 1$, Theorem 8 and the Handshaking Theorem imply that $(4a-2)|V(H)| \le \delta(H)|V(H)| \le \sum_{v \in V(H)} d_H(v) = 2|E(H)| \le 2((2a-1)|V(H)| - b) = (4a-2)|V(H)| - 2b < (4a-2)|V(H)|$. This leads to a contradiction. □

Similarly, we obtain the following consequence:

Corollary 10 *Let G be a $(a, 0)$-graph with $a \ge 1$. Let M be a matching of G. Then G_M is $(4a-2)$-degenerate.*

A matching M of a graph G is said to be *partitioned* into q strong matchings of G if $M = M_1 \cup M_2 \cup \cdots \cup M_q$ and $M_i \cap M_j = \emptyset$ for $i \ne j$ such that each M_i is a strong matching of G. Let $\rho_G(M)$ denote the least q such that M is partitioned into q strong matchings. By definition, $1 \le \rho_G(M) \le |M|$.

The following result is highly inspired from a result of [11] on the strong chromatic index of planar graphs. For the sake of completeness, we here give the detailed proof.

Lemma 11 *Let G be a graph and M be a matching of G. Then $\rho_G(M) \le \chi(G_M)$.*

Proof: Let $V(G_M) = S_1 \cup S_2$, where S_1 is the set of vertices in G_M formed from G by contracting edges in M and $S_2 = V(G) \setminus V(M)$. Set $k = \chi(G_M)$. Then G_M admits a proper k-coloring $\phi : V(G_M) \to \{1, 2, \ldots, k\}$. For $1 \leq i \leq k$, let V_i denote the set of vertices in G_M with the color i. In G, for $1 \leq i \leq k$, let

$$E_i^* = \{e \in M \mid e \text{ is contracted to some vertex } v_e \in S_1 \text{ with } \phi(v_e) = i\}.$$

Let $e_1, e_2 \in E_i^*$ be any two edges. Since $e_1, e_2 \in M$, e_1 and e_2 are not adjacent in G. We claim that no edge $e \in E(G)$ is simultaneously adjacent to both e_1 and e_2. Assume to the contrary, there exists $e = xy \in E(G)$ adjacent to e_1 and e_2. Without loss of generality, we may suppose that $e_1 = xx'$ and $e_2 = yy'$. Let v_{e_1} and v_{e_2} denote the corresponding vertices of e_1 and e_2 in S_1, respectively. Indeed, x is v_{e_1}, and y is v_{e_2}. Since $xy \notin M$, it follows that $xy \in E(G_M)$ and thus x is adjacent to y in G_M. By the definition of ϕ, $\phi(x) \neq \phi(y)$. Let $\phi(x) = p$ and $\phi(y) = q$. Then $e_1 \in E_p^*$ and $e_2 \in E_q^*$ with $p \neq q$, which contradicts the assumption that $e_1, e_2 \in E_i^*$. So, each of $E_1^*, E_2^*, \ldots, E_k^*$ is a strong matching of G. This confirms that $\rho_G(M) \leq k = \chi(G_M)$. □

4 Strong chromatic index

In this section, we will discuss the strong edge coloring of some graphs by using the previous preliminary results.

4.1 An upper bound

We first establish an upper bound of strong chromatic index for a general graph G, which reveals a relation between the strong chromatic index, chromatic index and maximum average degree of G.

Lemma 12 *Let H be a subgraph of a graph G. Then $|E(H)| \leq \frac{1}{2}\bar{d}(G)|V(H)|$.*

Proof: For any subgraph $H \subseteq G$, it follows from the definition of $\bar{d}(G)$ that $\frac{2|E(H)|}{|V(H)|} \leq \bar{d}(G)$. Consequently, $|E(H)| \leq \frac{1}{2}\bar{d}(G)|V(H)|$. □

Theorem 13 *Every graph G has $\chi'_s(G) \leq (2\bar{d}(G) - 1)\chi'(G)$.*

Proof: Let $k = \chi'(G)$. Then G has an edge k-coloring $(E_1, E_2, \ldots, E_k)$, where each E_i is a matching of G. Let G_i be the graph obtained from G by contracting each of edges in E_i. By Lemma 12 and Corollary 10, G_i is $(2\bar{d}(G) - 2)$-degenerate. By Lemma 1, $\chi(G_i) \leq 2\bar{d}(G) - 1$. By Lemma 11, $\chi'_s(G) \leq (2\bar{d}(G) - 1)k = (2\bar{d}(G) - 1)\chi'(G)$. □

By Theorems 3, 4 and 13, the following two corollaries hold automatically.

Corollary 14 *Every graph G has $\chi'_s(G) \leq (2\bar{d}(G) - 1)(\Delta + 1)$.*

Corollary 15 *Every bipartite graph G has $\chi'_s(G) \leq (2\bar{d}(G) - 1)\Delta$.*

Corollary 16 *If G is a graph with $\Delta \geq 2\bar{d}(G)$, then $\chi'_s(G) \leq (2\bar{d}(G) - 1)\Delta$.*

Proof: Since G is $\bar{d}(G)$-degenerate and $\Delta \geq 2\bar{d}(G)$, Theorem 7 asserts that $\chi'(G) = \Delta$. By Theorem 13, $\chi'_s(G) \leq (2\bar{d}(G) - 1)\Delta$. □

4.2 1-planar graphs

Recently, Liu et al. [22] investigated the existence of light edges in a 1-planar graph with minimum degree at least three. For our purpose, we here list one of their results as follows:

Theorem 17 ([22]) *Every* 1*-planar graph* G *with* $\delta(G)=7$ *contains two adjacent* 7*-vertices.*

With a greedy coloring procedure, it can be constructively shown that the strong chromatic index of a simple graph G is at most $2\Delta(\Delta-1)+1$.

Theorem 18 *If* G *is a* 1*-planar graph, then* $\chi'_{\rm s}(G)\le 14\Delta$.

Proof: The proof is split into the following cases, depending on the size of Δ.

Case 1: $\Delta\le 7$.

It is easy to check that $2\Delta(\Delta-1)+1\le 14\Delta$ and henceforth the result follows.

Case 2: $\Delta=8$.

Since G is 7-degenerate, it follows from the result of [27] that $\chi'_{\rm s}(G)\le(4\times 7-2)\Delta-2\times 7^2+1=26\Delta-97=111<112=14\Delta$.

Case 3: $\Delta=9$.

The proof is given by induction on the number of edges in G. If $|E(G)|\le 14\Delta=126$, the result holds trivially, since we may color all edges of G with distinct colors. Let G be a 1-planar graph with $\Delta=9$ and $|E(G)|>126$. Without loss of generality, assume that G is connected, hence $\delta(G)\ge 1$. We have to consider two subcases as follows.

Case 3.1: $\delta(G)\le 6$.

Let $u\in V(G)$ with $d_G(u)=\delta(G)\ge 1$. Let $u_0,u_1,\ldots,u_{s-1}$ denote the neighbors of u in a cyclic order, where $1\le s=\delta(G)\le 6$. For $0\le i\le s-1$, let $x_i^1,x_i^2,\ldots,x_i^{p_i}$ denote the neighbors of u_i other than u. Consider the graph $H=G-u$. Then H is a 1-planar graph with $\Delta(H)\le 9$ and $|E(H)|<|E(G)|$. By the induction hypothesis or Cases 1 and 2, H admits a strong edge coloring ϕ using the color set $C=\{1,2,\ldots,126\}$. For a vertex $v\in V(H)$, let $C(v)$ denote the set of colors assigned to the edges incident with v. For $i=0,1,\ldots,s-1$, define a list $L(uu_i)$ of available colors for the edge uu_i as follows:

$$L(uu_i)=C-\bigcup_{0\le j\le s-1;\ j\ne i}C(u_j)-\bigcup_{1\le t\le p_i}C(x_i^t).$$

It is easy to calculate that

$$\begin{aligned}|L(uu_i)| &\ge |C|-|\bigcup_{0\le j\le s-1;\ j\ne i}C(u_j)|-|\bigcup_{1\le t\le p_i}C(x_i^t)|\\ &\ge 126-(s-1)(\Delta-1)-(\Delta-1)\Delta\\ &\ge 126-(6-1)\times(9-1)-(9-1)\times 9\\ &= 14.\end{aligned}$$

Based on ϕ, we color uu_0 with a color $a_0 \in L(uu_0)$, uu_1 with a color $a_1 \in L(uu_1) \setminus \{a_0\}$, $\cdots$, uu_{s-1} with a color $a_{s-1} \in L(uu_{s-1}) \setminus \{a_0, a_1, \ldots, a_{s-2}\}$. It is easy to testify that ϕ is extended to whole graph G.

Case 3.2: $\delta(G) = 7$.

By Theorem 17, G contains two adjacent 7-vertices. Let u be a 7-vertex of G with neighbors $u_0, u_1, \ldots, u_6$ such that $d_G(u_0) = 7$ and $d_G(u_i) \leq 9$ for $i = 1, 2, \ldots, 6$. Similarly to Case 3.1, for $0 \leq i \leq 6$, let $x_i^1, x_i^2, \ldots, x_i^{p_i}$ denote the neighbors of u_i other than u. Note that $p_0 = 6$ and $p_i \leq 8$ for $i \geq 1$. Let $H = G - u$, which has a strong edge coloring ϕ using the color set $C = \{1, 2, \ldots, 126\}$, by the induction hypothesis or Cases 1 and 2. For each $0 \leq i \leq 6$, we define similarly a list $L(uu_i)$ of available colors. It is easy to check that

$$\begin{aligned} |L(uu_0)| &\geq |C| - |\bigcup_{1\leq j\leq 6} C(u_j)| - |\bigcup_{1\leq t\leq 6} C(x_0^t)| \\ &\geq 126 - 6(\Delta - 1) - 6\Delta \\ &= 24. \end{aligned}$$

For $1 \leq i \leq 6$,

$$\begin{aligned} |L(uu_i)| &\geq |C| - |C(u_0)| - |\bigcup_{1\leq j\leq 6;\, j\neq i} C(u_j)| - |\bigcup_{1\leq t\leq p_i} C(x_i^t)| \\ &\geq 126 - 6 - 5(\Delta - 1) - 8\Delta \\ &= 8. \end{aligned}$$

Based on ϕ, we color uu_0 with a color $a_0 \in L(uu_0)$, uu_1 with a color $a_1 \in L(uu_1) \setminus \{a_0\}$, $\cdots$, uu_6 with a color $a_6 \in L(uu_6) \setminus \{a_0, a_1, \ldots, a_5\}$. It is easy to confirm that ϕ is extended to G.

Case 4: $\Delta \geq 10$.

By Theorem 6, G is of Class I. Let $(E_1, E_2, \ldots, E_\Delta)$ be an edge Δ-coloring of G, where each E_i is a matching of G. Let G_i be the graph obtained from G by contracting each edge in E_i. Note that each subgraph H of G is 1-planar and therefore $|E(H)| \leq 4|V(H)| - 8$. Taking $a = 4$ and $b = 8$ in Corollary 9, we deduce that G_i is 13-degenerate. By Lemma 1, $\chi(G_i) \leq 14$. Therefore $\chi'_{\mathrm{s}}(G) \leq 14\Delta$.

□

4.3 Special 1-planar graphs

Suppose that G is a 1-planar graph which is drawn in the plane so that each edge is crossed by at most one other edge. Let E' and E'' denote the set of non-crossing edges and crossing edges of G, respectively. Let $H_1 = G[E']$ and $H_2 = G[E'']$. That is, H_1 and H_2 are the subgraphs of G induced by non-crossing edges and crossing edges, respectively.

Theorem 19 *Let* G *be a* 1*-planar graph. Then* $\chi'_{\mathrm{s}}(G) \leq 6\chi'(H_1) + 14\chi'(H_2)$.

Proof: Let $k_1 = \chi'(H_1)$ and $k_2 = \chi'(H_2)$. Then $\chi'(G) \le k_1 + k_2$. Let $(E_1, E_2, \ldots, E_{k_1})$ be an edge k_1-coloring of H_1, and $(F_1, F_2, \ldots, F_{k_2})$ be an edge k_2-coloring of H_2. Then each of E_i's and F_j's is a matching in G. So, $(E_1, E_2, \ldots, E_{k_1}, F_1, F_2, \ldots, F_{k_2})$ is an edge $(k_1 + k_2)$-coloring of G. Similarly to the proof of Case 4 in Theorem 18, every F_j can be partitioned into 14 strong matchings of G. Moreover, for each $1 \le i \le k_1$, because G_{E_i} is a 1-planar graph, we derive that $\chi(G_{E_i}) \le 6$ by Theorem 2. By Lemma 11, E_i can be partitioned into 6 strong matchings. Consequently, $\chi'_s(G) \le 6k_1 + 14k_2$. □

An *IC-planar graph* is a 1-planar graph such that two pairs of crossing edges have no common end-vertices. Equivalently, each vertex of this kind of 1-planar graph is incident with at most one crossing edge. It is easy to verify that every IC-planar graph G has $|E(G)| \le 3.25|V(G)| - 6$ and this bound is attainable. Král and Stacho [20] showed that every IC-planar graph is vertex 5-colorable. Yang et al. [30] showed that every IC-planar graph is vertex 6-choosable. Furthermore, Dvořák et al. [9] proved that every graph drawn in the plane so that the distance between every pair of crossings is at least 15 is 5-choosable.

Using Theorem 19, we can establish the smaller upper bound for the strong chromatic index of IC-planar graphs.

Theorem 20 *Every IC-planar graph G has $\chi'_s(G) \le 6\Delta + 20$.*

Proof: If $\Delta \le 5$, then it is easy to obtain that $\chi'_s(G) \le 2\Delta(\Delta - 1) + 1 \le 6\Delta + 20$ and therefore the theorem holds. So assume that $\Delta \ge 6$. Let H_1 and H_2 denote the graphs induced by non-crossing edges and crossing edges of G, respectively. Since no two crossing-edges of G are adjacent, H_2 is a matching of G. Thus, $\chi'(H_2) \le 1$. Note that H_1 is a planar graph with $\Delta(H_1) \le \Delta$. If $\Delta(H_1) \ge 7$, then $\chi'(H_1) = \Delta(H_1) \le \Delta$ by Theorem 5. So, by Theorem 19, $\chi'_s(G) \le 6\chi'(H_1) + 14\chi'(H_2) \le 6\Delta + 14$. Otherwise, we have to consider two cases as follows:

- $\Delta(H_1) = 6$. Then $6 \le \Delta \le 7$. By Theorem 3, $\chi'(H_1) \le 7$. By Theorem 19, $\chi'_s(G) \le 6\chi'(H_1) + 14\chi'(H_2) \le 6 \times 7 + 14 = 56 \le 6\Delta + 20$.
- $\Delta(H_1) = 5$. Then $\Delta = 6$ by the assumption. By Theorem 3, $\chi'(H_1) \le 6$. By Theorem 19, $\chi'_s(G) \le 6\chi'(H_1) + 14\chi'(H_2) \le 6 \times 6 + 14 = 50 = 6\Delta + 20$. □

A 1-planar graph G is called *optimal* if $|E(G)| = 4|V(G)| - 8$. A *plane quadrangulation* is a plane graph such that each face of G is of degree 4. It is not hard to show that a 3-connected plane quadrangulation is a bipartite plane graph with minimum degree 3. Suzuki [25] showed that every simple optimal 1-planar graph G can be obtained from a 3-connected plane quadrangulation by adding a pair of crossing edges to each face of G. So an optimal 1-planar graph is an Eulerian graph, i.e., each vertex is of even degree. It was shown in [21] that every optimal 1-planar graph G can be edge-partitioned into two planar graphs G_1 and G_2 such that $\Delta(G_2) \le 4$.

Theorem 21 *Every optimal 1-planar graph G has $\chi'_s(G) \le 10\Delta + 14$.*

Proof: Let G be an optimal 1-planar graph. Let H_1 and H_2 denote the graphs induced by non-crossing edges and crossing edges of G, respectively. Then $G = H_1 \cup H_2$, where H_1 is a bipartite plane graph. For each vertex $v \in V(G)$, it is easy to see that $d_{H_1}(v) = d_{H_2}(v) = \frac{1}{2}d_G(v)$; in particular, we have $\Delta(H_1) = \Delta(H_2) = \frac{\Delta}{2}$.

Since H_1 is bipartite, $\chi'(H_1) = \Delta(H_1) = \frac{\Delta}{2}$ by Theorem 4. By Theorem 3, $\chi'(H_2) \le \Delta(H_2) + 1 = \frac{\Delta}{2} + 1$. By Theorem 19, $\chi'_s(G) \le 6\chi'(H_1) + 14\chi'(H_2) \le 6 \times \frac{\Delta}{2} + 14 \times (\frac{\Delta}{2} + 1) = 10\Delta + 14$. □

5 Concluding remarks

In this paper, we show that the strong chromatic index of every 1-planar graph is at most 14Δ. As for the lower bound of strong chromatic index, Bensmail et al. [3] showed that for each $\Delta \geq 5$, there exist 1-planar graphs with strong chromatic index $6\Delta - 12$. Based on these facts, we put forward the following:

Question 1. *What is the least constant* c_1 *such that every* 1*-planar graph* G *satisfies* $\chi'_s(G) \leq c_1\Delta$ *?*

The foregoing discussion asserts that $6 \leq c_1 \leq 14$. We think that it is very difficult to reduce further the value of c_1 by employing the method used in this paper.

This paper also involves the strong edge coloring of some special 1-planar graphs such as IC-planar graphs and optimal 1-planar graphs. In particular, we show that the strong chromatic index of every IC-planar graph is at most $6\Delta + 20$. For $\Delta \geq 4$, by attaching $\Delta - 4$ new pendant vertices to each vertex of the complete graph K_5, we get a graph H_Δ. Since K_5 is an IC-planar graph, so is H_Δ. It is easy to inspect that any two edges of H_Δ lie in a path of length 2 or 3. So it follows that $\chi'_s(H_\Delta) = |E(H_\Delta)| = 10 + 5(\Delta - 4) = 5\Delta - 10$.

Question 2. *What is the least constant* c_2 *such that every IC-planar graph* G *satisfies* $\chi'_s(G) \leq c_2\Delta$ *?*

Notice that $5 \leq c_2 \leq 6$. We conjecture that $c_2 = 5$.

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