Improved Product Structure for Graphs on Surfaces*

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Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [J. ACM 2020] proved that for every graph G with Euler genus g there is a graph H with treewidth at most 4 and a path P such that $G \subseteq H \boxtimes P \boxtimes K_{\max\{2g,3\}}$. We improve this result by replacing "4" by "3" and with H planar. We in fact prove a more general result in terms of so-called framed graphs. This implies that every (g, d)-map graph is contained in $H \boxtimes P \boxtimes K_{\ell}$, for some planar graph H with treewidth 3, where $\ell = \max\{2g\lfloor \frac{d}{2} \rfloor, d+3\lfloor \frac{d}{2} \rfloor -3\}$. It also implies that every (g, 1)-planar graph (that is, graphs that can be drawn in a surface of Euler genus g with at most one crossing per edge) is contained in $H \boxtimes P \boxtimes K_{\max\{4g,7\}}$, for some planar graph H with treewidth 3.

Keywords: product structure, graphs on surfaces

The motivation for this work is the following question: what is the global structure for graphs embeddable in a fixed surface? Dujmović et al. (2020b) answered this question for planar graphs⁽ⁱ⁾ in terms of products⁽ⁱⁱ⁾ of graphs of bounded treewidth⁽ⁱⁱⁱ⁾ (^{iv)}.

Theorem 1 ((Dujmović et al., 2020b)). Every planar graph is contained in $H \boxtimes P \boxtimes K_3$ for some planar graph H with treewidth 3 and for some path P.

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⁽i) A plane graph is a graph embedded in the plane with no crossings. A plane triangulation is a plane graph in which every face is bounded by a triangle (that is, has length 3). A plane near-triangulation is a plane graph, where the outer-face is a cycle, and every internal face is a triangle.

⁽ⁱⁱ⁾ For two graphs G and H, the *strong product* $G \boxtimes H$ is the graph with vertex-set $V(G) \times V(H)$ and an edge between two vertices (v, w) and (v', w') if and only if v = v' and $ww' \in E(H)$, or w = w' and $vv' \in E(G)$, or $vv' \in E(G)$ and $ww' \in E(H)$.

⁽iii) A *tree-decomposition* of a graph G is a collection $(B_x \subseteq V(G) : x \in V(T))$ of subsets of V(G) (called *bags*) indexed by the nodes of a tree T, such that (a) for every edge $uv \in E(G)$, some bag B_x contains both u and v, and (b) for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty subtree of T. The *width* of a tree-decomposition is the size of a largest bag minus 1. The *treewidth* of a graph G, denoted by tw(G), is the minimum width of a tree-decomposition of G. Treewidth is recognised as the most important measure of how similar a given graph is to a tree.

^(iv) A graph G is *contained* in a graph X if G is isomorphic to a subgraph of X. A multigraph G is *contained* in a graph X if the simple graph underlying G is contained in X.

This result, now known as the Planar Graph Product Structure Theorem, has been the key tool in solving several open problems regarding queue layouts (Dujmović et al., 2020b), non-repetitive colourings (Dujmović et al., 2020a), centred colourings (Dębski et al., 2021), clustered colourings (Dujmović et al., 2022), adjacency labellings (Bonamy et al., 2020; Dujmović et al., 2021; Esperet et al., 2020), vertex rankings (Bose et al., 2020), twin-width (Bonnet et al., 2022; Bekos et al., 2022) and infinite graphs (Huynh et al., 2021).

Dujmović et al. (2020b) generalised Theorem 1 for graphs embeddable in any fixed surface^(v) as follows. A graph H is *apex* if H - v is planar for some vertex v of H.

Theorem 2 ((Dujmović et al., 2020b)). Every graph with Euler genus g is contained in $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ for some apex graph H with treewidth 4 and for some path P.

This paper improves this bound on the treewidth of H from 4 to 3.

Theorem 3. Every graph with Euler genus g is contained in $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ for some planar graph H with treewidth 3 and for some path P.

The bound on the treewidth of H in Theorem 3 is optimal since Dujmović et al. (2020b) showed that for every integer $\ell \ge 0$ there is a planar graph G such that if G is contained in $H \boxtimes P \boxtimes K_{\ell}$, then H has treewidth at least 3.

We in fact prove a more general result in terms of so-called framed graphs. Let G be a multigraph embedded in a surface Σ without crossings, where each face is bounded by a cycle. For any integer $d \ge 3$, let $G^{(d)}$ be the multigraph embedded in Σ obtained from G as follows: for each face F of G bounded by a cycle C of length at most d, for all distinct non-adjacent vertices v, w in C, add an edge vw across F to $G^{(d)}$. We say that $G^{(d)}$ is a (Σ, d) -framed multigraph with frame G. If Σ has Euler genus at most g, then $G^{(d)}$ is a (g, d)-framed multigraph.

We prove the following theorem.

Theorem 4. For all integers $g \ge 0$ and $d \ge 3$, every (g, d)-framed multigraph is contained in $H \boxtimes P \boxtimes K_{\ell}$ for some planar graph H with treewidth 3 and for some path P, where $\ell = \max\{2g\lfloor \frac{d}{2} \rfloor, d+3\lfloor \frac{d}{2} \rfloor - 3\}$.

Framed graphs (for g = 0) were introduced by Bekos et al. (2020) and are useful because they include several interesting graph classes, as shown by the following three examples.

First, every graph with Euler genus g is a subgraph of a (g, 3)-framed multigraph. Thus Theorem 4 with d = 3 implies Theorem 3.

Now consider map graphs. Start with a graph G embedded in a surface Σ without crossings, with each face labelled a 'nation' or a 'lake', where each vertex of G is incident with at most d nations. Let M be the graph whose vertices are the nations of G, where two vertices are adjacent in G if the corresponding faces in G share a vertex. Then M is called a (Σ, d) -map graph. If Σ has Euler genus at most g, then M is called a (g, d)-map graph. Graphs embeddable in Σ are precisely the $(\Sigma, 3)$ -map graphs (Dujmović et al., 2017). So map graphs are a natural generalisation of graphs embeddable in surfaces.

We show that every (Σ, d) -map graph is a spanning subgraph of $G^{(d)}$ for some multigraph G embedded in Σ without crossings; see Lemma 11. Thus Theorem 4 implies that (g, d)-map graphs have the following product structure.

2

⁽v) The *Euler genus* of a surface with *h* handles and *c* cross-caps is 2h + c. The *Euler genus* of a graph *G* is the minimum integer $g \ge 0$ such that there is an embedding of *G* in a surface of Euler genus *g*; see Mohar and Thomassen (2001) for more about graph embeddings in surfaces. A *triangulation* of a surface Σ is a graph embedded in Σ with no crossings, such that every face is a triangle.

Theorem 5. Every (g, d)-map graph is contained in $H \boxtimes P \boxtimes K_{\ell}$ for some planar graph H with treewidth 3 and for some path P, where $\ell = \max\{2g\lfloor \frac{d}{2} \rfloor, d+3\lfloor \frac{d}{2} \rfloor - 3\}$.

A graph is *k*-planar if it has an embedding in the plane where each edge is involved in at most k crossings. This definition has a natural extension for other surfaces Σ . A graph is (Σ, k) -planar if it has an embedding in Σ where each edge is involved in at most k crossings. A graph is (g, k)-planar if it is (Σ, k) -planar for some surface Σ with Euler genus at most g. In the planar setting (g = 0), these graphs have been extensively studied; see Kobourov et al. (2017); Didimo et al. (2019) for surveys.

We show that every $(\Sigma, 1)$ -planar graph is contained in $G^{(4)}$ for some multigraph G embedded in Σ without crossings; see Lemma 12. Thus Theorem 4 implies the following product structure theorem.

Theorem 6. Every (g, 1)-planar graph is contained in $H \boxtimes P \boxtimes K_{\max\{4g,7\}}$ for some planar graph H with treewidth 3 and for some path P.

Dujmović et al. (2019) proved that every (g, k)-planar graph is contained in $H \boxtimes P \boxtimes K_{\ell}$, for some graph H with treewidth $\binom{k+4}{3} - 1$ where $\ell = \max\{2g, 3\}(6k^2 + 16k + 10)$. Hickingbotham and Wood (2021) improved ℓ to $2 \max\{2g, 3\}(k+1)^2$. In the k = 1 case, Theorem 6 is significantly stronger than both these results since H has treewidth 3 instead of treewidth 9. As mentioned above, treewidth 3 is best possible, even for planar graphs (Dujmović et al., 2020b). Note that Dujmović et al. (2019) previously proved Theorem 6 in the planar case (g = 0), and a similar result was independently obtained by Bekos et al. (2022).

1 Proofs

Undefined terms and notation can be found in Diestel's text (Diestel, 2018). A *partition* of a graph G is a set \mathcal{P} of non-empty sets of vertices in G such that each vertex of G is in exactly one element of \mathcal{P} . Each element of \mathcal{P} is called a *part*. The *quotient* of \mathcal{P} is the graph, denoted by G/\mathcal{P} , with vertex set \mathcal{P} where distinct parts $A, B \in \mathcal{P}$ are adjacent in G/\mathcal{P} if and only if some vertex in A is adjacent in G to some vertex in B. An *H*-partition of G is a partition $\mathcal{P} = (A_x : x \in V(H))$ where $H \cong G/\mathcal{P}$. For simplicity, we sometimes abuse notation and say $J \in \mathcal{P}$ where J is a subgraph of G with $V(J) \in \mathcal{P}$.

If T is a tree rooted at a vertex r, then a non-empty path P in T is vertical if the vertex of P closest to r in T is an end-vertex of P. If T is a rooted spanning tree in a graph G, then a tripod in G (with respect to T) consists of up to three pairwise vertex-disjoint vertical paths in T whose lower end-vertices form a clique in G.

A *layering* of a graph G is an ordered partition $\mathcal{L} := (L_0, L_1, ...)$ of V(G) such that for every edge $vw \in E(G)$, if $v \in L_i$ and $w \in L_j$, then $|i - j| \leq 1$. A *layered partition* $(\mathcal{P}, \mathcal{L})$ of a graph G consists of a partition \mathcal{P} and a layering \mathcal{L} of G. If $\mathcal{P} = (A_x : x \in V(H))$ is an H-partition, then $(\mathcal{P}, \mathcal{L})$ is a *layered H-partition* with *width* max{ $|A_x \cap L| : x \in V(H), L \in \mathcal{L}$ }. Layered partitions were introduced by Dujmović et al. (2020b) who observed the following connection to strong products (which follows directly from the definitions).

Observation 7 (Dujmović et al. (2020b)). For all graphs G and H, G is contained in $H \boxtimes P \boxtimes K_{\ell}$ for some path P if and only if G has a layered H-partition $(\mathcal{P}, \mathcal{L})$ with width at most ℓ .

We need the following lemma of Dujmović et al. (2019), which is a special case of their Lemma 24 (which is an extension of Lemma 17 from (Dujmović et al., 2020b)).

Lemma 8 ((Dujmović et al., 2019)). Let G^+ be a plane multigraph in which each face of G^+ is bounded by a cycle with length in $\{3, \ldots, d\}$. Let T be a spanning tree of G^+ rooted at some vertex r on the boundary of the outer-face of G^+ . Assume there is a vertical path P in T with end-vertices p_1 and p_2 such that the cycle C obtained from P by adding the edge p_1p_2 is a subgraph of $G^+ - r$. Let G be the plane graph consisting of all the vertices and edges of G^+ contained in C and the interior of C. Then $G^{(d)}$ has an H-partition P such that $P \in P$ and each part $S_i \in P \setminus \{P\}$ has a partition $\{X_i, Y_i\}$ where $|X_i| \leq d - 3$ and Y_i is the union of at most three vertical paths in T, and H is planar with treewidth at most 3.

The next lemma is the heart of our proof.

Lemma 9. Let G be a connected multigraph embedded in a surface of Euler genus g without crossings, where each face of G is bounded by a cycle. Then for every spanning tree T of G and every integer $d \ge 3$, $G^{(d)}$ has an H-partition \mathcal{P} such that one part $Z \in \mathcal{P}$ is the union of at most 2g vertical paths in T and each part $S_i \in \mathcal{P} \setminus \{Z\}$ has a partition $\{X_i, Y_i\}$ where $|X_i| \le d - 3$ and Y_i is the union of at most three vertical paths in T, and H is planar with treewidth at most 3.

Proof: We start by following the proof of (Dujmović et al., 2020b, Lemma 21), which is the heart of the proof of Theorem 2. Near the end of our proof we follow a different strategy to obtain the stronger result.

If g = 0, then the claim follows from Lemma 8 by considering an appropriate supergraph G^+ of G. Now assume that $g \ge 1$. Say G has n vertices, m edges, and f faces. By Euler's formula, n - m + f = 2 - g. Let D be the multigraph with vertex-set the set of faces in G, where for each edge e of $E(G) \setminus E(T)$, if f_1 and f_2 are the faces of G with e on their boundary, then there is an edge joining f_1 and f_2 in D. (Think of D as the spanning subgraph of the dual graph consisting of those edges that do not cross edges in T.) Note that |V(D)| = f = 2 - g - n + m and |E(D)| = m - (n - 1) = |V(D)| - 1 + g. Since T is a tree, D is connected; see (Dujmović et al., 2017, Lemma 11) for a proof. Let T^* be a spanning tree of D. Thus $|E(D) \setminus E(T^*)| = g$. Let $Q = \{a_1b_1, a_2b_2, \ldots, a_gb_g\}$ be the set of edges in G dual to the edges in $E(D) \setminus E(T^*)$. Let r be the root of T, and for $i \in \{1, 2, \ldots, g\}$, let Z_i be the union of the a_ir -path and the b_ir -path in T, plus the edge a_ib_i . Let $Z := Z_1 \cup Z_2 \cup \cdots \cup Z_g$. By construction, Z is a connected subgraph of G; see Figure 1 for an example. In fact, since r is contained in each of the 2g vertical paths, T[V(Z)] is connected. Say Z has p vertices and q edges. Since Z consists of a subtree of T plus the g edges in Q, we have q = p - 1 + g.

We now describe how to 'cut' along the edges of Z to obtain a new embedded graph \tilde{G} ; see Figure 2. First, each edge e of Z is replaced by two edges e' and e'' in \tilde{G} . Each vertex of G that is not contained in V(Z) is untouched. Consider a vertex $v \in V(Z)$ incident with edges e_1, e_2, \ldots, e_d in Z in clockwise order. In \tilde{G} replace v by new vertices v_1, v_2, \ldots, v_d , where v_i is incident with e'_i, e''_{i+1} and all the edges incident with v clockwise from e_i to e_{i+1} (exclusive). Here e_{d+1} means e_1 and e''_{d+1} means e''_1 . This operation defines a cyclic ordering of the edges in \tilde{G} incident with each vertex (where e''_{i+1} is followed by e'_i in the cyclic order at v_i). This in turn defines an embedding of \tilde{G} in some orientable surface^(vi). Let Z' be the set of vertices introduced in \tilde{G} by cutting through vertices in Z.

We now show that \hat{G} is connected. Consider vertices x_1 and x_2 of \tilde{G} . Select faces f_1 and f_2 of \tilde{G} respectively incident to x_1 and x_2 that are also faces of G. Let P be a path joining f_1 and f_2 in the dual tree T^* . Then the edges of G dual to the edges in P were not split in the construction of \tilde{G} . Therefore an x_1x_2 -walk in \tilde{G} can be obtained by following the boundaries of the faces corresponding to vertices in P. Hence \tilde{G} is connected.

^(vi) If G is embedded in a non-orientable surface, then the edge signatures for G are ignored in the embedding of \tilde{G} .



Fig. 1: Example of the construction in the proof of Lemma 9, where brown edges are in T, red edges are in Q, and blue edges are in T and in Z - E(Q).

Say \tilde{G} has n' vertices and m' edges, and the embedding of \tilde{G} has f' faces and Euler genus g'. Each vertex with degree d in Z is replaced by d vertices in \tilde{G} . Each edge in Z is replaced by two edges in \tilde{G} , while each edge of E(G) - E(Z) is maintained in \tilde{G} . Thus

$$n' = n - p + \sum_{v \in V(Z)} \deg_Z(v) = n + 2q - p = n + 2(p - 1 + g) - p = n + p - 2 + 2g$$

and m' = m + q = m + p - 1 + g. Each face of G is preserved in \tilde{G} . Say s new faces are created by the cutting. Thus f' = f + s. Since \tilde{G} is connected, n' - m' + f' = 2 - g' by Euler's formula. Thus (n + p - 2 + 2g) - (m + p - 1 + g) + (f + s) = 2 - g', implying (n - m + f) - 1 + g + s = 2 - g'. Hence (2 - g) - 1 + g + s = 2 - g', implying g' = 1 - s. Since $g' \ge 0$, we have $s \le 1$. Since $g \ge 1$, by construction, $s \ge 1$. Thus s = 1 and g' = 0. Hence \tilde{G} is plane and all the vertices in Z' are on the boundary of a single face, F, of \tilde{G} . Moreover, the boundary of F is a cycle C_F and $V(C_F) = Z'$. Consider F to be the outer-face of \tilde{G} .

Now construct a supergraph G^+ of \tilde{G} by adding a vertex r^+ in F and edges from r^+ to each vertex in Z'. Then G^+ is a plane multigraph where each face of G^+ is bounded by a cycle.

We now depart from the proof of Dujmović et al. (2020b, Lemma 21). Let P^+ be an arbitrary path such that $V(P^+) = V(C_F)$ and let $v^+ \in V(P^+)$ be an end-vertex of P^+ . Let T^+ be the following spanning tree of G^+ rooted at r^+ . Initialise T^+ to be the path P^+ plus the edge r^+v^+ . Let $E' := \{vw \in E(T) : v \in Z, w \in V(G) \setminus V(Z)\}$ and h := |E'|. Observe that T - V(Z) is a forest with h components. For each edge $vw \in E'$, w is adjacent to exactly one vertex $v_i \in V(Z')$ introduced when cutting v. Add the edge $v_i w$ to T^+ . Finally, add the induced forest T - V(Z) to T^+ ; see Figure 3. Then T^+ is connected since each component of T - V(Z) is adjacent in T^+ to some vertex in $V(P^+)$. Furthermore, since $|V(T^+)| = |V(P^+)| + |V(G) \setminus V(Z)|$ and $|E(T^+)| = |E(P^+)| + h + (|V(G) \setminus V(Z)| - h) =$ $|V(P^+)| + |V(G) \setminus V(Z)| - 1$, it follows that T^+ is indeed a spanning tree of G^+ . Consider each component of T - V(Z) to be a subtree of T^+ .



Fig. 2: Cutting the blue edges in Z at each vertex.

Now every vertical path in T^+ contained in $V(G) \setminus V(Z)$ corresponds to a vertical path in T. Every maximal vertical path in T^+ consists of the edge r^+v^+ , a subpath of P^+ , some edge v_iw (where $w \in V(G) \setminus V(Z)$), followed by a path in T - V(Z) from w to a leaf in T. Since every vertical path P in T^+ is contained in some maximal vertical path in T^+ , it follows that $P \cap (V(G) \setminus V(Z))$ is a vertical path in T. Thus every vertical path in T^+ that is contained in $V(G) \setminus V(Z)$ is a vertical path in T.

Triangulate every face in G^+ whose facial cycle has length greater than d. Since r^+ is on the boundary of the outer-face of G^+ , $V(P^+) = V(C_F)$, every facial cycle has length in $\{3, \ldots, d\}$ and P^+ is a vertical



Fig. 3: Example of the spanning tree T^+ in the graph G^+ , where the edges in $E(P^+) \cup \{r^+v^+\}$ are red and the edges that are either in E(T - V(Z)) or of the form $v_i w$ are orange.

path of T^+ , Lemma 8 is applicable. Let \mathcal{P}' be the H-partition of $\tilde{G}^{(d)}$ given by Lemma 8. Therefore, H is planar with treewidth at most 3, where $P^+ \in \mathcal{P}'$ and each part in $S_i \in \mathcal{P}' \setminus \{P^+\}$ has a partition $\{X_i, Y_i\}$ where $|X_i| \leq d-3$ and Y_i is the union of at most three vertical paths in T'. Let \mathcal{P} be the partition of $G^{(d)}$ obtained by replacing P^+ by Z. Since $V(P^+) = V(Z')$ and all the split vertices of G are in Z, we have $G^{(d)}/\mathcal{P} \cong \tilde{G}^{(d)}/\mathcal{P}' \cong H$. Hence \mathcal{P} is also an H-partition where H is planar with treewidth at most 3. In addition, since each vertical path in T^+ that is disjoint from $V(Z') \cup \{r^+\}$ is a vertical path in T, each part $S_i \in \mathcal{P} \setminus \{Z\}$ has a partition $\{X_i, Y_i\}$ where $|X_i| \leq d-3$ and Y_i is the union of at most three vertical paths in T, as required.

Theorem 4 is an immediate consequence of Observation 7 and the next lemma.

Lemma 10. Let G be a multigraph embedded in a surface of Euler genus g without crossings, where each face is bounded by a cycle. Then $G^{(d)}$ has a layered H-partition $(\mathcal{P}, \mathcal{L})$ with width at most $\max\{2g\lfloor \frac{d}{2} \rfloor, d+3\lfloor \frac{d}{2} \rfloor -3\}$, such that H is planar with treewidth at most 3.

Proof: Since each face of G is bounded by a cycle, G is connected. Let T be a BFS-spanning tree of G with corresponding BFS-layering^(vii) (V_0, V_1, \ldots) . By Lemma 9, $G^{(d)}$ has an H-partition \mathcal{P} such that one part $Z \in \mathcal{P}$ is the union of at most 2g vertical paths in T and each part $S_i \in \mathcal{P} \setminus \{Z\}$ has a partition $\{X_i, Y_i\}$ where $|X_i| \leq d-3$ and Y_i is the union of at most three vertical paths in T, and H is planar with treewidth at most 3. It remains to adjust the layering of G to obtain a layering of $G^{(d)}$. If $uv \in E(G^{(d)})$ then $dist_G(u, v) \leq \lfloor \frac{d}{2} \rfloor$. Thus if $u \in V_i$ and $v \in V_j$ then $|i - j| \leq \lfloor \frac{d}{2} \rfloor$. For each $j \in \mathbb{N}$, let $L_j = V_j \lfloor \frac{d}{2} \rfloor \cup \cdots \cup V_{(j+1) \lfloor \frac{d}{2} \rfloor - 1}$. Then $(\mathcal{P}, \mathcal{L} = (L_0, L_1, \ldots))$ is a layered H-partition of $G^{(d)}$ with width at most $\max\{2g \lfloor \frac{d}{2} \rfloor, d + 3 \lfloor \frac{d}{2} \rfloor - 3\}$, as required.

We conclude by showing that (Σ, d) -map graphs and $(\Sigma, 1)$ -planar graphs are contained in framed graphs.

^(vii) If G is a connected graph and T is a spanning tree of G rooted at vertex r, then T is BFS if $dist_T(v, r) = dist_G(v, r)$ for every $v \in V(G)$. A layering $(L_0, L_1, ...)$ of a graph G is BFS if $L_0 = \{r\}$ for some root vertex $r \in V(G)$ and $L_i = \{v \in V(G) : dist_G(v, r) = i\}$ for all $i \ge 1$.

Dujmović et al. (2019) proved the following result in the case of plane map graphs (and similar results were previously known in the literature (Chen et al., 2006; Brandenburg, 2019, 2020)). An analogous proof works for arbitrary surfaces, which we include for completeness. Together with Theorem 4, this implies Theorem 5.

Lemma 11. For every surface Σ and integer $d \ge 3$, every (Σ, d) -map graph is a subgraph of $G^{(d)}$ for some multigraph G embedded in Σ without crossings, where each face of G is bounded by a cycle.

Proof: Let G_0 be a graph embedded in Σ , with each face labelled a nation or a lake, and where each vertex of G_0 is incident with at most d nations. Let M be the corresponding map graph.

If G_0 has a face F of length 2, then add a new vertex inside F adjacent to both vertices on the boundary of F, which creates two new triangular faces F_1 and F_2 . If F is a lake, then make F_1 and F_2 lakes. If Fis a nation, then make F_1 a nation and make F_2 a lake. The resulting map graph is still M. So we may assume that G_0 is an edge-maximal multigraph embedded in Σ with no face of length 2 (and with each face labelled a nation or a lake), such that M is the corresponding map graph. This is well-defined since the assumption of having no face of length 2 implies that $|E(G_0)| \leq 3(|V(G)| + g - 2)$, where g is the Euler genus of Σ .

Suppose that some face f of G_0 has a disconnected boundary. Let v and w be vertices in distinct components of the boundary of f. Add the edge vw to G_0 across f. The corresponding map graph is unchanged, which contradicts the edge-maximality of G_0 . Thus each face of G_0 has a connected boundary. Suppose that some face f of G_0 has a repeated vertex v in the boundary walk of f. Let u, v, w be consecutive vertices on the boundary of f. So u, v, w are distinct. Add the edge uw inside f so that uvw bounds a disk. Label the resulting face uvw as a lake. Since v appears elsewhere in the boundary of f, the corresponding map graph is unchanged, which contradicts the edge-maximality of G_0 . Thus no facial walk of G_0 has a repeated vertex. Since each facial walk is connected, every face of G_0 is bounded by a cycle.

Let G_0^* be the dual multigraph of G_0 . So the vertices of G_0^* correspond to faces of G_0 , and each vertex of G_0^* is a nation vertex or a lake vertex. Since every face of G_0 is bounded by a cycle, every face of G_0^* is bounded by a cycle.

Let x be a vertex of G_0 , let F_x be the corresponding face of G_0^* , and let (v_1, \ldots, v_s) be the facial cycle of F_x . Let $C_x := (w_1, \ldots, w_r)$ be the circular subsequence of (v_1, \ldots, v_s) consisting of only the nation vertices. Since x is incident to at most d nations, $r \leq d$.

Let G be the supergraph of G_0^* obtained by adding an edge between each pair of consecutive vertices in $C_x = (w_1, \ldots, w_r)$ for each vertex x of G_0 . The graph consisting of C_x plus these added edges is called the *nation cycle* (of x). Note that if r = 1 then the nation cycle has no edges, and if r = 2 then the nation cycle has one edge. Since every face of G_0^* is bounded by a cycle, every face of G is bounded by a cycle. Moreover, each nation cycle of length at least 3 is now a facial cycle of G with length at most d. By construction, G embeds in Σ with no crossings. Let $G^{(d)}$ be the d-framed graph whose frame is G.

By definition, $V(M) \subseteq V(G^{(d)})$. To prove the claim, it suffices to show that $E(M) \subseteq E(G^{(d)})$. Indeed, if $vw \in E(M)$ then the nation faces corresponding to v and w have a common vertex x on their boundary. The vertex x corresponds to a face F_x in G_0^* and the facial cycle of F_x contains v and w. Therefore, the nation cycle C_x of F_x contains v and w. If C_x has length 2 then $vw \in E(G) \subseteq E(G^{(d)})$. If C_x has length at least 3 then it has length at most d and it bounds a face in G. So $vw \in E(G^{(d)})$.

Dujmović et al. (2019) proved the following result in the case of 1-planar graphs (and similar results were previously known in the literature (Chen et al., 2006; Bekos et al., 2020; Brandenburg, 2019, 2020)).

An analogous proof works for arbitrary surfaces, which we include for completeness. Together with Theorem 4, this implies Theorem 6.

Lemma 12. Every $(\Sigma, 1)$ -planar graph G with at least three vertices is contained in $G_0^{(4)}$ for some multigraph G_0 embedded in Σ with no crossings where each face of G_0 is bounded by a cycle.

Proof: We may assume that G is embedded in Σ with at most one crossing on each edge, such that no two edges of G incident to a common vertex cross, since such a crossing can be removed by a local modification to obtain an embedding of G in which the two edges do not cross.

Initialise G' := G. Add edges to G' to obtain an edge-maximal multigraph embedded in Σ such that each edge is in at most one crossing, no two edges incident to a common vertex cross, and no face is bounded by two parallel edges. The final condition ensures that G' is well-defined, since it follows from Euler's formula that if G has k crossings, then $|E(G')| \leq 3(|V(G)| + k + g - 2) - 2k$.

Consider crossing edges $e_1 = vw$ and $e_2 = xy$ in G'. So v, w, x, y are distinct. Since e_1 is the only edge that crosses e_2 and e_2 is the only edge that crosses e_1 , by the edge-maximality of G', there is a cycle C = (v, x, w, y) in G' that bounds a disc whose interior intersects no edge of G' except e_1 and e_2 .

Let G_0 be the embedded multigraph obtained from G' by deleting each pair of crossing edges. Thus the above-defined cycle C bounds a face of G_0 . By the edge-maximality of G', every other face of G_0 (that is, not arising from a pair of deleted crossing edges) is a triangular face of G'. Thus, G_0 is a multigraph embedded in Σ with no crossings, such that each face of G_0 is bounded by a 3-cycle or a 4-cycle, and G is contained in $G_0^{(4)}$.

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