

The number of $\{1243, 2134\}$ -avoiding permutations

David Callan

Department of Statistics, University of Wisconsin-Madison, USA

received 15th Mar. 2019, revised 25th July 2022, 18th Jan. 2023, accepted 6th Mar. 2023.

We show that the counting sequence for permutations avoiding both of the (classical) patterns 1243 and 2134 has the algebraic generating function supplied by Vaclav Kotesovec for sequence A164651 in The On-Line Encyclopedia of Integer Sequences.

Keywords: pattern avoidance, permutation diagram, start-small

1 Introduction

Several authors have developed methods to count permutations avoiding a given set of patterns; see, for example, the references in the Wikipedia entry [1]. In particular, Vatter [2] and Pudwell [3] have devised enumeration schemes for automated counting. When successful, an automated method produces an enumeration scheme that yields the initial terms of the counting sequence (perhaps 20 or more) and sometimes these terms appear to have an algebraic generating function that does not follow readily from the enumeration scheme. Here, we treat one such case. The counting sequence for permutations that avoid both of the (classical) patterns 1243 and 2134 begins 1, 2, 6, 22, 87, 354, . . . , sequence A164651 in The On-Line Encyclopedia of Integer Sequences [4]. In a comment on this sequence dated Oct 24 2012, Vaclav Kotesovec observed that the generating function

$$\frac{3x^2 - 9x + 2 + x(1-x)\sqrt{1-4x}}{2(x-1)(x^2+4x-1)}$$

fits the known terms of the sequence.

We show that $\{1243, 2134\}$ -avoiders do indeed have this generating function. Defining a *start-small* permutation to be one that does not start with its largest entry, the proof rests on a bijection ϕ from start-small $\{1243, 2134\}$ -avoiders of length n to lists of start-small 123-avoiders whose total length is $n - 1$

plus the length of the list. Note that a start-small permutation has length at least 2. The Catalan numbers count 123-avoiders and the combinatorial interpretation of the Invert transform permits counting these lists of start-small 123-avoiders. The generating function for start-small $\{1243, 2134\}$ -avoiders thus obtained readily yields the generating function for unrestricted $\{1243, 2134\}$ -avoiders.

In Section 2, we show that ϕ is given by iteration of a more basic bijection ψ , which is presented in Section 3. Finally, Section 4 gives the bookkeeping details to obtain the desired generating function from ϕ .

2 The bijection ϕ

A *mid-123* entry in a permutation is an entry that serves as the “2” in a 123 pattern. A *key* mid-123 entry is a mid-123 entry b whose immediate predecessor is either less than b or a right-to-left maximum (max for short). For example, the mid-123 entries in 1 6 3 2 4 5 are 3, 2, 4 but only 3 and 4 are key.

Lemma 1. *A permutation with no key mid-123 entries has no mid-123 entries at all and so is a 123-avoider.*

Proof: We suppose that π has no key mid-123 entry and π_j is the first mid-123 entry, and deduce a contradiction. Say $\pi_i\pi_j$ begins a 123 pattern. Since π_j is not key, we have (i) $\pi_{j-1} > \pi_j$ and (ii) π_{j-1} is not a right-to-left max. Now (i) implies $i \neq j - 1$ and (ii) implies $\pi_\ell > \pi_{j-1}$ for some $\ell > j - 1$. But then $\pi_i\pi_{j-1}\pi_\ell$ is a 123, contradicting the minimality of j . \square

Lemma 2. *Suppose b is the last mid-123 entry in a $\{1243, 2134\}$ -avoider. Then there is only one entry that occurs after b in π and is larger than b .*

Proof: Suppose abc and abc' are 123 patterns with $c \neq c'$, say $c < c'$. If c precedes c' in the permutation, then c is a mid-123, violating the hypothesis on b . If c follows c' , then $abc'c$ is an instance of a forbidden 1243 pattern. \square

Let \mathcal{A}_n denote the set of start-small $\{1243, 2134\}$ -avoiders on $[n]$, and $\mathcal{A}_{n,k}$ the subset with k key mid-123 entries. To produce the promised bijection ϕ from \mathcal{A}_n to lists of start-small 123-avoiders whose total length is $n - 1$ plus the length of the list, it suffices to exhibit, for $0 \leq k \leq n - 2$, a bijection

$$\mathcal{A}_{n,k} \rightarrow \{(\sigma_1, \dots, \sigma_{k+1}) : \text{each } \sigma_i \text{ is a start-small 123-avoider,} \\ \text{lengths of the } \sigma_i\text{'s sum to } n + k\}.$$

For $k = 0$, naturally the bijection is $\pi \rightarrow (\pi)$, a singleton list, because, by Lemma 1, π is already 123-avoiding. For $1 \leq k \leq n - 2$ and $k + 1 \leq j \leq n - 1$ let

$$\mathcal{A}_{n,k,j} = \{\pi \in \mathcal{A}_{n,k} : \text{the last mid-123 entry of } \pi \text{ is in position } j\}.$$

In the next section, for $1 \leq k < j < n$, we present a bijection ψ between $\mathcal{A}_{n,k,j}$ and $\mathcal{A}_{j,k-1} \times \mathcal{A}_{n+1-j,0}$. Iteration of ψ then gives the required bijection ϕ .

3 The bijection ψ

To *standardize* a list τ of distinct integers means to replace the smallest by 1, the next smallest by 2, and so on, and the result is denoted $\text{St}(\tau)$. We define ψ as follows: given $\pi \in \mathcal{A}_{n,k,j}$, we need to reversibly produce $\sigma_1 \in \mathcal{A}_{j,k-1}$ and $\sigma_2 \in \mathcal{A}_{n+1-j,0}$. Write π as $\tau_1 b \tau_2$ where b is the last mid-123 entry in π , and let abc be the 123 pattern in π with smallest a (c is uniquely determined by Lemma 2). Concatenate a and τ_2 and standardize to get the desired $\sigma_2 = \text{St}(a\tau_2)$ with no 123's.

Concatenate τ_1 and c to get a $\{1243, 2134\}$ -avoider ρ —a candidate (after standardization) for σ_1 . This ρ may need further processing because of two glitches: ρ may still have k key mid-123's instead of the required $k-1$ and ρ cannot end with its smallest entry (which must be possible in σ_1). But these glitches cancel out.

If b is a key mid-123 in π , then ρ has $k-1$ key mid-123's because b has been lost. In this case, just standardize ρ to get σ_1 . If b is not key, the last entry of τ_1 exceeds b but is not a right-to-left max of π . Delete from ρ the longest right factor (terminal string) of τ_1 that is decreasing but does not start with a right-to-left max of π , equivalently, does not start with an entry $> c$; say $t \geq 1$ entries are deleted. Add t to each remaining entry of ρ , append the entries $t, t-1, \dots, 1$ and standardize to obtain σ_1 .

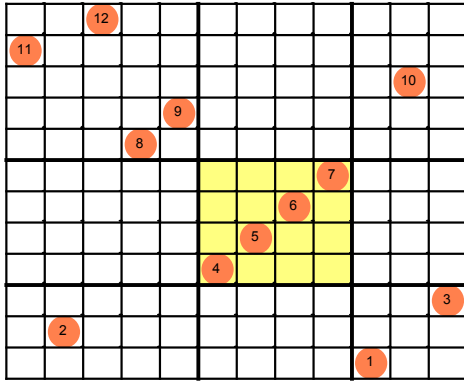
Before proving the invertibility of ψ , we give 3 examples according as

1. $\pi_j = b$ is key and $\pi_{j-1} < b$,
2. π_j is key and π_{j-1} is a right-to-left max,
3. π_j is not key.

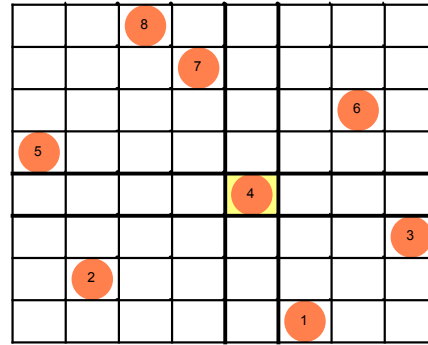
These examples are illustrated in Figure 1 below (the yellow box in the figures is explained in Figure 2 further below).

1. $\pi = 11\ 2\ 12\ 8\ 9\ 4\ 5\ 6\ 7\ 1\ 10\ 3$ decomposes as $\tau_1\ 7\ \tau_2$ with $a = 2$, $b = 7$, $c = 10$. So $\sigma_1 = \text{St}(11\ 2\ 12\ 8\ 9\ 4\ 5\ 6\ 10) = 8\ 1\ 9\ 5\ 6\ 2\ 3\ 4\ 7$ and $\sigma_2 = \text{St}(2\ 1\ 10\ 3) = 2\ 1\ 4\ 3$.
2. $\pi = 5\ 2\ 8\ 7\ 4\ 1\ 6\ 3$ decomposes as $\tau_1\ 4\ \tau_2$ with $a = 2$, $b = 4$, $c = 6$. So $\sigma_1 = \text{St}(5\ 2\ 8\ 7\ 6) = 2\ 1\ 5\ 4\ 3$ and $\sigma_2 = \text{St}(2\ 1\ 6\ 3) = 2\ 1\ 4\ 3$.
3. $\pi = 10\ 3\ 12\ 7\ 8\ 9\ 6\ 5\ 2\ 1\ 11\ 4$ decomposes as $\tau_1\ 5\ \tau_2$ with $a = 3$, $b = 5$, $c = 11$. Here, b is not key and $t = 2$, so $\sigma_1 = \text{St}((2 + (10\ 3\ 12\ 7\ 8\ 11))\ 2\ 1) = 6\ 3\ 8\ 4\ 5\ 7\ 2\ 1$ and $\sigma_2 = \text{St}(3\ 2\ 1\ 11\ 4) = 3\ 2\ 1\ 5\ 4$.

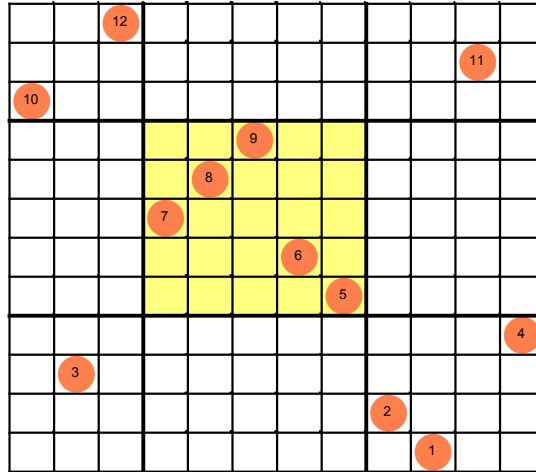
Next, we investigate the structure of start-small $\{1243, 2134\}$ -avoiders π to obtain a more explicit description of ψ . Recall $\pi_j = b$ is the last mid-123 entry in π , and abc is the 123 pattern in π with smallest a and c is uniquely determined by Lemma 2. Recall also that b is key if $\pi_{j-1} < b$ or π_{j-1} is a right-to-left max. Define i and k by $\pi_i = a$ and $\pi_k = c$.



A $\{1243, 2134\}$ -avoider π , last mid-123, $b = 7$, is key and its predecessor, 6, is smaller
 $\sigma_1 = 819562347$, $\sigma_2 = 2143$



A $\{1243, 2134\}$ -avoider π , last mid-123, $b = 4$, is key and its predecessor, 7, is a RL maximum
 $\sigma_1 = 21543$, $\sigma_2 = 2143$



A $\{1243, 2134\}$ -avoider π , last mid-123, $b = 5$, is not key
 $\sigma_1 = 63845721$, $\sigma_2 = 32154$

Fig. 1: Three examples of the action of ψ .

Lemma 3. Suppose b is key. Take ℓ minimal in $[i + 1, j]$ such that $\pi_\ell < \pi_{\ell+1} \cdots < \pi_j$ ($\ell = j$ if π_{j-1} is a right-to-left max). Then

- (i) π_ℓ, \dots, π_j are consecutive integers, and
- (ii) For $h < \ell$, $\pi_h < b$ implies $h = i$.

Proof: (i) If not, say $\pi_h < x < \pi_{h+1}$ with $\ell \leq h < j$ and consider the position of x in π . If x occurs after c , then $a\pi_h c x$ is a forbidden 1243. If x occurs between b and c , then $a x c$ is a 123 contradicting the assumption that b is the last mid-123. Otherwise, since $\pi_h, \pi_{h+1}, \dots, \pi_j = b$ are contiguous entries in π , x must occur before π_h and $x\pi_h\pi_{h+1}c$ is a forbidden 2134.

(ii) If not, by (i) there exists $h < \ell$ with $\pi_h < \pi_\ell$ and $h \neq i$. Now $\pi_{\ell-1} = a$ or $\pi_{\ell-1} > b$ by the definition of ℓ , and $\pi_h > a$ by the definition of a . In case $\pi_{\ell-1} = a$, $\pi_h a b c$ is a 2134, and in case $\pi_{\ell-1} > b$, $\pi_h a b c$ is a 2134 if π_h occurs before a and $a\pi_h\pi_{\ell-1}b$ is a 1243 if π_h occurs after a . \square

Lemma 4. *Suppose b is not key. Take $r \geq 1$ maximal such that π_{j-r} is not a right-to-left max and $\pi_{j-r} > \pi_{j-r+1} > \dots > \pi_j$. Then*

(i) $\pi_j, \pi_{j-1}, \dots, \pi_{j-r+1}$ are consecutive integers, and

(ii) the integers in the interval $\pi_{j-r+1} + 1, \dots, \pi_{j-r} - 1$ (if any) all occur immediately before π_{j-r} and in increasing order.

Proof: Take $u \geq 0$ maximal such that $\pi_j, \pi_{j-1}, \dots, \pi_{j-u}$ is an increasing sequence of consecutive integers $b, b+1, \dots, b+u$. Thus $\pi_{j-(u+1)} \neq b+u+1$. If $u \geq 1$ and either $\pi_{j-(u+1)} < \pi_{j-u}$ or $\pi_{j-(u+1)}$ is a right-to-left max, then the result holds with $r = u$ and no integers in the interval $[\pi_{j-r+1} + 1, \pi_{j-r} - 1] = [b+u, b+u-1]$ of part (ii). Otherwise, using the fact b is not key when $u = 0$, we have $\pi_{j-(u+1)} > b+u+1$, say $\pi_{j-(u+1)} = b+u+s$ with $s \geq 2$, and $b+u+s < c$ since $\pi_{j-(u+1)}$ is not a right-to-left max. Also, a occurs before $b+u+1$ for else $(b+u+1)a(b+u+s)c$ is a 2134. The integers $b+u+1, \dots, b+u+s-1$ all occur before $\pi_{j-(u+1)} = b+u+s$ by Lemma 2, and in increasing order for else $(b+u+s)c$ is the “34” of a 2134 pattern. Furthermore, $b+u+1, \dots, b+u+s-1$ all lie directly to the left of $b+u+s$ for else some $x \notin [b, b+u+s]$ lies between $b+u+1$ and $b+u+s$ in π ; if $x < b$, then $(b+u+1)x(b+u+s)c$ is a 2134 while if $x > b+u+s$, then $a(b+u+1)x(b+u+s)$ is a 1243. The result follows with $r = u+1$ and a nonempty interval in part (ii). \square

X		C
	B	
A		Z

Fig. 2: Matrix form of a $\{1243, 2134\}$ -avoider.

It follows from Lemmas 3 and 4 that the matrix diagram of π has the form shown in Figure 2 above where blank regions are empty, A contains the lone entry a , C contains the lone entry c , and, if b is key, B consists, left to right, of $s, s+1, \dots, b$, an interval of integers, for some $s \leq b$, while if b is not key, B

consists of $b+r, b+r+1, \dots, b+t, b+r-1, b+r-2, \dots, b$, for some $t \geq r$. Note that the central region B , colored yellow, is square-shaped and, since A and C each contain one entry, so are X and Z .

The construction of σ_1 and σ_2 to define ψ can now be described more explicitly: σ_2 is obtained from $a\tau_2$ by replacing c with the length of $a\tau_2$ and leaving all other entries intact, and σ_1 is obtained from τ_1c as follows.

- Case b is key. Here, B consists of $s, s+1, \dots, b$ with $2 \leq s \leq b$. Replace a with 1, subtract $s-1$ from each entry in X and from the sole entry c in C (to account for $s-1$ missing entries $\{b\} \cup (\tau_2 \setminus \{c\})$), and subtract $s-2$ from each of the entries $s, s+1, \dots, b-1$ in B (to account for $\tau_2 \setminus \{c\}$).
- Case b is not key. Here B consists of $b+r, b+r+1, \dots, b+t, b+r-1, b+r-2, \dots, b$. Replace a with $r+1$, subtract $b-1$ from each entry in X and from the sole entry c in C , subtract $b-2$ from each of the entries $b+r, b+r+1, \dots, b+t-1$ in B , and then append $r, r-1, \dots, 1$.

Proposition 5. *For $1 \leq k < j < n$, the mapping ψ is a bijection.*

Proof: In view of the preceding paragraph, to retrieve π from $\psi(\pi) = (\sigma_1, \sigma_2)$, we need to recover n, i, j, k, a, b, c , to determine whether b is key or not, and to find r, s, t in the appropriate cases. Then we need to distinguish in σ_1 the entries increased by $s-1$ from those increased by $s-2$ when b is key, and to distinguish the entries increased by $b-1$ from those increased by $b-2$ when b is not key. Clearly $n = |\sigma_1| + |\sigma_2| - 1$ and $j = |\sigma_1|$. The index i is the position of 1 in σ_1 . Also, $k = (\text{position in } \sigma_2 \text{ of its maximum entry}) + j - 1$, and a is the first entry of σ_2 . The cases (1) $\pi_j = b$ is key and $\pi_{j-1} < b$, (2) π_j is key and π_{j-1} is a right-to-left max, (3) π_j is not key, can be distinguished according as (1) σ_1 ends with an ascent, (2) σ_1 ends with a descent to a number other than 1, (3) σ_1 ends with a descent to 1. In all cases, the entries in π after b other than c are evident in σ_2 .

- Case b is key. Here, s is retrieved as the length of σ_2 and i is the position of 1 in σ_1 . Write σ_1 as $\sigma_{11}\sigma_{12}$ where σ_{12} is the longest increasing right factor of σ_1 . Then $b = s-1 + (\text{length of } \sigma_{12})$. In σ_1 , add $s-1$ to each entry of σ_{11} and to the last entry of σ_{12} and add $s-2$ to all other entries of σ_{12} to obtain τ_1c .
- Case b is not key. Here, b is retrieved as the length of σ_2 , r is the length of the longest right factor of σ_1 of the form $r(r-1)\dots 1$, and i is the position of $r+1$ in σ_1 . Write σ_1 as $\sigma'_1 r(r-1)\dots 1$ and write σ'_1 as $\sigma'_{11}\sigma'_{12}$ where σ'_{12} is the longest increasing right factor of σ'_1 . Then $t = r + |\sigma'_{12}| - 1$. In σ'_1 , add $b-1$ to each entry of σ'_{11} , add $b-2$ to all but the last entry of σ'_{12} , and append $b+t, b+r-1, b+r-2, \dots, b+1$ to get τ_1 . Add $b-1$ to the last entry of σ'_{12} to get c .

Thus ψ is invertible. □

4 Putting it all together

It is well known that 123-avoiders are counted by the Catalan numbers $C_n = \binom{2n}{n} - \binom{2n}{n-1}$ and so the number of start-small 123-avoiders of length n is $C_n - C_{n-1}$ for $n \geq 1$, with generating function $C(x) - 1 - xC(x) = x^2C(x)^3$ where $C(x) = \sum_{n \geq 0} C_n x^n$ is the generating function for the Catalan numbers. Say a k -list is a list of length k and the *weight* of a start-small 123-avoider is one less than its length.

From the bijection ϕ , the number of start-small {1243, 2134}-avoiders of length n is

$$\begin{aligned} & \sum_{k=1}^{n-1} \# k\text{-lists of start-small 123-avoiders of total length } n-1+k \\ = & \# \text{ lists (of arbitrary length) of start-small 123-avoiders of total weight } n-1. \end{aligned} \quad (1)$$

Recall that if \mathcal{A} is a class (species) of combinatorial structures with a_n structures of weight n ($n \geq 1$), and a *compositional* \mathcal{A} -structure of weight n is one obtained by taking a composition (n_1, n_2, \dots, n_k) of n and forming a k -list of \mathcal{A} -structures of respective weights n_1, n_2, \dots, n_k , then the counting sequence $(b_n)_{n \geq 1}$ for compositional \mathcal{A} -structures by total weight has generating function $B(x) := \sum_{n \geq 1} b_n x^n$ given by

$$1 + B(x) = \frac{1}{1 - A(x)},$$

where $A(x)$ is the generating function for \mathcal{A} -structures (this is the INVERT transform). Applying it to the class of nonempty start-small 123-avoiders with weight measured as “length minus 1” so that $A(x) = xC(x)^3$, we get

$$B(x) = \frac{1}{1 - xC(x)^3} - 1. \quad (2)$$

From (1) and (2), the generating function $G(x)$ for nonempty start-small {1243, 2134}-avoiders (with x marking length) is $G(x) = xB(x)$. Now let $(u_n)_{n \geq 0} = (1, 1, 2, 6, \dots)$ and $(v_n)_{n \geq 0} = (1, 0, 1, 4, \dots)$ denote the counting sequences for {1243, 2134}-avoiders and start-small {1243, 2134}-avoiders respectively. Clearly, $v_n = u_n - u_{n-1}$ for $n \geq 1$, (consider deletion of the first entry from a {1243, 2134}-avoider on $[n]$ that starts n). So the generating functions $F(x) = \sum_{n \geq 0} u_n x^n$ and $G(x) = \sum_{n \geq 1} v_n x^n$ are related by $F(x) = (1 + G(x))/(1 - x)$. Thus

$$F(x) = \frac{1 + G(x)}{1 - x} = \frac{1 + \frac{x}{1 - xC(x)^3} - x}{1 - x}$$

which, after expansion, agrees with Kotesovec’s formula.

Losonczy [5] has counted permutations that avoid 3421, 4312 and 4321 or equivalently (by reversal) both of the patterns treated here and 1234.

Acknowledgement

I thank one anonymous referee for their helpful comments and another for numerous suggestions that greatly improved the presentation.

References

- [1] Enumerations of specific permutation classes, Wikipedia.
- [2] Vincent Vatter, Enumeration schemes for restricted permutations, *Combinatorics, Probability and Computing*, **17** Issue 1 (2008), 137–159, available at <http://www.math.ufl.edu/~vatter/publications/wilfplus/>
- [3] Lara Pudwell, Enumeration Schemes for Permutations Avoiding Barred Patterns, *Electronic J. Combinatorics*, **17** Issue 1 (2010), R29.
- [4] The On-Line Encyclopedia of Integer Sequences, published electronically at <http://oeis.org>, 2022.
- [5] Jozsef Losonczy, Maximally clustered elements and Schubert varieties, *Annals of Combinatorics* **11** Issue 2 (2007), 195–212.