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The variance and the asymptotic distribution of the length of longest k-alternating subsequences

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We obtain an explicit formula for the variance of the number of k-peaks in a uniformly random permutation. This is then used to obtain an asymptotic formula for the variance of the length of longest k-alternating subsequence in random permutations. Also a central limit is proved for the latter statistic.

Keywords: Alternating subsequences, k-alternating subsequences, Peak, central limit theorem

1 Introduction

Letting $(a_i)_{i=1}^n$ be a sequence of real numbers, a subsequence a_{i_k}, where $1 \leq i_1 < \ldots < i_k \leq n$, is called an *alternating subsequence* if $a_{i_1} > a_{i_2} < a_{i_3} > \cdots$. The *length of the longest alternating subsequence of* $(a_i)_{i=1}^n$ is defined to be the largest integer q such that $(a_i)_{i=1}^n$ has an alternating subsequence of length q. Denoting the symmetric group on n letters by S_n, an alternating subsequence of a permutation $\sigma \in S_n$ refers to an alternating subsequence corresponding to the sequence $\sigma(1), \sigma(2), \ldots, \sigma(n)$. See Stanley (2008) for a survey on the topic.

The purpose of this manuscript is to study a generalization of the length of longest alternating subsequences in uniformly random permutations. Letting $\sigma \in S_n$, a subsequence $1 \leq i_1 < i_2 < \ldots < i_t \leq n$ is said to be k-*alternating* for σ if

$$\sigma(i_1) \geq \sigma(i_2) + k, \quad \sigma(i_2) + k \leq \sigma(i_3), \quad \sigma(i_3) \geq \sigma(i_4) + k, \cdots .$$

In other words, the subsequence is k-alternating if it is alternating and additionally

$$|\sigma(i_j) - \sigma(i_{j+1})| \geq k, \quad j \in [t-1],$$

where we set $[m] = \{1, \ldots, m\}$ for $m \in \mathbb{N}$. Below the length of the longest k-alternating subsequence of $\sigma \in S_n$ is denoted by $\mathrm{as}_{n,k}(\sigma)$, or simply $\mathrm{as}_{n,k}$.

Let us also define k-peaks and k-valleys which will be intermediary tools to understand the longest k-alternating subsequences. Let $\sigma = \sigma(1) \ldots \sigma(n) \in S_n$. We say that a section $\sigma(i) \ldots \sigma(j)$ of the permutation σ is a k-*up* (k-*down*, resp.) if $i < j$ and $\sigma(j) - \sigma(i) \geq k$ ($\sigma(i) - \sigma(j) \geq k$, resp.). We say that the section is k-*ascending* if it satisfies:

- $\sigma(i) = \min\{\sigma(i), \ldots, \sigma(j)\}$ and $\sigma(j) = \max\{\sigma(i), \ldots, \sigma(j)\}$, and
- the section $\sigma(i) \ldots \sigma(j)$ is a k-up, and
- there is no k-down in $\sigma(i) \ldots \sigma(j)$, i.e. for any $i \leq s < t \leq j$, we have $\sigma(s) - \sigma(t) < k$.

If also there is no k-ascending section that contains $\sigma(i) \ldots \sigma(j)$, it is called a *maximal* k-*ascending section*. In this case, $\sigma(i)$, $\sigma(j)$ are called a k-*valley* and a k-*peak* of σ, respectively.

A maximal k-descending section $\sigma(i) \ldots \sigma(j)$ can be defined similarly, and this time $\sigma(i)$, $\sigma(j)$ are called a k-peak and a k-valley of σ, respectively. An alternative description can be given as in Cai (2015).

Proposition 1.1 *Let $\sigma = \sigma(1)\sigma(2) \ldots \sigma(n) \in S_n$, $i \in [n]$ and $1 \leq k \leq n-1$. Then $\sigma(i)$ is a k-peak if and only if it satisfies both of the following two properties:*
(i) If there is an $s > i$ with $\sigma(s) > \sigma(i)$, then there is a k-down $\sigma(i) \ldots \sigma(j)$ in $\sigma(i) \ldots \sigma(s)$.
(ii) If there is an $s < i$ with $\sigma(s) > \sigma(i)$, then there is a k-up $\sigma(j) \ldots \sigma(i)$ in $\sigma(s) \ldots \sigma(i)$.

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Considering the case where σ is a uniformly random permutation, our purpose in present paper is to study $\mathrm{Var}(\mathrm{as}_{n,k})$ and to show that $\mathrm{as}_{n,k}$ satisfies a central limit theorem. The statistic $\mathrm{Var}(\mathrm{as}_{n,k})$ is well understood for the case $k = 1$. Indeed, Stanley proved in Stanley (2008) that

$$\mathbb{E}[\mathrm{as}_{n,1}] = \frac{4n+1}{6} \quad \text{and} \quad \mathrm{Var}[\mathrm{as}_{n,1}] = \frac{8n}{45} - \frac{13}{180}.$$

It was later shown in Houdré and Restrepo (2010) and Romik (2011) that $\mathrm{as}_{n,1}$ satisfies a central limit theorem, and convergence rates for the normal approximation were obtained in Islak (2018). All these limiting distribution results rely on the simple fact that $\mathrm{as}_{n,1}$ can be represented as a sum of m-dependent random variables (namely, the indicators of local extrema) and they then use the well-established theory of such sequences.

Regarding the general k, Armstrong conjectured in Armstrong (2014) that $\mathbb{E}[\mathrm{as}_{n,k}] = \frac{4(n-k)+5}{6}$. Pak and Pemantle Pak and Pemantle (2015) then used probabilistic methods to prove that $\mathbb{E}[\mathrm{as}_{n,k}]$ is asymptotically $\frac{2(n-k)}{3} + O\left(n^{2/3}\right)$.

Let us very briefly mention their approach. For $x \in (0,1)$, a vector $\mathbf{y} = (y_1, \ldots, y_n) \in [0,1]^n$ is said to be *x-alternating* if $(-1)^{j+1}(y_j - y_{j+1}) \geqslant x$ for all $1 \leqslant j \leqslant n-1$. Given a vector $\mathbf{y} = (y_1, \ldots, y_n) \in [0,1]^n$, a subsequence $1 \leq i_1 < i_2 < \ldots < i_r \leq n$ is said to be *x-alternating for* $\mathbf{y}$ if

$$|y_{i_j} - y_{i_{j+1}}| \geq x, \quad j \in [r-1].$$

Denoting the length of the longest $x-$alternating subsequence of a random vector $\mathbf{y}$, with Lebesgue measure on $[0,1]^n$ as its distribution, by $\mathrm{as}_{\mathrm{n,x}}(\mathbf{y})$, their main observation was: If Z is a binomial random variable with parameters n and $1-x$, then

$$\mathrm{as}_{n,x}(\mathbf{y}) \stackrel{\mathcal{D}}{=} \mathrm{as}_{Z,1}$$

(Here, $\stackrel{\mathcal{D}}{=}$ means equality in distribution). That is, they concluded that $\mathrm{as}_{n,x}(\mathbf{y})$ has the same distribution as the length of the longest ordinary alternating subsequence of a random permutation on S_Z. Afterwards, using $\mathbb{E}[\mathrm{as}_{n,1}] = \frac{4n+1}{6}$ and $\mathrm{Var}(\mathrm{as}_{n,1}) = \frac{8n}{45} - \frac{13}{180}$, they proved

$$\mathbb{E}[\mathrm{as}_{n,x}] = \frac{2}{3}n(1-x) + \frac{1}{6} \quad \text{and} \quad \mathrm{Var}(\mathrm{as}_{n,x}) = (1-x)(2+5x)\frac{4n}{45}.$$

Further, for suitable x_1 and x_2, they showed that $\mathbb{E}[\mathrm{as}_{n,x_2}] \leqslant \mathbb{E}[\mathrm{as}_{n,k}] \leqslant \mathbb{E}[\mathrm{as}_{n,x_1}]$ and in this way they are able to bound $\mathbb{E}[\mathrm{as}_{n,k}]$.

A closely related problem to the longest alternating subsequence problem is that of calculating the longest zigzagging subsequence. For a given permutation σ, denoting its vertical flip by $\tilde{\sigma}$, a subsequence is said to be *zigzagging* if it is alternating for either σ or $\tilde{\sigma}$. The k-zigzagging case is defined similarly. We will be using the notation $\mathrm{zs}_{n,k}$ for the length of the longest k-zigzagging subsequence in the sequel. Note that in exactly half of the permutations, $\mathrm{as}_{n,k}$ and $\mathrm{zs}_{n,k}$ are equal to each other, and in the other half the length of the k-zigzagging subsequence is exactly one more than the length of the k-alternating subsequence. This is seen via the involution map $I : \sigma(1)\sigma(2)\ldots\sigma(n) \to (n+1-\sigma(1))(n+1-\sigma(2))\ldots(n+1-\sigma(n))$ as noted in Cai (2015). Therefore

$$\mathbb{E}[\mathrm{zs}_k] = \mathbb{E}[\mathrm{as}_k] + 1/2. \tag{1}$$

Cai proved in 2015 that $\mathbb{E}[\mathrm{zs}_k] = \frac{2(n-k)+4}{3}$, and then combining this with (1), solved the Armstrong conjecture Cai (2015).

Our first result in this paper is an asymptotic formula for $\mathrm{Var}(\mathrm{as}_{n,k})$. Namely, we will prove

$$\mathrm{Var}(\mathrm{as}_{n,k}) = \frac{8(n-k)}{45} + O(\sqrt{n}).$$

In order to obtain this result, we first study the number of k-peaks P in random permutations and show that

$$\mathrm{Var}(P) = \frac{2(n-k)+4}{45}.$$

Our second result is a central limit theorem for $\mathrm{as}_{n,k}$:

$$\frac{\mathrm{as}_{n,k} - \mathbb{E}[\mathrm{as}_{n,k}]}{\sqrt{\mathrm{Var}(\mathrm{as}_{n,k})}} \longrightarrow_d \mathcal{G},$$

where $\mathcal{G}$ is the standard normal distribution and where $\rightarrow_d$ is used for convergence in distribution.

The rest of the paper is organized as follows. Next section proves our formulas for the variances of P and $\mathrm{as}_{n,k}$. In Section 3, we prove the central limit theorem for $\mathrm{as}_{n,k}$.

2 The variances of P and $\mathrm{as}_{n,k}$

Next result gives an exact formula for the variance of the number of k-peaks P in a uniformly random permutation.

Theorem 2.1 *Let P be the number of k-peaks in a uniformly random permutation in S_n. We have*

$$\mathrm{Var}(P) = \frac{2(n-k)+4}{45}.$$

We will prove Theorem 2.1 after providing a corollary related to the length of longest k-alternating subsequence of a uniformly random permutation. Note that we have $\mathrm{as}_{n,k} = 2P + E$ where $|E| \leq 1$ for any n, k. Thus, $\mathrm{Var}(\mathrm{as}_{n,k}) = 4\,\mathrm{Var}(P) + \mathrm{Var}(E) + 2\,\mathrm{Cov}(P,E)$. Here, clearly $\mathrm{Var}(E) \leq 1$ and by Cauchy-Schwarz inequality $|\,\mathrm{Cov}(P,E)| \leq 2\sqrt{\mathrm{Var}(P)}\sqrt{\mathrm{Var}(E)} \leq C_0\sqrt{n}$ where C_0 is a constant independent of n and k. We now obtain the following.

Corollary 2.1 *Let $\mathrm{as}_{n,k}$ be the length of longest k-alternating subsequence of a uniformly random permutation in S_n. Then,*

$$\mathrm{Var}(\mathrm{as}_{n,k}) = \frac{8(n-k)}{45} + O(\sqrt{n}).$$

In particular, when $k = o(n)$, $\mathrm{Var}(\mathrm{as}_{n,k}) \sim \frac{8n}{45}$ as $n \to \infty$.

Remark 2.1 *In setting of Corollary 2.1, we conjecture that $\mathrm{Var}(\mathrm{as}_{n,k}) = \dfrac{8(n-k)}{45} + \dfrac{19}{180}$. Although we have a heuristic derivation of this equality, we were not able to justify it rigorously.*

Now, let us proceed to the proof of Theorem 2.1.

Proof of Theorem 2.1. Below P_i is the indicator of i being a k-peak[(i)], i.e.

$$P_i := \begin{cases} 1, & i \text{ is a k-peak,} \\ 0, & \text{otherwise.} \end{cases}$$

In particular,

$$P = \sum_{i=1}^{n} P_i.$$

We are willing to compute

$$\mathrm{Var}(P) = \mathrm{Var}\left(\sum_{i=1}^{n} P_i\right) = \mathbb{E}\left[\left(\sum_{i=1}^{n} P_i\right)^2\right] - \left(\mathbb{E}\left[\sum_{i=1}^{n} P_i\right]\right)^2.$$

Recall from Cai (2015) that

$$\mathbb{E}\left[\sum_{i=1}^{n} P_i\right] = \mathbb{E}[P] = \frac{1}{2}\mathbb{E}[\mathrm{zs}_k] = \frac{n-k+2}{3}. \tag{2}$$

Let us next analyze

$$\mathbb{E}\left[\left(\sum_{i=1}^{n} P_i\right)^2\right] = \sum_{i=1}^{n} \mathbb{E}[P_i^2] + 2\sum_{i<j} \mathbb{E}[P_i P_j].$$

Denoting the probability that i is a k-peak by $p_{n,k}(i)$ and the probability that both i, j are k-peaks by $p_{n,k}(i,j)$, we may rewrite this last equation as

$$\mathbb{E}\left[\left(\sum_{i=1}^{n} P_i\right)^2\right] = \sum_{i=1}^{n} p_{n,k}(i) + 2\sum_{i<j} p_{n,k}(i,j).$$

We already know from (2) that the first sum on the right-hand side is $\frac{n-k+2}{3}$. We are then left with calculating $p_{n,k}(i,j)$.

(i) Note that when we say i is a k-peak, we consider i to be an element in the image of the permutation, not an element of the domain of the permutation. If the position i is considered in domain of the permutation, we will be emphasizing it there.

With the definition of k-peaks in mind, for given i and j, we can divide $[n] \setminus \{i\}$ and $[n] \setminus \{j\}$ into three sets according to the following partitions respectively. The first partition is with respect to i:

$$\begin{aligned} A_i =& \{\ell : 1 \leq \ell \leq i-k\}, \\ B_i =& \{\ell : i-k+1 \leq \ell \leq i-1\}, \\ C_i =& \{\ell : i+1 \leq \ell \leq n\}, \end{aligned}$$

and the second partition is with respect to j:

$$\begin{aligned} A_j =& \{\ell : 1 \leq \ell \leq j-k\}, \\ B_j =& \{\ell : j-k+1 \leq \ell \leq j-1\}, \\ C_j =& \{\ell : j+1 \leq \ell \leq n\}. \end{aligned}$$

Assuming without loss of generality that $i < j$, observe

$$\begin{aligned} i < j \implies A_i \subset A_j \\ i < j \implies C_j \subset C_i. \end{aligned}$$

By Proposition 1.1, we observe that for i to be a k-peak, there should be at least one element from A_i between any element of C_i and i, and similarly for j to be a k-peak, there should be at least one element from A_j between any element of C_j and j. To ensure these two properties, we will place the elements accordingly.

Our procedure for placing the elements starts with placing $A_i \cup \{i\}$ in a row $a_1 a_2 \ldots a_{i-k+1}$ arbitrarily. Leaving the insertion of the elements in $A_j \setminus A_i$ to the end of the argument, we will next focus on placing the elements of C_i and C_j. Note that by the observation in previous paragraph, in order to have i and j as k-peaks, the two places next to i are not available for the elements in $C_i \setminus C_j$, and the four places next to i and j are not available for the elements in $C_i \cap C_j = C_j$.

Now, let us focus on the elements of $C_i \setminus C_j = \{i+1, \ldots, j\}$. There are $|A_i \cup \{i\}| = i-k+1$ elements that are placed in a row. Thus, we have $i-k+2$ vacant spots for the element $i-k+2$ to be inserted into the row $a_1 a_2 \ldots a_{i-k+1}$. Since the two places next to i are prohibited, we see that

$$\mathbb{P}(\{i+1\} \text{ does not prevent } i, j \text{ being a } k\text{-peak}) = \frac{i-k}{i-k+2}.$$

Now, we have $i+k+3$ vacant spots for the element $i+2$, and the two places next to i are prohibited, and so,

$$\mathbb{P}(\{i+2\} \text{ does not prevent } i, j \text{ being a } k\text{-peak}) = \frac{i-k+1}{i-k+3}.$$

Continuing in this manner, we see that when we arrive at j, which is the last element to be inserted in from the set $C_i \setminus C_j$, we have $i-k+(j-i+1) = j-k+1$ many vacant places, and the two places next to i are prohibited, and then

$$\mathbb{P}(\{j\} \text{ does not prevent } i, j \text{ being a } k\text{-peak}) = \frac{j-k-1}{j-k+1}.$$

More generally, for $t = 1, \ldots, j-i$, we have

$$\mathbb{P}\left(\{i+t\} \text{ does not prevent } i, j \text{ being a } k\text{-peak}\right) = \frac{i-k+t-1}{i-k+t+1}.$$

Therefore,

$$\begin{aligned} \mathbb{P}\left(C_i \setminus C_j \text{ does not prevent } i, j \text{ being a } k\text{-peak}\right) &= \mathbb{P}\left(\bigcap_{t=1}^{j-i} \{i+t\} \text{ does not prevent } i, j \text{ being a } k\text{-peak}\right) \\ &= \prod_{t=1}^{j-i} \frac{i-k+t-1}{i-k+t+1} \\ &= \frac{(i-k)(i-k+1)}{(j-k)(j-k+1)}. \end{aligned}$$

Now, let us focus on the elements of $C_i \cap C_j = C_j = \{j+1,\dots,n\}$. Recall that there are four prohibited places for these elements to be inserted. We have $j-k+2$ many vacant places to insert $j+1$ into but four of these are prohibited. Thus,

$$\mathbb{P}(\{j+1\} \text{ does not prevent } i,j \text{ being a } k\text{-peak}) = \frac{j-k-2}{j-k+2}.$$

Similar to the analysis in $C_i \setminus C_j$, continuing in this manner, we have $n = j+(n-j)$, and in the end we will have $j-k+(n-j+1) = n-k+1$ many vacant places to insert n, and four of these are prohibited. So,

$$\mathbb{P}(n \text{ does not prevent } i,j \text{ being a } k\text{-peak}) = \frac{n-k-3}{n-k+1}.$$

We may generalize this to obtain

$$\mathbb{P}\left(\{j+t\} \text{ does not prevent } i,j \text{ being a } k\text{-peak}\right) = \frac{j-k+t-3}{j-k+t+1}$$

for $t = 1,\dots,n-j$. We then obtain

$$\begin{aligned}
\mathbb{P}(C_i \cap C_j \text{ does not prevent } i,j \text{ being a } k\text{-peak}) &= \mathbb{P}\left(\bigcap_{t=1}^{n-j}\{j+t\} \text{ does not prevent } i,j \text{ being a } k\text{-peak}\right) \\
&= \prod_{t=1}^{n-j}\frac{j-k+t-3}{j-k+t+1} \\
&= \frac{(j-k-2)(j-k-1)(j-k)(j-k+1)}{(n-k-2)(n-k-1)(n-k)(n-k+1)}.
\end{aligned}$$

Note that we can multiply the probabilities (here, and above in the case of $C_i \setminus C_j$), since in essence what we are doing is conditioning on the event that the previous added elements do not prevent i,j being a k-peak. Now, clearly, the elements of $A_j \setminus A_i$ are in $B_i \cup C_i$. Since the elements that are in C_i have been inserted, we will then be done once we insert the elements of B_i and B_j. But the elements in the sets B_i and B_j have no effect on i and j being a k-peak (once the elements from C_i and C_j are placed), and so we may insert them in any place. Thus, overall, we have

$$\begin{aligned}
p_{n,k}(i,j) &= \mathbb{P}(\text{the set } C_i \cup C_j \text{ does not prevent } i,j \text{ being a } k\text{-peak}) \\
&= \mathbb{P}(C_i \setminus C_j \text{ does not prevent } i,j \text{ being a } k\text{-peak}) \\
&\quad \times \mathbb{P}(C_i \cap C_j \text{ does not prevent } i,j \text{ being a } k\text{-peak}) \\
&= \frac{(i-k)(i-k+1)}{(j-k)(j-k+1)}\frac{(j-k-2)(j-k-1)(j-k)(j-k+1)}{(n-k-2)(n-k-1)(n-k)(n-k+1)} \\
&= \frac{(i-k)(i-k+1)(j-k-2)(j-k-1)}{(n-k-2)(n-k-1)(n-k)(n-k+1)}.
\end{aligned}$$

These add up to

$$\begin{aligned}
\sum_{i<j} p_{n,k}(i,j) &= \sum_{i=k+1}^{n}\sum_{j=i+1}^{n}\frac{(i-k)(i-k+1)(j-k-2)(j-k-1)}{(n-k-2)(n-k-1)(n-k)(n-k+1)} \\
&= \frac{1}{90}(5k-5n+3)(k-n-2),
\end{aligned}$$

where the sum is computed fairly easily noting that essentially we are summing the consecutive integers and squares of consecutive integers. Therefore we obtain

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=1}^{n} P_i\right)^2\right] &= \sum_{i=1}^{n} p_{n,k}(i) + 2\sum_{i<j} p_{n,k}(i,j) = \frac{n-k+2}{3} + \frac{1}{45}(5k-5n+3)(k-n-2) \\
&= \frac{n-k+2}{3}\left(1+\frac{1}{15}(5n-5k-3)\right) = \frac{1}{45}(n-k+2)(5n-5k+12).
\end{aligned}$$

Using this we arrive at

$$\begin{aligned}\mathrm{Var}(P) &= \mathbb{E}\left[\left(\sum_{i=1}^{n} P_i\right)^2\right] - \left(\mathbb{E}\left[\sum_{i=1}^{n} P_i\right]\right)^2 \\ &= \frac{1}{45}(n-k+2)(5n-5k+12) - \left(\frac{n-k+2}{3}\right)^2 = \frac{2(n-k)+4}{45}\end{aligned}$$

as asserted in Theorem 2.1. □

3 A Central Limit theorem

In this section, we will prove the following central limit theorem.

Theorem 3.1 *Let k be a fixed positive integer. Then the length of the longest k-alternating subsequence $\mathrm{as}_{n,k}$ of a uniformly random permutation satisfies a central limit theorem,*

$$\frac{\mathrm{as}_{n,k} - \mathbb{E}[\mathrm{as}_{n,k}]}{\sqrt{\mathrm{Var}(\mathrm{as}_{n,k})}} \longrightarrow_d \mathcal{G},$$

where $\mathcal{G}$ is the standard normal distribution.

The proof involves a suitable truncation argument that allows us to reduce the problem to proving a central limit theorem for sums of locally dependent random variables for which a theory is already available. Since the length of the longest k alternating sequence differs from twice the number of k peaks by at most 1, we may focus on the number of peaks. For any i, let P_i be the random variable that is 1 if the value i is a k-peak and zero otherwise as before. Also recall $P = P_1 + \cdots + P_n$. We know that $P_i = 1$ precisely when

- Scanning to the right of the value i, we encounter an element in $[i-k]$ before we encounter an element in $[i+1, n]$. It is permitted that we do not encounter an element from $[i+1, n]$ at all.
- Scanning to the left of the value i, we encounter an element in $[i-k]$ before we encounter an element in $[i+1, n]$. It is permitted that we do not encounter an element from $[i+1, n]$ at all.

Our approach to getting a central limit theorem is to define a suitable truncation that can be computed using local data. There are a number of theorems that establish central limit behaviour for variables with only local correlations and this approach has been employed in a number of situations.

Note that the condition on $P_i = 1$ can be restated as

- There is an index $j > \sigma^{-1}(i)$ such that $i - k \geq \sigma(j)$ and such that

$$i = \max\nolimits_{s \in [\sigma^{-1}(i), j]} \sigma(s), \quad \sigma(j) = \min\nolimits_{s \in [\sigma^{-1}(i), j]} \sigma(s).$$

- There is an index $j < \sigma^{-1}(i)$ such that $i - k \geq \sigma(j)$ and such that

$$i = \max\nolimits_{s \in [j, \sigma^{-1}(i)]} \sigma(s), \quad \sigma(j) = \min\nolimits_{s \in [j, \sigma^{-1}(i)]} \sigma(s).$$

Note that we might need to scan far to the left and right in order to determine whether a value is a k-peak or not and thus we will have long range dependence. We will show that ignoring long range interactions does not change the statistic very much.

Fix a number m that we will specify later. Let $Y_i = 1$ if we can determine that i is a k-peak by only looking at m positions to the left and right of i. Precisely, let $Y_i = 1$ if

- There is an index $j \in [\sigma^{-1}(i), \sigma^{-1}(i) + m]$ such that $i - k \geq \sigma(j)$ and such that

$$i = \max\nolimits_{s \in [\sigma^{-1}(i), j]} \sigma(s), \quad \sigma(j) = \min\nolimits_{s \in [\sigma^{-1}(i), j]} \sigma(s).$$

- There is an index $j \in [\sigma^{-1}(i) - m, \sigma^{-1}(i)]$ such that $i - k \geq \sigma(j)$ and such that

$$i = \max\nolimits_{s \in [j, \sigma^{-1}(i)]} \sigma(s), \quad \sigma(j) = \min\nolimits_{s \in [j, \sigma^{-1}(i)]} \sigma(s).$$

If $Y_i = 1$, we call it a *local k-peak* (suppressing the reference to m). Note that any local k-peak is a k-peak and thus, $Y_i \le P_i$. We should next understand the case where $Y_i = 0$ and $P_i = 1$. Note that if $i \le k$, then $P_i = Y_i = 0$.

If $\sigma^{-1}(i) \in [m+1]$, there is no issue when scanning to the left. However, if we scan to the right and this event happens, then the m indices to the right should have values in $[i-k+1, i-1]$. The probability of this is at most $\left(\frac{k-1}{n-1}\right)^m$. Similarly, the probability of this event when $\sigma^{-1}(i) \in [n-m, n]$ is at most $\left(\frac{k-1}{n-1}\right)^m$.

If $\sigma^{-1}(i) \in [m+2, n-m-2]$, the event can only happen if the $2m$ positions, m to the left and m to the right take values in $[i-k-1, i-1]$ and the probability of this is at most $\left(\frac{k-1}{n-1}\right)^{2m}$.

Putting these together, recalling $Y_i \le P_i$, and denoting the total variation distance by d_{TV}, we see that

$$\begin{aligned} d_{TV}(P_i, Y_i) &= \frac{1}{2}\sum_{j=0}^{1} |\mathbb{P}(P_i = j) - \mathbb{P}(Y_i = j)| \\ &= \frac{1}{2}(\mathbb{P}(Y_i = 0) - \mathbb{P}(P_i = 0) + \mathbb{P}[P_i = 1] - \mathbb{P}(Y_i = 1)) \\ &= \frac{1}{2}(2(\mathbb{P}(P_i = 1) - \mathbb{P}(Y_i = 1))) \\ &= \mathbb{P}(Y_i = 0,\, P_i = 1) \\ &\le \frac{2m+2}{n}\left(\frac{k-1}{n-1}\right)^m + \frac{n-2m-2}{n}\left(\frac{k-1}{n-1}\right)^{2m}. \end{aligned}$$

This implies

$$\begin{aligned} d_{TV}(P_1 + \ldots + P_n, Y_1 + \ldots + Y_n) \le &\frac{(2m+2)(n)}{n}\left(\frac{k-1}{n-1}\right)^m \\ &+ \frac{(n-2m-2)(n)}{n}\left(\frac{k-1}{n-1}\right)^{2m}, \\ \le &(2m+2)\left(\frac{k}{n}\right)^m + n\left(\frac{k}{n}\right)^{2m}, \\ \le &3n\left(\frac{k}{n}\right)^m. \end{aligned} \tag{3}$$

When k is fixed, taking $m = 3$ suffices for our purpose. Note in particular that

$$\mathbb{P}(Y_1 + \ldots + Y_n < P_1 + \cdots + P_n) = o\left(\frac{1}{n}\right), \tag{4}$$

when m is chosen appropriately.

Next we will show that $Y = Y_1 + \cdots + Y_n$ satisfies a central limit theorem. Let $Z_i = 1$ if the position i is a local k-peak and 0 otherwise. It is immediate that $Z = Z_1 + \cdots + Z_n$ and Y have the same distribution. We let Z be such a random variable for which (P, Z) and (P, Y) have the same distribution. Further, note that the variables Z_i have the property that Z_i and Z_j are independent if $|i-j| > 2m$.

There are a number of related theorems that guarantee central limit behaviour for sums of locally dependent variables. A result due to Rinott Rinott (1994) will suffice for our purpose. The version we give is a slight variation of the one discussed in Raic (2003).

Theorem 3.2 *Let $U_1, \ldots, U_n$ be random variables such that U_i and U_j are independent when $|i-j| > 2m$. Setting $U = U_1 + \cdots + U_n$, we have*

$$d_K\left(\frac{U - \mathbb{E}[U]}{\sqrt{\mathrm{Var}(U)}}, \mathcal{G}\right) \le C(2m+1)\sqrt{\frac{\sum_{i=1}^n \mathbb{E}|U_i|^3}{(\mathrm{Var}(U))^{3/2}}},$$

where d_K is the Kolmogorov distance.

We will now apply this result for $Z = Z_1 + \cdots + Z_n$. For this purpose we need a lower bound on the variance of the random variable Z. Recall that the variance of P is $\Omega(n)$ and let us show that the same holds for Z.

We have

$$\begin{aligned}
\sqrt{\mathrm{Var}(Z)} \geq & \sqrt{\mathrm{Var}(P)} - \sqrt{\mathrm{Var}(P-Z)} \\
\geq & \sqrt{\mathrm{Var}(P)} - \sqrt{\mathbb{E}[(P-Z)^2]} \\
\geq & \sqrt{\mathrm{Var}(P)} - \sqrt{n}\sqrt{\mathbb{E}|P-Z|} \\
\geq & \sqrt{\mathrm{Var}(P)} - \sqrt{n}\sqrt{\mathbb{E}[|P-Z| \mid P \neq Z]\mathbb{P}(P \neq Z)} \\
\geq & \sqrt{\mathrm{Var}(P)} - \sqrt{n}\sqrt{n}\sqrt{o\left(\frac{1}{n}\right)} && \text{using (4)} \\
= & \Omega(\sqrt{n}) - o(\sqrt{n}) \\
= & \Omega(\sqrt{n}).
\end{aligned}$$

Also, the Z_i are Bernoulli random variables and thus $\sum_{i=1}^{n} \mathbb{E}|Z_i|^3 = O(n)$. This shows that

$$d_K\left(\frac{Z - \mathbb{E}[Z]}{\sqrt{\mathrm{Var}(Z)}}, \mathcal{G}\right) \leq O\left(\frac{m}{n^{1/4}}\right),$$

proving that when k is fixed, we have a central limit theorem,

$$\frac{Z - \mathbb{E}[Z]}{\sqrt{\mathrm{Var}(Z)}} \longrightarrow_d \mathcal{G}.$$

Together with the total variation distance bound between P and Z, and noting that convergence in d_{TV} implies convergence in d_K, we conclude that P satisfies a central limit theorem. Since $\mathrm{as}_{n,k}$ differs from $2P$ by at most 1, the same holds for it as well after proper centering and scaling.

Remark 3.1 *The arguments given in this section carry over to certain cases where k grows with n. For example, considering the case $k = \gamma n$ for constant γ, the quantity $3n\left(\frac{k}{n}\right)^m$ in (3) can be made $o(1/n)$ by choosing m suitably. To see this, letting $\alpha > 1$, suppose $\frac{1}{n^\alpha} = 3n\left(\frac{k}{n}\right)^m$. Since $\gamma = \frac{k}{n}$, we then have*

$$n^{-1-\alpha} = 3(\gamma)^m,$$

and then $m = \frac{(-1-\alpha)\log(n) - \log(3))}{\log(\gamma)}$. We can choose $\alpha = 2$ so that $m = \dfrac{-3\log(n)}{\log(\gamma)}$. Note that $m > 0$ since $\log\left(\frac{k}{n}\right) < 0$.

Remark 3.2 *In notation of the Introduction, if we were to prove a central limit theorem for $\mathrm{as}_{\mathrm{n,x}}$, then that would be straightforward. This is thanks to the fact that it can be written as a random sum (where the number of summands is binomial) of locally dependent variables, and that central limit theorem for such cases are already available. See, for example, Islak (2016).*

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