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Rainbow vertex pair-pancyclicity of strongly edge-colored graphs

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An edge-colored graph is *rainbow* if no two edges of the graph have the same color. An edge-colored graph G^c is called *properly colored* if every two adjacent edges of G^c receive distinct colors in G^c. A *strongly edge-colored* graph is a proper edge-colored graph such that every path of length 3 is rainbow. We call an edge-colored graph G^c *rainbow vertex pair-pancyclic* if any two vertices in G^c are contained in a rainbow cycle of length ℓ for each ℓ with $3 \leq \ell \leq n$. In this paper, we show that every strongly edge-colored graph G^c of order n with minimum degree $\delta \geq \frac{2n}{3} + 1$ is rainbow vertex pair-pancyclicity.

Keywords: edge-coloring; strongly edge-colored graph; rainbow cycle; rainbow vertex pair-pancyclicity.

1 Introduction

In this paper, we only consider finite, undirected and simple graphs. Let G be a graph consisting of a vertex set $V(G)$ and an edge set $E = E(G)$. We use $d(v)$ to denote the number of edges incident with vertex v in G and $\delta(G) = min\{d(v) : v \in E\}$. An *edge-coloring* of G is a mapping $c : \ E(G) \to S$, where S is a set of colors. A graph G with an edge-coloring c is called an *edge-colored* graph, and denoted by G^c. For any $e \in E(G)$, e has color k if $c(e) = k$. For any subset $E_1 \subseteq E$, $c(E_1)$ is the set $\{c(e) : e \in E_1\}$. We use $d^c_G(v)$ (or briefly $d^c(v)$) to denote the number of different colors on the edges incident with vertex v in G^c and $\delta^c(G) = min\{d^c(v) : v \in V(G^c)\}$. An edge-colored graph G^c is called *properly colored* if every two adjacent edges of G^c receive distinct colors in G^c. Edge-colored graph G^c is $rainbow$ if no two edges of G^c have the same color. A *strongly edge-colored graph* is a proper edge-colored graph such that every path of length 3 is rainbow. It is clearly that $d(v) = d^c(v)$ for all $v \in V(G^c)$ in a strongly edge-colored graph G^c, or equivalently, for every vertex v in strongly edge-colored graph G^c, the colors on the edges incident with v are pairwise distinct. An edge-colored graph G^c is called *rainbow Hamiltonian* if G^c contains a rainbow Hamiltonian cycle and *rainbow vertex(edge)-pancyclic* if every vertex (edge) in G^c is contained in a rainbow cycle of length l for each l with $3 \leq l \leq n$. We call an edge-colored graph G^c *rainbow vertex pair-pancyclic* if any two vertices in G^c are contained a rainbow cycle of length l for each l with $3 \leq l \leq n$. further, we call a cycle C *l-cycle* if the length of the cycle C is l. For notation and terminology not defined here, we refer the reader to Bondy and Murty (2008).

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The classical Dirac's theorem states that every graph G is Hamiltonian if $\delta(G) \geq \frac{n}{2}$. Inspired by this famous theorem, Hendry (1990) show that every graph G of order n with minimum degree $\delta \geq \frac{n+1}{2}$ is vertex-pancyclic. During the past few decades, the existence of cycles in graphs have been extensively studied in the literatures. We recommend Abouelaoualim et al. (2010); Chen (2018); Chen and Li (2021, 2022); Chen et al. (2019); Czygrinow et al. (2021); Ehard and Mohr (2020); Fujita et al. (2019); Guo et al. (2022); Kano and Li (2008); Li et al. (2022) for more results.

For edge-colored graphs, Lo (2014) proved the following asymptotic theorem about properly colored cycles.

Theorem 1.1 (Lo (2014)) *For any $\varepsilon > 0$, there exists an integer n_0 such that every edge-colored graph G^c with n vertices and $\delta^c(G) \geq (\frac{2}{3} + \varepsilon)n$ and $n \geq n_0$ contains a properly edge-colored cycle of length l for all $3 \leq l \leq n$, where $\delta^c(G)$ is the minimum number of distinct colors of edges incident with a vertex in G^c.*

Cheng et al. (2019) considered the existence of rainbow Hamiltonian cycles in strongly edge-colored graph and proposed the following two conjectures.

Conjecture 1.2 (Cheng et al. (2019)) *Every strongly edge-colored graph G^c with n vertices and degree at least $\frac{n+1}{2}$ has a rainbow Hamiltonian cycle.*

Conjecture 1.3 (Cheng et al. (2019)) *Every strongly edge-colored graph G^c with n vertices and degree at least $\frac{n}{2}$ has a rainbow Hamiltonian path.*

To support the above two conjectures, they presented the following theorem.

Theorem 1.4 (Cheng et al. (2019)) *Let G^c be a strongly edge-colored graph with minimum degree δ, if $\delta \geq \frac{2|G|}{3}$, then G^c has a rainbow Hamiltonian cycle.*

Wang and Qian (2021) showed that every strongly edge-colored graph G^c on n vertices is rainbow vertex-pancyclic if $\delta \geq \frac{2n}{3}$. Li and Li (2022) further considered the rainbow edge-pancyclicity of strongly edge-colored graphs and proposed the following theorem.

Theorem 1.5 (Li and Li (2022)) *Let G^c be a strongly edge-colored graph on n vertices. If $\delta(G^c) \geq \frac{2n+1}{3}$, then G^c is rainbow edge-pancyclic. Furthermore, for every edge e of G^c, one can find a rainbow l-cycle containing e for each l with $3 \leq l \leq n$ in polynomial time.*

In this paper, we consider the rainbow vertex pair-pancyclicity of strongly edge-colored graph. Our main result is as follows.

Theorem 1.6 *Let G^c be a strongly edge-colored graph with n vertices and minimum degree δ. If $\delta \geq \frac{2n}{3} + 1$, then G^c is rainbow vertex pair-pancyclicity.*

2 Proof of Theorem 1.6

First, we introduce some useful notations. Given a rainbow cycle C in graph G^c, a color s is called a *C-color* (resp., *$\widetilde{C}$-color*) if $s \in c(C)$ (resp., $s \notin c(C)$). Correspondingly, we call an edge e a *C-color edge* (resp., *$\widetilde{C}$-color edge*) if $c(e) \in c(C)$ (resp., $c(e) \notin c(C)$). Two adjacent vertices u and v are called *C-adjacent* (resp., *$\widetilde{C}$-adjacent*) if $c(uv) \in c(C)$ (resp., $c(uv) \notin c(C)$). For two disjoint adjacent subsets V_1 and V_2 of $V(G)$, let $E(V_1, V_2)$ denote the set of edges between V_1 and V_2. We denote the subsets of $E(V_1, V_2)$ consisting of the C-color edges (resp., $\widetilde{C}$-color edges) by $E_C(V_1, V_2)$ (resp., $E_{\widetilde{C}}(V_1, V_2)$). Similarly, for two subgraphs H_1 and H_2, we denote the set of C-color edges (resp., $\widetilde{C}$-color edges) between $V(H_1)$ and $V(H_2)$ by $E_C(H_1, H_2)$ (resp., $E_{\widetilde{C}}(H_1, H_2)$). For any two vertices v_i and v_j of cycle $C = v_1v_2 \ldots v_lv_1$, we identify the two subscripts i and j if $i \equiv j \ (mod\ l)$. Let $v_iC^+v_j$ be the path $v_iv_{i+1}...v_{j-1}v_j$ and $v_iC^-v_j$ the path $v_iv_{i-1}...v_{j+1}v_j$, respectively. For any vertex $v \in V(G^c)$, let $CN(v)$ be the set of colors used by the edges incident with v.

From the definition of strongly edge-coloring, we can easily get the following observation.

Obervation 2.1 *Each cycle of length at most 5 in a strongly edge-colored graph is rainbow.*

Proof of Theorem 1.6: Recall that the colors on the edges incident with v are pairwise distinct for each vertex v of a strongly edge-colored graph. So we do not distinguish the colors of adjacent edges in the following. If $n \leq 8$, G is complete since $\delta \geq \frac{2n}{3} + 1$, and so the result clearly holds. Thus we suppose that $n \geq 9$. Let a and b be two arbitrary vertices of G. If a and b are adjacent, then a and b are contained in a rainbow cycle of length l for each l with $3 \leq l \leq n$ from Theorem 1.5. So we consider that a and b are not adjacent. Since $\delta \geq \frac{2n}{3} + 1$, we have that a and b are contained in a 4-cycle which is rainbow from Observation 2.1. Suppose to the contrary that the result is not true. Then there is an integer l with $4 \leq l \leq n-1$ such that there is a rainbow l-cycle containing a and b, but there is no rainbow $(l+1)$-cycle containing both a and b. Let $C := v_1v_2 \ldots v_lv_1$ be a rainbow l-cycle containing a and b.

Without loss of generality, we assume that $c(v_iv_{i+1}) = i$ for $1 \leq i \leq l$. For $1 \leq i \leq l$, let N_i be the set of the vertices of C which are adjacent to v_i, that is, $N_i = N(v_i) \cap V(C)$. We then proof the following claim.

Claim 1 $l \geq \frac{n+12}{3}$. *In particular,* $l \geq 7$ *when* $n \geq 9$.

Proof. Since G^c is strongly edge-colored, for any $v_j \in N_1$, the color j does not occur in $CN(v_1)$. So the number of C-colors not contained in $CN(v_1)$ is at least $|N_1| - 1$, and therefore, the number of C-colors contained in $CN(v_1)$ is at most $l - (|N_1| - 1)$. Since 1 and l are C-colors in $CN(v_1)$, we have that the number of C-colors contained in $E(v_1, V(G) \setminus V(C))$ is at most $l - (|N_1| - 1) - 2 = l - |N_1| - 1$. Hence, we have $|E_C(v_1, V(G) \setminus V(C))| \leq l - |N_1| - 1$. Since $|E(v_1, V(G) \setminus V(C))| \geq \delta - |N_1|$, we have that

$$\begin{aligned} |E_{\widetilde{C}}(v_1, V(G) \setminus V(C))| &= |E(v_1, V(G) \setminus V(C))| - |E_C(v_1, V(G) \setminus V(C))| \\ &\geq (\delta - |N_1|) - (l - |N_1| - 1) \\ &= \delta - l + 1. \end{aligned}$$

Similarly, we can also deduce that $|E_{\widetilde{C}}(v_i, V(G) \setminus V(C))| \geq \delta - l + 1$ for all $1 \leq i \leq l$. For any two vertices v_i and v_{i+1} with $1 \leq i \leq l$, if there exists a vertex $w \in V(G) \setminus V(C)$ such that both v_iw and $v_{i+1}w$ are $\widetilde{C}$-color edges, then both a and b are contained in a rainbow $(l+1)$-cycle $C' := v_iwv_{i+1}C^+v_i$, a contradiction. Thus, for any common neighbor $w \in V(G) \setminus V(C)$ of v_i and v_{i+1}, either v_iw or $v_{i+1}w$ is not a $\widetilde{C}$-color edge. Then we have that $|E_{\widetilde{C}}(v_i, w)| + |E_{\widetilde{C}}(v_{i+1}, w)| \leq 1$. Therefore, we have

$$n \geq |E_{\widetilde{C}}(v_i, V(G) \setminus V(C))| + |E_{\widetilde{C}}(v_{i+1}, V(G) \setminus V(C))| + l \geq 2(\delta - l + 1) + l = 2\delta - l + 2.$$

Hence,

$$l \geq 2\delta - n + 2 \geq 2 \cdot (\frac{2n}{3} + 1) - n + 2 = \frac{n+12}{3}.$$

This completes the claim. □

Let $H = K_k$ be the maximal rainbow complete graph in $G^c[V(G) \setminus V(C)]$ such that every edge in H is $\widetilde{C}$-colored, and let $R = G^c[V(G) - (V(C) \cup V(H))]$. It is clearly that for any $w \in V(H)$, if there is a vertex $v_i \in V(C)$ such that $v_i w$ is a $\widetilde{C}$-color edge, then $c(v_i w) \notin c(H)$ since G^c is a strongly edge-colored graph.

For two $\widetilde{C}$-color edges $v_i w_1$ and $v_j w_2$ with $w_1, w_2 \in V(H)$ and $1 \leq i < j \leq l$, if $w_1 = w_2$ and $j - i = 1$, we say $v_i w_1$ and $v_j w_2$ are *forbidden pair of type 1*; if $w_1 \neq w_2$, both a and b are contained in $v_i C^- v_j$, and $2 \leq j - i \leq k$, we say $v_i w_1$ and $v_j w_2$ are *forbidden pair of type 2*. Clearly, if $E_{\widetilde{C}}(C, H)$ has a forbidden pair of type 1, then there exists a rainbow $(l+1)$-cycle $C' := v_i w_1 v_j C^+ v_i$ containing both a and b, and if $E_{\widetilde{C}}(C, H)$ has a forbidden pair of type 2, then there exist a rainbow $(l+1)$-cycle $C' := v_i w_1 H w_2 v_j C^+ v_i$ containing both a and b, where $w_1 H w_2$ is a path of length $|E(v_i C^+ v_j)| - 1$ with endpoints w_1 and w_2 in H.

Claim 2 $k \geq 3$.

Proof. For each $w \in V(H)$, let

$$\widetilde{s}_w = |E_{\widetilde{C}}(w, C)|, s_w = |E_C(w, C)|,$$
$$\widetilde{t}_w = |E_{\widetilde{C}}(w, R)|, t_w = |E_C(w, R)|.$$

We have

$$\widetilde{s}_w + s_w + \widetilde{t}_w + t_w + (k - 1) \geq \delta. \tag{1}$$

If there is an integer i with $1 \leq i \leq l$ such that $v_i w \in E(G^c)$, then the colors $i - 1$ and i can not appear in $CN(w)$. Thus the number of C-colors not contained in $CN(w)$ is at least $\widetilde{s}_w + s_w$, which implies that

$$s_w + t_w \leq l - (\widetilde{s}_w + s_w),$$

and so, we have

$$\widetilde{s}_w + 2s_w + t_w \leq l. \tag{2}$$

Let $v_{i_1}, v_{i_2}, ..., v_{i_{\widetilde{s}_w}}$ be the vertices on C which are $\widetilde{C}$-adjacent to w. Without loss of generality, we suppose that $1 \leq i_1 < i_2 < ... < i_{\widetilde{s}_w} \leq l$. Then $i_{j+1} - i_j \geq 2$ for each $1 \leq j \leq \widetilde{s}_w - 1$ and $i_{\widetilde{s}_w} - i_1 \leq l - 2$. Let $I = \{i_1 - 1, i_1, i_2 - 1, i_2, ..., i_{\widetilde{s}_w} - 1, i_{\widetilde{s}_w}\}$. Clearly, we have $|I| = 2\widetilde{s}_w$ and $I \cap CN(w) = \phi$. Thus, we can deduce that

$$2\widetilde{s}_w + s_w + t_w = |I| + s_w + t_w \leq l. \tag{3}$$

Since $|V(R)| = n - l - k$, we have $t_w + \widetilde{t}_w \leq n - l - k$. Together with inequalities (2) and (3), we have

$$3\widetilde{s}_w + 3s_w + 3t_w + \widetilde{t}_w \leq l + l + n - l - k = n + l - k.$$

Let

$$\widetilde{S} = \sum_{w \in V(H)} \widetilde{s}_w, S = \sum_{w \in V(H)} s_w, \widetilde{T} = \sum_{w \in V(H)} \widetilde{t}_w, T = \sum_{w \in V(H)} t_w.$$

Then,

$$3\widetilde{S} + 3S + 3T + \widetilde{T} \leq k(n + l - k). \tag{4}$$

Since k is maximal, each vertex of R has at most $k-1$ number of $\widetilde{C}$-color edges to H, which implies that

$$\widetilde{T} = \sum_{w \in V(H)} \widetilde{t}_w \leq (k-1)(n-l-k). \tag{5}$$

Recall that $w \in V(H)$. By (1) and the arbitrariness of w, we have

$$\begin{aligned} k\delta &\leq \sum_{w \in V(H)} (\widetilde{s}_w + s_w + \widetilde{t}_w + t_w + (k-1)) \\ &= \widetilde{S} + S + \widetilde{T} + T + k(k-1). \end{aligned} \tag{6}$$

Combining inequalities (4), (5) and (6), we can get the following inequality

$$\begin{aligned} 3k\delta &\leq 3\widetilde{S} + 3S + 3T + 3\widetilde{T} + 3k(k-1) \\ &\leq k(n+l-k) + 2(k-1)(n-l-k) + 3k(k-1) \\ &\leq n(3k-2) + l(2-k) - k. \end{aligned}$$

If $k = 1$, then $l > n$, a contradiction. If $k = 2$, then $\delta \leq \frac{2n-1}{3}$, again a contradiction. So we have $k \geq 3$. Claim 2 follows. □

Since H is a rainbow complete graph, we can deduce that

$$S + T \leq l. \tag{7}$$

Claim 3 $\widetilde{S} \geq l + 1$.

Proof. Suppose, by way of contradiction, that $\widetilde{S} \leq l$. Combining with inequality (6), we can get that

$$k\delta \leq \widetilde{S} + S + \widetilde{T} + T + k(k-1) \leq l + l + (k-1)(n-l-k) + k(k-1),$$

which implies that $k(n-l-\delta) \geq n-3l$. Since $\delta \geq \frac{2n}{3} + 1$ and $l \geq \frac{n+12}{3}$ from Claim 1, we have $n-l-\delta \leq 0$. Thus we have $3(n-l-\delta) \geq k(n-l-\delta) \geq n-3l$ from Claim 2, and therefore $\delta \leq \frac{2n}{3}$, a contradiction. Claim 3 follows. □

Without loss of generality, we suppose that $a = v_1$ and $b = v_m$, where $2 \leq m \leq l-1$, and let $P^1 = aC^+b$. Then we design an algorithm to generate a sequence of disjoint sub-paths $P_1^1, P_2^1, ..., P_{h_1}^1$ of C respect to P^1 and H.

Algorithm AI

Input: a strongly edge-colored graph G^c, a rainbow cycle $C = v_1v_2\ldots v_lv_1$, a path $P^1 = v_1v_2...v_m$ and a rainbow complete subgraph $H = K_k$ of $G^c - V(C)$.
Output: a sequence of disjoint paths $P_1^1, P_2^1, ..., P_{h_1}^1$ such that P_i^1 is a subgraph of C.
1: **Set** $i = 1$
2: **While** $V(P^1) \neq \phi$ **do**
  If $E_{\widetilde{C}}(P^1, H) = \phi$
    stop
  Else Set d be the smallest subscript such that $E_{\widetilde{C}}(v_d, H) \neq \phi$
    If $d + k \geq m$ **then**
      Set $P_i^1 = v_dv_{d+1}...v_m$
      stop
    Else **If** $|E_{\widetilde{C}}(v_d, H)| \geq 2$ **then**
        Set $P_i^1 = v_dv_{d+1}...v_{d+k}$
      If $|E_{\widetilde{C}}(v_d, H)| = 1$ **then**
        Set $P_i^1 = v_dv_{d+1}...v_{d+k+1}$
    Set $P^1 = P^1 \setminus P_i^1$
    Set $i = i + 1$
3: **return** $P_1^1, P_2^1, ..., P_{h_1}^1$

Claim 4 *$|E_{\widetilde{C}}(P_i^1, H)| \leq |V(P_i^1)| - 1$ for any $1 \leq i \leq h_1 - 1$, $|E_{\widetilde{C}}(P_{h_1}^1, H)| \leq k$ if $|V(P_{h_1}^1)| \in \{1, 2\}$, and $|E_{\widetilde{C}}(P_{h_1}^1, H)| \leq k + 1$ if $3 \leq |V(P_{h_1}^1)| \leq k + 1$.*

Proof. For $1 \leq i \leq h_1 - 1$, we distinguish the following two cases.
Case 1. $|E_{\widetilde{C}}(v_d, H)| \geq 2$. Then we have $P_i^1 = v_dv_{d+1}...v_{d+k}$. Let w_1 and w_2 be two vertices in H such that $v_dw_1, v_dw_2 \in E_{\widetilde{C}}(v_d, H)$. Since there exist no forbidden pairs of type 1 for any vertex $w \in V(H)$, then we have $|E_{\widetilde{C}}(v_d, H)| + |E_{\widetilde{C}}(v_{d+1}, H)| \leq k$. For any j with $d + 2 \leq j \leq d + k$, if w_1 and v_j are $\widetilde{C}$-adjacent, then v_jw_1 and v_dw_2 form a forbidden pair of type 2; if w_2 and v_j are $\widetilde{C}$-adjacent, then v_jw_2 and v_dw_1 form a forbidden pair of type 2; if v_j and w are $\widetilde{C}$-adjacent for some w with $w \neq w_1$ and $w \neq w_2$, then v_jw and v_dw_1 form a forbidden pair of type 2. Therefore, we have $|E_{\widetilde{C}}(v_j, H)| = 0$. Thus,

$$\begin{aligned}|E_{\widetilde{C}}(P_i^1, H)| &= \sum_{j=d}^{d+k} |E_{\widetilde{C}}(v_j, H)| \\ &= |E_{\widetilde{C}}(v_d, H)| + |E_{\widetilde{C}}(v_{d+1}, H)| \\ &\leq k \\ &= |V(P_i^1)| - 1.\end{aligned}$$

Case 2. $|E_{\widetilde{C}}(v_d, H)| = 1$. Then we have $P_i^1 = v_dv_{d+1}...v_{d+k+1}$. Let w_1 be a vertex in H such that $v_dw_1 \in E_{\widetilde{C}}(v_d, H)$. We further distinguish the following three cases.
Case 2.1. $|E_{\widetilde{C}}(v_{d+1}, H)| = 0$. For any $w \in V(H) \setminus \{w_1\}$, we have that v_j and w cannot be $\widetilde{C}$-adjacent for any $d + 2 \leq j \leq d + k + 1$ since otherwise v_jw and v_dw_1 form a forbidden pair of type 2. Thus, we

have $|E_{\widetilde{C}}(v_j, H)| \leq 1$ and $\sum_{j=d+2}^{d+k+1} |E_{\widetilde{C}}(v_j, H)| \leq k-1$. Therefore,

$$\begin{aligned}
|E_{\widetilde{C}}(P_i^1, H)| &= \sum_{j=d}^{d+k+1} |E_{\widetilde{C}}(v_j, H)| \\
&= |E_{\widetilde{C}}(v_d, H)| + |E_{\widetilde{C}}(v_{d+1}, H)| + \sum_{j=d+2}^{d+k+1} |E_{\widetilde{C}}(v_j, H)| \\
&\leq 1 + 0 + (k-1) \\
&= k \\
&\leq |V(P_i^1)| - 1.
\end{aligned}$$

Case 2.2. $|E_{\widetilde{C}}(v_{d+1}, H)| = 1$. Let w_2 be a vertex in H such that $v_{d+1}w_2 \in E_{\widetilde{C}}(v_d, H)$. Clearly, $w_1 \neq w_2$. If v_{d+2} and w_2 are $\widetilde{C}$-adjacent, we have that $v_{d+2}w_2$ and $v_d w_1$ form a forbidden pair of type 2, a contradiction. If v_{d+2} and w are $\widetilde{C}$-adjacent for some $w \in V(H)$ with $w \neq w_1$ and $w \neq w_2$, then $v_{d+2}w$ and $v_d w_1$ form a forbidden pair of type 2, again a contradiction. So, $|E_{\widetilde{C}}(v_{d+2}, H)| \leq 1$. For any j with $d+3 \leq j \leq d+k+1$, if w_1 and v_j are $\widetilde{C}$-adjacent, then $v_j w_1$ and $v_{d+1}w_2$ form a forbidden pair of type 2; if w_2 and v_j are $\widetilde{C}$-adjacent, then $v_j w_2$ and $v_d w_1$ form a forbidden pair of type 2; if v_j and w are $\widetilde{C}$-adjacent for some $w \in V(H)$ with $w \neq w_1$ and $w \neq w_2$, then $v_j w$ and $v_d w_1$ form a forbidden pair of type 2. We obtain a contradiction in the above three cases, and therefore, we have $\sum_{j=d+3}^{d+k+1} |E_{\widetilde{C}}(v_j, H)| = 0$. Therefore,

$$\begin{aligned}
|E_{\widetilde{C}}(P_i^1, H)| &= \sum_{j=d}^{d+k+1} |E_{\widetilde{C}}(v_j, H)| \\
&= |E_{\widetilde{C}}(v_d, H)| + |E_{\widetilde{C}}(v_{d+1}, H)| + |E_{\widetilde{C}}(v_{d+2}, H)| + \sum_{j=d+3}^{d+k+1} |E_{\widetilde{C}}(v_j, H)| \\
&\leq 1 + 1 + 1 + 0 \\
&\leq k \\
&\leq |V(P_i^1)| - 1.
\end{aligned}$$

Case 2.3. $|E_{\widetilde{C}}(v_{d+1}, H)| \geq 2$. Let $Q_i^1 = P_i^1 \setminus \{v_d\} = v_{d+1}v_{d+2}...v_{d+k+1}$. Similar to the discussion of Case 1, we have that $|E_{\widetilde{C}}(Q_i^1, H)| \leq |V(Q_i^1)| - 1 = (k+1) - 1 = k$. Thus, $|E_{\widetilde{C}}(P_i^1, H)| = |E_{\widetilde{C}}(v_d, H)| + |E_{\widetilde{C}}(Q_i^1, H)| \leq 1 + k = |V(P_i^1)| - 1$.

Then we analysis the value of $|E_{\widetilde{C}}(P_{h_1}^1, H)|$. If $|V(P_{h_1}^1)| = 1$, the inequality $|E_{\widetilde{C}}(P_{h_1}^1, H)| \leq k$ clearly holds. If $|V(P_{h_1}^1)| = 2$, that is, $P_{h_1}^1 = v_d v_{d+1}$, we have $|E_{\widetilde{C}}(v_d, H)| + |E_{\widetilde{C}}(v_{d+1}, H)| \leq k$ since v_d and v_{d+1} are adjacent. Therefore, $|E_{\widetilde{C}}(P_{h_1}^1, H)| = E_{\widetilde{C}}(v_d, H)| + |E_{\widetilde{C}}(v_{d+1}, H)| \leq k$. If $3 \leq |V(P_{h_1}^1)| \leq k+1$, we have $|E_{\widetilde{C}}(P_{h_1}^1, H)| \leq k$ when $|E_{\widetilde{C}}(v_d, H)| \geq 2$ by the similar analysis of the above Case 1 (taking m as $d+k$), and $|E_{\widetilde{C}}(P_{h_1}^1, H)| \leq k+1$ when $|E_{\widetilde{C}}(v_d, H)| = 1$ by the similar analysis of the above Case 2 (taking m as $d+k+1$). The proof is thus completed. □

Let $P^2 = aC^-b$. Then we design another algorithm to generate a sequence of disjoint sub-paths $P_1^2, P_2^2, ..., P_{h_2}^2$ of C respect to P^2 and H in the following.

Algorithm AII

Input: a strongly edge-colored graph G, a rainbow cycle $C = v_1 v_2 \dots v_l v_1$, $P^2 = aC^-b = v_{l+1}v_l v_{l-1}...v_m$ and a rainbow complete subgraph $H = K_k$ of $G^c - V(C)$.
Output: a sequence of disjoint paths $P_1^2, P_2^2, ..., P_{h_2}^2$ such that P_i^2 is a subgraph of C.
1: **Set** $i = 1$
2: **While** $V(P^2) \neq \phi$ **do**
  If $E_{\widetilde{C}}(P^2, H) = \phi$
    stop
  Else Set d be the biggest subscript for which $E_{\widetilde{C}}(v_d, H) \neq \phi$
    If $d - k \leq m$ **then**
      Set $P_i^2 = v_d v_{d-1}...v_m$
      stop
    Else **If** $|E_{\widetilde{C}}(v_d, H)| \geq 2$ **then**
        Set $P_i^2 = v_d v_{d-1}...v_{d-k}$
      If $|E_{\widetilde{C}}(v_d, H)| = 1$ **then**
        Set $P_i^2 = v_d v_{d-1}...v_{d-k-1}$
    Set $P^2 = P^2 \setminus P_i^2$
    Set $i = i + 1$
3: **return** $P_1^2, P_2^2, ..., P_{h_2}^2$

Similar to Claim 4, we can get the following Claim.

Claim 5 $|E_{\widetilde{C}}(P_i^2, H)| \leq |V(P_i^2)| - 1$ *for all* $1 \leq i \leq h_2 - 1$, $|E_{\widetilde{C}}(P_{h_2}^2, H)| \leq k$ *if* $|V(P_{h_2}^2)| \in \{1, 2\}$ *and* $|E_{\widetilde{C}}(P_{h_2}^2, H)| \leq k + 1$ *if* $3 \leq |V(P_{h_2}^2)| \leq k + 1$.

According to the above claims, we have

$$\begin{aligned} |E_{\widetilde{C}}(C, H)| &= |E_{\widetilde{C}}(aC^+b, H)| + |E_{\widetilde{C}}(aC^-b, H)| - |E_{\widetilde{C}}(a, H)| - |E_{\widetilde{C}}(b, H)| \\ &\leq \sum_{i=1}^{h_1-1} |V(P_i^1)| - (h_1 - 1) + |E_{\widetilde{C}}(P_{h_1}^1, H)| \\ &\quad + \sum_{i=1}^{h_2-1} |V(P_i^2)| - (h_2 - 1) + |E_{\widetilde{C}}(P_{h_2}^2, H)| \\ &\quad - |E_{\widetilde{C}}(a, H)| - |E_{\widetilde{C}}(b, H)| \\ &\leq [l - |V(P_{h_1}^1)| - |V(P_{h_2}^2)| + 1] - (h_1 + h_2) + 2 \\ &\quad + |E_{\widetilde{C}}(P_{h_1}^1, H)| + |E_{\widetilde{C}}(P_{h_2}^2, H)| - |E_{\widetilde{C}}(a, H)| - |E_{\widetilde{C}}(b, H)| \\ &= l - (|V(P_{h_1}^1)| + |V(P_{h_2}^2)|) - (h_1 + h_2) + 3 \\ &\quad + |E_{\widetilde{C}}(P_{h_1}^1, H)| + |E_{\widetilde{C}}(P_{h_2}^2, H)| - |E_{\widetilde{C}}(a, H)| - |E_{\widetilde{C}}(b, H)|. \end{aligned} \tag{8}$$

Claim 6 $\widetilde{S} \leq l + 2k - 4$.

Proof. We show that $\widetilde{S} \leq max\{2k+2, l+k-1, l+2k-4\}$, which implies $\widetilde{S} \leq l+2k-4$ since $l \geq 7$ from Claim 1 and $k \geq 3$ from Claim 2.

Let $h = h_1 + h_2$. By symmetry, we suppose $h_1 \geq h_2$ and $|V(P^1_{h_1})| \geq |V(P^2_{h_2})|$. From Claim 3, we have $h \geq 1$. Then we proceed our proof by distinguishing the following four cases.

Case 1. $h_1 = 1$ and $h_2 = 0$. From Algorithm AII, we have $E_{\widetilde{C}}(aC^-b, H) = \phi$. Thus, $E_{\widetilde{C}}(a, H) = \phi$ and $E_{\widetilde{C}}(b, H) = \phi$. From Algorithm AI, we have $|V(P^1_{h_1})| \geq 2$. If $|V(P^1_{h_1})| = 2$, let u be the vertex distinct from b in C such that $E_{\widetilde{C}}(u, H) \neq \phi$. Thus we have $\widetilde{S} = |E_{\widetilde{C}}(u, H)| \leq k < 2k+2$. If $|V(P^1_{h_1})| \geq 3$, from Claim 4, we have $\widetilde{S} = E_{\widetilde{C}}(P^1_{h_1}, H) \leq k+1 < 2k+2$. The claim follows.

Case 2. $h_1 \geq 2$ and $h_2 = 0$. From Algorithm AI and AII, we have $E_{\widetilde{C}}(a, H) = \phi$, $E_{\widetilde{C}}(b, H) = \phi$ and $|V(P^1_{h_1})| \geq 2$. If $|V(P^1_{h_1})| = 2$, since $E_{\widetilde{C}}(b, H) = \phi$, we have $|E_{\widetilde{C}}(P^1_{h_1}, H)| + |E_{\widetilde{C}}(P^2_{h_2}, H)| - |E_{\widetilde{C}}(b, H)| = |E_{\widetilde{C}}(P^1_{h_1}, H)| \leq k$. Applying inequality (8), we have $\widetilde{S} \leq l-2-2+3+k+0 = l+k-1$. If $|V(P^1_{h_1})| \geq 3$, from Claim 4, we have $|E_{\widetilde{C}}(P^1_{h_1}, H)| + |E_{\widetilde{C}}(P^2_{h_2}, H)| - |E_{\widetilde{C}}(b, H)| = |E_{\widetilde{C}}(P^1_{h_1}, H)| \leq k+1$. Thus, by inequality (8), we have $\widetilde{S} \leq l-3-2+3+k+1+0 = l+k-1$. The claim follows.

Case 3. $h_1 = 1$ and $h_2 = 1$. By Claim 4 and 5, if $|V(P^1_{h_1})| \in \{1, 2\}$ and $|V(P^2_{h_2})| \in \{1, 2\}$, we have $\widetilde{S} \leq |E_{\widetilde{C}}(P^1_{h_1}, H)| + |E_{\widetilde{C}}(P^2_{h_2}, H)| \leq 2k < 2k+2$. If $|V(P^1_{h_1})| \geq 3$ and $|V(P^2_{h_2})| \in \{1, 2\}$, we have $\widetilde{S} \leq |E_{\widetilde{C}}(P^1_{h_1}, H)| + |E_{\widetilde{C}}(P^2_{h_2}, H)| \leq 2k+1 < 2k+2$. If $|V(P^1_{h_1})| \geq 3$ and $|V(P^2_{h_2})| \geq 3$, we have $\widetilde{S} \leq |E_{\widetilde{C}}(P^1_{h_1}, H)| + |E_{\widetilde{C}}(P^2_{h_2}, H)| \leq 2k+2$. The claim holds.

Case 4. $h \geq 3$ and $h_2 \geq 1$. We consider the following six cases.

Case 4.1. $|V(P^1_{h_1})| = 1$ and $|V(P^2_{h_2})| = 1$. It is clearly that

$$V(P^1_{h_1}) = V(P^2_{h_2}) = \{b\}$$

and

$$|E_{\widetilde{C}}(P^1_{h_1}, H)| + |E_{\widetilde{C}}(P^2_{h_2}, H)| - |E_{\widetilde{C}}(b, H)| = |E_{\widetilde{C}}(b, H)| \leq k.$$

By inequality (8), we have

$$\widetilde{S} = |E_{\widetilde{C}}(C, H)| \leq l-2-3+3+k+0 = l+k-2 < l+k-1.$$

Case 4.2. $|V(P^1_{h_1})| = 2$ and $|V(P^2_{h_2})| = 1$. It is clearly that $V(P^2_{h_2}) = \{b\}$. From Claim 4, we have

$$|E_{\widetilde{C}}(P^1_{h_1}, H)| + |E_{\widetilde{C}}(P^2_{h_2}, H)| - |E_{\widetilde{C}}(b, H)| = |E_{\widetilde{C}}(P^1_{h_1}, H)| \leq k.$$

By inequality (8) and $h \geq 3$, we have

$$\widetilde{S} \leq l-3-3+3+k+0 = l+k-3 < l+k-1.$$

Case 4.3. $|V(P^1_{h_1})| \geq 3$ and $|V(P^2_{h_2})| = 1$. It is clearly that $V(P^2_{h_2}) = \{b\}$. From Claim 4, we have

$$|E_{\widetilde{C}}(P^1_{h_1}, H)| + |E_{\widetilde{C}}(P^2_{h_2}, H)| - |E_{\widetilde{C}}(b, H)| = |E_{\widetilde{C}}(P^1_{h_1}, H)| \leq k+1.$$

By inequality (8) and $h \geq 3$, we have

$$\widetilde{S} = |E_{\widetilde{C}}(C, H)| \leq l-4-3+3+k+1+0 = l+k-3 < l+k-1.$$

Case 4.4. $|V(P^1_{h_1})| = 2$ and $|V(P^2_{h_2})| = 2$. From Claim 4 and 5, we have

$$|E_{\widetilde{C}}(P^1_{h_1}, H)| + |E_{\widetilde{C}}(P^2_{h_2}, H)| - |E_{\widetilde{C}}(b, H)| \leq 2k.$$

By inequality (8) and $h \geq 3$, we have

$$\widetilde{S} = |E_{\widetilde{C}}(C, H)| \leq l - 4 - 3 + 3 + 2k + 0 = l + 2k - 4 < l + k - 1.$$

Case 4.5. $|V(P^1_{h_1})| \geq 3$ and $|V(P^2_{h_2})| = 2$. It is clearly that

$$|E_{\widetilde{C}}(P^1_{h_1}, H)| + |E_{\widetilde{C}}(P^2_{h_2}, H)| - |E_{\widetilde{C}}(b, H)| \leq k + k + 1 = 2k + 1.$$

By inequality (8) and $h \geq 3$, we have

$$\widetilde{S} = |E_{\widetilde{C}}(C, H)| \leq l - 5 - 3 + 3 + 2k + 1 + 0 = l + 2k - 4.$$

Case 4.6. $|V(P^1_{h_1})| \geq 3$ and $|V(P^2_{h_2})| \geq 3$. From Claim 4 and 5, we have

$$|E_{\widetilde{C}}(P^1_{h_1}, H)| + |E_{\widetilde{C}}(P^2_{h_2}, H)| - |E_{\widetilde{C}}(b, H)| \leq k + 1 + k + 1 = 2k + 2.$$

By inequality (8), we have

$$\widetilde{S} = |E_{\widetilde{C}}(C, H)| \leq l - 6 - 3 + 3 + 2k + 2 + 0 = l + 2k - 4.$$

The Claim follows. □

From Claim 6, inequalities (5) (6) and (7), we can deduce that

$$\begin{aligned} k\delta &\leq \widetilde{S} + S + \widetilde{T} + T + k(k-1) \\ &\leq l + 2k - 4 + l + (k-1)(n-l-k) + k(k-1) \\ &= l + 2k - 4 + k(n-l) + 2l - n. \end{aligned}$$

Therefore, we have $k(n - l - \delta + 2) \geq n - 3l + 4$. Since $l \geq \frac{n+12}{3}$ from Claim 1 and $\delta \geq \frac{2n}{3} + 1$, we have $n - l - \delta + 2 < 0$. Then from Claim 2, we have

$$3(n - l - \delta + 2) \geq k(n - l - \delta + 2) \geq n - 3l + 4,$$

which implies that $\delta \leq \frac{2n+2}{3}$, a contradiction. We complete the proof of Theorem 1.6. □

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