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# *Several Roman domination graph invariants on Kneser graphs*

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This paper considers the following three Roman domination graph invariants on Kneser graphs: Roman domination, total Roman domination, and signed Roman domination. For Kneser graph $K_{n,k}$, we present exact values for Roman domination number $\gamma_R(K_{n,k})$ and total Roman domination number $\gamma_{tR}(K_{n,k})$ proving that for $n \geqslant k(k+1)$, $\gamma_R(K_{n,k}) = \gamma_{tR}(K_{n,k}) = 2(k+1)$. For signed Roman domination number $\gamma_{sR}(K_{n,k})$, the new lower and upper bounds for $K_{n,2}$ are provided: we prove that for $n \geqslant 12$, the lower bound is equal to 2, while the upper bound depends on the parity of $n$ and is equal to 3 if $n$ is odd, and equal to 5 if $n$ is even. For graphs of smaller dimensions, exact values are found by applying exact methods from literature.

**Keywords:** Kneser graphs, Roman domination, total Roman domination, signed Roman domination

## 1 Introduction

Let $G = (V, E)$ be a simple connected graph, with a set of vertices $V$, a set of edges $E$ and its order $|V|$. For an arbitrary vertex $v \in V$, open neighborhood $N(v)$ is defined as set $\{u \in V \mid uv \in E\}$, while closed neighborhood $N[v]$ is set $N[v] = N(v) \cup \{v\}$.

*The domination set* $S$ of graph $G$ is defined as the subset of set $V$ such that

$$(\forall u \in V \setminus S)(\exists v \in S) \ uv \in E.$$

The minimum cardinality $\gamma(G)$ of a domination set is called *the domination number* of graph $G$.

*Roman domination function* (RDF) on graph $G$, formally introduced by Cockayne et al. (2004), is defined as function $f : V \to \{0, 1, 2\}$ which satisfies the condition

$$(\forall v \in V_0)(\exists u \in V_2) \ uv \in E, \tag{1}$$

where $V_i = \{v \in V \mid f(v) = i\}$, $i = 0, 1, 2$. The *weight* of function $f$ is value $f(V) = \sum_{v \in V} f(v)$. The minimum value of the weights of all RDFs on graph $G$, denoted with $\gamma_R(G)$, is called the *Roman domination number* (RDN).

The basic relation between domination and Roman domination numbers is given in the following property.

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**Property 1.** *Cockayne et al. (2004) For any graph $G$, it holds $\gamma(G) \leq \gamma_R(G) \leq 2 \cdot \gamma(G)$.*

*Total Roman domination function* (TRDF), introduced by Liu and Chang (2013), is defined as function $f : V \to \{0, 1, 2\}$, i.e., by the partition $(V_0, V_1, V_2)$ of set $V$, which satisfies conditions (1) and

$$(\forall v \in V) \sum_{u \in N(v)} f(u) \geqslant 1. \tag{2}$$

In literature, condition (2) is also introduced with an equivalent property that the subgraph of graph $G$ induced with vertices with a positive label has no isolated vertices.

*The total Roman domination number* (TRDN) of graph $G$, denoted with $\gamma_{tR}(G)$, is the minimum weight $f(V) = \sum_{v \in V} f(v)$ of all TRDFs $f$ on $G$. As each TRDF satisfies condition (1), it is also an RDF. Therefore, the following observation is straightforward.

**Observation 1.** *Ahangar et al. (2016) For each graph $G$ without isolated vertices, it holds $\gamma_R(G) \leqslant \gamma_{tR}(G)$.*

*Signed Roman domination function* (SRDF) is a function $f : V \to \{-1, 1, 2\}$ for which it holds

$$(\forall v \in V_{-1})(\exists u \in V_2) \ uv \in E, \tag{3}$$

where $V_i = \{v \in V \mid f(v) = i\}$, $i \in \{-1, 1, 2\}$ and

$$(\forall v \in V) \sum_{u \in N[v]} f(u) \geqslant 1. \tag{4}$$

For proving new results, the following equivalent of the last condition is introduced. For $v \in V$, let $\alpha_v, \beta_v$ and $\gamma_v$ represent cardinalities $|N(v) \cap V_2|, |N(v) \cap V_1|$ and $|N(v) \cap V_{-1}|$, respectively. Then condition (4) is equivalent to condition (5)

$$(\forall v \in V) \ 2\alpha_v + \beta_v - \gamma_v + f(v) \geqslant 1. \tag{5}$$

*The signed Roman domination number* (SRDN) $\gamma_{sR}(G)$ of graph $G$ is the minimum weight of all SRDFs on graph $G$.

The concept of *the Kneser graph* $K_{n,k}, n, k \in \mathbb{N}$ is introduced by Kneser (1955). The set of vertices of graph $K_{n,k}$ is set of all $k$-element subsets of set $\{1, 2, \ldots, n\}$ and two vertices are adjacent if corresponding sets are disjoint. Its order is $\binom{n}{k}$ and this is a type of regular graph with degree of each vertex equal to $\binom{n-k}{k}$. If $n < 4$, graph $K_{n,2}$ is edgeless. $K_{n,1}$ is the complete graph, while $K_{2k,k}$ for $k > 1$ is not connected. Therefore, in the rest of the paper we suppose that $n > 2k$ and $k > 1$. An example of the Kneser graph for $n = 5$ and $k = 2$ ($K_{5,2}$) is given in Figure 1. It can be noticed that this graph is isomorphic to the Petersen graph.

## 1.1 Previous work

Graph $G$ is said to be a Roman graph if $\gamma_R(G) = 2\gamma(G)$. Several classes of Roman graphs were studied by Cockayne et al. (2004); Henning (2002); Yero and Rodríguez-Velázquez (2013); Xueliang et al. (2009). The exact result for the RDN of generalized Petersen graphs was given by Wang et al. (2011). Some more results regarding RDN can be found in Mobaraky and Sheikholeslami (2008); Liu and Chang (2012);

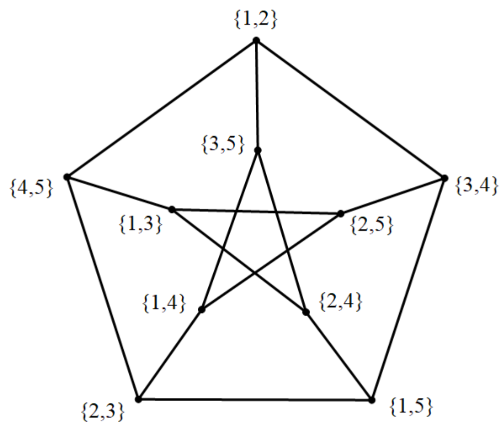


**Fig. 1:** The Kneser graph $K_{5,2}$

Favaron et al. (2009); Kartelj et al. (2021); Li (2021), for example. A detailed review of results on many variants of RDN is out of the scope of this paper and can be found in Chellali et al. (2020, 2021).

The relation between TRDN and (total) domination number as well as with RDN was studied by Martínez et al. (2020); Ahangar et al. (2016). Several bounds on SRDN in terms of graph order, size, minimum and maximum vertex degree and (signed) domination number were explained by Ahangar et al. (2014). The authors also gave the exact value of a SRDN for some special graph classes: $\gamma_{sR}(K_3) = 2$ and $\gamma_{sR}(K_n) = 1, n \neq 3$, $\gamma_{sR}(K_{1,n-1}) = 1, n = 2l$ and $\gamma_{sR}(K_{1,n-1}) = 2, n = 2l+1, l \in \mathbb{N}$, $\gamma_{sR}(C_n) = \lceil \frac{2n}{3} \rceil, n \geqslant 3$ and $\gamma_{sR}(P_n) = \lfloor \frac{2n}{3} \rfloor, n \geqslant 1$ . The SRDN was also considered for: digraphs by Sheikholeslami and Volkmann (2015), trees by Henning and Volkmann (2015), the join of graphs by Behtoei et al. (2014) and planar graphs by Zec et al. (2021).

The value of the domination number for the Kneser graph is determined in Theorem 1.

**Theorem 1.** *Ostergard et al. (2014) For $n \geq k \cdot (k+1)$, it holds $\gamma(K_{n,k}) = k+1$.*

Domination problems are quite an attractive research domain which has captivated researchers from various fields over the past few decades, including mathematicians and computer scientists. It is known that, for example, determining the Roman domination number in case of general graphs is NP–hard Cockayne et al. (2004). That implies that a successful application of provenly strong exact computational paradigms is not expected for arbitrary large graphs. Thus, general widely–applied exact methods, such as the branch-and-bound framework Lawler and Wood (1966), are usually restricted to a successful application onto small to middle-sized graphs. However, these techniques still serve here in several ways $(i)$ to determine Roman domination-type numbers on small-sized graphs, and $(ii)$ to get an insight into these numbers in case of some graph classes w.r.t. graph parameters. Please note that from the theoretical point of view, these techniques do not provide any proof on established Roman domination numbers. In this work, some exact methods based on Integer linear programming (ILP) techniques Graver (1975) are used for solving the corresponding problems for some Kneser graphs of small dimensions. More precisely, the model given by Burger et al. (2013) is used to obtain the results presented in Remark 1 and Remark 2, and the model exposed by Filipović et al. (2022) is used to obtain the results presented in Remark 3. The formulations of these ILP models are given in Appendix A.

# 2 New results for Kneser graphs

## 2.1 (Total) Roman domination for Kneser graphs

In this section we present exact values for (total) Roman domination numbers for Kneser graphs.

**Theorem 2.** *For $n \geq k \cdot (k+1), k > 1$, it holds $\gamma_{tR}(K_{n,k}) = 2(k+1)$.*

**Proof:** **Step 1:** $\gamma_{tR} \geqslant 2(k+1)$**.**
First, let us show that for an arbitrary TRDF $\overline{f}$ it holds

$$\overline{f}(V) \geqslant 2(k+1). \tag{6}$$

Suppose that $\overline{f}$ is defined by partition $(V_0, V_1, V_2)$. We consider all possible values for $|V_2|$.

**Case 1:** $|V_2| = 0$.
Since $\overline{f}$ is TRDF and $V_2$ is empty, then $V_0$ is empty as well, so $\overline{f}(V) = |V_1|$. Therefore

$$\begin{aligned} |V_1| = |V| = \binom{n}{k} &= \frac{n(n-1)\dots(n-k+1)}{k!} \\ &\geqslant \frac{(k^2+k)(k^2+k-1)\dots(k^2+1)}{k!} > \frac{k^{2k}}{k!}. \end{aligned}$$

For $k = 2$ we have $\frac{k^{2k}}{k!} = 8 > 6 = 2(k+1)$. If $k \geqslant 3$, we get $\frac{k^{2k}}{k!} = \frac{k^k}{k!} k^k \geqslant k^k \geqslant 2(k+1)$. So, in this case $\overline{f}(V) \geqslant 2(k+1)$ holds.

**Case 2:** $1 \leqslant |V_2| \leqslant k-1$.
At most $|V_2| \cdot k$ different numbers (from set $\{1, \dots n\}$) are used to form vertices from set $V_2$. Then at least $n - |V_2| \cdot k \geqslant k^2 + k - (k-1) \cdot k = 2k$ different numbers do not appear in any vertex from set $V_2$. Let $X$ denote set of these numbers. Let us identify $2k$ vertices not belonging to set $V_2$ and not adjacent to any vertex from set $V_2$. Let $\{s_1, s_2, \dots, s_{k-1}\}$ be set of some $k-1$ different numbers which are chosen such that every vertex from $V_2$ contains at least one of these numbers $s_i$, $i = 1, \dots, k-1$. An illustration of this procedure is shown in Example 1. Now, let $Y = \{\{s_1, s_2, \dots, s_{k-1}, s\} : s \in X\}$. It holds that $Y \cap V_2 = \emptyset$. In addition, any vertex from $Y$ is not adjacent to any vertex from set $V_2$. Since $\overline{f}$ is a TRDF, we conclude that $Y \subseteq V_1$. Therefore, $|V_1| \geqslant |Y| = 2k$, so using assumption $|V_2| \geqslant 1$, we get

$$\overline{f}(V) = 2|V_2| + |V_1| \geqslant 2|V_2| + 2k \geqslant 2|V_2| + 2k - 2|V_2| + 2 = 2(k+1).$$

**Case 3:** $|V_2| = k$.
Similarly, as in Case 2, at most $|V_2| \cdot k = k^2$ different numbers are used to form vertices from set $V_2$. Let us consider the numbers which do not appear in any vertex from set $V_2$ and denote set of such numbers with $X$. Therefore, it holds $|X| \geqslant n - k^2 \geqslant k^2 + k - k^2 = k$. We here analyze two subcases.

*Subcase 3.1:* All vertices from set $V_2$ are adjacent to each other.
Then, by choosing one number per each vertex, we can identify total $k^k$ vertices, such that neither of them

is adjacent to any vertex from $V_2$. Also notice that none of these vertices belong to $V_2$. Therefore we conclude that all these vertices belong to set $V_1$. Thus, we get $|V_1| \geqslant k^k \geqslant 2k$ and

$$\overline{f}(V) = 2|V_2| + |V_1| \geqslant 2|V_2| + 2k \geqslant 2|V_2| + 2k - 2|V_2| + 2 \geqslant 2k + 2 = 2(k+1).$$

*Subcase 3.2:* There exists at least one pair of non-adjacent vertices in set $V_2$.
Let $u, v \in V_2$ be vertices such that $u \cap v \neq \emptyset$ and $s_1 \in u \cap v$. If we choose numbers $s_2, s_3 \ldots, s_{k-1}$ such that each of the remaining $(k-2)$ vertices from set $V_2$ contains at least one of these numbers, then set $\{s_1, s_2, ..., s_{k-1}\}$ has the same properties like the appropriate one from Case 2. Similarly as above, we conclude that each vertex of form $\{s_1, s_2, \ldots, s_{k-1}, s\}$, where $s \in X$ , belongs to set $V_1$. Therefore, $|V_1| \geqslant k \geqslant 2$ and again

$$\overline{f}(V) = 2|V_2| + |V_1| \geqslant 2k + 2 = 2(k+1),$$

which concludes Case 3.

**Case 4:** $|V_2| \geqslant k + 1$.
In this case, it trivially holds that $\overline{f}(V) \geqslant 2(k+1)$.

**Step 2:** $\gamma_{tR} \leqslant 2(k+1)$**.**
Following the idea from Ostergard et al. (2014), we construct function $f$ as follows. Let $V_2$ be a collection of $k+1$ disjoint $k$-sets defined as

$$V_2 = \{\{1, 2, \ldots, k\}, \{k+1, k+2, \ldots, 2k\}, \ldots, \{k^2+1, k^2+2, \ldots, k^2+k\}\}.$$

Let $V_1 = \emptyset$ and $V_0 = V \backslash V_2$. The weight of the function $f$ is

$$f(V) = 2|V_2| = 2(k+1).$$

Let us show that $f$ is a TRDF. Each vertex, i.e., $k$-element set from $V_0$ is non-disjoint with at most $k$ vertices from set $V_2$, so it is disjoint with at least one vertex from $V_2$. Therefore, condition (1) is satisfied.

Given the previous consideration, for each vertex $v \in V_0$, $\sum_{u \in N(v)} f(u) \geqslant 1$ also holds. From the construction of set $V_2$, each vertex $v \in V_2$ is adjacent to all other vertices from $V_2$. So, for all $v \in V_2$, it holds that $\sum_{u \in N(v)} f(u) = 2k \geqslant 1$. Condition (2) is thus satisfied, which concludes the proof. □

**Example 1.** *By this example we illustrate the procedure shown in Case 2 of the previous theorem.*
*Let* $n = 20,\ k = 4$ *and let* $f$ *be a TRDF for* $K_{20,4}$*, such that*

$$V_2 = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{5, 6, 9, 10\}\}.$$

*Notice that* $n = k^2 + k$ *and* $|V_2| = k - 1$.

*Let us take three different numbers:* $s_1, s_2,$ *and* $s_3$ *such that for each* $v \in V_2$ *it holds* $v \cap \{s_1, s_2, s_3\} \neq \emptyset$*. For example, let* $s_1 = 1, s_2 = 5,$ *and* $s_3 = 8$.

*The vertices from set* $V_2$ *contain* $10$ *different elements, which is less than* $|V_2| \cdot k = 12$.

*Let* $X = \{13, 14, \ldots, 20\}$ *and* $Y = \{\{s_1, s_2, s_3, s\} : s \in X\} \subset V$*. It is obvious that for every two vertices* $u \in Y$ *and* $v \in V_2$*, it holds* $u \cap v \neq \emptyset$*, i.e., in graph* $K_{20,4}$ *no vertex from set* $Y$ *has a neighbor from set* $V_2$*. Also, for each* $u \in Y$*, it holds that* $u \notin V_2$*. Therefore, since* $f$ *is TRDF, every such vertex must belong to set* $V_1$*, so* $|V_1| \geqslant |Y| = 8 = 2k$.

The result for the Roman domination number follows straightforwardly.

**Corollary 1.** *For* $n \geqslant k \cdot (k+1), k > 1$, *it holds* $\gamma_R(K_{n,k}) = 2(k+1)$.

**Proof:**

**Step 1:** $\gamma_R(K_{n,k}) \geqslant 2(k+1)$**.**

One can notice that the complete proof for the lower bound of TRDN in Step 1 of Theorem 2 is based only on using property (1). Since each RDF must also satisfy that property, the same lower bound holds for RDN.

**Step 2:** $\gamma_R(K_{n,k}) \leqslant 2(k+1)$**.**

This inequality is a straightforward consequence of Theorem 2 and Observation 1.

□

**Observation 2.** *The inequality in Step 2 of Corollary 1 also follows from Theorem 1 and Property 1.*

The following two remarks (Remark 1 and Remark 2) contain results for Kneser graphs $K_{n,2}$ and $K_{n,3}$, which are not covered by Corollary 1. As previously mentioned, we used the ILP model from Burger et al. (2013) to find RDN of these graphs. The RDFs which correspond to these solutions and ILP model details are presented in Appendix A.

**Remark 1.** *It holds*

$$\gamma_R(K_{5,2}) = 6.$$

**Remark 2.** *It holds*

$$\gamma_R(K_{n,3}) = \begin{cases} 14, & n = 7, 8, 9, \\ 12, & n = 10, \\ 10, & n = 11. \end{cases}$$

## 2.2 Signed Roman domination for Kneser graphs

In this section we present new lower and upper bounds for the signed Roman domination number for Kneser graphs $K_{n,2}$.

**Theorem 3.** *For* $n \geqslant 12$ *it holds:*

- $2 \leqslant \gamma_{sR}(K_{n,2}) \leqslant 3,\ n$ is odd,
- $2 \leqslant \gamma_{sR}(K_{n,2}) \leqslant 5,\ n$ is even.

**Proof:** **Step 1:** $\gamma_{sR}(K_{n,2}) \geqslant 2$.

Let $\overline{f}$ be an arbitrary SRDF, defined as $(V_{-1}, V_1, V_2)$. Then for every vertex $v \in V$, inequality in condition (4) holds. By summing up all the inequalities from condition (4), we get

$$\sum_{v \in V} \sum_{u \in N[v]} \overline{f}(u) \geqslant \binom{n}{2} \tag{7}$$

Since each vertex $v \in V$ has the degree $\binom{n-2}{2}$, at the left hand side of inequality (7), value $\overline{f}(v)$ appears exactly $\binom{n-2}{2} + 1$ times. Thus,

$$\left(\binom{n-2}{2}+1\right)\sum_{v\in V}\overline{f}(v)\geqslant\binom{n}{2} \;\Leftrightarrow \left(\binom{n-2}{2}+1\right)\overline{f}(V)\geqslant\binom{n}{2} \;\Leftrightarrow$$
$$\overline{f}(V)\geqslant\frac{n(n-1)}{(n-2)(n-3)+2}.$$

Expression $\frac{n(n-1)}{(n-2)(n-3)+2}$ is greater than 1 and since the SRDN must be an integer, it holds that $\gamma_{sR}(K_{n,2})\geqslant 2$, which concludes the proof of Step 1.

**Step 2**: $\gamma_{sR}(K_{n,2})\leqslant 3,\ n$ is odd and $\gamma_{sR}(K_{n,2})\leqslant 5, n$ is even.

**Case 1:** $n$ is odd.

Let us partition the set $\{1,2,\dots,n\}$ on sets $A_n$ and $B_n$ and the set $V$ on sets $A_{n,2}, B_{n,2}$ and $C_{n,2}$, as shown in Tab. 1. We introduce the function $f=(V_{-1},V_1,V_2)$, where sets $V_{-1},V_1$ and $V_2$ are given in the last three rows of Tab. 1. We show that $f(V)=3$ and $f$ is SRDF.

| | |
|---|---|
| $A_n$ | $\{1,2,\dots,\frac{n-3}{2}\}$ |
| $B_n$ | $\{\frac{n-1}{2},\frac{n+1}{2},\dots,n\}$ |
| $A_{n,2}$ | $\{\{a,b\}\|a,b\in A_n\}$ |
| $B_{n,2}$ | $\{\{a,b\}\|a,b\in B_n\}$ |
| $C_{n,2}$ | $\{\{a,b\}\|a\in A_n,b\in B_n\}$ |
| $V_2$ | $\{\{1,2\},\{2,3\},\{3,4\},\dots,\{\frac{n-5}{2},\frac{n-3}{2}\},\{1,\frac{n-3}{2}\}\}$ |
| $V_{-1}$ | $C_{n,2}$ |
| $V_1$ | $V\setminus(V_2\cup V_{-1})$ |

**Tab. 1:** The construction of SDRF $f$ for which $f(V)=3$

Notice that $V_2\subset A_{n,2}$ and $V_1=(A_{n,2}\setminus V_2)\cup B_{n,2}$.

It holds $|A_n|=\frac{n-3}{2},\ |B_n|=\frac{n+3}{2},\ |A_{n,2}|=\binom{(n-3)/2}{2}, |B_{n,2}|=\binom{(n+3)/2}{2}$ and $|C_{n,2}|=\frac{n-3}{2}\cdot\frac{n+3}{2}=\frac{n^2-9}{4}$.

We have $|V_2|=\frac{n-3}{2},\ |V_1|=\binom{(n-3)/2}{2}-\frac{n-3}{2}+\binom{(n+3)/2}{2}=\frac{n^2-4n+15}{4}$ and $|V_{-1}|=\frac{n^2-9}{4}$, so $f(V)=2|V_2|+|V_1|-|V_{-1}|=3$.

Let us now prove that $f$ is an SRDF.

Let $v=\{a,b\}\in V_{-1}$ be an arbitrary vertex. W.l.o.g. suppose that $a\in A_n$. From the definition of sets $V_{-1}$ and $V_2$, it follows that $a$ occurs in exactly two vertices of set $V_2$, so $v$ has exactly $|V_2|-2=\frac{n-7}{2}>0$ neighbors labeled by 2. The conclusion is that condition (3) is satisfied.

Let us now prove that condition (5) is satisfied.

(*i*) First let $v=\{a,b\}$ be an arbitrary vertex from set $V_2$.

Notice that $a,b\in A_n$. From the definition of $V_2$, it follows that $a$ and $b$ occur in exactly 3 vertices in set $V_2$, including vertex $v$.

Let $\{a,e\}$ and $\{b,f\}$ be the other two vertices from $V_2$ which contain $a$ and $b$, respectively. So $\alpha_v=|V_2|-|\{v,\{a,e\},\{b,f\}\}|=\frac{n-3}{2}-3=\frac{n-9}{2}$.

To calculate $\beta_v$, we now observe those vertices from set $V_1$ which are not adjacent to $v$, i.e., those which contain $a$ or $b$. These vertices are from set $A_{n,2}\setminus V_2$ of the form $\{a,c\}$, where $c\in A_n\setminus$

$\{a,b,e\}$, or of the form $\{b,d\}$, where $d \in A_n \setminus \{a,b,f\}$. The total number of such vertices is $2 \cdot \left(\frac{n-3}{2} - 3\right)$. Therefore, $\beta_v = |V_1| - 2 \cdot \left(\frac{n-3}{2} - 3\right) = \frac{n^2-8n+51}{4}$.

As vertices from set $V_{-1}$ which are not adjacent to $v$ are those of form $\{a,c\}$ and $\{b,c\}$ for each $c \in B_n$, vertex $v$ has $\gamma_v = |V_{-1}| - 2 \cdot \frac{n+3}{2} = \frac{n^2-4n-21}{4}$ neighbors in this set.

Finally, for $v \in V_2$, it holds $2\alpha_v + \beta_v - \gamma_v + f(v) = 11$, the conclusion being that condition (5) is satisfied for vertices from set $V_2$.

*(ii)* For $v = \{a,b\} \in V_1$, we have two possibilities: $v \in A_{n,2} \setminus V_2$ or $v \in B_{n,2}$.

- $v \in A_{n,2} \setminus V_2$.
  Here $a, b \in A_n$ and these elements are contained in exactly 4 vertices, namely $\{a,e\}, \{a,f\}, \{b,g\}$ and $\{b,h\}$, which are all labeled with 2.
  This implies $\alpha_v = |V_2 \setminus \{\{a,e\},\{a,f\},\{b,g\},\{b,h\}\}| = |V_2| - 4 = \frac{n-11}{2}$.
  To calculate $\beta_v$ for this case, we again observe the vertices from set $V_1$ which are not adjacent to $v$. Such vertices form set
  $$\{v\} \cup \{\{a,c\} | c \in A_n \setminus \{a,b,e,f\}\} \cup \{\{b,c\} | c \in A_n \setminus \{a,b,g,h\}\}.$$
  The cardinality of this set is equal to $1 + 2 \cdot \left(\frac{n-3}{2} - 4\right) = n - 10$. Therefore, we get $\beta_v = |V_1| - (n-10) = \frac{n^2-8n+55}{4}$. The set of neighbors in set $V_{-1}$ which are not adjacent to $v$ are of the form $\{a,c\}$ and $\{b,c\}$ for each $c \in B_n$. So, $\gamma_v = |V_{-1}| - 2 \cdot \frac{n+3}{2} = \frac{n^2-4n-21}{4}$. So, for this case, we conclude $2\alpha_v + \beta_v - \gamma_v + f(v) = 9$.
- $v \in B_{n,2}$.
  Here $a, b \in B_n$, so neither $a$ or $b$ are contained in any vertex from $V_2$, which gives $\alpha_v = |V_2|$. Neighbors of $v$ from $V_1$ form set $(A_{n,2} \setminus V_2) \cup \{\{c,d\} | c,d \in B_n \setminus \{a,b\}\}$, with its cardinality equal to
  $$\beta_v = \binom{(n-3)/2}{2} - |V_2| + \binom{(n+3)/2 - 2}{2} = \frac{n^2-8n+15}{4}.$$
  Now it is left to calculate $\gamma_v$. All vertices labeled with $-1$, which are not adjacent to $v$ form set $\{\{a,c\} | c \in A_n\} \cup \{\{b,c\} | c \in A_n\}$. This gives $\gamma_v = |V_{-1}| - 2 \cdot \frac{n-3}{2} = \frac{n^2-4n+3}{4}$.
  Thus for this case we get $2\alpha_v + \beta_v - \gamma_v + f(v) = 1$. So, condition (5) is satisfied for all vertices labeled by 1.

*(iii)* Let $v = \{a,b\}, a \in A_n, b \in B_n$ be an arbitrary vertex from set $V_{-1}$. Let $\{a,e\}$ and $\{a,f\}$ be two vertices from $V_2$, which are not adjacent to $v$. All other vertices from $V_2$ are adjacent to $v$, so $\alpha_v = |V_2| - 2 = \frac{n-7}{2}$. Vertices from $V_1$ which are not adjacent to $v$ form set $\{\{a,c\} | c \in A_n \setminus \{a,e,f\}\} \cup \{\{b,c\} | c \in B_n \setminus \{b\}\}$. This gives $\beta_v = |V_1| - \left(\frac{n-3}{2} - 3\right) - \left(\frac{n+3}{2} - 1\right) = \frac{n^2-8n+31}{4}$. To calculate $\gamma_v$, we consider the vertices from set $V_{-1}$ which are not adjacent to $v$. These vertices form set $\{v\} \cup \{\{a,c\} | c \in B_n \setminus \{b\}\} \cup \{\{b,c\} | c \in A_n \setminus \{a\}\}$. So, $\gamma_v = |V_{-1}| - \left(1 + \left(\frac{n+3}{2} - 1\right) + \left(\frac{n-3}{2} - 1\right)\right) = \frac{n^2-4n-5}{4}$. Thus, if $v \in V_{-1}$, it holds $2\alpha_v + \beta_v - \gamma_v + f(v) = 1$, i.e., condition (5) is satisfied if $v \in V_{-1}$.

As we analyzed every possible case for $v \in V$, we conclude that $f$ satisfies condition (5). Therefore, the proof of the theorem for Case 1: $n$ is odd is finished.

**Case 2:** $n$ is even.

We define SRDF $f$ for which $f(V) = 5$. We consider two subcases: $n \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$.

*Subcase 2.1:* $n \equiv 0 \pmod 4$.

| | |
|---|---|
| $A_n$ | $\{1, 2, \ldots, \frac{n-2}{2}\}$ |
| $B_n$ | $\{\frac{n}{2}, \frac{n+2}{2}, \ldots, n\}$ |
| $A_{n,2}$ | $\{\{a, b\} \vert a, b \in A_n\}$ |
| $B_{n,2}$ | $\{\{a, b\} \vert a, b \in B_n\}$ |
| $C_{n,2}$ | $\{\{a, b\} \vert a \in A_n, b \in B_n\}$ |
| $V_2$ | $\{\{1, 2\}, \{3, 4\}, \ldots, \{\frac{n-6}{2}, \frac{n-4}{2}\}\} \cup \{\{\frac{n}{2}, \frac{n+2}{2}\}, \{\frac{n+4}{2}, \frac{n+6}{2}\}, \ldots, \{n-2, n-1\}\} \cup \{\{1, 3\}, \{2, 4\}\},$ |
| $V_{-1}$ | $C_{n,2} \setminus \{\frac{n-2}{2}, n\}$ |
| $V_1$ | $V \setminus (V_2 \cup V_{-1})$ |

**Tab. 2:** The construction of SDRF $f$ for which $f(V) = 5$

Similarly, as in Case 1 we introduce sets $A_n, B_n, A_{n,2}, B_{n,2}$ and $C_{n,2}$, as well as function $f = (V_{-1}, V_1, V_2)$, given in Tab. 2. Notice that $|A_n| = \frac{n-2}{2}$, $|B_n| = \frac{n+2}{2}$, $|A_{n,2}| = \binom{(n-2)/2}{2}$, $|B_{n,2}| = \binom{(n+2)/2}{2}$, $|C_{n,2}| = \frac{n-2}{2} \cdot \frac{n+2}{2} = \frac{n^2-4}{4}$. Also, $V_1 = (A_{n,2} \setminus V_2) \cup (B_{n,2} \setminus V_2) \cup \{\frac{n-2}{2}, n\}$.

We have $|V_2| = \frac{n-4}{4} + \frac{n}{4} + 2 = \frac{n+2}{2}$, $|V_{-1}| = \frac{n^2-4}{4} - 1 = \frac{n^2-8}{4}$ and $|V_1| = \binom{n}{2} - \left(\frac{n+2}{2} + \frac{n^2-8}{4}\right) = \frac{n^2-4n+4}{4}$, so $f(V) = 2|V_2| + |V_1| - |V_{-1}| = 5$.

Let us now check whether the condition (3) is satisfied.

Let $v = \{a, b\} \in V_{-1}$ be an arbitrary vertex and w.l.o.g. suppose that $a \in A_n$, $b \in B_n$. Since $n \geqslant 12$, $\{1, 2\}, \{3, 4\} \in V_2$.

- If $a \geqslant 3$, then $\{1, 2\} \cap v = \emptyset$, i.e. $\{1, 2\}$ and $v$ are adjacent. Therefore, $v$ has a neighbor in set $V_2$, which implies that the condition (3) is satisfied.
- If $a \leqslant 2$, then $\{3, 4\} \cap v = \emptyset$. Similarly, we conclude the condition (3) is satisfied.

Let us prove that condition (5) is satisfied.

For $n = 12$, the values $2\alpha_v + \beta_v - \gamma_v + f(v)$ are calculated for all vertices and the results are shown in Tab. 3. From the last column of Tab. 3 one can see that condition (5) holds.

Let now $n \geqslant 16$.

The proof that condition (5) is satisfied is similar to the corresponding proof in Case 1. It should be noted that in this case there are more subcases depending on the definitions of sets $V_2, V_1$ and $V_{-1}$. For that reason we shortened the proof, still covering all possible cases.

*(i)* Let $v = \{a, b\} \in V_2$. We consider two possibilities.

- $v \in A_{n,2}$.

  The lowest value for $\alpha_v$ is obtained for $v \in \{\{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}\}$ and it is equal to $|V_2| - 3 = \frac{n-4}{2}$. This also shows that the minimum value of $\beta_v$ is obtained for $v \notin$

| $v$ | $\alpha_v$ | $\beta_v$ | $\gamma_v$ | $f(v)$ | $2\alpha_v+\beta_v-\gamma_v+f(v)$ |
|---|---|---|---|---|---|
| $v \in V_2 \cap A_{12,2}$ | 4 | 21 | 20 | 2 | 11 |
| $v \in V_2 \cap B_{12,2}$ | 6 | 15 | 24 | 2 | 5 |
| $\{1,4\},\{2,3\}$ | 3 | 22 | 20 | 1 | 9 |
| $\{1,5\},\{2,5\},\{3,5\},\{4,5\}$ | 5 | 19 | 21 | 1 | 9 |
| $\{5,12\}$ | 7 | 14 | 24 | 1 | 5 |
| $\{a,b\} \in V_1 \cap B_{12,2}, b \neq 12$ | 5 | 19 | 21 | 1 | 9 |
| $\{a,12\} \in V_1 \cap B_{12,2}, a \neq 5$ | 6 | 14 | 25 | 1 | 2 |
| $\{a,b\} \in V_{-1} \cap C_{12,2}, a \neq 5, b \neq 12$ | 4 | 18 | 23 | -1 | 2 |
| $\{a,12\} \in V_{-1} \cap C_{12,2}$ | 5 | 16 | 24 | -1 | 1 |
| $\{5,b\} \in V_{-1} \cap C_{12,2}$ | 6 | 15 | 24 | -1 | 2 |

**Tab. 3:** The values $2\alpha_v+\beta_v-\gamma_v+f(v)$ for all vertices of graph $K_{12,2}$

$\{\{1,2\},\{3,4\},\{1,3\},\{2,4\}\}$ and it is equal to $|V_1|-(n-6)=\frac{n^2-8n+28}{4}$. For each $v \in A_{n,2}$, we have $\gamma_v=\frac{n^2-4n-16}{4}$.
Now $2\alpha_v+\beta_v-\gamma_v+f(v) \geqslant 2\cdot\frac{n-4}{2}+\frac{n^2-8n+28}{4}-\frac{n^2-4n-16}{4}+2=9$.

- $v \in B_{n,2}$.
  For each $v \in B_{n,2}$, it holds that $\alpha_v=\frac{n}{2}$, $\beta_v=\frac{n^2-8n+12}{4}$ and $\gamma_v=\frac{n^2-4n}{4}$.
  Therefore, $2\alpha_v+\beta_v-\gamma_v+f(v)=5$.

We hereby showed that for each $v \in V_2$, the inequality from condition (5) is satisfied.

*(ii)* For $v=\{a,b\} \in V_1$ we observe three possibilities.

- $v \in A_{n,2} \setminus V_2$.
  The lowest value for $\alpha_v$ is obtained for $v \in \{\{1,4\},\{2,3\}\}$ and it is equal to $|V_2|-4=\frac{n-6}{2}$.
  Further, the lowest value for $\beta_v$ is obtained when either $a$ or $b$ belong to set $A_n \setminus \{1,2,3,4,\frac{n-2}{2}\}$ and the other one is equal to $\frac{n-2}{2}$.
  Here we get $\beta_v=|V_1|-\left(\frac{n-2}{2}-2+\frac{n-2}{2}-2\right)-1=\frac{n^2-8n+24}{4}$.
  The greatest value for $\gamma_v$ is obtained when one of the numbers $a$ or $b$ is equal to $\frac{n-2}{2}$ and $\gamma_v=|V_{-1}|-\left(2\cdot\frac{n+2}{2}-1\right)=\frac{n^2-4n-12}{4}$.
  Thus, $2\alpha_v+\beta_v-\gamma_v+f(v) \geqslant 2\cdot\frac{n-6}{2}+\frac{n^2-8n+24}{4}-\frac{n^2-4n-12}{4}+1=4$.
- $v \in B_{n,2} \setminus V_2$.
  For $a,b \in B_n \setminus \{n\}$, we get: $\alpha_v=|V_2|-2=\frac{n-2}{2}$, $\beta_v=|V_1|-\left(\frac{n+2}{2}-2+\frac{n+2}{2}-3\right)=\frac{n^2-8n+16}{4}$ and $\gamma_v=|V_{-1}|-2\cdot\frac{n-2}{2}=\frac{n^2-4n}{4}$, so $2\alpha_v+\beta_v-\gamma_v+f(v)=3$.
  If one of $a$ or $b$ equals $n$ then: $\alpha_v=|V_2|-1=\frac{n}{2}$, $\beta_v=|V_1|-\left(\frac{n+2}{2}-2+\frac{n+2}{2}-2\right)-1=\frac{n^2-8n+8}{4}$ and $\gamma_v=|V_{-1}|-\left(2\cdot\frac{n-2}{2}-1\right)=\frac{n^2-4n+4}{4}$, so $2\alpha_v+\beta_v-\gamma_v+f(v)=2$.
- $v=\{\frac{n-2}{2},n\}$.
  For this vertex we get: $\alpha_v=|V_2|=\frac{n+2}{2}$, $\beta_v=|V_1|-\left(\frac{n-2}{2}-1+\frac{n+2}{2}-1\right)-1=\frac{n^2-8n+8}{4}$, $\gamma_v=|V_{-1}|-\left(\frac{n+2}{2}-1+\frac{n-2}{2}-1\right)=\frac{n^2-4n}{4}$, which gives $2\alpha_v+\beta_v-\gamma_v+f(v)=5$.

We hereby proved that the inequality from condition (5) is satisfied for $v \in V_1$.

*(iii)* For $v = \{a, b\} \in V_{-1}$, we also consider three cases.

- $a \in A_n \setminus \{\frac{n-2}{2}\}$ and $b \in B_n \setminus \{n\}$.
  The smallest value for $\alpha_v$ is obtained for $a \in \{1, 2, 3, 4\}$, where $a$ and $b$ occur in exactly three vertices in set $V_2$, so $\alpha_v = |V_2| - 3 = \frac{n-4}{2}$.
  The smallest value of $\beta_v$ is achieved for $a \notin \{1, 2, 3, 4\}$ and it is equal to $\beta_v = |V_1| - \left(\frac{n-2}{2} - 2 + \frac{n+2}{2} - 2\right) = \frac{n^2-8n+20}{4}$.
  For each vertex $v$, in this case we get $\gamma_v = |V_{-1}| - \left(\frac{n+2}{2} + \frac{n-2}{2} - 1\right) = \frac{n^2-4n-4}{4}$.
  Therefore, $2\alpha_v + \beta_v - \gamma_v + f(v) \geqslant 2 \cdot \frac{n-4}{2} + \frac{n^2-8n+20}{4} - \frac{n^2-4n-4}{4} - 1 = 1$.
- $a = \frac{n-2}{2}$ and $b \in B_n \setminus \{n\}$.
  In this case $v$ is not adjacent to only one vertex from $V_2$ which contains $b$, so $\alpha_v = |V_2| - 1 = \frac{n}{2}$.
  Further, we get $\beta_v = |V_1| - \left(\frac{n-2}{2} - 1 + \frac{n+2}{2} - 2\right) - 1 = \frac{n^2-8n+12}{4}$
  and $\gamma_v = |V_{-1}| - \left(\frac{n+2}{2} - 1 - \frac{n-2}{2} - 1\right) = \frac{n^2-4n}{4}$.
  Therefore, $2\alpha_v + \beta_v - \gamma_v + f(v) = 2$.
- $a \in A_n \setminus \{\frac{n-2}{2}\}$ and $b = n$.
  If $a \in \{1, 2, 3, 4\}$, we get: $\alpha_v = \frac{n-2}{2}$, $\beta_v = \frac{n^2-8n+16}{4}$, $\gamma_v = \frac{n^2-4n}{4}$ and $2\alpha_v + \beta_v - \gamma_v + f(v) = 1$.
  If $a \notin \{1, 2, 3, 4\}$, then holds: $\alpha_v = \frac{n}{2}$, $\beta_v = \frac{n^2-8n+12}{4}$, $\gamma_v = \frac{n^2-4n}{4}$ and $2\alpha_v + \beta_v - \gamma_v + f(v) = 2$.

This proves that the inequality from condition (5) is satisfied for $v \in V_{-1}$.

Since we covered all possible cases for $n \equiv 0 \pmod 4$, the constructed function $f$ is an SDRF and this part of the theorem is proved.

*Subcase 2.2:* $n \equiv 2 \pmod 4$.
Let sets $A_n, B_n, A_{n,2}, B_{n,2}$ and $C_{n,2}$ be constructed as in Subcase 2.1. The definition of function $f = (V_{-1}, V_1, V_2)$ such that $f(V) = 5$ is given in Tab 4.

| | |
|---|---|
| $V_2$ | $\{\{1,2\},\{3,4\},\ldots,\{\frac{n-4}{2},\frac{n-2}{2}\}\} \cup \{\{\frac{n}{2},\frac{n+2}{2}\},\{\frac{n+4}{2},\frac{n+6}{2}\},\ldots,$ $\{n-1,n\}\} \cup \{\{1,3\},\{2,4\},\{\frac{n}{2},\frac{n+4}{2}\}\},$ |
| $V_{-1}$ | $C_{n,2}$ |
| $V_1$ | $V \setminus (V_2 \cup V_{-1})$ |

**Tab. 4:** The construction of SDRF $f$ for which $f(V) = 5$

Notice that $V_1 = (A_{n,2} \setminus V_2) \cup (B_{n,2} \setminus V_2)$.

The cardinalities of these sets are equal to: $|V_2| = \frac{n-2}{4} + \frac{n+2}{4} + 3 = \frac{n+6}{2}$, $|V_{-1}| = \frac{n-2}{2} \cdot \frac{n+2}{2} = \frac{n^2-4}{4}$ and $|V_1| = \binom{n}{2} - \left(\frac{n+6}{2} + \frac{n^2-4}{4}\right) = \frac{n^2-4n-8}{4}$. Thus, $f(V) = 2|V_2| + |V_1| - |V_{-1}| = 5$.

Let us prove that condition (3) is satisfied.

Since $n \geqslant 14$, we have that $\{1,2\},\{3,4\} \in V_2$. Therefore, similarly as in Subcase 2.1. it can be shown that condition (3) holds.

Let us now prove that condition (5) is satisfied.

For $n = 14$ the values $2\alpha_v + \beta_v - \gamma_v + f(v), v \in V$ are given in Tab. 5. One can see that condition (5) holds in this case.

| $v$ | $\alpha_v$ | $\beta_v$ | $\gamma_v$ | $f(v)$ | $2\alpha_v + \beta_v - \gamma_v + f(v)$ |
|---|---|---|---|---|---|
| $\{1,2\},\{1,3\},\{2,4\},\{3,4\}$ | 7 | 27 | 32 | 2 | 11 |
| $\{5,6\}$ | 7 | 27 | 32 | 2 | 11 |
| $\{7,8\},\{9,10\}$ | 8 | 22 | 36 | 2 | 4 |
| $\{11,12\},\{13,14\}$ | 9 | 21 | 36 | 2 | 5 |
| $\{7,9\}$ | 7 | 23 | 36 | 2 | 3 |
| $\{1,4\},\{2,3\}$ | 6 | 28 | 32 | 1 | 9 |
| $\{a,b\}, a \in \{1,2,3,4\}, b \in \{5,6\}$ | 7 | 27 | 32 | 1 | 10 |
| $\{a,b\}, a,b \in B_{14} \setminus \{7,9\}$ | 8 | 22 | 36 | 1 | 3 |
| $\{a,b\} \in V_1 \cap B_{14,2}, a \in \{7,9\}$ | 7 | 23 | 36 | 1 | 2 |
| $\{a,b\}, a \in \{1,2,3,4\}, b \in \{7,9\},$ | 6 | 25 | 35 | -1 | 1 |
| $\{a,b\}, a \in \{1,2,3,4\}, b \in B_{14} \setminus \{7,9\},$ | 7 | 24 | 35 | -1 | 2 |
| $\{a,b\}, a \in \{5,6\}, b \in B_{14} \setminus \{7,9\},$ | 8 | 23 | 35 | -1 | 3 |
| $\{5,7\},\{5,9\},\{6,7\},\{6,9\}$ | 7 | 24 | 35 | -1 | 2 |

**Tab. 5:** The values $2\alpha_v + \beta_v - \gamma_v + f(v)$ for all vertices of graph $K_{14,2}$

Let now $n \geqslant 18$.

*(i)* Let $v = \{a,b\} \in V_2$. Similar to the previous subcase, we differ two cases.

- $v \in A_{n,2}$.
  The minimum value of $\alpha_v$ is obtained for
  $v \in \{\{1,2\},\{3,4\},\{1,3\},\{2,4\}\}$, where $\alpha_v = |V_2| - 3 = \frac{n}{2}$.
  This implies that the minimum value of $\beta_v$ is obtained for
  $v \notin \{\{1,2\},\{3,4\},\{1,3\},\{2,4\}\}$ and equals $\beta_v = |V_1| - 2 \cdot \left(\frac{n-2}{2} - 2\right) = \frac{n^2-8n+16}{4}$.
  Further, we get $\gamma_v = |V_{-1}| - 2 \cdot \frac{n+2}{2} = \frac{n^2-4n-12}{4}$.
  This gives $2\alpha_v + \beta_v - \gamma_v + f(v) \geqslant 2 \cdot \frac{n}{2} + \frac{n^2-8n+16}{4} - \frac{n^2-4n-12}{4} + 2 = 9$.
- $v \in B_{n,2}$.
  In this case $\alpha_v$ is minimal for vertex $v \in \{\{\frac{n}{2}, \frac{n+4}{2}\}\}$ for which $\alpha_v = |V_2| - 3 = \frac{n}{2}$.
  The value $\beta_v$ is minimal for $a, b \notin \{\frac{n}{2}, \frac{n+4}{2}\}$ for which $\beta_v = |V_1| - 2 \cdot \left(\frac{n+2}{2} - 2\right) = \frac{n^2-8n}{4}$.
  Further, we get $\gamma_v = |V_{-1}| - 2 \cdot \frac{n-2}{2} = \frac{n^2-4n+4}{4}$.
  This gives $2\alpha_v + \beta_v - \gamma_v + f(v) \geqslant 2 \cdot \frac{n}{2} + \frac{n^2-8n}{4} - \frac{n^2-4n+4}{4} + 2 = 1$.

Therefore the inequality from condition (5) is fulfilled for every $v \in V_2$.

(*ii*) For $v \in V_1$ we consider two cases.

- $v \in A_{n,2} \setminus V_2$.
Value $\alpha_v$ is the smallest for $v \in \{\{1,4\},\{2,3\}\}$ and equals $|V_2| - 4 = \frac{n-2}{2}$.
The minimum value of $\beta_v$ is obtained when $a, b \in A_n \setminus \{1,2,3,4\}$ and it equals $\beta_v = |V_1| - \left(\frac{n-2}{2} - 2 + \frac{n-2}{2} - 3\right) = \frac{n^2-8n+20}{4}$.
We also get that $\gamma_v = |V_{-1}| - 2 \cdot \frac{n+2}{2} = \frac{n^2-4n-12}{4}$.
So in this case $2\alpha_v + \beta_v - \gamma_v + f(v) \geqslant 2 \cdot \frac{n-2}{2} + \frac{n^2-8n+20}{4} - \frac{n^2-4n-12}{4} + 1 = 7$.
- $v \in B_{n,2} \setminus V_2$.
Value $\alpha_v$ is minimal when $a$ or $b$ belong to set $\{\frac{n}{2}, \frac{n+4}{2}\}$, where $\alpha_v = |V_2| - 3 = \frac{n}{2}$.
The lowest value of $\beta_v$ is obtained for $a, b \notin \{\frac{n}{2}, \frac{n+4}{2}\}$, when $\beta_v = |V_1| - \left(\frac{n+2}{2} - 2 + \frac{n+2}{2} - 3\right) = \frac{n^2-8n+4}{4}$.
Here it holds that $\gamma_v = |V_{-1}| - 2 \cdot \frac{n-2}{2} = \frac{n^2-4n+4}{4}$.
Therefore, $2\alpha_v + \beta_v - \gamma_v + f(v) \geqslant 2 \cdot \frac{n}{2} + \frac{n^2-8n+4}{4} - \frac{n^2-4n+4}{4} + 1 = 1$.

The conclusion is the inequality from condition (5) holds for each $v \in V_1$.

(*iii*) For an arbitrary vertex $v = \{a, b\} \in V_{-1}$, we get the following results:

- If $a \in \{1,2,3,4\}$ and $b \in \{\frac{n}{2}, \frac{n+4}{2}\} : \alpha_v = \frac{n-2}{2}$,
$\beta_v = |V_1| - \left(\frac{n-2}{2} - 3 + \frac{n+2}{2} - 3\right) = \frac{n^2-8n+16}{4}$, $\gamma_v = \frac{n^2-4n}{4}$, $2\alpha_v + \beta_v - \gamma_v + f(v) = 1$.
- If $a \in \{1,2,3,4\}$ and $b \notin \{\frac{n}{2}, \frac{n+4}{2}\} : \alpha_v = \frac{n}{2}$,
$\beta_v = |V_1| - \left(\frac{n-2}{2} - 3 + \frac{n+2}{2} - 2\right) = \frac{n^2-8n+12}{4}$, $\gamma_v = \frac{n^2-4n}{4}$, $2\alpha_v + \beta_v - \gamma_v + f(v) = 2$.
- If $a \notin \{1,2,3,4\}$ and $b \in \{\frac{n}{2}, \frac{n+4}{2}\} : \alpha_v = \frac{n}{2}$,
$\beta_v = |V_1| - \left(\frac{n-2}{2} - 2 + \frac{n+2}{2} - 3\right) = \frac{n^2-8n+12}{4}$, $\gamma_v = \frac{n^2-4n}{4}$, $2\alpha_v + \beta_v - \gamma_v + f(v) = 2$.
- If $a \notin \{1,2,3,4\}$ and $b \notin \{\frac{n}{2}, \frac{n+4}{2}\} : \alpha_v = \frac{n+2}{2}$,
$\beta_v = |V_1| - \left(\frac{n-2}{2} - 2 + \frac{n+2}{2} - 2\right) = \frac{n^2-8n+8}{4}$, $\gamma_v = \frac{n^2-4n}{4}$, $2\alpha_v + \beta_v - \gamma_v + f(v) = 3$.

It follows that the inequality from condition (5) holds for each $v \in V_{-1}$. Therefore, the function $f$ introduced in this subcase is also an SRDF, which finally proves the theorem.

□

We used the ILP model from Filipović et al. (2022) to find SRDN for some special cases of Kneser graphs which are provided in Remark 3. The SRDFs which correspond to these solutions and ILP model details are presented in Appendix A.

**Remark 3.** *It holds*

$$\gamma_{sR}(K_{n,2}) = \begin{cases} 5, & n = 5,6,7,8, \\ 4, & n = 10, \\ 3, & n = 9, 11. \end{cases}$$

It can be observed that $\gamma_{sR}(K_{9,2}) = \gamma_{sR}(K_{11,2}) = 3$, which is in line with the proposed upper bound proposed in Theorem 3 for odd $n$. Also, $\gamma_{sR}(K_{8,2}) = 5$, which is equal to the upper bound for graphs with greater even dimensions, considered in Theorem 3.

# 3 Conclusions

This article considered the (total) Roman domination problem for Kneser graphs $K_{n,k}$, $n \geqslant k(k+1)$ and the signed Roman domination problem for $K_{n,2}$. We proved that $\gamma_{tR}(K_{n,k}) = \gamma_R(K_{n,k}) = 2(k+1)$, if $n \geqslant k(k+1)$. For all $n \geqslant 12$ the lower and upper bounds for SRDN were given for even $n$, $2 \leqslant \gamma_{sR}(K_{n,2}) \leqslant 5$, while for odd $n$, $2 \leqslant \gamma_{sR}(K_{n,2}) \leqslant 3$.

Finding a more tighter bounds for SDRNs in cases $k = 2, 3$, or even the exact values could be a promising direction for future work. Also, finding the bounds for (T)RDN, when $2k < n < k(k+1)$, as well as the bounds of SRDN for $k \geqslant 3$ remains open. Investigating the other graph invariants on Kneser graphs, such as Roman $k-$domination Kammerling and Volkmann (2009), double Roman domination Beeler et al. (2016), signed double Roman domination Ahangar et al. (2019), strong Roman domination Álvarez-Ruiz et al. (2017), etc. could be a challenge for further work.

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# A Results on small Kneser graphs

## *A.1 Results on small Kneser graphs for RDP*

The ILP model from Burger et al. (2013) was implemented in Cplex solver Lima and Seminar (2010) to obtain the RDN of some Kneser graphs of small sizes. It is stated as follows. The set of variables is defined by:

$$x_v = \begin{cases} 1, & f(v) = 1, \\ 0, & otherwise. \end{cases}$$

$$y_v = \begin{cases} 1, & f(v) = 2, \\ 0, & otherwise. \end{cases}$$

The ILP model for RDP is formulated as:

$$\begin{aligned} &\min \sum_{v \in V} (x_v + 2y_v) \\ &\text{s.t.} \\ &x_v + y_v + \sum_{u \in N(v)} y_u \geqslant 1, \forall\, v \in V, \\ &x_v + y_v \leqslant 1,\ \forall v \in V, \\ &x_v, y_v \in \{0, 1\},\ \forall v \in V. \end{aligned}$$

In Tab. 6 we present the obtained ILP solutions for RDFs with minimum weight. The first two columns contain basic parameters for graph $K(n, k)$. The third column contains value of RDN obtained by solving the corresponding ILP model. The last three columns contain detailed information about sets $(V_2, V_0, V_1)$, respectively, which corresponds to the exact solution obtained by the ILP model.

| $n$ | $k$ | $f(V)$ | $V_2$ | $V_0$ | $V_1$ |
|---|---|---|---|---|---|
| $4,5$ | $2$ | $6$ | $\{\{1,2\},\{1,3\},\{2,3\}\}$ | $V\setminus V_2$ | $\emptyset$ |
| $6$ | $3$ | $20$ | $\{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,3,4\},$ $\{1,3,5\},\{1,3,6\},\{2,3,4\},\{2,3,5\},\{2,3,6\}\}$ | $V\setminus V_2$ | $\emptyset$ |
| $7,8,9$ | $3$ | $14$ | $\{\{1,2,5\},\{1,3,6\},\{1,4,7\},\{2,3,4\},\{2,6,7\},$ $\{3,5,7\},\{4,5,6\}\}$ | $V\setminus V_2$ | $\emptyset$ |
| $10$ | $3$ | $12$ | $\{\{1,2,8\},\{1,4,8\},\{2,4,10\},\{3,5,9\},\{3,6,7\}$ $\{5,6,9\}\}$ | $V\setminus V_2$ | $\emptyset$ |
| $11$ | $3$ | $10$ | $\{\{1,5,9\},\{1,7,9\},\{2,3,8\},\{4,5,7\},\{6,10,11\}\}$ | $V\setminus V_2$ | $\emptyset$ |

**Tab. 6:** The solutions obtained by solving the ILP on small Kneser graphs

## A.2 Results on small Kneser graphs for SRDP

The ILP model from Filipović et al. (2022) was implemented in Cplex solver Lima and Seminar (2010) to obtain relation between TRDN and domination number as well as with RDN values of SRDN for small Kneser graphs. It is stated as follows.

The set of variables is given by:

$$x_v = \begin{cases} 1, & f(v)=1 \\ 0, & otherwise. \end{cases}$$

$$y_v = \begin{cases} 1, & f(v)=2 \\ 0, & otherwise. \end{cases}$$

The ILP model for SRDP is formulated as:

$$\begin{aligned}
&\min \sum_{v\in V}(2x_v+3y_v-1) \\
&\text{s.t.} \\
&x_v+y_v \le 1,\ \forall u\in V, \\
&x_v+y_v+\sum_{u\in N(v)} y_u \ge 1,\ \forall v\in V, \\
&\sum_{u\in N[v]}(2x_u+3y_u-1)\ge 1,\ \forall v\in V, \\
&x_v,y_v\in\{0,1\},\ \forall v\in V.
\end{aligned}$$

Tab. 7 contains the SRDFs of the minimum weight which are obtained by solving the aforementioned ILP model for SRDP on small Kneser graphs $K(n,2)$. The table is organized similarly as Tab. 6, with the exception that column $k$ is omitted since $k=2$ in all cases. The last three columns carry the information about sets $V_2, V_1$, and $V_{-1}$, respectively, in the corresponding partition.

| $n$ | $f(V)$ | $V_2$ | $V_1$ | $V_{-1}$ |
|---|---|---|---|---|
| 4 | 3 | $\{\{1,2\},\{1,3\},\{2,3\}\}$ | $\emptyset$ | $V \setminus V_2$ |
| 5 | 5 | $\{\{1,3\},\{1,4\},\{3,4\}\}$ | $\{\{2,4\},\{2,5\},\{4,5\}\}$ | $V \setminus (V_2 \cup V_1)$ |
| 6 | 5 | $\{\{1,2\},\{1,4\},\{1,5\},$ $\{2,5\},\{3,6\},\{4,5\}\}$ | $\{\{2,4\}\}$ | $V \setminus (V_2 \cup V_1)$ |
| 7 | 5 | $\{\{2,5\},\{2,6\},\{3,4\},$ $\{5,6\}\}$ | $\{\{1,2\},\{1,5\},\{1,6\},\{1,7\},$ $\{2,7\},\{5,7\},\{6,7\}\}$ | $V \setminus (V_2 \cup V_1)$ |
| 8 | 5 | $\{\{1,7\},\{2,5\},\{2,8\},$ $\{3,4\},\{3,6\},\{4,6\},$ $\{5,8\}\}$ | $\{1,2\},\{1,5\},\{1,8\},$ $\{2,7\},\{5,7\},\{7,8\}\}$ | $V \setminus (V_2 \cup V_1)$ |
| 9 | 3 | $\{\{3,4\},\{3,8\},\{4,8\}\}$ | $\{\{1,2\},\{1,5\},\{1,6\},\{1,7\},$ $\{1,9\},\{2,5\},\{2,6\},\{2,7\},$ $\{2,9\},\{5,6\},\{5,7\},\{5,9\},$ $\{6,7\},\{6,9\},\{7,9\}\}$ | $V \setminus (V_2 \cup V_1)$ |
| 10 | 4 | $\{\{1,2\},\{3,5\},\{4,8\},$ $\{6,7\},\{6,10\},\{7,9\},$ $\{9,10\}\}$ | $\{\{1,3\},\{1,4\},\{1,5\},\{1,8\},$ $\{2,3\},\{2,4\},\{2,5\},\{2,8\},$ $\{3,4\},\{3,8\},\{4,5\},\{5,8\},$ $\{6,9\},\{7,10\}\}$ | $V \setminus (V_2 \cup V_1)$ |
| 11 | 3 | $\{\{3,6\},\{3,11\},\{4,6\},$ $\{4,11\}\}$ | $\{\{1,2\},\{1,5\},\{1,7\},\{1,8\},$ $\{1,9\},\{1,10\},\{2,5\},\{2,7\},$ $\{2,8\},\{2,9\},\{2,10\},\{3,4\},$ $\{5,7\},\{5,8\},\{5,9\},\{5,10\},$ $\{6,11\},\{7,8\},\{7,9\},\{7,10\},$ $\{8,9\},\{8,10\},\{9,10\}\}$ | $V \setminus (V_2 \cup V_1)$ |

**Tab. 7:** The solutions obtained by solving the ILP on small Kneser graphs

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