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Gossiping with interference in radio ring networks

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In this paper, we study the problem of gossiping with interference constraint in radio ring networks. Gossiping (or total exchange information) is a protocol where each node in the network has a message and is expected to distribute its own message to every other node in the network. The gossiping problem consists in finding the minimum running time (makespan) of a gossiping protocol and algorithms that attain this makespan. We focus on the case where the transmission network is a ring network. We consider synchronous protocols where it takes one unit of time (step) to transmit a unit-length message. During one step, a node receives at most one message only through one of its two neighbors. We also suppose that, during one step, a node cannot be both a sender and a receiver (half duplex model). Moreover communication is subject to interference constraints. We use a primary node interference model where, if a node receives a message from one of its neighbors, its other neighbor cannot send at the same time. With these assumptions we completely solve the problem for ring networks. We first show lower bounds and then give gossiping algorithms which meet these lower bounds and so are optimal. The number of rounds depends on the congruences of n modulo 12.

Keywords: Gossiping, Radio Networks, Interference, Rings

1 Introduction and Notations

This paper answers a problem considered in Gąsieniec and Potapov (2002) where we refer the readers for motivations and more references. Our aim is to design optimal gossiping (or total exchange information) protocols for ring networks with neighboring interferences. More precisely our transmission network is a symmetric cycle called here a ring C_n of length n. The nodes are labeled with the set of integers modulo n $\mathbb{Z}_n : 0, 1, ..., n-1$. The arcs represent the possible communications and are of the form $(i, i+1)$ and $(i, i-1)$. Each node i has a message also denoted by i.

The network is assumed to be synchronous and the time is slotted into *steps*. During a step, a node receives at most one message only through one of its two neighbors. One important feature of our model

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is the assumption that a node can either transmit or receive at most one message per step. In particular, we do not allow concatenation of messages.

We will consider only *useful* calls in which the sender sends a message to a receiver only if it is unknown to the receiver. We can have two types of sendings as follows:

(a) via a (regular) call $(i, i+1)$ (resp. $(i, i-1)$), in which the node i sends to the right (resp. to the left) i.e. to the node $i+1$ (resp. $i-1$) one message which is not known to the node $i+1$ (resp. $i-1$)

(b) via a 2-call $\{i : i-i, i+1\}$, in which the node i sends at the same time to both nodes $i-1$ and $i+1$ one message which is not known to both nodes and so the message must be i.

We suppose that each device is equipped with a half duplex interface, i.e. a node can receive or send, but cannot both receive and send during the same step. Furthermore, in order to model neighboring interferences, we use a primary node interference model like the one used in Bermond et al. (2009, 2006, 2020); Gąsieniec and Potapov (2002); Klasing et al. (2008). In this model, when one node is transmitting, its own power prevents any other signal to be properly received in its neighbors (near-far effect of antennas). Note that the binary interference model used is a simplified version of the reality, where the Signal-to-Noise-and-Interferences Ratio (the ratio of the received power from the source of the transmission to the sum of the noise and the received powers of all other simultaneously transmitting nodes) has to be above a given threshold for a transmission to be successful. However, the values of the completion times that we obtain in the above problem will lead to lower bounds on the corresponding real life values. In this model, two calls (s, r) and (s', r') interfere if $d(s, r') \leq 1$ or $d(s', r) \leq 1$. For example call $(i, i+1)$ will interfere with all the following calls $(i-2, i-1)$, $(i-1, i)$, $(i+1, i)$, $(i+1, i+2)$, $(i+2, i+1)$ and $(i+2, i+3)$. Two non-interfering calls will be called compatible. Therefore the two calls (s, r) and (s', r') are compatible if $d(s, r') > 1$ and $d(s', r) > 1$. For example call $(i, i+1)$ is compatible with calls $(i-1, i-2)$ and $(i+3, i+2)$. Only non-interfering (or compatible) calls can be performed in the same step and we will define a (valid) *round* as a set of compatible calls. In this article we will omit the word valid before round.

Gossiping (also called All To All communication or total exchange) is a protocol where each node in the network has a message and is expected to distribute its own message to every other node in the network. The gossiping problem consists in finding the minimum running time (makespan) of a gossiping protocol, i.e. the minimum number R_n of rounds needed to complete the gossiping and to find efficient algorithms that attain this makespan.

On problems related to information dissemination, we refer to the survey in Bonifaci et al. (2009). The gossiping problem has been mainly studied in both full duplex and half duplex models (i.e. without interferences) with unbounded size of messages. Limited size of the messages was considered in Bermond et al. (1998) where results are obtained in particular for the full duplex model (without interferences). In particular in a ring network with n nodes, the minimum gossiping time is $n-1$ if n is even or n if n is odd (in full duplex model). A survey for gossiping with primary node interference model has been done in Gąsieniec (2010) but most of the results concern unbounded size of messages (i.e. concatenation is allowed).

The gossiping problem with unit length messages and neighboring primary node model interference (our model) was first studied in Gąsieniec and Potapov (2002). The authors established that the makespans of gossiping protocols in chain (called line) and ring networks with n nodes are $3n + \Theta(1)$ and $2n + \Theta(1)$ respectively. They gave for general graphs an upper bound of $O(n \log^2 n)$. This bound was improved in Manne and Xin. (2006) to $O(n \log n)$ with the help of probabilistic argument. In Bermond et al. (2020), we completely solved the gossiping problem in radio chain networks with the same model and proved that the makespan is $3n-5$ for $n > 3$.

More precisely, in Gąsieniec and Potapov (2002) the authors gave a general lower bound of $2n-2$ which is not always correct as when $n = 12p$ we will give a gossiping protocol with makespan $2n-3$. The lower bound of $2n-2$ can be proved only if there are no 2-calls. However, the authors wrote that in their model "a message transmitted by processor v reaches all its neighbors in the same time step". Therefore 2-calls are allowed in their model and should be take in consideration. They also gave an upper-bound of $2n+9$. To our best knowledge, no improvement has been done on these bounds since 2002 and the determination of the exact value of R_n has not been solved until this paper.

In this article, we determine exactly the minimum number R_n of rounds needed to complete the gossiping when transmission network is a ring C_n on n nodes based on the model described above (see Theorem 1). Our results depend on the congruence of n modulo 12. We first give a non trivial proof of the lower bounds. The tools developed in Bermond et al. (2020) to design optimal protocols for chain networks cannot be used for rings. Though an optimal protocol can easily be found for $n \equiv 0 \pmod{12}$, for other congruences the problem is not easy as one of the difficulties is to ensure that a sender has a useful message to transmit. We succeeded in designing an optimal protocol by developing new sophisticated tools.

Theorem 1 *The minimum number of rounds R needed to achieve a gossiping in a ring network C_n ($n \geq 3$), with the above model is :*

$$\begin{cases} 2n-3 & \text{if } n \equiv 0 \pmod{12} \\ 2n-2 & \text{if } n \equiv 4, 8 \pmod{12} \\ 2n-1 & \text{if } n \text{ is odd, except when } n = 3 \text{ for which } R = 3 \text{ and } n = 5 \text{ for which } R = 10 \\ 2n+1 & \text{if } n \equiv 2 \pmod{4} \text{ except for } n = 6 \text{ for which } R = 12 \end{cases}$$

2 Lower bounds

The determination of the lower bound is based on two facts. Firstly we need $n(n-1)/2$ calls to transmit all the messages. Therefore to minimize the number of rounds, we have to use rounds with the maximum number of calls and that depends on the number of 2-calls included in this round. Secondly we have the constraint that each node can be involved in at most one useful 2-call and so the number of 2-calls in any protocol is bounded by n.

We first determine the maximum number of calls $f_\alpha(n)$ that a round with α 2-calls can contain. Note that we have $n \geq 3\alpha$.

Lemma 1 *For $n \geq 3\alpha$, $f_\alpha(n) = \lfloor (n+\alpha)/2 \rfloor - \epsilon_\alpha(n)$, where $\epsilon_\alpha(n) = 1$ if $n - 3\alpha \equiv 2 \pmod{4}$ and 0 otherwise.*

Proof: We note that, if we have α 2-calls, the maximum number of single calls which can be put between the remaining $n-3\alpha$ nodes is $\lfloor (n-3\alpha)/2 \rfloor$ and so the maximum number of possible calls satisfies $f_\alpha(n) \leq 2\alpha + \lfloor (n-3\alpha)/2 \rfloor = \lfloor (n+\alpha)/2 \rfloor$.

If $n - 3\alpha \not\equiv 2 \pmod{4}$ (i.e. $\epsilon_\alpha(n) = 0$), we claim that the bound is attained. Indeed if $n = 3\alpha + 4p$ or $n = 3\alpha + 4p + 1$, a solution is given by letting the α 2-calls be $(3i+1 : 3i, 3i+2)$ for $0 \leq i \leq \alpha - 1$

and the $2p$ single calls be $(3\alpha+4j+1, 3\alpha+4j)$ and $(3\alpha+4j+2, 3\alpha+4j+3)$ for $0 \le j \le p-1$. This round is valid as there is no interference and so in that case $f_\alpha(n) = 2\alpha+2p = \lfloor (n+\alpha)/2 \rfloor$. If $n = 3\alpha+4p+3$, we add to the preceding solution the single call $(3\alpha+4p+1, 3\alpha+4p)$ and so we get $f_\alpha(n) = 2\alpha+2p+1 = \lfloor (n+\alpha)/2 \rfloor$.

Let us now prove that if there is no idle vertex, then $n - 3\alpha \equiv 0 \pmod 4$. Let us denote by s a sender and r a receiver and let (s_0, r_0) be a call. If the vertex preceding s_0 is a receiver then it should receive from s_0 otherwise there will be an interference and so we have a group of the form rsr corresponding to a 2-call. If the vertex preceding s_0 is a sender then the other neighbor of it should be a receiver and so we have a group of the form $rssr$. So if there is no idle vertex we can partition the vertices in α groups of size 3 and groups of size 4. Therefore $n - 3\alpha \equiv 0 \pmod 4$. In particular if $n - 3\alpha \equiv 2 \pmod 4$ there is at least an idle vertex and so at most $\lfloor (n-3\alpha-1)/2 \rfloor = (n-3\alpha)/2 - 1$ single calls. So $f_\alpha(n) \le 2\alpha + (n-3\alpha)/2 - 1 = (n+\alpha)/2 - 1$.

Finally when $n = 3\alpha+4p+2$, we can take the solution obtained for $3\alpha+4p$, with $2\alpha+2p = (n+\alpha)/2 - 1$ calls reaching the lower bound. □

Now we will be able to determine the lower bounds. For that purpose we first state three simple but useful relations. Let x_α be the number of rounds with α 2-calls and R be the number of rounds of a valid solution.

$$\sum_\alpha x_\alpha = R \tag{1}$$

$$\sum_\alpha \alpha x_\alpha \le n \tag{2}$$

$$\sum_\alpha f_\alpha(n) x_\alpha \ge n(n-1) \tag{3}$$

For the above three relations, (1) follows from the definitions; (2) follows from the fact that on one side the number of 2-calls is $\sum_\alpha \alpha x_\alpha$, while on the other side as noted before each node can be involved in at most one useful 2-call and so the number of 2-calls is bounded by n; (3) follows from the fact that in each round with α 2-calls, there are at most $f_\alpha(n)$ calls and so the number of calls in the protocol is at most $\sum_\alpha f_\alpha(n) x_\alpha$, and it should be at least $n(n-1)$.

Remark 1 *Note that equality in (2) implies that each node is a sender of a 2-call. In particular, if in the first round there are h single calls, then the senders of these calls can no more be senders of useful calls and so we get $\sum_\alpha \alpha x_\alpha \le n - h$.*

Equality in (3) implies that each round has exactly the maximum number $f_\alpha(n)$ of calls.

Theorem 2 *For $n \ge 3$, the lower bound of the minimum number of rounds R is:*

$$R \ge \begin{cases} 2n-3 & \text{if } n \equiv 0 \pmod{12} \\ 2n-2 & \text{if } n \equiv 4,8 \pmod{12} \\ 2n-1 & \text{if } n \text{ is odd, except when } n=3 \text{ for which } R=3 \text{ and } n=5 \text{ for which } R=10 \\ 2n+1 & \text{if } n \equiv 2 \pmod 4 \text{ except for } n=6 \text{ for which } R=12 \end{cases}$$

Proof:

- Case $n \equiv 0 \pmod 4$:
  In that case, $f_\alpha(n) = n/2 + \lfloor \alpha/2 \rfloor - \epsilon_\alpha$ where $\epsilon_\alpha = 1$ if $\alpha \equiv 2 \pmod 4$ and 0 otherwise.

  Using (1), inequality (3) becomes $nR/2 + \sum_\alpha (\lfloor \alpha/2 \rfloor - \epsilon_\alpha)x_\alpha \geq n(n-1)$.

  So $R \geq (2n-3) + \frac{1}{n}(n - \sum_\alpha \alpha x_\alpha + \sum_\alpha F(\alpha)x_\alpha)$ where $F(\alpha) = \alpha - 2\lfloor \alpha/2 \rfloor + 2\epsilon_\alpha$.

  For any α, $\alpha \geq 2\lfloor \alpha/2 \rfloor$ and hence $F(\alpha) \geq 0$. As, by (2), $n \geq \sum_\alpha \alpha x_\alpha$, we get $R \geq 2n-3$.

  The equality holds only if we have equality everywhere. We have $F(\alpha) > 0$, when α is odd as $\alpha > 2\lfloor \alpha/2 \rfloor$ and when $\alpha \equiv 2 \pmod 4$ as $\epsilon_\alpha = 1$. Therefore $F(\alpha) = 0$ only when $\alpha \equiv 0 \pmod 4$ that is $3\alpha \equiv 0 \pmod{12}$. But then, when $n \equiv 4, 8 \pmod{12}$, we get $n - 3\alpha \geq 4$ and so the number of single calls is $(n-3\alpha)/2 \geq 2$. By the remark concerning the equality in (2) we get $n > \sum_\alpha \alpha x_\alpha$ and therefore $R > 2n-3$ or $R \geq 2n-2$.

- Case $n \equiv 2 \pmod 4$:
  Here $f_\alpha(n) = (n-2)/2 + \lfloor (\alpha + 2/2) \rfloor - \epsilon_\alpha$, where $\epsilon_\alpha = 1$ if $\alpha \equiv 0 \pmod 4$ and 0 otherwise.

  By (1) and (3), we get $(n-2)R/2 + \sum_\alpha (\lfloor (\alpha+2)/2 \rfloor - \epsilon_\alpha)x_\alpha \geq n(n-1)$, and so

  $R \geq 2n + \frac{2}{n-2}(n - \sum_\alpha \alpha x_\alpha + \sum_\alpha F(\alpha)x_\alpha)$ where $F(\alpha) = \alpha - \lfloor (\alpha+2)/2 \rfloor + \epsilon_\alpha$. Hence, $F(\alpha) = 0$ for $\alpha = 0, 1, 2$ (for $\alpha = 0$ it follows from the fact that $\epsilon_0 = 1$) and $F(\alpha) > 0$ for $\alpha \geq 3$. Therefore, $R > 2n$, except if $x_\alpha = 0$ for $\alpha \geq 3$ and furthermore by the above remark concerning the equality in (3) each round with α 2-calls ($\alpha = 0, 1, 2$), has exactly $f_\alpha(n)$ calls. If $n \geq 10$ there are in the first round $(n-3\alpha)/2 \geq 2$ single calls and so by the remark concerning the equality in (2) we get $n > \sum_\alpha \alpha x_\alpha$ and therefore and $R > 2n$ or $R \geq 2n+1$.

  For $n = 6$, we can have equality everywhere with $x_2 = 3$ and so $x_0 = 9$ and $R = 12$. This bound is attained with the following protocol. The first three rounds consist of the two 2-calls: $(r : r-1, r+1)$ and $(r+3 : r+2, r+4)$ for $r = 1, 2, 3$. At the end of these three rounds each node has exactly 3 messages $\{i-1, i, i+1\}$. Then we complete with the nine rounds consisting of two single calls, $(r, r-1)$ and $(r+2, r+1)$ for $4 \leq r \leq 12$.

- Case n odd :
  Here $f_\alpha(n) = (n-1)/2 + \lfloor (\alpha+1)/2 \rfloor - \epsilon_\alpha$, where, for $n \equiv 1 \pmod 4$, $\epsilon_\alpha = 1$ for $\alpha \equiv 1 \pmod 4$ and 0 otherwise; while for $n \equiv 3 \pmod 4$ $\epsilon_\alpha = 1$ for $\alpha \equiv 3 \pmod 4$ and 0 otherwise.

  By (1) and (3), we get $(n-1)R/2 + \sum_\alpha (\lfloor (\alpha+1)/2 \rfloor - \epsilon_\alpha)x_\alpha \geq n(n-1)$, and so

  $R \geq 2n - 2 + \frac{2}{n-1}(n - 1 - \sum_\alpha \alpha x_\alpha + \sum_\alpha F(\alpha)x_\alpha)$, where $F(\alpha) = \alpha - \lfloor (\alpha+1)/2 \rfloor + \epsilon_\alpha$.

  For $\alpha = 0$, $F(\alpha) = 0$. For $\alpha = 1$, $F(\alpha) = 1$ if $n \equiv 1 \pmod 4$ and $F(\alpha) = 0$ if $n \equiv 3 \pmod 4$. For $\alpha = 2$, $F(\alpha) = 1$. For $\alpha = 3$, $F(\alpha) = 1$ if $n \equiv 1 \pmod 4$ and $F(\alpha) = 2$ if $n \equiv 3 \pmod 4$. Finally for $\alpha \geq 4$, $F(\alpha) \geq 2$.

  If $n \equiv 1 \pmod 4$, except if $x_\alpha = 0$ for $\alpha \geq 4$ and $x_1 + x_2 + x_3 \leq 1$, $\sum_\alpha F(\alpha)x_\alpha \geq 2$ and by (2), $n - \sum_\alpha \alpha x_\alpha \geq 0$, then $R > 2n-2$. When $x_\alpha = 0$ for $\alpha \geq 4$ and $x_1 + x_2 + x_3 \leq 1$, then $\sum_\alpha \alpha x_\alpha \leq 3$ and as $n \geq 5, n - \sum_\alpha \alpha x_\alpha \geq 2$ and so as $F(\alpha) \geq 0$, $R > 2n-2$.

  If $n \equiv 3 \pmod 4$, except if $x_\alpha = 0$ for $\alpha \geq 3$ and $x_2 \leq 1$, we have $F(\alpha) \geq 2$ and as $n - \sum_\alpha \alpha x_\alpha \geq 0$ we get $R > 2n-2$.

  For $n \geq 7$, consider the case where $x_\alpha = 0$ for $\alpha \geq 3$ and $x_2 \leq 1$.

If $x_2 = 1$, there should be equality everywhere otherwise we have $R > 2n - 2$. In particular by the remark concerning equality in (3), each round with α 2-calls has exactly $f_\alpha(n)$ calls and so there are $(n - 3\alpha)/2$ single calls. Therefore, if $n \geq 11$ or $n = 7$ and the first round has only one 2-call, then there are at least 2 single calls and so by the remark concerning the equality in (2), we get $n > \sum_\alpha \alpha x_\alpha$. For $n = 7$, if the first round contains two 2-calls, as $x_2 = 1$, the second round contains at most one 2-call and so at least two single calls. Furthermore, as the senders of the 2-calls of the first round have no new message to transmit, the senders of these useful single calls are different from the senders of the 2-calls. So the two senders of single calls in the second round cannot be senders of 2-calls and we again get $n > \sum_\alpha \alpha x_\alpha$. Therefore we have strict inequality and so $R > 2n - 2$ or $R \geq 2n - 1$.

If $x_2 = 0$, then all the rounds have exactly $f_\alpha(n)$ calls except perhaps one round which might have $f_\alpha(n) - 1$ calls. Otherwise we will have a gap of 2 in the inequality (3) and then $R \geq 2n - 2 + \frac{2}{n-1}(n - 1 - \sum_\alpha \alpha x_\alpha + 2) > 2n - 2$. If there are at least 2 single calls by the remark concerning the equality in (2), we get $n \geq \sum_\alpha \alpha x_\alpha + 2$ and so $R > 2n - 2$. That is the case for $n \geq 11$ and $n = 7$ except if the first round has one 2-call and only one single call (the exceptional round). In this latter case, $n - \sum_\alpha x_\alpha \geq 1$ and $R \geq 2n - 2 + \frac{2}{n-1}(n - 1 - \sum_\alpha \alpha x_\alpha + 1) > 2n - 2$.

In summary in all the cases for n odd ≥ 7, $R > 2n - 2$ or $R \geq 2n - 1$.

For $n = 3$, we can have $x_1 = 3$ and $R = 3$ which is optimal (the 3 rounds consisting of a 2-call). For $n = 5$ we can improve the lower bound. Indeed we have $f_\alpha(5) = 2$ and so inequality (3) becomes $R \geq 10$. An optimal solution is obtained as follows: the first 5 rounds consists of the 2-calls $(i : i-1, i+1)$ for $1 \leq i \leq 5$, while the last 5 rounds consists of 2 single calls $(i, i+1), (i+3, i+2)$ for $1 \leq i \leq 5$.

□

3 Upper bounds

3.1 Symmetric rounds and matchings

Recall that the nodes are assigned to the integers modulo n labeled from 0 to $n - 1$. We will use i to denote the message of node i. To an edge $\{i, i + 1\}$ of the cycle we can associate two calls $(i, i + 1)$ and $(i + 1, i)$. Let us call two rounds *symmetric* if, when $(i, i + 1)$ is a call in one round, then $(i + 1, i)$ is a call in the other round. Therefore, to a matching we can associate two sets of symmetric rounds. However these rounds are not necessarily valid as they might have interference. If we want to obtain two symmetric (valid) rounds the following condition should be satisfied: if we have in the matching two consecutive edges $\{i, i + 1\}$ and $\{i + 2, i + 3\}$, then to avoid interference one round should contain the calls $(i + 1, i)$ and $(i + 2, i + 3)$ and the other round the calls $(i, i + 1)$ and $(i + 3, i + 2)$. This condition can be satisfied for any matching except for a perfect matching when $n \equiv 2 \pmod 4$. Indeed we have seen in the proof of Lemma 1 that in this case there is no valid round with $n/2$ calls.

3.2 Sketch of the proof

The protocol will consist of three phases. The first phase consists of a small number of rounds (between 0 and 5) which will contain mainly 2-calls. In the second or main phase we will perform sequences of four rounds associated to two matchings whose union is either a Hamilton cycle (case $n \equiv 0 \pmod 4$) or a Hamilton path (case n odd) or a path of length $n-2$ (case $n \equiv 2 \pmod 4$). In such a sequence of four rounds, all the nodes covered by the two matchings will receive one message from the left and one from the right. The last phase will consist of a few rounds to ensure that at the end of the protocol, the nodes will have all received all messages.

The section is organized as follows. We first give optimal protocols when $n \equiv 0 \pmod 4$. We illustrate the protocol with the example of $n = 8$. Then we introduce some definitions and notations which will be useful to describe our protocols for the other cases of congruences. In these cases, the proofs are more involved and in some special cases we have to slightly modified the last rounds to attain the lower bounds.

3.3 Design of an optimal protocol when $n \equiv 0 \pmod 4$

Now we construct protocols which will match the lower bounds given in the previous section when $n \equiv 0 \pmod 4$. In that case there are two perfect matchings that we denote M and M', where $M = \{\{2j, 2j+1\}, 0 \le j \le (n-2)/2\}$ and $M' = \{\{2j+1, 2j+2\}, 0 \le j \le (n-2)/2\}$. To M we associate the two symmetric (valid) rounds $R_1 = \{(4k+1, 4k), (4k+2, 4k+3), 0 \le k \le (n-4)/4\}$ and $R_2 = \{(4k, 4k+1), (4k+3, 4k+2), 0 \le k \le (n-4)/4\}$. Similarly to M' we associate the two symmetric (valid) rounds $R'_1 = \{(4k+2, 4k+1), (4k+3, 4k+4), 0 \le k \le (n-4)/4\}$ and $R'_2 = \{(4k+1, 4k+2), (4k+4, 4k+3), 0 \le k \le (n-4)/4\}$.

The main phase will consist of repeated sequences of 4 rounds R_1, R_2, R'_1, R'_2. These two sets of symmetric rounds associated to M and M' can be viewed as associated to the Hamilton cycle formed by the union of M and M'. During such a sequence of 4 rounds we note that node i receives exactly one message from node $i+1$ and one from node $i-1$, but we have to prove that the calls are useful (i.e. that $i-1$ and $i+1$ have messages unknown to i to transmit).

3.3.1 Optimal Protocol for $n = 12p + 4, n = 12p + 8$

In these cases there is no first phase. In the main phase, we execute $(n-2)/2$ times the sequence of 4 rounds R_1, R_2, R'_1, R'_2 consisting of the symmetric rounds associated to M and M'. Then in the last phase we perform the two symmetric rounds associated to M. The reader can follow the protocol for $n = 8$ in Table 1, where columns correspond to the nodes and each row corresponds to a round. An arrow in a cell indicates the direction of a call, and a value indicates the message received by the corresponding node. We also indicate the matching associated to two symmetric rounds.

Theorem 3 *The above protocol is an optimal gossiping protocol for* $n \equiv 4$ *or* $8 \pmod{12}$ *with* $2n-2$ *rounds.*

Proof: We claim that at the end of round $4h, 0 \le h \le (n-2)/2$, each node i knows the messages of nodes at distance at most h from it namely messages j with $i-h \le j \le i+h$. The proof is by induction. It is true for $h = 0$ as node i knows it own message. Suppose it is true for h. When we perform after round $4h$ a sequence of 4 rounds R_1, R_2, R'_1, R'_2, node i receives from $i+1$ (resp. $i-1$) the message $(i+1)+h$ (resp. $(i-1)-h$). By the induction hypothesis, these messages were acquired by $i+1$ (resp. $i-1$) during the preceding sequence and are unknown to i. So, the claim is true for $h+1$. Thus, at the end of round $2n-4$, node i knows all the messages except $i+n/2$, but this message is already acquired by both nodes $i-1$ and $i+1$. Therefore in the two more symmetric rounds associated to the matching

round \ node	matching	0	1	2	3	4	5	6	7
1	M	1	$\leftarrow$	$\rightarrow$	2	5	$\leftarrow$	$\rightarrow$	6
2		$\rightarrow$	0	3	$\leftarrow$	$\rightarrow$	4	7	$\leftarrow$
3	M'	7	2	$\leftarrow$	$\rightarrow$	3	6	$\leftarrow$	$\rightarrow$
4		$\leftarrow$	$\rightarrow$	1	4	$\leftarrow$	$\rightarrow$	5	0
5	M	2	$\leftarrow$	$\rightarrow$	1	6	$\leftarrow$	$\rightarrow$	5
6		$\rightarrow$	7	4	$\leftarrow$	$\rightarrow$	3	0	$\leftarrow$
7	M'	6	3	$\leftarrow$	$\rightarrow$	2	7	$\leftarrow$	$\rightarrow$
8		$\leftarrow$	$\rightarrow$	0	5	$\leftarrow$	$\rightarrow$	4	1
9	M	3	$\leftarrow$	$\rightarrow$	0	7	$\leftarrow$	$\rightarrow$	4
10		$\rightarrow$	6	5	$\leftarrow$	$\rightarrow$	2	1	$\leftarrow$
11	M'	5	4	$\leftarrow$	$\rightarrow$	1	0	$\leftarrow$	$\rightarrow$
12		$\leftarrow$	$\rightarrow$	7	6	$\leftarrow$	$\rightarrow$	3	2
13	M	4	$\leftarrow$	$\rightarrow$	7	0	$\leftarrow$	$\rightarrow$	3
14		$\rightarrow$	5	6	$\leftarrow$	$\rightarrow$	1	2	$\leftarrow$

Tab. 1: An optimal protocol for $n = 8$

M of the last phase, node i receives it from one of its neighbors ($i+1$ if i is even or $i-1$ if i is odd). In total, $2n-2$ rounds are used in this protocol, which is optimal as the number of rounds matches the lower bound. □

3.3.2 Optimal Protocol for $n = 12p$

The first phase consists of three rounds where we use all 2-calls. More precisely in round $r = 1, 2, 3$ we perform the 2-calls: $\{(3j + r : 3j + r - 1, 3j + r + 1), j = 0, 1, 2, \ldots, 4p - 1\}$. At the end of these three rounds each node i knows the messages $i - 1, i, i + 1$. The rest of the protocol is identical to the preceding cases. In the main phase we repeat $(n - 2)/2$ times the sequence of 4 rounds R_1, R_2, R'_1, R'_2 associated to M and M'. Then in the last phase we perform the two symmetric rounds associated to M.

Theorem 4 *The above protocol is an optimal gossiping protocol for* $n \equiv 0 \pmod{12}$ *with* $2n - 3$ *rounds.*

Proof: We claim that at the end of round $4h + 3, 0 \leq h \leq (n - 4)/2$, each node knows the messages of nodes at distance at most $h + 1$ from it and they are messages j with $i - h - 1 \leq j \leq i + h + 1$. The proof is by induction. It is true for $h = 0$, as noted above. Then the proof of the induction step is exactly the same as in the proof of Theorem 3. In total, $2n - 3$ rounds are used in this protocol, which is optimal as the number of rounds matches the lower bound. □

Remark 2 *This protocol can be modified for the case n=12p+8 by using four rounds containing all the 2-calls. This also gives an optimal protocol.*

3.4 Labels

For n odd or $n \equiv 2 \pmod 4$, the main phase uses also symmetric rounds associated to matchings. But the matchings are not perfect and so nodes do not receive the same number of messages. Therefore we have to check carefully that the calls in a round are all useful. In fact we do not need to know exactly what are the messages a node knows but only the number of the known messages. We will introduce a notion of labels which will greatly simplify the proofs.

Sets L_i^t and R_i^t and their sizes l_i^t and r_i^t: We define L_i^t (resp. R_i^t) as the set of messages that have already been received at the end of the round t at the node i from *the left* that is via the call $(i-1, i)$ (resp. from *the right* that is via the call $(i+1, i)$). By convention, as at the beginning each node knows its own message, we have $L_i^0 = R_i^0 = \{i\}$.

Note that the messages in L_i^t (resp. R_i^t) arrive only via a call $(i-1, i)$ (resp. $(i+1, i)$) and so $L_i^t - \{i\} \subseteq L_{i-1}^{t-1}$ (resp. $R_i^t - \{i\} \subseteq R_{i+1}^{t-1}$). We will use l_i^t and r_i^t to denote the sizes of L_i^t and R_i^t, respectively. Recall that we consider only useful calls in which the sender transmits a message unknown to the receiver. The usefulness of a call is expressed in the following remark.

Lemma 2 *A call $(i-1, i)$ is useful in round $t+1$ if and only if $l_{i-1}^t \geq l_i^t$. Similarly a call $(i+1, i)$ is useful in round $t+1$ if and only if $r_{i+1}^t \geq r_i^t$.*

Proof: Node i knows its own messages plus $l_i^t - 1$ messages transmitted by $i-1$ and so it can receive a new message if and only if $l_{i-1}^t > l_i^t - 1$. Similarly it can receive a message from $i+1$ if and only if $r_{i+1}^t > r_i^t - 1$. □

Order to send the messages: When $l_{i-1}^t > l_i^t$ or $r_{i+1}^t > r_i^t$, a node has more than one message (unknown to the receiver) to transmit to its neighbor. In the protocols we will send the one which arrives the earliest (FIFO). More precisely, in the call $(i-1, i)$ (resp. $(i+1, i)$), the node $i-1$ (resp. $i+1$) will send among the messages unknown to i that of the nearest node of i. Note that in such protocols, the values of l_i^t (resp. r_i^t) precisely determine the sets L_i^t (resp. R_i^t) and so provide all the messages acquired by the node i at the end of t rounds. Indeed, if $l_i^t = k_1$ and $r_i^t = k_2$, then at the end of t rounds, $L_i^t = \{i-k_1+1, i-k_1+2, \ldots, i-1, i\}$ and $R_i^t = \{i, i+1, \ldots, i+k_2-2, i+k_2-1\}$. Therefore, the set of messages acquired at the node i is $\{i-k_1+1, i-k_1+2, \ldots, i-1, i, i+1, \ldots, i+k_2-2, i+k_2-1\}$.

Table 2 gives the ordered pairs of values (l_i^t, r_i^t) for node i at the end of round t in the case of the example for $n = 8$ (Table 1).

Balanced protocols and labels: We will design "balanced protocols" where, at the end of each round, the nodes have received almost the same number of messages from left and right. More precisely, at the end of any round t, the values of l_i^t and r_i^t at each node i will consist of at most 3 consecutive values denoted $\alpha - 1$, α and $\alpha + 1$. In order to facilitate the proof of the protocol presented, we will attach some labels to the nodes which reflect how the values of l_i^t and r_i^t behave in terms of the value α.

We will mainly use 6 types of labels assigned to the nodes: $B, B^+, R^+, R^-, L^+, L^-$. The labels are defined as below. We use memo-technic letters where a B stands for balanced (l_i^t and r_i^t are equal), and L or R when the values of the l_i^t and r_i^t are different. A superscript $+$ (resp. $-$) stands for the value concerned being $\alpha + 1$ (resp. $\alpha - 1$).

round \ node	0	1	2	3	4	5	6	7
initial state	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)
1	(1,2)			(2,1)	(1,2)			(2,1)
2		(2,1)	(1,2)			(2,1)	(1,2)	
3	(2,2)	(2,2)			(2,2)	(2,2)		
4			(2,2)	(2,2)			(2,2)	(2,2)
5	(2,3)			(3,2)	(2,3)			(3,2)
6		(3,2)	(2,3)			(3,2)	(2,3)	
7	(3,3)	(3,3)			(3,3)	(3,3)		
8			(3,3)	(3,3)			(3,3)	(3,3)
9	(3,4)			(4,3)	(3,4)			(4,3)
10		(4,3)	(3,4)			(4,3)	(3,4)	
11	(4,4)	(4,4)			(4,4)	(4,4)		
12			(4,4)	(4,4)			(4,4)	(4,4)
13	(4,5)			(5,4)	(4,5)			(5,4)
14		(5,4)	(4,5)			(5,4)	(4,5)	

Tab. 2: Values of (l_i^t, r_i^t) for $n = 8$

Let the label be B, if $l_i^t = r_i^t = \alpha$, B^+ if $l_i^t = r_i^t = \alpha + 1$, L^+ if $l_i^t = \alpha + 1, r_i^t = \alpha$, R^+ if $l_i^t = \alpha, r_i^t = \alpha + 1$, L^- if $l_i^t = \alpha - 1, r_i^t = \alpha$, and R^- if $l_i^t = \alpha, r_i^t = \alpha - 1$.

Table 3 gives for $n = 8$ the labels of the nodes and the value of α at the end of each round t and one can see the regularity in the pattern of the labels.

We can express the condition of usefulness of Lemma 2 in terms of labels.

Proposition 1 *A call $(i, i+1)$ is useful for round $t+1$, if and only if the labels of nodes i and $i+1$ are in one of the following situations at the end of round t:*

- *Node i is labeled B^+ or L^+.*
- *Node i is labeled B, R^+ or R^- and node $i+1$ is labeled B, R^+, L^- or R^-.*
- *Node i is labeled L^- and node $i+1$ is labeled L^-.*

Similarly, a call $(i+1, i)$ is useful for round $t+1$, if and only if the labels of nodes $i+1$ and i are in one of the following situations at the end of round t:

- *Node $i+1$ is labeled B^+ or R^+,*
- *Node i is labeled B, L^+, L^- or R^-, and node $i+1$ is labeled B, L^+ or L^-.*
- *Node i is labeled R^- and node $i+1$ is labeled R^-.*

The next Proposition will be used heavily to prove that two symmetric rounds associated to a matching contain useful calls. We list only the cases that we will use (there are other possible cases not mentioned).

node / round, matching	0	1	2	3	4	5	6	7	α
initial state	B	B	B	B	B	B	B	B	0
M	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		
R_2	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	0
M'	$\rightarrow$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$	
R_4	B	B	B	B	B	B	B	B	1
M	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		
R_6	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	1
M'	$\rightarrow$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$	
R_8	B	B	B	B	B	B	B	B	2
M	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		
R_{10}	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	2
M'	$\rightarrow$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$	
R_{12}	B	B	B	B	B	B	B	B	3
M	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		
R_{14}	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	3

Tab. 3: Matchings and labels for the case $n = 8$

Proposition 2 *The two symmetric calls associated to an edge $\{i, i+1\}$ are useful when the end nodes i and $i+1$ have the following pair of possible labels: L^+R^+, BB, B^+B^+, L^+B, L^+B^+, BR^+, B^+R^+ and R^-L^-.*

Proof: In all the case mentioned in this proposition, the usefulness conditions of Proposition 1 are satisfied for both calls $(i, i+1)$ and $(i+1, i)$. $\square$

One can see in Table 3 that, before the execution of the rounds associated to the matching M, the end nodes of any edge of the matching have labels BB while before the execution of the rounds associated to the matching M', the end nodes of any edge of the matching have labels L^+R^+ and so all calls of the rounds are useful.

3.5 Construction of the first 5 rounds for odd n and $n \equiv 2 \pmod 4$

In the cases n odd and $n \equiv 2 \pmod 4$, the first phase of the protocol consists of designing 5 rounds in such a way that the list of labels of all nodes at this stage is as indicated in the next proposition.

Proposition 3 *We can design the first five rounds of the protocol such that, at the end of these five rounds, the label list of the nodes consists of a sequence (starting at node 0) of*

$$\begin{cases} \frac{n-3}{2}\ R^+L^+ \text{ followed by } BBB & \text{if } n \equiv 1,3 \pmod 6 \\ \frac{n-7}{2}\ R^+L^+ \text{ followed by } BR^+L^+BL^+R^+B & \text{if } n \equiv 5 \pmod 6 \\ \frac{n-4}{2}\ R^+L^+ \text{ followed by } BBBB & \text{if } n \equiv 2 \pmod 4 \end{cases}$$

We leave the proof of Proposition 3 to the Appendix A. The proof consists in first designing the first five rounds for the values of $n_0 = 7, 9, 11, 13, 15, 17$ and $n_0 = 10, 14, 18$ and then in extending the construction to $n = 12p + n_0$.

3.6 Optimal protocols for $n \equiv 2 \pmod 4$

We will now give the main phase of the construction for the case $n \equiv 2 \pmod 4$. Each round will consist of the maximum number of single calls which is $(n-2)/2$ and will be associated to a matching with $(n-2)/2$ edges. In what follows we will denote by $M_{i,i+1}$ the matching which does not contain nodes i and $i+1$. So $M_{i,i+1} = \{\{i+2j+2, i+2j+3\}, 0 \le j \le (n-4)/2\}$. The idea of the protocol consists in applying repeatedly $(n-2)/2$ sequences of 4 rounds associated to the two matchings $M_{i,i+1}$ and $M_{i+1,i+2}$ (whose union is a path of length $n-2$).

Protocol: We first perform the sequence of 5 rounds as given in Proposition 3. Then in the main phase we perform the $(n-2)/2$ sequences of 4 rounds associated to the two matchings $M_{2h-1,2h}$ and $M_{2h,2h+1}$ (taken in this order), where $h = 0, 1, \ldots, (n-4)/2$, except one modification in the last round associated to the matching $M_{n-4,n-3}$, in which we replace the call $(n-1, n-2)$ with the call $(n-3, n-4)$. The construction is illustrated for $n = 10$ in Table 4.

round \ node	0	1	2	3	4	5	6	7	8	9	α
R_5	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B	B	1
$M_{9,0}$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\times$	
R_7	R^+	B^+	B^+	B^+	B^+	B^+	L^+	R^+	L^+	B	1
$M_{0,1}$	$\times$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		
R_9	L^-	B	R^+	L^+	R^+	L^+	B	B	B	R^-	2
$M_{1,2}$	$\rightarrow$	$\times$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$	
R_{11}	B	B	R^+	B^+	B^+	B^+	L^+	R^+	L^+	B	2
$M_{2,3}$	$\longleftrightarrow$		$\times$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		
R_{13}	L^-	R^-	L^-	B	R^+	L^+	B	B	B	R^-	3
$M_{3,4}$	$\rightarrow$	$\longleftrightarrow$		$\times$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$	
R_{15}	B	B	B	B	R^+	B^+	L^+	R^+	L^+	B	3
$M_{4,5}$	$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		
R_{17}	L^-	R^-	L^-	R^-	L^-	B	B	B	B	R^-	4
$M_{5,6}$	$\rightarrow$	$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\times$	$\longleftrightarrow$		$\leftarrow$	
R_{19}	B	B	B	B	B	B	B	R^+	L^+	B	4
$M_{6,7}$	$\longleftarrow$		$\longrightarrow$		$\longleftarrow$		$\times$	$\times$	$\longrightarrow$		
$*$	$\longrightarrow$		$\longleftarrow$		$\longrightarrow$		$\longleftarrow$		$\times$	$\times$	
R_{21}	L^-	R^-	L^-	R^-	L^-	R^-	L^-	L^-	R^-	R^-	5

Tab. 4: Construction for $n = 10$

Theorem 5 *The above protocol is an optimal gossiping protocol for* $n \equiv 2 \pmod 4$ $(n > 6)$ *with* $2n+1$ *rounds.*

Proof: We will prove by induction on h the following property called P_h: at the end of round $5+4h$ $(0 \leq h \leq (n-4)/2)$ the label list starting at node $2h$ consists of $(n-4)/2-h$ pairs of R^+L^+, then three B, followed by h pairs of R^-L^- and one B. From that we will deduce that all the calls are useful. Note that at the end of round 5, property P_0 is satisfied by Proposition 3. Suppose the property is true for some $h < (n-4)/2$. Then we first perform 2 rounds associated to $M_{2h-1,2h}$. The calls of these rounds are useful by Proposition 2; indeed the labels of the end nodes of the edges of this matching are successively $((n-6)/2-h)$ L^+R^+, L^+B, BB and h R^-L^-. Furthermore, at the end of round $5+4h+2$ the label list starting at node $2h+1$ consists of $((n-6)/2-h)$ pairs of B^+B^+, a pair of B^+L^+ and a pair of R^+L^+, followed by h pairs of BB and a pair of BR^+ (corresponding to the two idle nodes $2h-1$ and $2h$). Then we perform the 2 rounds associated to $M_{2h,2h+1}$. The calls of these rounds are useful by Proposition 2; indeed the labels of the end nodes of the edges of this matching are successively $((n-6)/2-h)$ B^+B^+, L^+R^+, L^+B and h BB. Furthermore, at the end of round $5+4h+4$ the label list satisfies property P_{h+1} and so the induction step is proved.

At the end of round $2n-3$, the label list starting at the node $n-5$ consists of two pairs of BB followed by $(n-4)/2$ pairs of R^-L^-. So the calls of the two rounds associated to the matching $M_{n-5,n-4}$ are useful. Now at round $2n-1$ the label list starting at node $n-3$ consists of one pair of R^+L^+ and $(n-2)/2$ pairs of BB. So nodes $n-3$ and $n-2$ know all the messages and the other nodes are missing one message. Furthermore, the calls of the two rounds associated to the matching $M_{n-4,n-3}$ are useful except the call $(n-1, n-2)$ which has no new message to transmit to $n-2$ (which already knows all messages). So, we delete the call $(n-1, n-2)$. All the nodes will know via the other calls of the matching all the messages except node $n-4$. But using the fact that nodes $n-4$ and $n-3$ were not involved in the matching, we can add the call $(n-3, n-4)$. The round is still valid as it contains the call $(n-6, n-5)$. In summary at the end of round $2n+1$ every node knows all the messages. As the lower bound on the number of rounds is $2n+1$ by Theorem 2, this protocol is optimal. □

3.7 Optimal protocols for odd n

We will now give the main phase of the protocol for odd n. In what follows we will denote by M_i the near-perfect matching which does not contain node i. So $M_i = \{\{i+2j+1, i+2j+2\}, 0 \leq j \leq (n-3)/2\}$. The idea of the protocol consists in applying repeatedly $(n-3)/2$ sequences of 4 rounds associated to two near-perfect matchings M_i and M_{i+1} whose union is a Hamilton path. In each sequence we will be able to transform a pair R^+L^+ of the label list of Proposition 3 into BB and so the protocol will be completed in $2n-1$ rounds. The main difficulty will be to choose the suitable nodes i and $i+1$ and to find the right order of the near-perfect matchings M_i and M_{i+1}. We first deal with the case $n \equiv 1$ or $3 \pmod 6$.

3.7.1 Optimal protocol for $n \equiv 1$ or $3 \pmod 6$ $(n > 5)$

Protocol: We first perform the sequence of 5 rounds as given in Proposition 3. Then in the main phase we perform the $(n-3)/2$ sequences of 4 rounds associated to the two near-perfect matchings M_{n-4-2h} and M_{n-5-2h} (taken in this order), where $h = 0, 1, \ldots, (n-5)/2$.

The construction is illustrated for $n = 9$ in Table 5.

round \ node	0	1	2	3	4	5	6	7	8
R_5	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B
M_5	$\rightarrow$	$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\leftarrow$
R_7	B^+	B^+	B^+	B^+	B^+	L^+	R^+	L^+	R^+
M_4	$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$	
R_9	R^+	L^+	R^+	L^+	B	B	B	B	B
M_3	$\rightarrow$	$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
R_{11}	B^+	B^+	B^+	L^+	R^+	L^+	R^+	L^+	R^+
M_2	$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$	
R_{13}	R^+	L^+	B	B	B	B	B	B	B
M_1	$\rightarrow$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
R_{15}	B^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+
M_0	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$	
R_{17}	B	B	B	B	B	B	B	B	B

Tab. 5: Call matchings and labels for $n = 9$

Theorem 6 *The above protocol is an optimal gossiping protocol for* $n \equiv 1$ *or* $3 \pmod 6$ *(*$n > 3$*) with* $2n - 1$ *rounds.*

Proof: We will prove by induction on h the following property called Q_h: at the end of round $5+4h$ ($0 \leq h \leq (n-3)/2$), the label list (starting at node 0) consists of $(n-3-2h)/2$ pairs of R^+L^+ followed by $(3+2h)$ B (with value $\alpha = h+1$). From that we will deduce that all the calls are useful. Note that, at round 5, the label list satisfies property Q_0.

So suppose that at the end of the round $5+4h$ the label list satisfies property Q_h. We first perform the two symmetric rounds associated to the near-perfect matching $M_{n-4-2h} = \{\{n-4-2h+2j+1, n-4-2h+2j+2\}, 0 \leq j \leq (n-3)/2\}$. The labels of the end nodes of the edges of the matching are successively BB for $0 \leq j \leq h$, then BR^+ for $j = h+1$, then L^+R^+ for $h+2 \leq j \leq (n-3)/2$. Therefore, the calls of the two rounds associated to the near-perfect matching M_{n-4-2h} are useful by Proposition 2. The label of the node $n-4-2h$ stays unchanged at L^+. But the pairs of BB have changed to R^+L^+, the pair of BR^+ to R^+B^+, and the pairs of L^+R^+ to B^+B^+. So, at the end of round $4+5h+2$, the label list consists of $n-2h-4$ B^+ followed by $h+2$ pairs L^+R^+. Therefore the labels of the end nodes of the edges of the matching M_{n-5-2h} are either L^+R^+ or B^+B^+. So by Proposition 2 the calls of the two rounds associated to the near-perfect matching M_{n-5-2h} are useful. Furthermore at the end of round $5+4h+4$ the label list satisfies property Q_{h+1}. We note that after a sequence of 4 rounds, the labels of the nodes have been unchanged (i.e. they are the same with parameter $\alpha = h+1$) except those of $n-5-2h$ and $n-4-2h$ which were respectively R^+ and L^+ and are now both B's. For $h = (n-3)/2$ that is at the end of round $5+2(n-3) = 2n-1$, all the nodes have labels B with $\alpha = (n-1)/2$ and so every node knows all the messages. As the lower bound on the number of rounds is $2n-1$ by Theorem 2, this protocol is optimal. □

Remark: Note that the order of the matchings is important. For example, we cannot perform first the rounds associated to the near-perfect matching M_{n-5-2h}; indeed the calls will not all be useful as the end labels of some edges are R^+L^+.

3.7.2 Optimal protocol for $n \equiv 5 \pmod 6$ $(n > 5)$

The construction is similar to that of the preceding case, but we have to do small modifications. Recall that by Proposition 3, the label list consists of $(n-7)/2$ pairs of R^+L^+ followed by $BR^+L^+BL^+R^+B$.

Protocol: We perform the sequence of 5 rounds as given in Proposition 3. Then in the main phase we first perform the two symmetric rounds associated to the near-perfect matching M_{n-6} and then those associated to M_{n-5} (in order to replace the labels R^+L^+ of nodes $n-6$ and $n-5$ with BB). Then we perform $(n-7)/2$ sequences of 4 rounds associated to two near-perfect matchings M_{n-8-2h} and M_{n-9-2h} (taken in this order), where $h = 0, 1, \ldots, (n-9)/2$ (in order to replace like in the case $n \equiv 1$ or $3 \pmod 6$ the $(n-7)/2$ pairs of R^+L^+ with BB). Then we perform the 4 symmetric rounds associated to two near-perfect matchings M_{n-3} and M_{n-2} except we modify one call in two rounds as the labels of $n-3$ and $n-2$ are L^+R^+. More precisely in the round associated to M_{n-3} which contains the call $(n-1, n-2)$, we delete the call $(n-1, n-2)$ and add for $n \equiv 11 \pmod{12}$ the call $(n-3, n-2)$ and for $n \equiv 5 \pmod{12}$ the call $(n-2, n-3)$. In the round associated to M_{n-2} which contains the call $(n-4, n-3)$, we delete the call $(n-4, n-3)$ and add for $n \equiv 11 \pmod{12}$ the call $(n-2, n-3)$ and for $n \equiv 5 \pmod{12}$ the call $(n-3, n-2)$. The construction is illustrated for $n = 11$ in Table 6 and for $n = 17$ in Table 7.

round \ node	0	1	2	3	4	5	6	7	8	9	10
R_5	R^+	L^+	R^+	L^+	B	R^+	L^+	B	L^+	R^+	B
M_5	$\rightarrow$	$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
R_7	B^+	B^+	B^+	B^+	L^+	R^+	B^+	L^+	B^+	B^+	R^+
M_6	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$	
R_9	R^+	L^+	R^+	L^+	B	B	B	B	L^+	R^+	B
M_3	$\rightarrow$	$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
M_2	$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$	
R_{13}	R^+	L^+	B	B	B	B	B	B	L^+	R^+	B
M_1	$\rightarrow$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
M_0	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$	
R_{17}	B	B	B	B	B	B	B	B	L^+	R^+	B
M_8	5	$\leftarrow$	$\rightarrow$	9	9	$\leftarrow$	$\rightarrow$	2	$\times$		5
*	$\rightarrow$	7	7	$\leftarrow$	$\rightarrow$	0	0	$\leftarrow$	$\rightarrow$	4	$\times$
M_9	$\leftarrow$	$\rightarrow$	8	8	$\leftarrow$	$\rightarrow$	1	1	$\leftarrow$	$\times$	4
*	6	6	$\leftarrow$	$\rightarrow$	10	10	$\leftarrow$	$\times$	2	$\leftarrow$	$\rightarrow$
R_{21}	B	B	B	B	B	B	B	B	B	B	B

Tab. 6: Construction for $n = 11$

Theorem 7 *The above protocol is an optimal gossiping protocol for* $n \equiv 5 \pmod 6$ $(n > 5)$ *with* $2n-1$ *rounds.*

round \ node	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
R_5	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	B	R^+	L^+	B	L^+	R^+	B
M_{11}	$\rightarrow$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
R_7	B^+	B^+	B^+	B^+	B^+	B^+	B^+	B^+	B^+	B^+	L^+	R^+	B^+	L^+	B^+	B^+	R^+
M_{12}	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$	
R_9	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B	B	L^+	R^+	B
M_9	$\rightarrow$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
M_8	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$	
R_{13}	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B	B	B	B	L^+	R^+	B
								$\vdots$									
R_{29}	B	B	B	B	B	B	B	B	B	B	B	B	B	B	L^+	R^+	B
M_{14}	8	$\leftarrow$	$\rightarrow$	12	12	$\leftarrow$	$\rightarrow$	16	16	$\leftarrow$	$\rightarrow$	3	3	$\leftarrow$	$\times$	$\rightarrow$	8
*	$\rightarrow$	10	10	$\leftarrow$	$\rightarrow$	14	14	$\leftarrow$	$\rightarrow$	1	1	$\leftarrow$	$\rightarrow$	5	5	$\leftarrow$	$\times$
M_{15}	9	9	$\leftarrow$	$\rightarrow$	13	13	$\leftarrow$	$\rightarrow$	0	0	$\leftarrow$	$\rightarrow$	4	4	$\leftarrow$	$\times$	$\rightarrow$
*	$\leftarrow$	$\rightarrow$	11	11	$\leftarrow$	$\rightarrow$	15	15	$\leftarrow$	$\rightarrow$	2	2	$\leftarrow$	$\times$	$\rightarrow$	7	7
R_{33}	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B

Tab. 7: Construction for $n = 17$

Proof: Recall that in that case by Proposition 3, the label list consists of $(n-7)/2$ pairs of R^+L^+ followed by $BR^+L^+BL^+R^+B$. Therefore the calls of the two rounds associated to M_{n-6} are useful by Proposition 2. Furthermore, at the end of these two rounds the label list consists of $(n-7)$ B^+ followed by $L^+R^+B^+L^+B^+B^+B^+R^+$. Therefore the calls of the two rounds associated to M_{n-5} are useful by Proposition 2. Finally at the end of round 9, the label list consists of $(n-7)/2$ pairs of R^+L^+ followed by $BBBBL^+R^+B$.

Similarly to the case $n \equiv 1$ or $3 \pmod 6$, we say that a label list satisfies property Q'_h if, at the end of round $9+4h$, it consists of $(n-7-2h)/2$ pairs of R^+L^+ followed by $BBBBL^+R^+B$. As we have seen just above the property is satisfied for $h = 0$. Then we prove exactly in the same manner as in the proof of Theorem 6 that the calls of two symmetric rounds associated to the near-perfect matching M_{n-8-2h} and then of those associated to M_{n-9-2h} are useful and that at the end of the round $9+4(h+1)$ the label list satisfies property Q'_{h+1}.

Unfortunately we can not perform without modifying the 4 symmetric rounds associated to the two near-perfect matchings M_{n-3} and M_{n-2}. Indeed the call $(n-1, n-2)$ in the round associated to M_{n-3} is not necessarily useful. But thanks to the fact that node $n-3$ is neither a sender nor a receiver, it suffices to replace (as described in the protocol) this call with the useful call $(n-3, n-2)$ (resp. $(n-2, n-3)$) if $n \equiv 11 \pmod{12}$ (resp. $n \equiv 5 \pmod{12}$). Furthermore the round is valid as, when $n \equiv 11 \pmod{12}$ (resp. $n \equiv 5 \pmod{12}$), the call $(n-3, n-2)$ (resp. $(n-2, n-3)$) does not interfere with the call $(n-4, n-5)$ (resp. $(n-5, n-4)$) of this round. Similarly the replacement of the call $(n-4, n-3)$ with the useful call $(n-2, n-3)$ (resp. $(n-3, n-2)$) gives a valid round. Therefore, at the end of round $5 + 2(n-3) = 2n-1$ all the nodes have labels B with $\alpha = (n-1)/2$ and so every node knows all the messages. As the lower bound on the number of rounds is $2n-1$ by Theorem 2, this protocol is optimal. □

4 Conclusion

In this article we have determined the exact minimum gossiping time in a ring network with n nodes under the hypothesis of unit length messages and a primary node interference model. One can also try to determine the exact gossiping time for other simple topologies. In Bermond et al. (2020), we completely solved the gossiping problem in radio chain networks with the same model proving that the makespan is $3n - 5$ for $n > 3$. The case of grids might be solvable. It will also be interesting to consider stronger interferences. We can use a binary asymmetric model of interference based on the distance in the communication digraph like in Bermond et al. (2009, 2006); Klasing et al. (2008). Let $d(u, v)$ denote the distance, that is the length of a shortest directed path from u to v in G and d_I be a non negative integer. We assume that when a node u transmits, all nodes v such that $d(u, v) \leq d_I$ are subject to the interference from u's transmission. Equivalently, two calls (s, r) and (s', r') do not interfere if and only if $d(s, r') > d_I$ and $d(s', r) > d_I$. Our model corresponds to $d_I = 1$. For example, for $d_I = 2$ we will have to use instead induced matchings (in the case of rings it is a matching such that between two edges of the matching there is at least one uncovered node). In this case in a round without 2-calls we can have at most $n/3$ single calls. So, we get a general lower bound of $3n - 5$. For the protocols we can use the union of 3 matchings, each with at most $n/3$ edges and so the gossiping time is $3n + O(1)$. In order to determine the exact value we need to consider many cases according to the congruences modulo 24. Similarly for any d_I, we will have a gossiping time of $(d_I + 2)n + O(1)$.

References

J.-C. Bermond, L. Gargano, A. Rescigno, and U. Vaccaro. Fast gossiping by short messages. *SIAM Journal on Computing*, 27(4):917–941, 1998.

J.-C. Bermond, J. Galtier, R. Klasing, N. Morales, and S. Perennes. Hardness and approximation of gathering in static radio networks. *Parallel Processing Letters*, 16(2):165–183, 2006.

J.-C. Bermond, R. Correa, and M.-L. Yu. Optimal gathering protocols on paths under interference constraints. *Discrete Mathematics*, 309(18):5574–5587, Sept. 2009.

J.-C. Bermond, T. Kodate, and J. Yu. Gossiping with Interference Constraints in Radio Chain Networks. *Journal of Information Processing*, 28:889–902, 2020.

V. Bonifaci, R. Klasing, P. Korteweg, L. Stougie, and A. Marchetti-Spaccamela. *Graphs and Algorithms in Communication Networks*, chapter Data Gathering in Wireless Networks, pages 357–377. Springer, 2009.

L. Gąsieniec. On efficient gossiping in radio networks. In *International Colloquium on Structural Information and Communication Complexity – SIROCCO, 2009*, pages 2–14. Lectures Notes in Computer science, Springer Verlag, vol 5869:, 2010.

L. Gąsieniec and I. Potapov. Gossiping with unit messages in known radio networks. In *Proc. 2nd IFIP Int. Conference on Theoretical Computer Science*, pages 193–205, 2002.

R. Klasing, N. Morales, and S. Perennes. On the complexity of bandwidth allocation in radio networks. *Theoretical Computer Science*, 406(3):225–239, 2008.

F. Manne and Q. Xin. Optimal gossiping with unit size in known topology radio networkss. In *Proc. Workshop on Combinatorial and algorithmic aspects of networking, CAAN 2006*, pages 125–134. Lectures Notes in Computer science, Springer Verlag, vol 4235:, 2006.

A Proof of Proposition 3

We give in this appendix the proof of Proposition 3.

Proposition 3 *We can design the first five rounds of the protocol such that at the end of these five rounds the label list of the nodes consists of a sequence (starting at node* 0*) of*

$$\begin{cases} \frac{n-3}{2}\ R^+L^+ \textit{ followed by } BBB & \textit{if } n \equiv 1,3 \pmod 6 \\ \frac{n-7}{2}\ R^+L^+ \textit{ followed by } BR^+L^+BL^+R^+B & \textit{if } n \equiv 5 \pmod 6 \\ \frac{n-4}{2}\ R^+L^+ \textit{ followed by } BBBB & \textit{if } n \equiv 2 \pmod 4 \end{cases}$$

We design in the next subsection the first five rounds for the values of $n_0 = 7, 9, 11, 13, 15, 17$ and $n_0 = 10, 14, 18$. Then we show how to extend the construction for $n = 12p + n_0$.

A.1 *Table of the first five rounds for* $n_0 = 7, 9, 11, 13, 15, 17$ *and* $n_0 = 10, 14, 18$

The nodes are labeled from 0 to $n_0 - 1$ and we have built the rounds in such a way they have the following properties

- Property 1 : Round 1 contains the 2-call emitted by node 1 : $\{(1 : 0, 2)\}$.
- Property 2 : Round 2 contains the 2-call emitted by node $n_0 - 1$: $\{(n_0 - 1 : 0, n_0 - 2)\}$.
- Property 3 : Round 3 contains the 2-call emitted by node 0 : $\{(0 : 1, n_0 - 1)\}$.
- Property 4 : In round 4, node 0 receives a message from node 1 (namely message 2), and node $n_0 - 1$ does not send any message.
- Property 5 : In round 5, node 0 sends a message to node 1 namely message $n_0 - 1$ and node $n_0 - 1$ does not receive any message.
- Property 6 : the label list consists of a sequence of R^+L^+ followed by BBB if $n_0 = 7, 9, 13, 15$ or $BR^+L^+BL^+R^+B$ if $n_0 = 11, 17$ or R^+BBB if $n_0 = 10, 18$ or $BBBB$ if $n_0 = 14$.

A.2 *Extension of the first five rounds for* $n = 12p + n_0$

Let $n = 12p + n_0$. The nodes are labeled from 0 to $n - 1$. We add to the nodes of the examples for n_0, the $12p$ nodes $n_0 + i$ for $0 \le i \le 12p - 1$. We now describe the calls added in the first five rounds (one can follow the extension on the Table 17 which shows how the rounds for $n = 19$ are obtained from those for $n_0 = 7$).

- Round 1 consists of the calls of the example for n_0, plus the 2-calls $\{(n_0 + 3j + 1 : n_0 + 3j, n_0 + 3j + 2), j = 0, 1, 2, \ldots, 4p - 1\}$.

round \ node	0	1	2	3	4	5	6
1	1		1	4		4	$\times$
2	6	2		2	$\times$	6	
3		0	3		3	$\times$	0
4	2			1	5		5
5		6	4		$\times$	$\times$	$\times$
label	R^+	L^+	R^+	L^+	B	B	B

Tab. 8: First five rounds for $n_0 = 7$

round \ node	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1		1	4		4	7		7	10		10	$\times$
2	12	2		2	5		5	8		8	$\times$	12	
3		0	3		3	6		6	9		9	$\times$	0
4	2			1	6			5	10		$\times$		11
5		12	4			3	8			7	11		$\times$
label	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B

Tab. 9: First five rounds for $n_0 = 13$, no 2-call from the node 11

round \ node	0	1	2	3	4	5	6	7	8
1	1		1	4		4	7		7
2	8	2		2	5		5	8	
3		0	3		3	6		6	0
4	2			1	6		$\times$	$\times$	$\times$
5		8	4			3	$\times$	$\times$	$\times$
label	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B

Tab. 10: First five rounds for $n_0 = 9$

round \ node	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1		1	4		4	7		7	10		10	13		13
2	14	2		2	5		5	8		8	11		11	14	
3		0	3		3	6		6	9		9	12		12	0
4	2			1	6			5	10			9	$\times$	$\times$	$\times$
5		14	4			3	8			7	12		$\times$	$\times$	$\times$
label	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B

Tab. 11: First five rounds for $n_0 = 15$

round \ node	0	1	2	3	4	5	6	7	8	9	10
1	1		1	4		4	7		7	$\times$	$\times$
2	10	2		2	5		5	$\times$	$\times$	10	
3		0	4		$\times$		4	8		8	0
4	2			1	$\times$	6		6	9		9
5		10	3		3	7			6	0	
label	R^+	L^+	R^+	L^+	B	R^+	L^+	B	L^+	R^+	B

Tab. 12: First five rounds for $n_0 = 11$

round \ node	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1		1	4		4	7		7	10		10	13		13	$\times$	$\times$
2	16	2		2	5		5	8		8	11		11	$\times$	$\times$	16	
3		0	3		3	6		6	10		$\times$		10	14		14	0
4	2			1	6			5	9		9	12		12	15		15
5		16	4			3	8			7	$\times$	13			12	0	
label	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	B	R^+	L^+	B	L^+	R^+	B

Tab. 13: First five rounds for $n_0 = 17$

node / round	0	1	2	3	4	5	6	7	8	9
1	1		1	4		4	7		7	×
2	9	2		2	5		5	×	9	
3		0	3		3	6		6	×	0
4	2			1	6		×	8		8
5		9	4			3	×	×	×	×
label	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B	B

Tab. 14: First five rounds for $n_0 = 10$

node / round	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1		1	4		4	7		7	10		10	×	×
2	13	2		2	5		5	8		8	×	×	13	
3		0	3		3	6		6	9		9	×	×	0
4	2			1	6			5	10		×	12		12
5		13	4			3	8			7	11		11	×
label	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B	B

Tab. 15: First five rounds for $n_0 = 14$

node / round	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	1		1	4		4	7		7	10		10	13		13	16		16
2	17	2		2	5		5	8		8	11		11	14		14	17	
3		0	3		3	6		6	9		9	12		12	15		15	0
4	2			1	6			5	10			9	14		×	×	×	×
5		17	4			3	8			7	12			11	×	×	×	×
label	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B	B

Tab. 16: First five rounds for $n_0 = 18$

- Round 2 consists of the calls of the example for n_0, except we delete the 2-call $\{(n_0-1 : 0, n_0-2)\}$, which exists by Property 2, and replace it by the 2-call $\{(n_0 - 1 : n_0, n_0 - 2)\}$. Then we add the 2-calls $\{(n_0+3j+2 : n_0+3j+1, n_0+3j+3), j = 0, 1, 2, \ldots, 4p-1\}$ (recall the node n_0+12p means the node 0).
- Round 3 consists of the calls of the example for n_0, except we delete the 2-call $\{(0 : 1, n_0 - 1)\}$ which exists by Property 3, and replace it by the 2-call $\{(0 : 1, n_0 + 12p - 1)\}$. Then we add the 2-calls $\{(n_0 + 3j : n_0 + 3j - 1, n_0 + 3j + 1), j = 0, 1, 2, \ldots, 4p - 1\}$.
- Round 4 consists of the calls of the example for n_0, plus the calls $\{(n_0 + 4k + 1, n_0 + 4k), (n_0 + 4k + 2, n_0 + 4k + 3), k = 0, 1, 2, \ldots, 3p - 1\}$.
- Round 5 consists of the calls of the example for n_0, plus the calls $\{(n_0 + 4k, n_0 + 4k + 1), (n_0 + 4k + 3, n_0 + 4k + 2), k = 0, 1, 2, \ldots, 3p - 1\}$. Also we have to impose that node 0 is now sending to node 1 the message $n_0 + 12p - 1$ (and not $n_0 - 1$).

round \ node	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1		1	4		4	×	8		8	11		11	14		14	17		17
2	18	2		2	×	6		6	9		9	12		12	15		15	18	
3		0	3		3	×	7		7	10		10	13		13	16		16	0
4	2			1	5		5	9			8	13			12	17			16
5		18	4		×	×	×		6	11			10	15			14	18	
label	R^+	L^+	R^+	L^+	B	B	B	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+

Tab. 17: First five rounds for $n = 19$ (from $n_0 = 7$)

First we note that these five rounds are valid. There is no interference between the calls added and not between these calls and the ones used for n_0 due to Properties $1, 4, 5$. Indeed in rounds 1 and 4, nodes n_0 and $n_0 + 12p - 1$ are both receiving while by Property 1 or 4, node 0 is receiving and node $n_0 - 1$ is not emitting. In round 5, nodes n_0 and $n_0 + 12p - 1$ are both emitting while, by Property 5, nodes 0 and $n_0 - 1$ do not receive any message.

We also note that in this construction the number of messages received from the left and from the right by the nodes from 0 to $n_0 - 1$ remains the same. Indeed the calls are the same, except in round 2 where node 0 is receiving from $n_0 + 12p - 1$ instead of $n_0 - 1$ and in round 3 where node $n_0 - 1$ is receiving from n_0 instead of 0. Furthermore, during the first three rounds the $12p$ nodes added receive one message from the left and another from the right. During the rounds 4 and 5, nodes $n_0 + 2j (0 \le j \le 6p-1)$ have received a message from the right and so have label R^+, and nodes $n_0 + 2j + 1 (0 \le j \le 6p - 1)$ have received a message from the left and so have label L^+. So using these labels and the Property 6, the label list starting at n_0 satisfies Proposition 3. However, as we are in a cycle, the starting node of a sequence can be chosen arbitrarily. If we relabel node n_0: 0 and $n_0 + i$: i, we get Proposition 3 now starting node 0 (as announced at the beginning of the subsection).