Computing the number of h-edge spanning forests in complete bipartite graphs
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Let $f_{m,n,h}$ be the number of spanning forests with $h$ edges in the complete bipartite graph $K_{m,n}$. Kirchhoff’s Matrix Tree Theorem implies $f_{m,n,m} + n - 1 = mn - 1$ when $m \geq 1$ and $n \geq 1$, since $f_{m,n,m} + n - 1$ is the number of spanning trees in $K_{m,n}$. In this paper, we give an algorithm for computing $f_{m,n,h}$ for general $m, n, h$. We implement this algorithm and use it to compute all non-zero $f_{m,n,h}$ when $m \leq 50$ and $n \leq 50$ in under 2 days.

**MSC 2010:** 68R10, 05C85, 05C30, 05C38, 05C05

**Keywords:** complete bipartite graph, spanning forest

1 Introduction

A consequence of Kirchhoff’s Matrix Tree Theorem is that the number of spanning trees in the complete bipartite graph $K_{m,n}$ is $m^{n-1}n^{m-1}$ when $m \geq 1$ and $n \geq 1$. In this paper, we consider a generalisation of this counting problem: the number $f_{m,n,h}$ of $h$-edge spanning forests in $K_{m,n}$. The number of spanning trees in $K_{m,n}$ is $f_{m,n,m+n-1}$ when $m \geq 1$ and $n \geq 1$, since spanning trees of $K_{m,n}$ have $m + n - 1$ edges.

The numbers $f_{m,n,h}$ arise in relation to the Tutte polynomial of $K_{m,n}$. For a general graph $G = (V, E)$, the Tutte polynomial is

$$T_G(x, y) = \sum_{A \subseteq E} (x - 1)^{k(A) - k(E)} (y - 1)^{k(A) + |A| - |V|}$$

where $k(A)$ is the number of connected components in the graph $(V, A)$. Importantly, for the Tutte polynomial, we take $0^0 = 1$. Thus, the number of spanning forests of $G$ is given by

$$T_G(2, 1) = \sum_{A \subseteq E} 0^{k(A) + |A| - |V|} = \# \{ A \subseteq E : k(A) + |A| = |V| \}$$

number of spanning forests of $G$, (1)
since \( k(A) + |A| = |V| \) if and only if \( A \) is a spanning forest.

In the present paper, we are interested in the case of when \( G = K_{m,n} \), and will look for \( h \)-edge spanning forests. In this situation,

\[
T_{K_{m,n}}(2, 1) = \sum_{h \geq 0} f_{m,n,h}.
\]

For general graphs \( G = (V, E) \), Myrvold \cite{10} gave an algorithm for computing the number of \( k \)-component spanning forests of \( G \). Bj"oklund et al. \cite{4} described an algorithm that could compute \( T_G(2, 1) \) in time \( 2^{|V|} |V|^{O(1)} \) by using Kirchhoff’s Matrix Tree Theorem for each of the \( 2^{|V|} \) subsets of \( V \), then combining the results together with Inclusion-Exclusion. Other relevant work for the enumeration of spanning forests includes \cite{9}. Porter \cite{11} gave an algorithm for generating the spanning trees of \( K_{m,n} \).

Farr and McDiarmid \cite{5} showed that computing the number of circuits in \( G \) is \#P-complete \cite{1}. Bounds for the number of spanning forests in graphs were given by Teranishi \cite{13}, from which we can deduce

\[
T_{K_{m,n}}(2, 1) \geq \sum_{k \geq 0} \left( \frac{\min(m, n)}{2} \right)^{m+n-k} \binom{m+n}{k}
\]

for all \( m \geq 0 \) and \( n \geq 0 \). Jin and Liu \cite{8} gave a simple formula for the number of rooted spanning forests in \( K_{m,n} \) (see also \cite{7} and \cite{12}).

This paper instead heads in a different direction to previous work: we derive algorithms for enumerating \( h \)-edge spanning forests in \( K_{m,n} \).

2 Basic results

To be clear, all forests in this paper will be labelled. Let \( \mathbb{N} = \{0, 1, 2, \ldots\} \). The following lemma gives the boundary conditions on \( f_{m,n,h} \).

**Lemma 1** Suppose \( m, n, h \in \mathbb{N} \). Then

- \( f_{m,n,0} = 1 \),
- if \( h \geq 1 \), then \( f_{0,n,h} = f_{m,0,h} = 0 \),
- if \( m \geq 1 \) and \( n \geq 1 \), then \( f_{m,n,h} > 0 \) if and only if \( 0 \leq h \leq m + n - 1 \),
- if \( m \geq 1 \) and \( n \geq 1 \), then \( f_{m,n,m+n-1} = n^{m-1}m^{n-1} \).

**Proof:** If \( h = 0 \), then there is exactly one subgraph of \( K_{m,n} \) with no edges (and it is a spanning forest), so \( f_{m,n,0} = 1 \). If \( m = 0 \) or \( n = 0 \), then \( K_{m,n} \) has no edges, and thus we obtain the second bulleted item.

Now assume \( m \geq 1 \) and \( n \geq 1 \). Since \( K_{m,n} \) is connected, it has a spanning tree, which must have exactly \( m + n - 1 \) edges. By deleting edges from this spanning tree, we find \( h \)-edge forests in \( K_{m,n} \) for all \( 0 \leq h \leq m + n - 1 \). Hence \( f_{m,n,h} > 0 \) when \( 0 \leq h \leq m + n - 1 \).

Now assume \( A \) is an \( h \)-subset of the edges in \( K_{m,n} \), where \( m \geq 1 \) and \( n \geq 1 \) and \( h \geq m + n \). To be a spanning forest, we need \( k(A) + h = m + n \) as per \cite{1}, but if \( h \geq m + n \), then \( k(A) \leq 0 \), giving a contradiction.

\(^{1)} \#P \) is the set of counting problems which ask for the number of “yes” instances for decision problems in NP. A \#P problem is in \#P-complete whenever any other problem in \#P can be reduced to it by a polynomial-time counting reduction.
The number of $h$-edge spanning forests in $K_{m,n}$

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**Fig. 1:** Small non-zero values of $f_{m,n,h}$.

The fourth bulleted item is from Kirchhoff’s Matrix Tree Theorem (already mentioned).

The following lemma is an important (but basic) consequence of the symmetry of the problem.

**Lemma 2** For all $m, n, h \in \mathbb{N}$, we have $f_{m,n,h} = f_{n,m,h}$.

In Figure 1 we list the non-zero values of $f_{m,n,h}$ when $0 \leq m \leq n \leq 6$. These values were generated using a straightforward backtracking algorithm.

## 3 Combinatorial equivalence

In this section, we will describe some combinatorial objects that are equivalent to $h$-edge spanning forests of $K_{m,n}$.

Let $M$ be an $m \times n \ (0,1)$-matrix. We define a cycle in $M$ to be a set $\{e_1, e_2, \ldots, e_t\}$ of entries of $M$, such that:

- each $e_i$ contains the symbol 1, and
- for $i \in \{1, 2, \ldots, t\}$, we have that (a) $e_{2i-1}$ belongs to the same row as $e_{2i}$ and (b) $e_{2i}$ belongs to the same column as $e_{2i+1}$ (where we take $e_{2t+1} = e_1$).

For example, the matrix

$$
\begin{pmatrix}
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0
\end{pmatrix}
$$
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has a cycle (consisting of the circled 1’s), whereas the matrices

\[
\begin{pmatrix}
0 & 1 & 1 \\
0 & 0 & 0 \\
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

do not have cycles.

Let \(A_{m,n,h}\) be the set of \((0,1)\)-matrices with exactly \(h\) elements equal to 1. Let \(p_{m,n,h}\) be the probability that an element of \(A_{m,n,h}\) chosen uniformly at random contains a cycle. An anonymous user of math.stackexchange.com asked for a formula for \(p_{m,n,h}\). In fact, this was the original motivation for the author to study this problem.

**Lemma 3** For all \(m,n,h \in \mathbb{N}\), we have \(f_{m,n,h}\) is the number of \(m \times n\) \((0,1)\)-matrices with \(h\) 1’s that do not contain a cycle, and hence

\[
f_{m,n,h} = (1 - p_{m,n,h}) \frac{m n}{h}.
\]

**Proof:** A matrix \(M \in A_{m,n,h}\) can be interpreted as the biadjacency matrix of an \(h\)-edge subgraph \(G = G(M)\) of \(K_{m,n}\). A cycle in \(M \in A_{m,n,h}\) corresponds to a cycle in \(G\). Hence

\[
f_{m,n,h} = (1 - p_{m,n,h}) |A_{m,n,h}| = (1 - p_{m,n,h}) \frac{m n}{h}.
\]

\(\square\)

A problem related to the \((0,1)\)-matrix problem above comes from the study of \((0,1)\)-matrices that do not have a submatrix which is the incidence matrix of any cycle of length at least 3; these are called “totally balanced matrices” (see e.g. [2]).

Another interpretation of matrices in \(A_{m,n,h}\) is as induced subgraphs of \(K_m \square K_n\), where \(\square\) represents the Cartesian product of graphs. The graph \(K_m \square K_n\) is sometimes called the “rook’s graph”, since the edges represent the legal moves of a rook on an \(m \times n\) chess board. The vertices in \(K_m \square K_n\) are \(\{(i,j) : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}\) and there is an edge between distinct vertices \((i,j)\) and \((i',j')\) whenever \(i = i'\) or \(j = j'\). The graph \(K_m \square K_n\) is also the line graph of \(K_{m,n}\).

There is a bijection between induced subgraphs \(H = H(M)\) of \(K_m \square K_n\) and \((0,1)\)-matrices \(M = (m_{ij}) \in A_{m,n,h}\): we include the vertex \((i,j)\) in \(H\) if and only if \(m_{ij} = 1\). For example, if we ignore the vertices marked 0 in

\[
\begin{array}{ccccccc}
0 & 1 & 1 & 0 & 1 & & \\
1 & 0 & 0 & 0 & 1 & & \\
1 & 0 & 0 & 0 & 0 & & \\
\end{array}
\]

\(\text{Full URL: } \text{http://math.stackexchange.com/q/24800}\)

\(\text{The biadjacency matrix of a bipartite graph } G \text{ on the vertex set } \{u_i\}_{1 \leq i \leq m} \cup \{v_j\}_{1 \leq j \leq n} \text{ is the } m \times n \text{ } (0,1)\)-matrix } M = (m_{ij}) \text{ with } m_{ij} = 1 \text{ if and only if } u_i v_j \text{ is an edge in } G.\)
we obtain an induced subgraph \( H(M) \) of \( K_3 \square K_5 \), corresponding to the matrix

\[
M = \begin{pmatrix}
0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

in \( A_{3,5,6} \). We see that cycles in \( M \) map from cycles \( C_k \) in \( H \) for some \( k \in \{4, 6, 8, \ldots \} \) (an example of a 6-cycle is highlighted). There are other cycles in \( H \) (e.g. \( C_5 \) in the above example; in fact, if \( M \) above had a row of 1’s, then \( H \) would have a \( K_5 \) subgraph). However, induced cycles \( C_k \) in \( H(M) \) of length \( k \in \{4, 6, 8, \ldots \} \) are in one-to-one correspondence with cycles in \( M \) via the above bijection.

**Lemma 4** For all \( m, n, h \in \mathbb{N} \), we have that \( f_{m,n,h} \) is the number of \( h \)-vertex induced subgraphs of \( K_m \square K_n \) that do not contain an induced \( C_k \) for any \( k \in \{4, 6, 8, \ldots \} \).

4 Simplifying the equation

4.1 A formula for \( f_{m,n,h} \)

We will look at two related ways of simplifying \( f_{m,n,h} \). Let \( K_{m,n} \) have the vertex bipartition \( M \cup N \), where \( M = \{u_1, u_2, \ldots, u_m\} \) and \( N = \{v_1, v_2, \ldots, v_n\} \). Let \( B_{m,n,h} \) be the set of \( h \)-edge spanning forests of \( K_{m,n} \). Hence \( f_{m,n,h} = |B_{m,n,h}| \). Define

\[
W_{m,n,h} = \{G \in B_{m,n,h} : G \text{ has no isolated vertices}\}
\]

and \( w_{m,n,h} = |W_{m,n,h}| \). We observe the following:

- Given any spanning forest in \( B_{m,n,h} \), we can delete the isolated vertices to obtain a spanning forest in \( W_{i,j,h} \) for some \( i \leq m \) and \( j \leq n \).

- Conversely, given any spanning forest in \( W_{i,j,h} \), we can add isolated vertices to it \( \binom{m}{m-i} \binom{n}{n-j} \) ways to obtain a spanning forest in \( B_{m,n,h} \).

In both of the operations above, we need to relabel the vertices, but we preserve the order of the indices of the non-isolated vertices within \( M \) and \( N \).

Hence

\[
f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} w_{i,j,h}.
\]

This formula eliminates the need to further account for isolated vertices.

We can decompose any \( G \in W_{m,n,h} \) into disjoint components, each of which belong to \( W_{i,j,k} \) for some \( i \leq m \) and \( j \leq n \) and \( k \leq h \). Hence any \( G \in W_{m,n,h} \) decomposes into the following.

- A partition \( P \) of \( M \) where \( x, y \in M \) belong to the same part if there is a path from \( x \) to \( y \) in \( G \).

- A partition \( Q \) of \( N \) where \( x', y' \in N \) belong to the same part if there is a path from \( x' \) to \( y' \) in \( G \).

- A bijection \( \alpha : P \to Q \) such that every edge \( ab \) in \( G \) has \( a \in p \) and \( b \in \alpha(p) \) for some \( p \in P \).

Hence we have \(|P| = |Q|\).
For each $p \in P$, a subgraph in $W_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$ induced by the vertices in $p \cup \alpha(p)$.

Regarding the final bulleted item, we observe that the subgraph induced by $p \cup \alpha(p)$ for some $p \in P$ must be a spanning tree on $K_{|p|,|\alpha(p)|}$. Any more edges would cause a cycle, while fewer edges would cause disjoint components (in which case $G$ could be further decomposed).

We give an example of a graph $G$ in $W_{3,5,6}$ below.

Here, the partitions $P$ and $Q$ are given by $P = \{ \{u_1, u_2\}, \{u_3\} \}$ and $Q = \{ \{v_1, v_2\}, \{v_3, v_4, v_5\} \}$ and we have the bijection $\alpha$ such that

$$\{u_1, u_2\} \mapsto \{v_1, v_2\},$$
$$\{u_3\} \mapsto \{v_3, v_4, v_5\}.$$

We see that $G$ decomposes into a graph in $W_{2,2,3}$ and a graph in $W_{1,3,3}$.

In general, for any $p \in P$, we must have exactly $|p| + |\alpha(p)| - 1$ edges in $p \cup \alpha(p)$. Summing this over all $p \in P$ gives $h = m + n - |P|$. We conclude that

$$|P| = m + n - h$$

(and thus $|Q| = m + n - h$).

We will now describe how to construct any graph in $W_{m,n,h}$ via its decomposition. Let $P$ be the set of partitions of $M$ and let $Q$ be the set of partitions of $N$. Given (a) a partition $P \in \mathcal{P}$ of size $|P| = m + n - h$, (b) a partition $Q \in \mathcal{Q}$ and (c) a bijection $\alpha : P \rightarrow Q$, we can construct $\prod_{p \in P} w_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$ distinct graphs $G \in W_{m,n,h}$ by the following steps:

1. Start with $G$ as the graph with vertex set $M \cup N$ and no edges.
2. For each $p \in P$ add one of the subgraphs in $W_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$ on the vertices $p \cup \alpha(p)$.

Since all graphs in $W_{m,n,h}$ can be constructed uniquely by the above steps we have the following theorem.

**Theorem 1** For $m, n, h \in \mathbb{N}$,

$$f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} w_{i,j,h}$$
The number of $h$-edge spanning forests in $K_{m,n}$

where

$$w_{m,n,h} = \sum_{P \in P} \sum_{Q \in Q} \sum_{\alpha: P \to Q} \prod_{p \in P} u_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$$

where

$$u_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1} = |p|^{\alpha(p)-1} |\alpha(p)|^{|p|-1}.$$

4.2 An improved formula for $f_{m,n,h}$

We will derive two further equations for $f_{m,n,h}$ (in Theorems 2 and 3, which are related to Theorem 1). They result in a slightly more complicated algorithm, but will allow us to compute $f_{m,n,h}$ faster than (2).

For all $m, n, h \in \mathbb{N}$, define

$$W'_{m,n,h} = \{G \in W_{m,n,h} : \text{the vertices in } N \text{ have degree } \geq 2\}.$$

Given any $G \in B_{m,n,h}$ we can delete the isolated vertices and the leaves (vertices of degree 1) in $N$ to obtain a spanning forest in $W'_{i,j,h-k}$ for some $i \leq m$ and $j \leq n$ and $k \in \mathbb{N}$. Conversely, given any $G \in W'_{i,j,h-k}$, we can add:

- $m - i$ isolated vertices to $M$ in $\binom{m}{m-i}$ ways, so as to increase $|M|$ to $m$, then
- $n - j$ isolated vertices to $N$ in $\binom{n}{n-j}$ ways, so as to increase $|N|$ to $n$, then
- $k$ edges to $G$ in $\binom{n-j}{k}$ ways, so as to increase the number of leaves in $N$ by $k$,

thereby obtaining a spanning forest in $B_{m,n,h}$. Similar to the derivation of (2), we need to relabel the vertices so as to preserve the order of the indices of the non-isolated vertices within $M$ and the non-isolated non-leaf vertices within $N$.

Hence

$$f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \min(n-j,h) \binom{m}{i} \binom{n}{j,k,n-j-k} m^{k} w'_{i,j,h-k}.$$

All of the steps involved for finding the formula (3) for $w_{m,n,h}$ are still valid for $w'_{m,n,h}$. Hence (3) remains true if we replace $w$ with $w'$. However, we can no longer make use of the simple formula for $u_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$. Nevertheless, we will be able to find a formula for $u'_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$ via Inclusion-Exclusion.

For $I \subseteq N$, define

$$A_I = \{G \in W_{m,n,m+n-1} : \text{vertices in } I \text{ are leaves}\}.$$

Note that spanning forests in $A_I$ might also have leaves outside of $I$. Hence $|A_0| = |W_{m,n,m+n-1}| = m^{n-1} n^{m-1}$. By symmetry, if $I, J \subseteq N$ and $|I| = |J|$ then $|A_I| = |A_J|$. So we will assume $I = \{n-i+1, n-i+2, \ldots, n\}$. We can construct the graphs in $A_I$ by adding $i$ leaves to the graphs in $W_{m,n-|I|,m+n-|I|-1}$ (i.e., the set of spanning trees of $K_{m,n-|I|}$), and these leaves can be added in $m^i$ ways. Hence

$$|A_I| = m^i \cdot \text{number of spanning trees of } K_{m,n-i}$$

$$= m^i m^{n-i-1}(n-i)^{m-1}$$

$$= m^{n-1}(n-i)^{m-1}.$$
when \( m \geq 1 \) and \( n \geq 1 \). Here we take \( 0^0 = 1 \), since we want to account for \( K_{1,0} \in W_{1,0,0} \). Hence, by Inclusion-Exclusion, we find

\[
\begin{align*}
w'_{m,n,m+n-1} &= |A_0| - \left| \bigcup_{I \subseteq N \atop I \neq \emptyset} A_I \right| \\
&= m^{n-1} n^{m-1} - \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} m^{n-1} (n-i)^{m-1} \\
&= m^{n-1} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^{m-1}
\end{align*}
\]

when \( m \geq 1 \) and \( n \geq 1 \). By definition, we also have \( w'_{m,n,m+n-1} = 0 \) when either \( m = 0 \) or \( n = 0 \), and \( w'_{m,n,m+n-1} \) is not defined when \( m = 0 \) and \( n = 0 \).

Before identifying formulae for \( w_{m,n,h} \) and \( w'_{m,n,h} \) in the next section, we make the following observation.

**Lemma 5** For all \( m, n, h \in \mathbb{N} \), we have \( n! \) divides \( w'_{m,n,h} \).

**Proof:** The symmetric group on \( N \) acts on \( W'_{m,n,h} \) by permuting the vertices in \( N \). Suppose a graph \( G \in W'_{m,n,h} \) has a non-trivial stabiliser subgroup under this action. Then there exists a non-identity permutation \( \alpha \) such that \( \alpha G = G \), and distinct vertices \( v, v' \in N \) for which \( \alpha(v) = v' \). Since \( G \in W'_{m,n,h} \), the degree of \( v \) is 2 or more, so assume \( v \) has distinct neighbours \( a, b \in M \). Then, since \( \alpha G = G \), we find that \( v' = \alpha(v) \) has the neighbours \( a \) and \( b \) too. Thus, \( \{v, v', a, b\} \) induces a 4-cycle, giving a contradiction. Hence, all stabiliser subgroups are trivial, and by the Orbit-Stabiliser Theorem, all orbits have size \( n! \). Hence \( n! \) divides \( w'_{m,n,h} \).

### 4.3 Formulae for \( w_{m,n,h} \) and \( w'_{m,n,h} \)

Now we will simplify (3), noting that our simplifications remain valid when we replace \( w \) with \( w' \).

For \( a \geq 1 \) and \( t \geq 1 \), let \( S_{a,t} \) be the set of (number) partitions of \( a \) into \( t \) non-zero parts. We will interpret elements of \( S_{a,t} \) as multisets; for example \( \{4,3,3,1,1\} \in S_{12,5} \). We will require \( S_{a,0} = \emptyset \) when \( t < 0 \), and \( S_{0,0} = \{\emptyset\} \) and \( S_{0,t} = \emptyset \) when \( t > 0 \). For \( z \in S_{a,t} \), let \( T[z] \) be the set of partitions of \( \{1,2,\ldots,a\} \) whose part sizes induce the number partition \( z \). For \( z \in S_{a,t} \), let \( z \) denote an arbitrary element of \( T[z] \). If \( P \in T[x] \) and \( Q \in T[z] \), then any bijection \( \alpha : P \to Q \) induces a bijection \( \beta : x \to z \).
Hence
\[ w_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \sum_{P \in T[x]} \prod_{\alpha : T[z] \to x \text{ is a bijection}} w_{P,|\alpha(P)|,|P|+|\alpha(P)|-1} \]
\[ = \sum_{x,z \in S_{m,m+n-h}} \sum_{P \in T[x]} \prod_{\alpha : T[z] \to x \text{ is a bijection}} w_{P,|\alpha(P)|,|P|+|\alpha(P)|-1} \]
\[ = \sum_{x,z \in S_{m,m+n-h}} \sum_{P \in T[x]} \prod_{\alpha : T[z] \to x \text{ is a bijection}} w_{P,|\alpha(P)|,|P|+|\alpha(P)|-1} \]
\[ = \sum_{x,z \in S_{m,m+n-h}} |T[x]| \cdot |T[z]| \sum_{\beta : x \to z} \prod_{r \in x} w_{r,|\beta(r)|,|r|+|\beta(r)|-1}. \]

If \( z \in S_{a,t} \), then
\[ |T[z]| = \frac{a!}{\prod_{i \geq 1} \frac{1}{i^{s_i(z)}} s_i(z)!} \]
where \( s_i(z) \) denotes the number of parts \( i \) in \( z \) (see e.g. [1] Theorem 13.2). Hence we have the following theorems.

**Theorem 2** For \( m, n, h \in \mathbb{N} \),
\[ f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} w_{i,j,h} \]
where
\[ w_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \sum_{\beta : x \to z} \prod_{r \in x} \frac{m!}{\prod_{i \geq 1} \frac{1}{i^{s_i(z)}} s_i(z)!} \prod_{i \geq 1} \frac{1}{i^{s_i(z)}} s_i(z)! \prod_{r \in x} w_{r,\beta(r),|r|+\beta(r)-1}. \quad (4) \]

**Theorem 3** For \( m, n, h \in \mathbb{N} \),
\[ f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{\min(n-j,h)} \binom{m}{i} \binom{n}{j,k} m^{j,k} w_{i,j,h-k}^{t} \]
where
\[ w_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \sum_{\beta : x \to z} \prod_{r \in x} \frac{m!}{\prod_{i \geq 1} \frac{1}{i^{s_i(z)}} s_i(z)!} \prod_{i \geq 1} \frac{1}{i^{s_i(z)}} s_i(z)! \prod_{r \in x} w_{r,\beta(r),|r|+\beta(r)-1}^{t} \]
where
\[ w_{m,n,m+n-1}^{t} = m^{n-1} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^{m-1}. \]

The main advantage of Theorems 2 and 3 over Theorem 1 is the summation is over number partitions (rather than set partitions).
5 Implementation

The author has implemented the two formulae for \( f_{m,n,h} \), described in the preceding section, along with a simple backtracking algorithm. The results between all three implementations concur, giving confidence in the accuracy of the code. The C source code is available as supplementary material to this document. The GMP library was used for arbitrary precision arithmetic [6].

The formula involving \( w'_{m,n,h} \) instead of \( w_{m,n,h} \) was (unsurprisingly) much faster, largely because many values of \( w_{m,n,h} \) equal 0, such as when \( m < n \), which can be used to drastically reduce the search tree.

5.1 Symmetry breaking

The run-time of the program was also improved through the use of symmetry breaking, which we will now describe. For \( x \in S_{m,m+n-h} \) and \( z \in S_{n,n+m-h} \), there are often many bijections \( \beta : x \rightarrow z \) which map the same elements to the same elements (since both \( x \) and \( z \) are multisets). In these instances, a na"ive algorithm would repeat the same computation unnecessarily. Let \( x \) and \( z \) be the multisets \( x = \{x_1, x_2, \ldots, x_{m+n-h}\} \) and \( z = \{z_1, z_2, \ldots, z_{m+n-h}\} \). We define the condition:

**Symmetry breaking condition 1:** We say \( \beta : x \rightarrow z \) is half-canonical if \( \beta^{-1}(z_{i-1}) < \beta^{-1}(z_i) \) whenever \( z_{i-1} = z_i \).

We need to add a multiplicative factor to adjust for the restriction to half-canonical bijections. Hence

\[
w_{m,n,h} = \sum_{x, z \in S_{m,m+n-h}} \sum_{\beta : x \rightarrow z} \left( \prod_{i \geq 1} \frac{m!}{i^{s_i(z)}} \right) \frac{m!}{\prod_{i \geq 1} \frac{i^{s_i(x)} s_i(x)!}{i^{s_i(z)} s_i(z)!}} \frac{n!}{\prod_{r \in x} w_{r,\beta(r), r+\beta(r)-1}}\]

and similarly with \( w' \) in place of \( w \). Using this assumption, we reduce the number of bijections by a factor of \( \prod_{i \geq 1} \frac{i^{s_i(z)}}{i^{s_i(x)}} \), which results in a substantial time saving.

Instead of the single symmetry breaking assumption, it is possible to utilise symmetry breaking using an additional condition:

**Symmetry breaking condition 2:** We say \( \beta : x \rightarrow z \) is canonical if it is half-canonical and \( \beta(x_{i-1}) < \beta(x_i) \) whenever \( x_i = x_{i-1} \).

Again, we find

\[
w_{m,n,h} = \sum_{x, z \in S_{m,m+n-h}} \sum_{\beta : x \rightarrow z} \Gamma_{x,z,\beta} \left( \prod_{i \geq 1} \frac{m!}{i^{s_i(x)} s_i(x)!} \right) \left( \prod_{i \geq 1} \frac{n!}{i^{s_i(z)} s_i(z)!} \right) \prod_{r \in x} w_{r,\beta(r), r+\beta(r)-1},\]

and similarly with \( w' \) in place of \( w \), where \( \Gamma_{x,z,\beta} \) is the number of bijections between the multisets \( x \) and \( z \) which map the same elements to the same elements (as \( \beta \)). A formula for \( \Gamma_{x,z,\beta} \) was given by e.g. [3], namely

\[
\Gamma_{x,z,\beta} = \prod_{i \geq 1} \frac{s_i(x)!}{s_i(z)!} \prod_{j \geq 1} \frac{s_i(j x, \beta)!}{s_i(j z, \beta)!},
\]

where
The number of $h$-edge spanning forests in $K_{m,n}$

- $s_i(z)$ denotes the number of parts $i$ in $z$ (as before), and
- $s_{i,j}(x, \beta)$ is the number of elements $(i, j)$ in the multiset $\{(r, \beta(r)) : r \in x\}$.

The author has implemented both of these symmetry breaking schemes in order to compare their performance (see Section 5.3).

5.2 Pseudo-code

Algorithm 1 gives a pseudo-code version of the C code used to implement the algorithm described by Theorem 3 using the half-canonical symmetry breaking condition. The partitions of $m$ and $n$ into $k$ parts were computed whenever needed and stored in memory. Iterating through the half-canonical bijections was performed “on the fly” using a backtracking algorithm.

While Theorems 2 and 3 are valid for all $m, n, h \in \mathbb{N}$, we need to set $f_{m,n,0} = 1$ separately in the C code.

5.3 Performance

The C code was run on a $2 \times 2.66$ GHz processor (although the code itself is not parallelised). The following table gives the run-times (in seconds) for the two algorithms under the two symmetry breaking schemes when computing all non-zero $f_{m,n,h}$ with $m, n \leq 19$.

<table>
<thead>
<tr>
<th></th>
<th>half-canonical</th>
<th>canonical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 2</td>
<td>27.1</td>
<td>6.8</td>
</tr>
<tr>
<td>Theorem 3</td>
<td>4.1</td>
<td>4.5</td>
</tr>
</tbody>
</table>

The fastest version is Theorem 3 using half-canonical symmetry breaking. Under these conditions, the code had the following run-times to find all non-zero values of $f_{m,n,h}$ with $m, n \leq t$:

<table>
<thead>
<tr>
<th>$t$</th>
<th>time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>20.5</td>
</tr>
<tr>
<td>26</td>
<td>27.7</td>
</tr>
<tr>
<td>27</td>
<td>35.4</td>
</tr>
<tr>
<td>28</td>
<td>46.0</td>
</tr>
<tr>
<td>29</td>
<td>62.2</td>
</tr>
<tr>
<td>30</td>
<td>83.0</td>
</tr>
</tbody>
</table>

This table indicates the scalability of the program, which is not overwhelming for these values of $t$. The author also ran the program to compute $f_{m,n,h}$ with $m, n \leq t$ where $t = 50$, which took under 2 days (to be precise, it took 1 day 15 hours and 17 minutes). The largest number encountered was $f_{50,50,101}$, which has 167 digits, and is equal to the number of spanning trees of $K_{50,50}$, which is $50^{50-1} \cdot 50^{50-1} = 50^{98}$.

5.4 Complexity

In the worst case, $x = z = \{1, 2, 3, \ldots, t\}$, where $\min(m, n) = 1 + 2 + \cdots + t = \frac{1}{2}t(t + 1)$ (so $t = O(\sqrt{\min(m, n)})$, in which case the program must iterate through all $t!$ bijections from $x$ to $z$. So, for a worst case analysis, we assume $m = n$ and $m + n - h = \lfloor \sqrt{m} \rfloor$. 

Algorithm 1 Implementation of Theorem 3 using half-canonical symmetry breaking

\[ f_{m,n,m+n+1} := 0; \text{ for all } m, n, h \]
\[ w'_{m,n,m+n+1} := 0; \text{ for all } m, n, h \]
\[ w'_{0,0,0} := 1; \]
for \( m = 0, 1, \ldots, \text{MAX}(m) \) do
  for \( n = 0, 1, \ldots, \text{MAX}(n) \) do
    if \( m > n \) and \( m > 0 \) and \( n > 0 \) then // Other values of \( w'_{m,n,h} \) are 0
      for \( i = 0, 1, \ldots, n \) do
        // Using the Inclusion-Exclusion formula
        \[ w'_{m,n,m+n-1} := w'_{m,n,m+n-1} + (-1)^i \binom{n}{i} (n-i)^{m-1}; \]
      end for
    end if
  end for
  for \( h = 1, 2, \ldots, m+n-2 \) do
    for \( i = 0, 1, \ldots, m \) do
      for \( j = 0, 1, \ldots, n \) do
        for \( k = 0, 1, \ldots, \min(n-j,h) \) do
          \[ f_{m,n,h} := f_{m,n,h} + \binom{m}{i} \binom{n}{j} (j,k,n-j-k) m^k w'_{i,j,h-k}; \]
        end for
      end for
    end for
  end for
  for \( h = 0, 1, \ldots, m+n-1 \) do
    Print \( f_{m,n,h} \);
  end for
end for
The number of $h$-edge spanning forests in $K_{m,n}$

Hence, given $x, z \in S_{m,m+n-h}$, there can be $O(\sqrt{m}!)$ canonical bijections from $x$ to $z$. Hardy and Ramanujan’s asymptotic formula for the number of partitions of $m$, namely

$$\frac{1}{4m\sqrt{3}} e^{\pi\sqrt{2m/3}},$$

gives a crude asymptotic upper bound on $|S_{m,\lfloor\sqrt{m}\rfloor}|^2$, specifically

$$|S_{m,\lfloor\sqrt{m}\rfloor}|^2 = O\left( \frac{e^{\text{const} \cdot m}}{m^2} \right)$$

as the number of ways of choosing $x$ and $z$. Hence, $|S_{m,\lfloor\sqrt{m}\rfloor}|^2$ is $O(\sqrt{m}m^{1.5} \sqrt{m} + \text{const})$ terms, by Stirling’s Approximation. In contrast, a backtracking algorithm would need to generate and check around $\left( \frac{m^2}{\lfloor\sqrt{m}\rfloor} \right) \leq \frac{1}{\lfloor\sqrt{m}\rfloor} m^2 \sqrt{m}$ graphs, which, when $m = n$ and $h = 2m - \sqrt{m}$, is $O(e^{\sqrt{m}m^{1.5\sqrt{m} + \text{const}}})$ iterations, by Stirling’s Approximation. Of course, when implementing these algorithms, we use pruning whenever possible to reduce the search space, which makes a drastic difference not accounted for in these approximations.

5.4.1 When $h$ is fixed

We conclude this paper with the observation that, when $h$ is fixed, computing $f_{m,n,h}$ is asymptotically “easy”. The underlying reason is that, for sufficiently large $m$ or $n$, we must have isolated vertices in graphs in $B_{m,n,h}$. Thus, $|B_{m,n,h}|$ contains only a finite number of non-zero terms.

Theorem 4 For fixed $h$, computing $f_{m,n,h}$ can be performed in time $O(\log(mn))$.

Proof: If $i > h$ or $j > h$, then $w_{i,j,h} = 0$ (since any graph in $B_{i,j,h}$ must have an isolated vertex). Hence, (2) is equivalent to

$$f_{m,n,h} = \sum_{i=0}^{h} \sum_{j=0}^{h} \binom{m}{i} \binom{n}{j} w_{i,j,h}.$$ 

For fixed $h$, there is a finite number of terms in this sum. Thus, for fixed $h$, we could write a program in which:

- we store a list of the pairs $(i, j)$ for which $w_{i,j,h}$ is non-zero, along with the value of $w_{i,j,h}$,
- we iterate through this list, computing $\binom{m}{i} \binom{n}{j} w_{i,j,h}$, and add it to a running total.

We can compute $\binom{m}{i} = \frac{1}{i!} m(m-1) \cdots (m-i+1)$ using $O(h)$ multiplications (since $i \leq h$), each of which takes time $O(\log m)$, and one division. Hence, $\binom{m}{i}$ can be computed in time $O(\log m)$ time (since $h$ is fixed). Similarly $\binom{n}{j}$ can be computed in $O(\log n)$ time. We conclude that the whole summation can be performed in time $O(\log(mn))$. $\square$
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References


