Computing the number of h-edge spanning forests in complete bipartite graphs

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Let $f_{m,n,h}$ be the number of spanning forests with $h$ edges in the complete bipartite graph $K_{m,n}$. Kirchhoff’s Matrix Tree Theorem implies $f_{m,n,m} + n - 1 = m n - 1$ when $m \geq 1$ and $n \geq 1$, since $f_{m,n,m} + n - 1$ is the number of spanning trees in $K_{m,n}$. In this paper, we give an algorithm for computing $f_{m,n,h}$ for general $m, n, h$. We implement this algorithm and use it to compute all non-zero $f_{m,n,h}$ when $m \leq 50$ and $n \leq 50$ in under 2 days.

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Keywords: complete bipartite graph, spanning forest

1 Introduction

A consequence of Kirchhoff’s Matrix Tree Theorem is that the number of spanning trees in the complete bipartite graph $K_{m,n}$ is $m^{n-1} n^{m-1}$ when $m \geq 1$ and $n \geq 1$. In this paper, we consider a generalisation of this counting problem: the number $f_{m,n,h}$ of $h$-edge spanning forests in $K_{m,n}$. The number of spanning trees in $K_{m,n}$ is $f_{m,n,m} + n - 1$ when $m \geq 1$ and $n \geq 1$, since spanning trees of $K_{m,n}$ have $m + n - 1$ edges.

The numbers $f_{m,n,h}$ arise in relation to the Tutte polynomial of $K_{m,n}$. For a general graph $G = (V, E)$, the Tutte polynomial is

$$T_G(x, y) = \sum_{A \subseteq E} (x-1)^{k(A)}(y-1)^{k(A)+|A|-|V|}$$

where $k(A)$ is the number of connected components in the graph $(V, A)$. Importantly, for the Tutte polynomial, we take $0^0 = 1$. Thus, the number of spanning forests of $G$ is given by

$$T_G(2, 1) = \sum_{A \subseteq E} 0^{k(A)+|A|-|V|}$$

$$= \# \{ A \subseteq E : k(A) + |A| = |V| \}$$

$$= \text{number of spanning forests of } G,$$

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since \( k(A) + |A| = |V| \) if and only if \( A \) is a spanning forest.

In the present paper, we are interested in the case of when \( G = K_{m,n} \), and will look for \( h \)-edge spanning forests. In this situation,

\[
T_{K_{m,n}}(2, 1) = \sum_{h \geq 0} f_{m,n,h}.
\]

For general graphs \( G = (V, E) \), Myrvold \cite{10} gave an algorithm for computing the number of \( k \)-component spanning forests of \( G \). Björklund et al. \cite{4} described an algorithm that could compute \( T_G(2, 1) \) in time \( 2^{|V|}|V|^{O(1)} \) by using Kirchhoff’s Matrix Tree Theorem for each of the \( 2^{|V|} \) subsets of \( V \), then combining the results together with Inclusion-Exclusion. Other relevant work for the enumeration of spanning forests includes \cite{9}. Porter \cite{11} gave an algorithm for generating the spanning trees of \( K_{m,n} \). Farr and McDiarmid \cite{5} showed that computing the number of circuits in \( G \) is \#P-complete \( \cite{i} \).

Bounds for the number of spanning forests in graphs were given by Teranishi \cite{13}, from which we can deduce

\[
T_{K_{m,n}}(2, 1) \geq \sum_{k \geq 0} \left( \frac{\min(m, n)}{2} \right)^{m+n-k} \binom{m+n}{k}
\]

for all \( m \geq 0 \) and \( n \geq 0 \). Jin and Liu \cite{8} gave a simple formula for the number of rooted spanning forests in \( K_{m,n} \) (see also \cite{7} and \cite{12}).

This paper instead heads in a different direction to previous work: we derive algorithms for enumerating \( h \)-edge spanning forests in \( K_{m,n} \).

2 Basic results

To be clear, all forests in this paper will be labelled. Let \( \mathbb{N} = \{0, 1, 2, \ldots \} \). The following lemma gives the boundary conditions on \( f_{m,n,h} \).

**Lemma 1** Suppose \( m, n, h \in \mathbb{N} \). Then

\begin{itemize}
  \item \( f_{m,n,0} = 1 \),
  \item if \( h \geq 1 \), then \( f_{0,n,h} = f_{m,0,h} = 0 \),
  \item if \( m \geq 1 \) and \( n \geq 1 \), then \( f_{m,n,h} > 0 \) if and only if \( 0 \leq h \leq m + n - 1 \),
  \item if \( m \geq 1 \) and \( n \geq 1 \), then \( f_{m,n,m+n-1} = m^{n-1}n^{m-1} \).
\end{itemize}

**Proof:** If \( h = 0 \), then there is exactly one subgraph of \( K_{m,n} \) with no edges (and it is a spanning forest), so \( f_{m,n,0} = 1 \). If \( m = 0 \) or \( n = 0 \), then \( K_{m,n} \) has no edges, and thus we obtain the second bulleted item.

Now assume \( m \geq 1 \) and \( n \geq 1 \). Since \( K_{m,n} \) is connected, it has a spanning tree, which must have exactly \( m + n - 1 \) edges. By deleting edges from this spanning tree, we find \( h \)-edge forests in \( K_{m,n} \) for all \( 0 \leq h \leq m + n - 1 \). Hence \( f_{m,n,h} > 0 \) when \( 0 \leq h \leq m + n - 1 \).

Now assume \( A \) is an \( h \)-subset of the edges in \( K_{m,n} \), where \( m \geq 1 \) and \( n \geq 1 \) and \( h \geq m + n \). To be a spanning forest, we need \( k(A) + h = m + n \) as per \cite{1}, but if \( h \geq m + n \), then \( k(A) \leq 0 \), giving a contradiction.

\( \#P \) is the set of counting problems which ask for the number of “yes” instances for decision problems in NP. A \#P problem is in \#P-complete whenever any other problem in \#P can be reduced to it by a polynomial-time counting reduction.
The number of h-edge spanning forests in $K_{m,n}$

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**Fig. 1:** Small non-zero values of $f_{m,n,h}$.

The fourth bulleted item is from Kirchhoff’s Matrix Tree Theorem (already mentioned). □

The following lemma is an important (but basic) consequence of the symmetry of the problem.

**Lemma 2** For all $m, n, h \in \mathbb{N}$, we have $f_{m,n,h} = f_{n,m,h}$.

In Figure 1 we list the non-zero values of $f_{m,n,h}$ when $0 \leq m \leq n \leq 6$. These values were generated using a straightforward backtracking algorithm.

### 3 Combinatorial equivalence

In this section, we will describe some combinatorial objects that are equivalent to $h$-edge spanning forests of $K_{m,n}$.

Let $M$ be an $m \times n$ $(0,1)$-matrix. We define a cycle in $M$ to be a set $\{e_1, e_2, \ldots, e_{2t}\}$ of entries of $M$, such that:

- each $e_i$ contains the symbol 1, and
- for $i \in \{1, 2, \ldots, t\}$, we have that (a) $e_{2i-1}$ belongs to the same row as $e_{2i}$ and (b) $e_{2i}$ belongs to the same column as $e_{2i+1}$ (where we take $e_{2t+1} = e_1$).

For example, the matrix

\[
\begin{bmatrix}
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]
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has a cycle (consisting of the circled 1’s), whereas the matrices

\[
\begin{pmatrix}
0 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

do not have cycles.

Let \( A_{m,n,h} \) be the set of \((0,1)\)-matrices with exactly \( h \) elements equal to 1. Let \( p_{m,n,h} \) be the probability that an element of \( A_{m,n,h} \) chosen uniformly at random contains a cycle. An anonymous user of \texttt{math.stackexchange.com} asked\(^{(ii)}\) for a formula for \( p_{m,n,h} \). In fact, this was the original motivation for the author to study this problem.

**Lemma 3** For all \( m, n, h \in \mathbb{N} \), we have

\[
f_{m,n,h} = \left(1 - p_{m,n,h}\right) \binom{mn}{h}.
\]

**Proof:** A matrix \( M \in A_{m,n,h} \) can be interpreted as the biadjacency matrix\(^{(iii)}\) of an \( h \)-edge subgraph \( G = G(M) \) of \( K_{m,n} \). A cycle in \( M \in A_{m,n,h} \) corresponds to a cycle in \( G \). Hence

\[
f_{m,n,h} = (1 - p_{m,n,h}) |A_{m,n,h}|
\]

\[
= (1 - p_{m,n,h}) \binom{mn}{h}.
\]

\(\square\)

A problem related to the \((0,1)\)-matrix problem above comes from the study of \((0,1)\)-matrices that do not have a submatrix which is the incidence matrix of any cycle of length at least 3; these are called “totally balanced matrices” (see e.g. [2]).

Another interpretation of matrices in \( A_{m,n,h} \) is as induced subgraphs of \( K_{m} \Box K_{n} \), where \( \Box \) represents the Cartesian product of graphs. The graph \( K_{m} \Box K_{n} \) is sometimes called the “rook’s graph”, since the edges represent the legal moves of a rook on an \( m \times n \) chess board. The vertices in \( K_{m} \Box K_{n} \) are \( \{(i,j) : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} \) and there is an edge between distinct vertices \((i,j)\) and \((i',j')\) whenever \( i = i' \text{ or } j = j' \). The graph \( K_{m} \Box K_{n} \) is also the line graph of \( K_{m,n} \).

There is a bijection between induced subgraphs \( H = H(M) \) of \( K_{m} \Box K_{n} \) and \((0,1)\)-matrices \( M = (m_{ij}) \in A_{m,n,h} \); we include the vertex \((i,j)\) in \( H \) if and only if \( m_{ij} = 1 \). For example, if we ignore the vertices marked 0 in

\[
\begin{array}{cccc}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{array}
\]

\(\text{Full URL: } \texttt{http://math.stackexchange.com/q/24800}\)

\(\text{(iii)}\) The biadjacency matrix of a bipartite graph \( G \) on the vertex set \( \{u_i\}_{1 \leq i \leq m} \cup \{v_j\}_{1 \leq j \leq n} \) is the \( m \times n \) \((0,1)\)-matrix \( M = (m_{ij}) \) with \( m_{ij} = 1 \) if and only if \( u_i v_j \) is an edge in \( G \).
we obtain an induced subgraph $H(M)$ of $K_3 \boxtimes K_5$, corresponding to the matrix

\[
M = \begin{pmatrix}
0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

in $A_{3,5,6}$. We see that cycles in $M$ map from cycles $C_k$ in $H$ for some $k \in \{4, 6, 8, \ldots \}$ (an example of a 6-cycle is highlighted). There are other cycles in $H$ (e.g. $C_3$ in the above example; in fact, if $M$ above had a row of $1$'s, then $H$ would have a $K_5$ subgraph). However, induced cycles $C_k$ in $H(M)$ of length $k \in \{4, 6, 8, \ldots \}$ are in one-to-one correspondence with cycles in $M$ via the above bijection.

**Lemma 4** For all $m, n, h \in \mathbb{N}$, we have that $f_{m,n,h}$ is the number of $h$-vertex induced subgraphs of $K_m \boxtimes K_n$ that do not contain an induced $C_k$ for any $k \in \{4, 6, 8, \ldots \}$.

## 4 Simplifying the equation

### 4.1 A formula for $f_{m,n,h}$

We will look at two related ways of simplifying $f_{m,n,h}$. Let $K_{m,n}$ have the vertex bipartition $M \cup N$, where $M = \{u_1, u_2, \ldots, u_m\}$ and $N = \{v_1, v_2, \ldots, v_n\}$. Let $B_{m,n,h}$ be the set of $h$-edge spanning forests of $K_{m,n}$. Hence $f_{m,n,h} = |B_{m,n,h}|$. Define

\[
W_{m,n,h} = \{ G \in B_{m,n,h} : G \text{ has no isolated vertices} \}
\]

and $w_{m,n,h} = |W_{m,n,h}|$. We observe the following:

- Given any spanning forest in $B_{m,n,h}$ we can delete the isolated vertices to obtain a spanning forest in $W_{i,j,h}$ for some $i \leq m$ and $j \leq n$.
- Conversely, given any spanning forest in $W_{i,j,h}$, we can add isolated vertices to it in $\binom{m}{m-i} \binom{n}{n-j}$ ways to obtain a spanning forest in $B_{m,n,h}$.

In both of the operations above, we need to relabel the vertices, but we preserve the order of the indices of the non-isolated vertices within $M$ and $N$.

Hence

\[
f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} w_{i,j,h}.
\]

This formula eliminates the need to further account for isolated vertices.

We can decompose any $G \in W_{m,n,h}$ into disjoint components, each of which belong to $W_{i,j,k}$ for some $i \leq m$ and $j \leq n$ and $k \leq h$. Hence any $G \in W_{m,n,h}$ decomposes into the following.

- A partition $P$ of $M$ where $x, y \in M$ belong to the same part if there is a path from $x$ to $y$ in $G$.
- A partition $Q$ of $N$ where $x', y' \in N$ belong to the same part if there is a path from $x'$ to $y'$ in $G$.
- A bijection $\alpha : P \to Q$ such that every edge $ab$ in $G$ has $a \in p$ and $b \in \alpha(p)$ for some $p \in P$. Hence we have $|P| = |Q|$.
For each $p \in P$, a subgraph in $W_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$ induced by the vertices in $p \cup \alpha(p)$.

Regarding the final bulleted item, we observe that the subgraph induced by $p \cup \alpha(p)$ for some $p \in P$ must be a spanning tree on $K_{|p|,|\alpha(p)|}$. Any more edges would cause a cycle, while fewer edges would cause disjoint components (in which case $G$ could be further decomposed).

We give an example of a graph $G$ in $W_{3,5,6}$ below.

Here, the partitions $P$ and $Q$ are given by $P = \{\{u_1, u_2\}, \{u_3\}\}$ and $Q = \{\{v_1, v_2\}, \{v_3, v_4, v_5\}\}$ and we have the bijection $\alpha$ such that

\[
\{u_1, u_2\} \mapsto \{v_1, v_2\}, \\
\{u_3\} \mapsto \{v_3, v_4, v_5\}.
\]

We see that $G$ decomposes into a graph in $W_{2,2,3}$ and a graph in $W_{1,3,3}$.

In general, for any $p \in P$, we must have exactly $|p| + |\alpha(p)| - 1$ edges in $p \cup \alpha(p)$. Summing this over all $p \in P$ gives $h = m + n - |P|$. We conclude that

\[|P| = m + n - h\]

(and thus $|Q| = m + n - h$).

We will now describe how to construct any graph in $W_{m,n,h}$ via its decomposition. Let $P$ be the set of partitions of $M$ and let $Q$ be the set of partitions of $N$. Given (a) a partition $P \in P$ of size $|P| = m + n - h$, (b) a partition $Q \in Q$ and (c) a bijection $\alpha : P \to Q$, we can construct $\prod_{p \in P} w_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$ distinct graphs $G \in W_{m,n,h}$ by the following steps:

1. Start with $G$ as the graph with vertex set $M \cup N$ and no edges.
2. For each $p \in P$ add one of the subgraphs in $W_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$ on the vertices $p \cup \alpha(p)$.

Since all graphs in $W_{m,n,h}$ can be constructed uniquely by the above steps we have the following theorem.

**Theorem 1** For $m, n, h \in \mathbb{N}$,

\[f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} w_{i,j,h}\]
The number of \( h \)-edge spanning forests in \( K_{m,n} \)

where

\[
    w_{m,n,h} = \sum_{P \in \mathcal{P}} \sum_{Q \in \mathcal{Q}} \sum_{\alpha : P \to Q} \prod_{p \in P} w_{\lvert p\rvert, \lvert \alpha(p)\rvert, \lvert p\rvert + \lvert \alpha(p)\rvert - 1}
\]

where

\[
    w_{\lvert p\rvert, \lvert \alpha(p)\rvert, \lvert p\rvert + \lvert \alpha(p)\rvert - 1} = \lvert p\rvert^{\lvert \alpha(p)\rvert - 1} \lvert \alpha(p)\rvert^{\lvert p\rvert - 1}.
\]

4.2 An improved formula for \( f_{m,n,h} \)

We will derive two further equations for \( f_{m,n,h} \) (in Theorems 2 and 3), which are related to Theorem 1. They result in a slightly more complicated algorithm, but will allow us to compute \( f_{m,n,h} \) faster than (2).

For all \( m, n, h \in \mathbb{N} \), define

\[
    W_{m,n,h}' = \{ G \in W_{m,n,h} : \text{the vertices in} \ N \ \text{have degree} \geq 2 \}.
\]

Given any \( G \in B_{m,n,h} \) we can delete the isolated vertices and the leaves (vertices of degree 1) in \( N \) to obtain a spanning forest in \( W_{i,j,h-k}' \) for some \( i \leq m \) and \( j \leq n \) and \( k \in \mathbb{N} \). Conversely, given any \( G \in W_{i,j,h-k}' \), we can add:

- \( m - i \) isolated vertices to \( M \) in \( m \choose m-i \) ways, so as to increase \( |M| \) to \( m \), then
- \( n - j \) isolated vertices to \( N \) in \( n \choose n-j \) ways, so as to increase \( |N| \) to \( n \), then
- \( k \) edges to \( G \) in \( (n-j) \choose k \) ways, so as to increase the number of leaves in \( N \) by \( k \), thereby obtaining a spanning forest in \( B_{m,n,h} \). Similar to the derivation of (2), we need to relabel the vertices so as to preserve the order of the indices of the non-isolated vertices within \( M \) and the non-isolated non-leaf vertices within \( N \). Hence

\[
    f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{\min(n-j,h)} \binom{m}{i} \binom{n}{j,k,n-j-k} m^k w_{i,j,h-k}'.
\]

All of the steps involved for finding the formula (3) for \( w_{m,n,h} \) are still valid for \( w_{m,n,h}' \). Hence \( f_{m,n,h} \) remains true if we replace \( w \) with \( w' \). However, we can no longer make use of the simple formula for \( w_{\lvert p\rvert, \lvert \alpha(p)\rvert, \lvert p\rvert + \lvert \alpha(p)\rvert - 1} \). Nevertheless, we will be able to find a formula for \( w_{\lvert p\rvert, \lvert \alpha(p)\rvert, \lvert p\rvert + \lvert \alpha(p)\rvert - 1} \) via Inclusion-Exclusion.

For \( I \subseteq N \), define

\[
    A_I = \{ G \in W_{m,n,m+n-1} : \text{vertices in} \ I \ \text{are leaves} \}.
\]

Note that spanning forests in \( A_I \) might also have leaves outside of \( I \). Hence \( |A_0| = |W_{m,n,m+n-1}| = m^{n-1}n^{m-1} \). By symmetry, if \( I, J \subseteq N \) and \( |I| = |J| \) then \( |A_I| = |A_J| \). So we will assume \( I = \{ n - i + 1, n - i + 2, \ldots, n \} \). We can construct the graphs in \( A_I \) by adding \( i \) leaves to the graphs in \( W_{m,n-|I|,m+n-|I|-1} \) (i.e., the set of spanning trees of \( K_{m,n-|I|} \)), and these leaves can be added in \( m^i \) ways. Hence

\[
    |A_I| = m^i \cdot \text{number of spanning trees of} \ K_{m,n-|I|}
    = m^i m^{n-1}(n-i)^{m-1}
    = m^{n-1}(n-i)^{m-1}.
\]
when \( m \geq 1 \) and \( n \geq 1 \). Here we take \( 0^0 = 1 \), since we want to account for \( K_{1,0} \in W_{1,0,0} \). Hence, by Inclusion-Exclusion, we find

\[
w'_{m,n,m+n-1} = |A_0| - \left| \bigcup_{I \subseteq N} A_I \right|
\]

\[
= m^{n-1}n^{m-1} - \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} m^{n-1}(n-i)^{m-1}
\]

when \( m \geq 1 \) and \( n \geq 1 \). By definition, we also have \( w'_{m,n,m+n-1} = 0 \) when either \( m = 0 \) or \( n = 0 \) (and \( w'_{m,n,m+n-1} \) is not defined when \( m = 0 \) and \( n = 0 \)).

Before identifying formulae for \( w_{m,n,h} \) and \( w'_{m,n,h} \) in the next section, we make the following observation.

**Lemma 5** For all \( m, n, h \in \mathbb{N} \), we have \( n! \) divides \( w'_{m,n,h} \).

**Proof:** The symmetric group on \( N \) acts on \( W'_{m,n,h} \) by permuting the vertices in \( N \). Suppose a graph \( G \in W'_{m,n,h} \) has a non-trivial stabiliser subgroup under this action. Then there exists a non-identity permutation \( \alpha \) such that \( \alpha G = G \), and distinct vertices \( v, v' \in N \) for which \( \alpha(v) = v' \). Since \( G \in W'_{m,n,h} \), the degree of \( v \) is 2 or more, so assume \( v \) has distinct neighbours \( a, b \in M \). Then, since \( \alpha G = G \), we find that \( v' = \alpha(v) \) has the neighbours \( a \) and \( b \) too. Thus, \( \{v, v', a, b\} \) induces a 4-cycle, giving a contradiction. Hence, all stabiliser subgroups are trivial, and by the Orbit-Stabiliser Theorem, all orbits have size \( n! \). Hence \( n! \) divides \( w'_{m,n,h} \).

\[
4.3 \text{ Formulae for } w_{m,n,h} \text{ and } w'_{m,n,h}
\]

Now we will simplify (3), noting that our simplifications remain valid when we replace \( w \) with \( w' \).

For \( a \geq 1 \) and \( t \geq 1 \), let \( S_{a,t} \) be the set of (number) partitions of \( a \) into \( t \) non-zero parts. We will interpret elements of \( S_{a,t} \) as multisets; for example \( \{4,3,1,1\} \in S_{12,5} \). We will require \( S_{a,t} = \emptyset \) when \( t < 0 \), and \( S_{0,0} = \{\emptyset\} \) and \( S_{0,t} = \emptyset \) when \( t > 0 \). For \( z \in S_{a,t} \), let \( T[z] \) be the set of partitions of \( \{1,2,\ldots,a\} \) whose part sizes induce the number partition \( z \). For \( z \in S_{a,t} \), let \( \hat{z} \) denote an arbitrary element of \( T[z] \). If \( P \in T[x] \) and \( Q \in T[z] \), then any bijection \( \alpha : P \rightarrow Q \) induces a bijection \( \beta : x \rightarrow \hat{z} \).
Hence
\[
\sum_{x,z \in S_{m,m+n-h}} \prod_{\alpha: P \rightarrow Q} w_{|\alpha(P)|,|\alpha(Q)| - 1} = \sum_{x,z \in S_{m,m+n-h}} \prod_{\alpha: \tilde{T}_\alpha \rightarrow \tilde{T}_{\tilde{\alpha}}} w_{|\alpha(P)|,|\alpha(Q)| - 1}
\]
\[
= \sum_{x,z \in S_{m,m+n-h}} |T[x]| \cdot |T[z]| \sum_{\beta: x \rightarrow z} \prod_{r \in x} w_{r,\beta(r),r+\beta(r)-1}.
\]
If \( z \in S_{n,t} \), then
\[
|T[z]| = \frac{a!}{\prod_{i \geq 1} i!^{s_i(z)} s_i(z)!}
\]
where \( s_i(z) \) denotes the number of parts \( i \) in \( z \) (see e.g. [1] Theorem 13.2)). Hence we have the following theorems.

**Theorem 2** For \( m, n, h \in \mathbb{N} \),
\[
f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} w_{i,j,h}
\]
where
\[
w_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \prod_{\beta: x \rightarrow z} m! \prod_{i \geq 1} i!^{s_i(x)} s_i(x)! \prod_{i \geq 1} i!^{s_i(z)} s_i(z)! \prod_{r \in x} r^{\beta(r)-1} \beta(r)^{r-1}. \tag{4}
\]

**Theorem 3** For \( m, n, h \in \mathbb{N} \),
\[
f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \min(n-j,k) \binom{m}{i} \binom{n}{j,k,n-j-k} m^k w'_{i,j,h-k}.
\]
where
\[
w'_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \prod_{\beta: x \rightarrow z} m! \prod_{i \geq 1} i!^{s_i(x)} s_i(x)! \prod_{i \geq 1} i!^{s_i(z)} s_i(z)! \prod_{r \in x} u_{r,\beta(r),r+\beta(r)-1}
\]
where
\[
w'_{m,n,m+n-1} = m^{n-1} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^{m-1}.
\]
The main advantage of Theorems 2 and 3 over Theorem 1 is the summation is over number partitions (rather than set partitions).
5 Implementation

The author has implemented the two formulae for $f_{m,n,h}$ described in the preceding section, along with a simple backtracking algorithm. The results between all three implementations concur, giving confidence in the accuracy of the code. The C source code is available as supplementary material to this document. The GMP library was used for arbitrary precision arithmetic [6].

The formula involving $w_{m,n,h}'$ instead of $w_{m,n,h}$ was (unsurprisingly) much faster, largely because many values of $w_{m,n,h}'$ equal 0, such as when $m \leq n$, which can be used to drastically reduce the search tree.

5.1 Symmetry breaking

The run-time of the program was also improved through the use of symmetry breaking, which we will now describe. For $x \in S_{m,m+n-h}$ and $z \in S_{m,n+m-h}$, there are often many bijections $\beta : x \rightarrow z$ which map the same elements to the same elements (since both $x$ and $z$ are multisets). In these instances, a naive algorithm would repeat the same computation unnecessarily. Let $x$ and $z$ be the multisets $x = \{x_1, x_2, \ldots, x_{m+n-h}\}$ and $z = \{z_1, z_2, \ldots, z_{m+n-h}\}$. We define the condition:

**Symmetry breaking condition 1:** We say $\beta : x \rightarrow z$ is half-canonical if $\beta^{-1}(z_{i-1}) < \beta^{-1}(z_i)$ whenever $z_{i-1} = z_i$.

We need to add a multiplicative factor to adjust for the restriction to half-canonical bijections. Hence

$$w_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \sum_{\beta : x \rightarrow z, \beta \text{ is a bijection}} \left( \prod_{i \geq 1} \frac{m!}{i^{\delta_i(z)}} \prod_{i \geq 1} \frac{n!}{i^{\delta_i(z)} s_i(z)} \prod_{r \in x} w_{r,\beta(r),r+\beta(r)-1} \right)$$

and similarly with $w'$ in place of $w$. Using this assumption, we reduce the number of bijections by a factor of $\prod_{i \geq 1} i^{\delta_i(z)}$, which results in a substantial time saving.

Instead of the single symmetry breaking assumption, it is possible to utilise symmetry breaking using an additional condition:

**Symmetry breaking condition 2:** We say $\beta : x \rightarrow z$ is canonical if it is half-canonical and $\beta(x_{i-1}) < \beta(x_i)$ whenever $x_i = x_{i-1}$.

Again, we find

$$w_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \sum_{\beta : x \rightarrow z, \beta \text{ is a bijection}} \Gamma_{x,z,\beta} \prod_{i \geq 1} \frac{m!}{i^{\delta_i(x)s_i(x)}} \prod_{i \geq 1} \frac{n!}{i^{\delta_i(z)s_i(z)}} \prod_{r \in x} w_{r,\beta(r),r+\beta(r)-1}$$

and similarly with $w'$ in place of $w$, where $\Gamma_{x,z,\beta}$ is the number of bijections between the multisets $x$ and $z$ which map the same elements to the same elements (as $\beta$). A formula for $\Gamma_{x,z,\beta}$ was given by e.g. [3], namely

$$\Gamma_{x,z,\beta} = \prod_{i \geq 1} \frac{s_i(x)!}{i^{\delta_i(x)} s_i(x)!} \prod_{j \geq 1} \frac{s_j(z)!}{i^{\delta_i(z)} s_j(z)!} \prod_{r \in x} w_{r,\beta(r),r+\beta(r)-1}$$

where
The number of \( h \)-edge spanning forests in \( K_{m,n} \)

- \( s_i(z) \) denotes the number of parts \( i \) in \( z \) (as before), and
- \( s_{i,j}(x,\beta) \) is the number of elements \((i,j)\) in the multiset \( \{(r,\beta(r)) : r \in x\} \).

The author has implemented both of these symmetry breaking schemes in order to compare their performance (see Section 5.3).

### 5.2 Pseudo-code

Algorithm 1 gives a pseudo-code version of the C code used to implement the algorithm described by Theorem 3 using the half-canonical symmetry breaking condition. The partitions of \( m \) and \( n \) into \( k \) parts were computed whenever needed and stored in memory. Iterating through the half-canonical bijections was performed “on the fly” using a backtracking algorithm.

While Theorems 2 and 3 are valid for all \( m,n,h \in \mathbb{N} \), we need to set \( f_{m,n,0} = 1 \) separately in the C code.

### 5.3 Performance

The C code was run on a 2 \( \times \) 2.66 GHz processor (although the code itself is not parallelised). The following table gives the run-times (in seconds) for the two algorithms under the two symmetry breaking schemes when computing all non-zero \( f_{m,n,h} \) with \( m,n \leq 19 \).

<table>
<thead>
<tr>
<th></th>
<th>half-canonical</th>
<th>canonical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 2</td>
<td>27.1</td>
<td>6.8</td>
</tr>
<tr>
<td>Theorem 3</td>
<td>4.1</td>
<td>4.5</td>
</tr>
</tbody>
</table>

The fastest version is Theorem 3 using half-canonical symmetry breaking. Under these conditions, the code had the following run-times to find all non-zero values of \( f_{m,n,h} \) with \( m,n \leq t \):

<table>
<thead>
<tr>
<th>( t )</th>
<th>time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>20.5</td>
</tr>
<tr>
<td>26</td>
<td>27.7</td>
</tr>
<tr>
<td>27</td>
<td>35.4</td>
</tr>
<tr>
<td>28</td>
<td>46.0</td>
</tr>
<tr>
<td>29</td>
<td>62.2</td>
</tr>
<tr>
<td>30</td>
<td>83.0</td>
</tr>
</tbody>
</table>

This table indicates the scalability of the program, which is not overwhelming for these values of \( t \). The author also ran the program to compute \( f_{m,n,h} \) with \( m,n \leq t \) where \( t = 50 \), which took under 2 days (to be precise, it took 1 day 15 hours and 17 minutes). The largest number encountered was \( f_{50,50,101} \), which has 167 digits, and is equal to the number of spanning trees of \( K_{50,50} \), which is \( 50^{50-1} \cdot 50^{50-1} = 50^{98} \).

### 5.4 Complexity

In the worst case, \( x = z = \{1,2,3,\ldots,t\}, \) where \( \min(m,n) = 1 + 2 + \cdots + t = \frac{1}{2}t(t + 1) \) (so \( t = O(\sqrt{\min(m,n)}) \)), in which case the program must iterate through all \( t! \) bijections from \( x \) to \( z \). So, for a worst case analysis, we assume \( m = n \) and \( m + n - h = \lfloor \sqrt{m} \rfloor \).
Algorithm 1: Implementation of Theorem 3 using half-canonical symmetry breaking

\[ f_{m,n,m+n-1} := 0; \text{ for all } m, n, h \]
\[ w'_{m,n,m+n-1} := 0; \text{ for all } m, n, h \]
\[ w'_{0,0,0} := 1; \]
for \( m = 0, 1, \ldots, \text{MAX}(m) \) do
  for \( n = 0, 1, \ldots, \text{MAX}(n) \) do
    if \( m > n \) and \( m > 0 \) and \( n > 0 \) then // Other values of \( w'_{m,n,h} \) are 0
      for \( i = 0, 1, \ldots, n \) do
        // Using the Inclusion-Exclusion formula
        \[ w'_{m,n,m+n-1} := w'_{m,n,m+n-1} + (-1)^i \binom{n}{i} (n-i)^{m-1}; \]
      end for
    end if
  end for
  for \( h = 0, 1, \ldots, m + n - 2 \) do
    \[ w'_{m,n,h} := 0; \]
    Compute all partitions of \( m \) into \( m + n - h \) parts and store in memory.
    Compute all partitions of \( n \) into \( m + n - h \) parts and store in memory.
    for partitions \( x \) of \( m \) into \( m + n - h \) parts do
      for half-canonical bijections \( \beta : x \to z \) do // via backtracking algorithm
        \[ \Gamma := \prod_{i \geq 1} i^{s_i(z)}; \]
        \[ w'_{m,n,h} := w'_{m,n,h} + \Gamma \cdot \frac{m!}{\prod_{i \geq 1} i^{s_i(x)} s_i(x)!} \prod_{i \geq 1} i^{s_i(z)} s_i(z)! \prod_{r \in x} w'_{r,\beta(r),r+\beta(r)-1}; \]
      end for
    end for
  end for
  end for
end for
end for
end for
end for
The number of $h$-edge spanning forests in $K_{m,n}$

Hence, given $x, z \in S_{m,m+n-h}$, there can be $O(\sqrt{m}!)$ canonical bijections from $x$ to $z$. Hardy and Ramanujan’s asymptotic formula for the number of partitions of $m$, namely

$$\frac{1}{4m\sqrt{3}} e^{\pi \sqrt{2m/3}},$$

gives a crude asymptotic upper bound on $|S_{m,\lfloor \sqrt{m} \rfloor}|^2$, specifically

$$|S_{m,\lfloor \sqrt{m} \rfloor}|^2 = O\left(\frac{e^{\text{const} \cdot m}}{m^2}\right)$$

as the number of ways of choosing $x$ and $z$. Hence, $|S_{m,\lfloor \sqrt{m} \rfloor}|^2$ terms, by Stirling’s Approximation. In contrast, a backtracking algorithm would need to generate and check around $(m^2 \lfloor \sqrt{m} \rfloor) \leq \frac{1}{\sqrt{m}}! m^2 \sqrt{m}$ graphs, which, when $m = n$ and $h = 2m - \sqrt{m}$, is

$$O\left(e^{\sqrt{m} \cdot \sqrt{m}/2 + \text{const}}\right)$$

iterations, by Stirling’s Approximation. Of course, when implementing these algorithms, we use pruning whenever possible to reduce the search space, which makes a drastic difference not accounted for in these approximations.

5.4.1 When $h$ is fixed

We conclude this paper with the observation that, when $h$ is fixed, computing $f_{m,n,h}$ is asymptotically “easy”. The underlying reason is that, for sufficiently large $m$ or $n$, we must have isolated vertices in graphs in $B_{m,n,h}$. Thus, (4) only contains a finite number of non-zero terms. Thus, (4) contains a finite number of non-zero terms.

**Theorem 4** For fixed $h$, computing $f_{m,n,h}$ can be performed in time $O\left(\log(mn)\right)$.

**Proof:** If $i > h$ or $j > h$, then $w_{i,j,h} = 0$ (since any graph in $B_{i,j,h}$ must have an isolated vertex). Hence, (4) is equivalent to

$$f_{m,n,h} = \sum_{i=0}^{h} \sum_{j=0}^{h} \binom{m}{i} \binom{n}{j} w_{i,j,h}.$$

For fixed $h$, there is a finite number of terms in this sum. Thus, for fixed $h$, we could write a program in which:

- we store a list of the pairs $(i, j)$ for which $w_{i,j,h}$ is non-zero, along with the value of $w_{i,j,h}$,
- we iterate through this list, computing $\binom{m}{i} \binom{n}{j} w_{i,j,h}$, and add it to a running total.

We can compute $\binom{m}{i} = \frac{1}{i} m(m-1) \cdots (m-i+1)$ using $O(h)$ multiplications (since $i \leq h$), each of which takes time $O(\log m)$, and one division. Hence, $\binom{m}{i}$ can be computed in time $O(\log m)$ time (since $h$ is fixed). Similarly $\binom{n}{j}$ can be computed in $O(\log n)$ time. We conclude that the whole summation can be performed in time $O(\log \log mn)$. \qed
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References


