Computing the number of h-edge spanning forests in complete bipartite graphs

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Let $f_{m,n,h}$ be the number of spanning forests with $h$ edges in the complete bipartite graph $K_{m,n}$. Kirchhoff’s Matrix Tree Theorem implies $f_{m,n,m+n-1} = n^{m-1}m^{n-1}$ when $m \geq 1$ and $n \geq 1$, since $f_{m,n,m+n-1}$ is the number of spanning trees in $K_{m,n}$. In this paper, we give an algorithm for computing $f_{m,n,h}$ for general $m, n, h$. We implement this algorithm and use it to compute all non-zero $f_{m,n,h}$ when $m \leq 50$ and $n \leq 50$ in under 2 days.

MSC 2010: 68R10, 05C85, 05C30, 05C38, 05C05

Keywords: complete bipartite graph, spanning forest

1 Introduction

A consequence of Kirchhoff’s Matrix Tree Theorem is that the number of spanning trees in the complete bipartite graph $K_{m,n}$ is $n^{m-1}m^{n-1}$ when $m \geq 1$ and $n \geq 1$. In this paper, we consider a generalisation of this counting problem: the number $f_{m,n,h}$ of $h$-edge spanning forests in $K_{m,n}$. The number of spanning trees in $K_{m,n}$ is $f_{m,n,m+n-1}$ when $m \geq 1$ and $n \geq 1$, since spanning trees of $K_{m,n}$ have $m+n-1$ edges.

The numbers $f_{m,n,h}$ arise in relation to the Tutte polynomial of $K_{m,n}$. For a general graph $G = (V,E)$, the Tutte polynomial is

$$T_G(x,y) = \sum_{A \subseteq E} (x-1)^{k(A)-k(E)}(y-1)^{k(A)+|A|-|V|}$$

where $k(A)$ is the number of connected components in the graph $(V,A)$. Importantly, for the Tutte polynomial, we take $0^0 = 1$. Thus, the number of spanning forests of $G$ is given by

$$T_G(2,1) = \sum_{A \subseteq E} 0^{k(A)+|A|-|V|}$$

$$= \# \{ A \subseteq E : k(A) + |A| = |V| \} \quad (1)$$

number of spanning forests of $G$.

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since \( k(A) + |A| = |V| \) if and only if \( A \) is a spanning forest.

In the present paper, we are interested in the case of when \( G = K_{m,n} \), and will look for \( h \)-edge spanning forests. In this situation,
\[
T_{K_{m,n}}(2, 1) = \sum_{h \geq 0} f_{m,n,h}.
\]

For general graphs \( G = (V,E) \), Myrvold \cite{10} gave an algorithm for computing the number of \( k \)-component spanning forests of \( G \). Bjöklund et al. \cite{4} described an algorithm that could compute \( T_{G}(2,1) \) in time \( 2^{|V|} |V|^{O(1)} \) by using Kirchhoff’s Matrix Tree Theorem for each of the \( 2^{|V|} \) subsets of \( V \), then combining the results together with Inclusion-Exclusion. Other relevant work for the enumeration of spanning forests includes \cite{9}. Porter \cite{11} gave an algorithm for generating the spanning trees of \( K_{m,n} \).

Farr and McDiarmid \cite{5} showed that computing the number of circuits in \( G \) is \#P-complete \cite{11}.

Bounds for the number of spanning forests in graphs were given by Teranishi \cite{13}, from which we can deduce
\[
T_{K_{m,n}}(2, 1) \geq \sum_{k \geq 0} \left( \frac{\min(m,n)}{2} \right)^{m+n-k} \binom{m+n}{k}
\]
for all \( m \geq 0 \) and \( n \geq 0 \). Jin and Liu \cite{8} gave a simple formula for the number of rooted spanning forests in \( K_{m,n} \) (see also \cite{7} and \cite{12}).

This paper instead heads in a different direction to previous work: we derive algorithms for enumerating \( h \)-edge spanning forests in \( K_{m,n} \).

\section{Basic results}

To be clear, all forests in this paper will be labelled. Let \( \mathbb{N} = \{0, 1, 2, \ldots\} \). The following lemma gives the boundary conditions on \( f_{m,n,h} \).

\begin{lemma}
Suppose \( m, n, h \in \mathbb{N} \). Then
\begin{itemize}
  \item \( f_{m,n,0} = 1 \),
  \item if \( h \geq 1 \), then \( f_{0,n,h} = f_{m,0,h} = 0 \),
  \item if \( m \geq 1 \) and \( n \geq 1 \), then \( f_{m,n,h} > 0 \) if and only if \( 0 \leq h \leq m+n-1 \),
  \item if \( m \geq 1 \) and \( n \geq 1 \), then \( f_{m,n,m+n-1} = m^{n-1} n^{m-1} \).
\end{itemize}
\end{lemma}

\begin{proof}
If \( h = 0 \), then there is exactly one subgraph of \( K_{m,n} \) with no edges (and it is a spanning forest), so \( f_{m,n,0} = 1 \). If \( m = 0 \) or \( n = 0 \), then \( K_{m,n} \) has no edges, and thus we obtain the second bulleted item.

Now assume \( m \geq 1 \) and \( n \geq 1 \). Since \( K_{m,n} \) is connected, it has a spanning tree, which must have exactly \( m+n-1 \) edges. By deleting edges from this spanning tree, we find \( h \)-edge forests in \( K_{m,n} \) for all \( 0 \leq h \leq m+n-1 \). Hence \( f_{m,n,h} > 0 \) when \( 0 \leq h \leq m+n-1 \).

Now assume \( h \) is a \( h \)-subset of the edges in \( K_{m,n} \), where \( m \geq 1 \) and \( n \geq 1 \) and \( h \geq m+n \). To be a spanning forest, we need \( k(A) + h = m+n \) as per \cite{1}, but if \( h \geq m+n \), then \( k(A) \leq 0 \), giving a contradiction.

\footnote{\#P is the set of counting problems which ask for the number of “yes” instances for decision problems in NP. A \#P problem is in \#P-complete whenever any other problem in \#P can be reduced to it by a polynomial-time counting reduction.}
\end{proof}
The number of $h$-edge spanning forests in $K_{m,n}$

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Fig. 1: Small non-zero values of $f_{m,n,h}$.

The fourth bulleted item is from Kirchhoff’s Matrix Tree Theorem (already mentioned).

The following lemma is an important (but basic) consequence of the symmetry of the problem.

**Lemma 2**  For all $m, n, h \in \mathbb{N}$, we have $f_{m,n,h} = f_{n,m,h}$.

In Figure 1 we list the non-zero values of $f_{m,n,h}$ when $0 \leq m \leq n \leq 6$. These values were generated using a straightforward backtracking algorithm.

## 3 Combinatorial equivalence

In this section, we will describe some combinatorial objects that are equivalent to $h$-edge spanning forests of $K_{m,n}$.

Let $M$ be an $m \times n \ (0, 1)$-matrix. We define a cycle in $M$ to be a set $\{e_1, e_2, \ldots, e_t\}$ of entries of $M$, such that:

- each $e_i$ contains the symbol 1, and
- for $i \in \{1, 2, \ldots, t\}$, we have that (a) $e_{2i-1}$ belongs to the same row as $e_{2i}$ and (b) $e_{2i}$ belongs to the same column as $e_{2i+1}$ (where we take $e_{2i+1} = e_1$).

For example, the matrix

```
0 1 0 1 1
0 0 1 0 1
1 0 0 0 1
1 1 0 0 0
```
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has a cycle (consisting of the circled 1’s), whereas the matrices

\[
\begin{array}{ccc}
0 & 1 & 1 \\
0 & 0 & 0 \\
\end{array} \quad \text{and} \quad \begin{array}{cccc}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
\end{array}
\]

do not have cycles.

Let \( A_{m,n,h} \) be the set of \((0,1)\)-matrices with exactly \( h \) elements equal to 1. Let \( p_{m,n,h} \) be the probability that an element of \( A_{m,n,h} \) chosen uniformly at random contains a cycle. An anonymous user of [math.stackexchange.com](http://math.stackexchange.com) asked for a formula for \( p_{m,n,h} \). In fact, this was the original motivation for the author to study this problem.

**Lemma 3** For all \( m, n, h \in \mathbb{N} \), we have \( f_{m,n,h} \) is the number of \( m \times n \) \((0,1)\)-matrices with \( h \) 1’s that do not contain a cycle, and hence

\[
f_{m,n,h} = (1 - p_{m,n,h}) \binom{mn}{h}.
\]

**Proof:** A matrix \( M \in A_{m,n,h} \) can be interpreted as the biadjacency matrix of an \( h \)-edge subgraph \( G = G(M) \) of \( K_{m,n} \). A cycle in \( M \in A_{m,n,h} \) corresponds to a cycle in \( G \). Hence

\[
f_{m,n,h} = (1 - p_{m,n,h}) |A_{m,n,h}| = (1 - p_{m,n,h}) \binom{mn}{h}.
\]

\( \Box \)

A problem related to the \((0,1)\)-matrix problem above comes from the study of \((0,1)\)-matrices that do not have a submatrix which is the incidence matrix of any cycle of length at least 3; these are called “totally balanced matrices” (see e.g. [2]).

Another interpretation of matrices in \( A_{m,n,h} \) is as induced subgraphs of \( K_m \square K_n \), where \( \square \) represents the Cartesian product of graphs. The graph \( K_m \square K_n \) is sometimes called the “rook’s graph”, since the edges represent the legal moves of a rook on an \( m \times n \) chess board. The vertices in \( K_m \square K_n \) are \( \{ (i,j) : 1 \leq i \leq m \text{ and } 1 \leq j \leq n \} \) and there is an edge between distinct vertices \((i,j)\) and \((i',j')\) whenever \( i = i' \) or \( j = j' \). The graph \( K_m \square K_n \) is also the line graph of \( K_{m,n} \).

There is a bijection between induced subgraphs \( H = H(M) \) of \( K_m \square K_n \) and \((0,1)\)-matrices \( M = (m_{ij}) \in A_{m,n,h} \); we include the vertex \((i,j)\) in \( H \) if and only if \( m_{ij} = 1 \). For example, if we ignore the vertices marked 0 in

\[
\begin{array}{cccc}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
\end{array}
\]

\(10\) Full URL: [http://math.stackexchange.com/q/24800](http://math.stackexchange.com/q/24800)

\(11\) The biadjacency matrix of a bipartite graph \( G \) on the vertex set \( \{ u_i \}_{1 \leq i \leq m} \cup \{ v_j \}_{1 \leq j \leq n} \) is the \( m \times n \) \((0,1)\)-matrix \( M = (m_{ij}) \) with \( m_{ij} = 1 \) if and only if \( u_i v_j \) is an edge in \( G \).
The number of $h$-edge spanning forests in $K_{m,n}$

we obtain an induced subgraph $H(M)$ of $K_3 \square K_5$, corresponding to the matrix

$$ M = \begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 
\end{pmatrix} $$

in $A_{3,5,6}$. We see that cycles in $M$ map from cycles $C_k$ in $H$ for some $k \in \{4, 6, 8, \ldots\}$ (an example of a 6-cycle is highlighted). There are other cycles in $H$ (e.g. $C_3$ in the above example; in fact, if $M$ above had a row of 1’s, then $H$ would have a $K_5$ subgraph). However, induced cycles $C_k$ in $H(M)$ of length $k \in \{4, 6, 8, \ldots\}$ are in one-to-one correspondence with cycles in $M$ via the above bijection.

**Lemma 4** For all $m, n, h \in \mathbb{N}$, we have that $f_{m,n,h}$ is the number of $h$-vertex induced subgraphs of $K_{m,n}$ that do not contain an induced $C_k$ for any $k \in \{4, 6, 8, \ldots\}$.

4 Simplifying the equation

4.1 A formula for $f_{m,n,h}$

We will look at two related ways of simplifying $f_{m,n,h}$. Let $K_{m,n}$ have the vertex bipartition $M \cup N$, where $M = \{u_1, u_2, \ldots, u_m\}$ and $N = \{v_1, v_2, \ldots, v_n\}$. Let $B_{m,n,h}$ be the set of $h$-edge spanning forests of $K_{m,n}$. Hence $f_{m,n,h} = |B_{m,n,h}|$. Define

$$ W_{m,n,h} = \{ G \in B_{m,n,h} : G \text{ has no isolated vertices} \} $$

and $w_{m,n,h} = |W_{m,n,h}|$. We observe the following:

- Given any spanning forest in $B_{m,n,h}$ we can delete the isolated vertices to obtain a spanning forest in $W_{i,j,h}$ for some $i \leq m$ and $j \leq n$.
- Conversely, given any spanning forest in $W_{i,j,h}$, we can add isolated vertices to it in $\binom{m}{i-j} \binom{n}{j} \binom{m}{i} \binom{n}{j}$ ways to obtain a spanning forest in $B_{m,n,h}$.

In both of the operations above, we need to relabel the vertices, but we preserve the order of the indices of the non-isolated vertices within $M$ and $N$.

Hence

$$ f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} w_{i,j,h}. \quad (2) $$

This formula eliminates the need to further account for isolated vertices.

We can decompose any $G \in W_{m,n,h}$ into disjoint components, each of which belong to $W_{i,j,k}$ for some $i \leq m$ and $j \leq n$ and $k \leq h$. Hence any $G \in W_{m,n,h}$ decomposes into the following.

- A partition $P$ of $M$ where $x, y \in M$ belong to the same part if there is a path from $x$ to $y$ in $G$.
- A partition $Q$ of $N$ where $x', y' \in N$ belong to the same part if there is a path from $x'$ to $y'$ in $G$.
- A bijection $\alpha : P \rightarrow Q$ such that every edge $ab$ in $G$ has $a \in p$ and $b \in \alpha(p)$ for some $p \in P$.

Hence we have $|P| = |Q|$. 

For each $p \in P$, a subgraph in $W_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$ induced by the vertices in $p \cup \alpha(p)$.

Regarding the final bulleted item, we observe that the subgraph induced by $p \cup \alpha(p)$ for some $p \in P$ must be a spanning tree on $K_{|p|,|\alpha(p)|}$. Any more edges would cause a cycle, while fewer edges would cause disjoint components (in which case $G$ could be further decomposed).

We give an example of a graph $G$ in $W_{3,5,6}$ below.

Here, the partitions $P$ and $Q$ are given by $P = \{\{u_1, u_2\}, \{u_3\}\}$ and $Q = \{\{v_1, v_2\}, \{v_3, v_4, v_5\}\}$ and we have the bijection $\alpha$ such that

$$\{u_1, u_2\} \mapsto \{v_1, v_2\}$$
$$\{u_3\} \mapsto \{v_3, v_4, v_5\}.$$

We see that $G$ decomposes into a graph in $W_{2,2,3}$ and a graph in $W_{1,3,3}$.

In general, for any $p \in P$, we must have exactly $|p| + |\alpha(p)| - 1$ edges in $p \cup \alpha(p)$. Summing this over all $p \in P$ gives $h = m + n - |P|$. We conclude that

$$|P| = m + n - h$$
(and thus $|Q| = m + n - h$).

We will now describe how to construct any graph in $W_{m,n,h}$ via its decomposition. Let $P$ be the set of partitions of $M$ and let $Q$ be the set of partitions of $N$. Given (a) a partition $P \in P$ of size $|P| = m + n - h$, (b) a partition $Q \in Q$ and (c) a bijection $\alpha : P \to Q$, we can construct $\prod_{p \in P} w_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$ distinct graphs $G \in W_{m,n,h}$ by the following steps:

1. Start with $G$ as the graph with vertex set $M \cup N$ and no edges.
2. For each $p \in P$ add one of the subgraphs in $W_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$ on the vertices $p \cup \alpha(p)$.

Since all graphs in $W_{m,n,h}$ can be constructed uniquely by the above steps we have the following theorem.

**Theorem 1** For $m, n, h \in \mathbb{N}$,

$$f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} w_{i,j,h}$$
where
\[
\sum_{P \in \mathcal{P}} \sum_{Q \in \mathcal{Q}} \sum_{\alpha : P \to Q} \prod_{p \in P} w[p,|\alpha(p)|,|p|+|\alpha(p)|-1]
\]

where
\[
w[p,|\alpha(p)|,|p|+|\alpha(p)|-1] = |p|^{\alpha(p) - 1} |\alpha(p)|^{p-1}.
\]

4.2 An improved formula for \( f_{m,n,h} \)

We will derive two further equations for \( f_{m,n,h} \) (in Theorems 2 and 3, which are related to Theorem 1). They result in a slightly more complicated algorithm, but will allow us to compute \( f_{m,n,h} \) faster than \( (3) \).

For all \( m, n, h \in \mathbb{N} \), define

\[
W'_{m,n,h} = \{ G \in W_{m,n,h} : \text{the vertices in } N \text{ have degree } \geq 2 \}.
\]

Given any \( G \in B_{m,n,h} \) we can delete the isolated vertices and the leaves (vertices of degree 1) in \( N \) to obtain a spanning forest in \( W'_{i,j,h-k} \) for some \( i \leq m \) and \( j \leq n \) and \( k \in \mathbb{N} \). Conversely, given any \( G \in W'_{i,j,h-k} \), we can add:

- \( m - i \) isolated vertices to \( M \) in \( \binom{m}{m-i} \) ways, so as to increase \( |M| \) to \( m \), then
- \( n - j \) isolated vertices to \( N \) in \( \binom{n}{n-j} \) ways, so as to increase \( |N| \) to \( n \), then
- \( k \) edges to \( G \) in \( \binom{n-j}{k} m^k \) ways, so as to increase the number of leaves in \( N \) by \( k \), thereby obtaining a spanning forest in \( B_{m,n,h} \). Similar to the derivation of \( (2) \), we need to relabel the vertices so as to preserve the order of the indices of the non-isolated vertices within \( M \) and the non-isolated non-leaf vertices within \( N \). Hence

\[
f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \min(n-j,h) \binom{m}{i} \binom{n}{j,k,n-j-k} m^k w'_{i,j,h-k}.
\]

All of the steps involved for finding the formula \( (3) \) for \( w_{m,n,h} \) are still valid for \( w'_{m,n,h} \). Hence \( (3) \) remains true if we replace \( w \) with \( w' \). However, we can no longer make use of the simple formula for \( w[p,|\alpha(p)|,|p|+|\alpha(p)|-1] \). Nevertheless, we will be able to find a formula for \( w'_{i,j,h-k} \) via Inclusion-Exclusion.

For \( I \subseteq N \), define

\[
A_I = \{ G \in W_{m,n,m+n-1} : \text{vertices in } I \text{ are leaves} \}.
\]

Note that spanning forests in \( A_I \) might also have leaves outside of \( I \). Hence \( |A_0| = |W_{m,n,m+n-1}| = m^{n-1} n^{m-1} \). By symmetry, if \( I, J \subseteq N \) and \( |I| = |J| \) then \( |A_I| = |A_J| \). So we will assume \( I = \{ n - i + 1, n - i + 2, \ldots, n \} \). We can construct the graphs in \( A_I \) by adding \( i \) leaves to the graphs in \( W_{m,n-|I|,m+n-|I|-1} \) (i.e., the set of spanning trees of \( K_{m,n-|I|} \)), and these leaves can be added in \( m^i \) ways. Hence

\[
|A_I| = m^i \cdot \text{number of spanning trees of } K_{m,n-i}
\]

\[
= m^i m^{n-i-1} (n-i)^{m-1}
\]

\[
= m^{n-1} (n-i)^{m-1}
\]
when \( m \geq 1 \) and \( n \geq 1 \). Here we take \( 0^0 = 1 \), since we want to account for \( K_{1,0} \in W_{1,0,0} \). Hence, by Inclusion-Exclusion, we find

\[
\begin{align*}
 w'_{m,n,m+n-1} &= |A_0| - \left| \bigcup_{I \subseteq N \atop I \neq \emptyset} A_I \right| \\
 &= m^{n-1} n^{m-1} - \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} m^{n-1} (n-i)^{m-1} \\
 &= m^{n-1} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^{m-1}
\end{align*}
\]

when \( m \geq 1 \) and \( n \geq 1 \). By definition, we also have \( w'_{m,n,m+n-1} = 0 \) when either \( m = 0 \) or \( n = 0 \) (and \( w'_{m,n,m+n-1} \) is not defined when \( m = 0 \) and \( n = 0 \)).

Before identifying formulae for \( w_{m,n,h} \) and \( w'_{m,n,h} \) in the next section, we make the following observation.

**Lemma 5** For all \( m, n, h \in \mathbb{N} \), we have \( n! \) divides \( w'_{m,n,h} \).

**Proof:** The symmetric group on \( N \) acts on \( W'_{m,n,h} \) by permuting the vertices in \( N \). Suppose a graph \( G \in W'_{m,n,h} \) has a non-trivial stabiliser subgroup under this action. Then there exists a non-identity permutation \( \alpha \) such that \( \alpha G = G \), and distinct vertices \( v, v' \in N \) for which \( \alpha(v) = v' \). Since \( G \in W'_{m,n,h} \), the degree of \( v \) is 2 or more, so assume \( v \) has distinct neighbours \( a, b \in M \). Then, since \( \alpha G = G \), we find that \( v' = \alpha(v) \) has the neighbours \( a \) and \( b \) too. Thus, \( \{v, v', a, b\} \) induces a 4-cycle, giving a contradiction. Hence, all stabiliser subgroups are trivial, and by the Orbit-Stabiliser Theorem, all orbits have size \( n! \). Hence \( n! \) divides \( w'_{m,n,h} \). \( \square \)

### 4.3 Formulae for \( w_{m,n,h} \) and \( w'_{m,n,h} \)

Now we will simplify (3), noting that our simplifications remain valid when we replace \( w \) with \( w' \).

For \( a \geq 1 \) and \( t \geq 1 \), let \( S_{a,t} \) be the set of (number) partitions of \( a \) into \( t \) non-zero parts. We will interpret elements of \( S_{a,t} \) as multisets; for example \( \{4,3,3,1,1\} \in S_{12,5} \). We will require \( S_{a,0} = \emptyset \) when \( t < 0 \), and \( S_{0,0} = \{\emptyset\} \) and \( S_{0,t} = \emptyset \) when \( t > 0 \). For \( z \in S_{a,t} \), let \( T[z] \) be the set of partitions of \( \{1,2,\ldots,a\} \) whose part sizes induce the number partition \( z \). For \( z \in S_{a,t} \), let \( \hat{z} \) denote an arbitrary element of \( T[z] \). If \( P \in T[x] \) and \( Q \in T[z] \), then any bijection \( \alpha : P \to Q \) induces a bijection \( \beta : x \to z \).
Hence
\[
\begin{align*}
\sum_{x,z} \sum_{P \in \mathcal{T}[x]} \sum_{Q \in \mathcal{T}[z]} \sum_{\alpha : P \rightarrow Q \atop \alpha \text{ is a bijection}} \prod_{p \in P} w_{|p|, |\alpha(p)|, |p| + |\alpha(p)| - 1} \\
\sum_{x,z} \sum_{P \in \mathcal{T}[x]} \sum_{Q \in \mathcal{T}[z]} \sum_{\alpha : z \rightarrow z} \prod_{p \in P} w_{|p|, |\alpha(p)|, |p| + |\alpha(p)| - 1} \\
\sum_{x,z} |T[x]| \cdot |T[z]| \sum_{\alpha : x \rightarrow z} \prod_{p \in P} w_{|p|, |\alpha(p)|, |p| + |\alpha(p)| - 1} \\
\sum_{x,z} |T[x]| \cdot |T[z]| \sum_{\beta : x \rightarrow z} \prod_{r \in P} w_{\beta(r), r + \beta(r) - 1}.
\end{align*}
\]

If \( z \in \mathcal{S}_{a,t} \), then
\[
|T[z]| = \frac{a!}{\prod_{i \geq 1} i! s_i(z)!},
\]
where \( s_i(z) \) denotes the number of parts \( i \) in \( z \) (see e.g. [1] Theorem 13.2]). Hence we have the following theorems.

**Theorem 2** For \( m, n, h \in \mathbb{N} \),
\[
f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} w_{i,j,h}
\]
where
\[
w_{m,n,h} = \sum_{x,z} \sum_{x \rightarrow z}^{\beta} \prod_{\alpha \in S_{m,n,h}} \prod_{i \geq 1} i! s_i(x)! \prod_{i \geq 1} i! s_i(z)! \prod_{r \in P} r^{\beta(r) - 1} (r + 1)^{-1}. \tag{4}
\]

**Theorem 3** For \( m, n, h \in \mathbb{N} \),
\[
f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \min(n-j,h) \binom{m}{i} \binom{n}{j,k} m^k w_{i,j,h-k}.
\]
where
\[
w_{m,n,h} = \sum_{x,z} \sum_{x \rightarrow z}^{\beta} \prod_{\alpha \in S_{m,n,h}} \prod_{i \geq 1} i! s_i(x)! \prod_{i \geq 1} i! s_i(z)! \prod_{r \in P} w_{\beta(r), r + \beta(r) - 1}.
\]

where
\[
w_{m,n,m+n} = m^{n-1} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n - i)^{m-1}.
\]

The main advantage of Theorems 2 and 3 over Theorem 1 is the summation is over number partitions (rather than set partitions).
5 Implementation

The author has implemented the two formulae for $f_{m,n,h}$ described in the preceding section, along with a simple backtracking algorithm. The results between all three implementations concur, giving confidence in the accuracy of the code. The C source code is available as supplementary material to this document. The GMP library was used for arbitrary precision arithmetic [6].

The formula involving $w'_{m,n,h}$ instead of $w_{m,n,h}$ was (unsurprisingly) much faster, largely because many values of $w'_{m,n,h}$ equal 0, such as when $m \leq n$, which can be used to drastically reduce the search tree.

5.1 Symmetry breaking

The run-time of the program was also improved through the use of symmetry breaking, which we will now describe. For $x \in S_{m,m+n−h}$ and $z \in S_{m,m+n−h}$, there are often many bijections $\beta : x \rightarrow z$ which map the same elements to the same elements (since both $x$ and $z$ are multisets). In these instances, a naïve algorithm would repeat the same computation unnecessarily. Let $x$ and $z$ be the multisets $x = \{x_1, x_2, \ldots, x_{m+n−h}\}$ and $z = \{z_1, z_2, \ldots, z_{m+n−h}\}$. We define the condition:

**Symmetry breaking condition 1**: We say $\beta : x \rightarrow z$ is half-canonical if $\beta^{-1}(z_{i−1}) < \beta^{-1}(z_i)$ whenever $z_{i−1} = z_i$.

We need to add a multiplicative factor to adjust for the restriction to half-canonical bijections. Hence

$$w_{m,n,h} = \sum_{x,z \in S_{m,m+n−h}} \sum_{\substack{\beta : x \rightarrow z \\beta \text{ is a bijection} \\beta(\beta^{-1}(x)) \text{ is half-canonical}}} \left( \prod_{i \geq 1} i^{t_i(z)} \right) \frac{m!}{\prod_{i \geq 1} t_i(x)^{s_i(x)}} \frac{n!}{\prod_{i \geq 1} t_i(z)^{s_i(z)}} \prod_{r \in x} w_{r,\beta(r),r+\beta(r)−1}$$

and similarly with $w'$ in place of $w$. Using this assumption, we reduce the number of bijections by a factor of $\prod_{i \geq 1} i^{t_i(z)}$, which results in a substantial time saving.

Instead of the single symmetry breaking assumption, it is possible to utilise symmetry breaking using an additional condition:

**Symmetry breaking condition 2**: We say $\beta : x \rightarrow z$ is canonical if it is half-canonical and $\beta(x_{i−1}) < \beta(x_i)$ whenever $x_i = x_{i−1}$.

Again, we find

$$w_{m,n,h} = \sum_{x,z \in S_{m,m+n−h}} \sum_{\substack{\beta : x \rightarrow z \\beta \text{ is a bijection} \\beta \text{ is canonical}}} \Gamma_{x,z,\beta} \frac{m!}{\prod_{i \geq 1} t_i(x)^{s_i(x)}} \frac{n!}{\prod_{i \geq 1} t_i(z)^{s_i(z)}} \prod_{r \in x} w_{r,\beta(r),r+\beta(r)−1}$$

and similarly with $w'$ in place of $w$, where $\Gamma_{x,z,\beta}$ is the number of bijections between the multisets $x$ and $z$ which map the same elements to the same elements (as $\beta$). A formula for $\Gamma_{x,z,\beta}$ was given by e.g. [3], namely

$$\Gamma_{x,z,\beta} = \frac{\prod_{i \geq 1} s_i(x)! \prod_{i \geq 1} s_i(z)!}{\prod_{i \geq 1} \prod_{j \geq 1} s_{i,j}(x,\beta)!}$$

where
The number of $h$-edge spanning forests in $K_{m,n}$

- $s_i(z)$ denotes the number of parts $i$ in $z$ (as before), and
- $s_{i,j}(x,\beta)$ is the number of elements $(i,j)$ in the multiset $\{(r,\beta(r)) : r \in x\}$.

The author has implemented both of these symmetry breaking schemes in order to compare their performance (see Section 5.3).

5.2 Pseudo-code

Algorithm 1 gives a pseudo-code version of the C code used to implement the algorithm described by Theorem 3 using the half-canonical symmetry breaking condition. The partitions of $m$ and $n$ into $k$ parts were computed whenever needed and stored in memory. Iterating through the half-canonical bijections was performed “on the fly” using a backtracking algorithm.

While Theorems 2 and 3 are valid for all $m,n,h \in \mathbb{N}$, we need to set $f_{m,n,0} = 1$ separately in the C code.

5.3 Performance

The C code was run on a $2 \times 2.66$ GHz processor (although the code itself is not parallelised). The following table gives the run-times (in seconds) for the two algorithms under the two symmetry breaking schemes when computing all non-zero $f_{m,n,h}$ with $m,n \leq 19$.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>half-canonical</th>
<th>canonical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 2</td>
<td>27.1</td>
<td>6.8</td>
</tr>
<tr>
<td>Theorem 3</td>
<td>4.1</td>
<td>4.5</td>
</tr>
</tbody>
</table>

The fastest version is Theorem 3 using half-canonical symmetry breaking. Under these conditions, the code had the following run-times to find all non-zero values of $f_{m,n,h}$ with $m,n \leq t$:

<table>
<thead>
<tr>
<th>$t$</th>
<th>time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>20.5</td>
</tr>
<tr>
<td>26</td>
<td>27.7</td>
</tr>
<tr>
<td>27</td>
<td>35.4</td>
</tr>
<tr>
<td>28</td>
<td>46.0</td>
</tr>
<tr>
<td>29</td>
<td>62.2</td>
</tr>
<tr>
<td>30</td>
<td>83.0</td>
</tr>
</tbody>
</table>

This table indicates the scalability of the program, which is not overwhelming for these values of $t$. The author also ran the program to compute $f_{m,n,h}$ with $m,n \leq t$ where $t = 50$, which took under 2 days (to be precise, it took 1 day 15 hours and 17 minutes). The largest number encountered was $f_{50,50,101}$, which has 167 digits, and is equal to the number of spanning trees of $K_{50,50}$, which is $50^{50-1} \cdot 50^{50-1} = 50^{98}$.

5.4 Complexity

In the worst case, $x = z = \{1,2,3,\ldots,t\}$, where $\min(m,n) = 1 + 2 + \cdots + t = \frac{1}{2}t(t + 1)$ (so $t = O(\sqrt{\min(m,n)})$, in which case the program must iterate through all $t!$ bijections from $x$ to $z$. So, for a worst case analysis, we assume $m = n$ and $m + n - h = \lceil \sqrt{m} \rceil$. 

Algorithm 1 Implementation of Theorem 3 using half-canonical symmetry breaking

\[ f_{m,n,m+n-1} := 0; \text{ for all } m,n,h \]
\[ w'_{m,n,m+n-1} := 0; \text{ for all } m,n,h \]
\[ w'_{0,0,0} := 1; \]

for \( m = 0, 1, \ldots, \text{MAX}(m) \) do
  for \( n = 0, 1, \ldots, \text{MAX}(n) \) do
    if \( m > n \) and \( m > 0 \) and \( n > 0 \) then // Other values of \( w'_{m,n,h} \) are 0
      for \( i = 0, 1, \ldots, n \) do
        // Using the Inclusion-Exclusion formula
        \[ w'_{m,n,m+n-1} := w'_{m,n,m+n-1} + (-1)^i \binom{n}{i} (n-i)^{m-1}; \]
      end for
    end if
    for \( h = 0, 1, \ldots, m+n-2 \) do
      \[ w'_{m,n,h} := 0; \]
      Compute all partitions of \( m \) into \( m+n-h \) parts and store in memory.
      Compute all partitions of \( n \) into \( m+n-h \) parts and store in memory.
      for partitions \( x \) of \( m \) into \( m+n-h \) parts do
        for partitions \( z \) of \( n \) into \( m+n-h \) parts do
          for half-canonical bijections \( \beta : x \to z \) do // via backtracking algorithm
            \[ \Gamma := \prod_{i \geq 1} i!^{h_i(x)}; \]
            \[ w'_{m,n,h} := w'_{m,n,h} + \Gamma \cdot \frac{m!}{\prod_{i \geq 1} i!^{h_i(x)} i!^{h_i(z)}} \prod_{r \in x} w'_{r,\beta(r),r+\beta(r)-1}; \]
          end for
        end for
      end for
    end for
  end for
end for

\[ f_{m,n,0} = 1; \]
for \( h = 1, 2, \ldots, m+n-1 \) do
  for \( i = 0, 1, \ldots, m \) do
    for \( j = 0, 1, \ldots, n \) do
      for \( k = 0, 1, \ldots, \min(n-j, h) \) do
        \[ f_{m,n,h} := f_{m,n,h} + \binom{m}{i} \binom{n}{j,k,n-j-k} m^k w'_{i,j,h-k}; \]
      end for
    end for
  end for
end for
for \( h = 0, 1, \ldots, m+n-1 \) do
  Print \( f_{m,n,h}; \)
end for
The number of $h$-edge spanning forests in $K_{m,n}$

Hence, given $x, z \in S_{m,m+n-h}$, there can be $O(\sqrt{m}!)$ canonical bijections from $x$ to $z$. Hardy and Ramanujan’s asymptotic formula for the number of partitions of $m$, namely

$$\frac{1}{4m\sqrt{3}} e^{\pi \sqrt{2m/3}},$$

gives a crude asymptotic upper bound on $|S_{m,\lfloor \sqrt{m} \rfloor}|^2$, specifically

$$|S_{m,\lfloor \sqrt{m} \rfloor}|^2 = O\left(\frac{e^{\text{const} \cdot m}}{m^2}\right)$$

as the number of ways of choosing $x$ and $z$. Hence, $f_{m,n,h}$ has

$$O\left(e^{\text{const} \cdot \sqrt{m} \cdot \sqrt{m}/2 + \text{const}}\right)$$

terms, by Stirling’s Approximation. In contrast, a backtracking algorithm would need to generate and check around $(\frac{m^2}{\sqrt{m}})^{1.5} \cdot \sqrt{m} \cdot \text{const}$ graphs, which, when $m = n$ and $h = 2m - \sqrt{m}$, is

$$O(e^{\sqrt{m} \cdot 1.5 \sqrt{m} + \text{const}})$$

iterations, by Stirling’s Approximation. Of course, when implementing these algorithms, we use pruning whenever possible to reduce the search space, which makes a drastic difference not accounted for in these approximations.

5.4.1 When $h$ is fixed

We conclude this paper with the observation that, when $h$ is fixed, computing $f_{m,n,h}$ is asymptotically “easy”. The underlying reason is that, for sufficiently large $m$ or $n$, we must have isolated vertices in graphs in $B_{m,n,h}$. Thus, $f_{m,n,h}$ only contains a finite number of non-zero terms.

**Theorem 4** For fixed $h$, computing $f_{m,n,h}$ can be performed in time $O(\log(mn))$.

**Proof:** If $i > h$ or $j > h$, then $w_{i,j,h} = 0$ (since any graph in $B_{i,j,h}$ must have an isolated vertex). Hence, $f_{m,n,h}$ is equivalent to

$$f_{m,n,h} = \sum_{i=0}^{h} \sum_{j=0}^{h} \binom{m}{i} \binom{n}{j} w_{i,j,h}.$$ 

For fixed $h$, there is a finite number of terms in this sum. Thus, for fixed $h$, we could write a program in which:

- we store a list of the pairs $(i, j)$ for which $w_{i,j,h}$ is non-zero, along with the value of $w_{i,j,h}$,
- we iterate through this list, computing $\binom{m}{i} \binom{n}{j} w_{i,j,h}$, and add it to a running total.

We can compute $\binom{m}{i} = \frac{1}{i} \cdot m(m-1) \cdots (m-i+1)$ using $O(h)$ multiplications (since $i \leq h$), each of which takes time $O(\log m)$, and one division. Hence, $\binom{m}{i}$ can be computed in time $O(\log m)$ time (since $h$ is fixed). Similarly $\binom{n}{j}$ can be computed in $O(\log n)$ time. We conclude that the whole summation can be performed in time $O(\log(mn))$. 

\qed
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