Computing the number of h-edge spanning forests in complete bipartite graphs

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Let $f_{m,n,h}$ be the number of spanning forests with $h$ edges in the complete bipartite graph $K_{m,n}$. Kirchhoff’s Matrix Tree Theorem implies $f_{m,n,m+n-1} = m^{n-1}n^{m-1}$ when $m \geq 1$ and $n \geq 1$, since $f_{m,n,m+n-1}$ is the number of spanning trees in $K_{m,n}$. In this paper, we give an algorithm for computing $f_{m,n,h}$ for general $m, n, h$. We implement this algorithm and use it to compute all non-zero $f_{m,n,h}$ when $m \leq 50$ and $n \leq 50$ in under 2 days.

MSC 2010: 68R10, 05C85, 05C30, 05C38, 05C05

Keywords: complete bipartite graph, spanning forest

1 Introduction

A consequence of Kirchhoff’s Matrix Tree Theorem is that the number of spanning trees in the complete bipartite graph $K_{m,n}$ is $m^{n-1}n^{m-1}$ when $m \geq 1$ and $n \geq 1$. In this paper, we consider a generalisation of this counting problem: the number $f_{m,n,h}$ of $h$-edge spanning forests in $K_{m,n}$. The number of spanning trees in $K_{m,n}$ is $f_{m,n,m+n-1}$ when $m \geq 1$ and $n \geq 1$, since spanning trees of $K_{m,n}$ have $m+n-1$ edges.

The numbers $f_{m,n,h}$ arise in relation to the Tutte polynomial of $K_{m,n}$. For a general graph $G = (V, E)$, the Tutte polynomial is

$$T_G(x, y) = \sum_{A \subseteq E} (x-1)^{k(A)-k(E)}(y-1)^{k(A)+|A|-|V|}$$

where $k(A)$ is the number of connected components in the graph $(V, A)$. Importantly, for the Tutte polynomial, we take $0^0 = 1$. Thus, the number of spanning forests of $G$ is given by

$$T_G(2, 1) = \sum_{A \subseteq E} 0^{k(A)+|A|-|V|} = \#\{A \subseteq E : k(A) + |A| = |V|\}$$

number of spanning forests of $G$.

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since $k(A) + |A| = |V|$ if and only if $A$ is a spanning forest.

In the present paper, we are interested in the case of when $G = K_{m,n}$, and will look for $h$-edge spanning forests. In this situation,

$$T_{K_{m,n}}(2, 1) = \sum_{h \geq 0} f_{m,n,h}.$$  

For general graphs $G = (V,E)$, Myrvold [10] gave an algorithm for computing the number of $k$-component spanning forests of $G$. Björklund et al. [4] described an algorithm that could compute $T_G(2, 1)$ in time $2^{|V|}|V|^{O(1)}$ by using Kirchhoff’s Matrix Tree Theorem for each of the $2^{|V|}$ subsets of $V$, then combining the results together with Inclusion-Exclusion. Other relevant work for the enumeration of spanning forests includes [9]. Porter [11] gave an algorithm for generating the spanning trees of $K_{m,n}$.

Farr and McDiarmid [5] showed that computing the number of circuits in $G$ is #P-complete

Bounds for the number of spanning forests in graphs were given by Teranishi [13], from which we can deduce

$$T_{K_{m,n}}(2, 1) \geq \sum_{k \geq 0} \left( \frac{\min(m,n)}{2} \right)^{m,n-k} \binom{m+n}{k}$$

for all $m \geq 0$ and $n \geq 0$. Jin and Liu [8] gave a simple formula for the number of rooted spanning forests in $K_{m,n}$ (see also [7] and [12]).

This paper instead heads in a different direction to previous work: we derive algorithms for enumerating $h$-edge spanning forests in $K_{m,n}$.

## 2 Basic results

To be clear, all forests in this paper will be labelled. Let $\mathbb{N} = \{0, 1, 2, \ldots \}$. The following lemma gives the boundary conditions on $f_{m,n,h}$.

**Lemma 1** Suppose $m, n, h \in \mathbb{N}$. Then

- $f_{m,n,0} = 1$,  
- if $h \geq 1$, then $f_{0,n,h} = f_{m,0,h} = 0$,  
- if $m \geq 1$ and $n \geq 1$, then $f_{m,n,h} > 0$ if and only if $0 \leq h \leq m + n - 1$,  
- if $m \geq 1$ and $n \geq 1$, then $f_{m,n,m+n-1} = m^{n-1}n^{m-1}$.

**Proof:** If $h = 0$, then there is exactly one subgraph of $K_{m,n}$ with no edges (and it is a spanning forest), so $f_{m,n,0} = 1$. If $m = 0$ or $n = 0$, then $K_{m,n}$ has no edges, and thus we obtain the second bulleted item.

Now assume $m \geq 1$ and $n \geq 1$. Since $K_{m,n}$ is connected, it has a spanning tree, which must have exactly $m + n - 1$ edges. By deleting edges from this spanning tree, we find $h$-edge forests in $K_{m,n}$ for all $0 \leq h \leq m + n - 1$. Hence $f_{m,n,h} > 0$ when $0 \leq h \leq m + n - 1$.

Now assume $A$ is an $h$-subset of the edges in $K_{m,n}$, where $m \geq 1$ and $n \geq 1$ and $h \geq m + n$. To be a spanning forest, we need $k(A) + h = m + n$ as per (1), but if $h \geq m + n$, then $k(A) \leq 0$, giving a contradiction.

---

(i) #P is the set of counting problems which ask for the number of “yes” instances for decision problems in NP. A #P problem is in #P-complete whenever any other problem in #P can be reduced to it by a polynomial-time counting reduction.
The number of \( h \)-edge spanning forests in \( K_{m,n} \)

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**Fig. 1:** Small non-zero values of \( f_{m,n,h} \).

The fourth bulleted item is from Kirchhoff’s Matrix Tree Theorem (already mentioned).

The following lemma is an important (but basic) consequence of the symmetry of the problem.

**Lemma 2** For all \( m, n, h \in \mathbb{N} \), we have \( f_{m,n,h} = f_{n,m,h} \).

In Figure 1, we list the non-zero values of \( f_{m,n,h} \) when \( 0 \leq m \leq n \leq 6 \). These values were generated using a straightforward backtracking algorithm.

## 3 Combinatorial equivalence

In this section, we will describe some combinatorial objects that are equivalent to \( h \)-edge spanning forests of \( K_{m,n} \).

Let \( M \) be an \( m \times n \) \((0,1)\)-matrix. We define a cycle in \( M \) to be a set \( \{e_1, e_2, \ldots, e_t\} \) of entries of \( M \), such that:

- each \( e_i \) contains the symbol 1, and
- for \( i \in \{1, 2, \ldots, t\} \), we have that (a) \( e_{2i-1} \) belongs to the same row as \( e_{2i} \) and (b) \( e_{2i} \) belongs to the same column as \( e_{2i+1} \) (where we take \( e_{2t+1} = e_1 \)).

For example, the matrix

\[
\begin{pmatrix}
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0
\end{pmatrix}
\]
Rebecca J. Stones has a cycle (consisting of the circled 1’s), whereas the matrices

\[
\begin{pmatrix}
0 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
do not have cycles.

Let \(A_{m,n,h}\) be the set of \((0,1)\)-matrices with exactly \(h\) elements equal to 1. Let \(p_{m,n,h}\) be the probability that an element of \(A_{m,n,h}\) chosen uniformly at random contains a cycle. An anonymous user of [math.stackexchange.com](http://math.stackexchange.com) asked [ii] for a formula for \(p_{m,n,h}\). In fact, this was the original motivation for the author to study this problem.

**Lemma 3** For all \(m,n,h \in \mathbb{N}\), we have \(f_{m,n,h}\) is the number of \(m \times n\) \((0,1)\)-matrices with \(h\) 1’s that do not contain a cycle, and hence

\[
f_{m,n,h} = (1 - p_{m,n,h}) \binom{mn}{h}.
\]

**Proof:** A matrix \(M \in A_{m,n,h}\) can be interpreted as the biadjacency matrix [iii] of an \(h\)-edge subgraph \(G = G(M)\) of \(K_{m,n}\). A cycle in \(M \in A_{m,n,h}\) corresponds to a cycle in \(G\). Hence

\[
f_{m,n,h} = (1 - p_{m,n,h}) |A_{m,n,h}| = (1 - p_{m,n,h}) \binom{mn}{h}.
\]

\[\square\]

A problem related to the \((0,1)\)-matrix problem above comes from the study of \((0,1)\)-matrices that do not have a submatrix which is the incidence matrix of any cycle of length at least 3; these are called “totally balanced matrices” (see e.g. [2]).

Another interpretation of matrices in \(A_{m,n,h}\) is as induced subgraphs of \(K_m \square K_n\), where \(\square\) represents the Cartesian product of graphs. The graph \(K_m \square K_n\) is sometimes called the “rook’s graph”, since the edges represent the legal moves of a rook on an \(m \times n\) chess board. The vertices in \(K_m \square K_n\) are \(\{i,j\} : 1 \leq i \leq m\) and \(1 \leq j \leq n\} \text{ and there is an edge between distinct vertices } (i,j) \text{ and } (i',j') \text{ whenever } i = i' \text{ or } j = j'\). The graph \(K_m \square K_n\) is also the line graph of \(K_{m,n}\).

There is a bijection between induced subgraphs \(H = H(M)\) of \(K_m \square K_n\) and \((0,1)\)-matrices \(M = (m_{ij}) \in A_{m,n,h}\); we include the vertex \((i,j)\) in \(H\) if and only if \(m_{ij} = 1\). For example, if we ignore the vertices marked 0 in

\[\begin{array}{cccccc}
0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{array}\]

\[\text{(ii) Full URL: } \text{http://math.stackexchange.com/q/24800}\]

\[\text{(iii) The biadjacency matrix of a bipartite graph } G \text{ on the vertex set } \{u_i\}_{1 \leq i \leq m} \cup \{v_j\}_{1 \leq j \leq n} \text{ is the } m \times n \text{ } (0,1)\text{-matrix } M = (m_{ij}) \text{ with } m_{ij} = 1 \text{ if and only if } u_i v_j \text{ is an edge in } G.\]
we obtain an induced subgraph $H(M)$ of $K_3 \Box K_5$, corresponding to the matrix

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

in $A_{3,5,6}$. We see that cycles in $M$ map from cycles $C_k$ in $H$ for some $k \in \{4, 6, 8, \ldots\}$ (an example of a 6-cycle is highlighted). There are other cycles in $H$ (e.g. $C_3$ in the above example; in fact, if $M$ above had a row of 1's, then $H$ would have a $K_5$ subgraph). However, induced cycles $C_k$ in $H(M)$ of length $k \in \{4, 6, 8, \ldots\}$ are in one-to-one correspondence with cycles in $M$ via the above bijection.

**Lemma 4** For all $m, n, h \in \mathbb{N}$, we have that $f_{m,n,h}$ is the number of $h$-vertex induced subgraphs of $K_m \Box K_n$ that do not contain an induced $C_k$ for any $k \in \{4, 6, 8, \ldots\}$.

## 4 Simplifying the equation

### 4.1 A formula for $f_{m,n,h}$

We will look at two related ways of simplifying $f_{m,n,h}$. Let $K_{m,n}$ have the vertex bipartition $M \cup N$, where $M = \{u_1, u_2, \ldots, u_m\}$ and $N = \{v_1, v_2, \ldots, v_n\}$. Let $B_{m,n,h}$ be the set of $h$-edge spanning forests of $K_{m,n}$. Hence $f_{m,n,h} = |B_{m,n,h}|$. Define

$$W_{m,n,h} = \{G \in B_{m,n,h} : G \text{ has no isolated vertices}\}$$

and $w_{m,n,h} = |W_{m,n,h}|$. We observe the following:

- Given any spanning forest in $B_{m,n,h}$ we can delete the isolated vertices to obtain a spanning forest in $W_{i,j,h}$ for some $i \leq m$ and $j \leq n$.

- Conversely, given any spanning forest in $W_{i,j,h}$, we can add isolated vertices to it in $\binom{m}{m-i} \binom{n}{n-j}$ ways to obtain a spanning forest in $B_{m,n,h}$.

In both of the operations above, we need to relabel the vertices, but we preserve the order of the indices of the non-isolated vertices within $M$ and $N$.

Hence

$$f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} w_{i,j,h}. \tag{2}$$

This formula eliminates the need to further account for isolated vertices.

We can decompose any $G \in W_{m,n,h}$ into disjoint components, each of which belong to $W_{i,j,k}$ for some $i \leq m$ and $j \leq n$ and $k \leq h$. Hence any $G \in W_{m,n,h}$ decomposes into the following.

- A partition $P$ of $M$ where $x, y \in M$ belong to the same part if there is a path from $x$ to $y$ in $G$.

- A partition $Q$ of $N$ where $x', y' \in N$ belong to the same part if there is a path from $x'$ to $y'$ in $G$.

- A bijection $\alpha : P \to Q$ such that every edge $ab$ in $G$ has $a \in p$ and $b \in \alpha(p)$ for some $p \in P$.

Hence we have $|P| = |Q|$. 

• For each $p \in P$, a subgraph in $W_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$ induced by the vertices in $p \cup \alpha(p)$.

Regarding the final bulleted item, we observe that the subgraph induced by $p \cup \alpha(p)$ for some $p \in P$ must be a spanning tree on $K_{|p|,|\alpha(p)|}$. Any more edges would cause a cycle, while fewer edges would cause disjoint components (in which case $G$ could be further decomposed).

We give an example of a graph $G$ in $W_{3,5,6}$ below.

Here, the partitions $P$ and $Q$ are given by $P = \{\{u_1, u_2\}, \{u_3\}\}$ and $Q = \{\{v_1, v_2\}, \{v_3, v_4, v_5\}\}$ and we have the bijection $\alpha$ such that

$$\{u_1, u_2\} \mapsto \{v_1, v_2\}, \quad \{u_3\} \mapsto \{v_3, v_4, v_5\}.$$

We see that $G$ decomposes into a graph in $W_{2,2,3}$ and a graph in $W_{1,3,3}$.

In general, for any $p \in P$, we must have exactly $|p| + |\alpha(p)| - 1$ edges in $p \cup \alpha(p)$. Summing this over all $p \in P$ gives $h = m + n - |P|$. We conclude that

$$|P| = m + n - h$$

(and thus $|Q| = m + n - h$).

We will now describe how to construct any graph in $W_{m,n,h}$ via its decomposition. Let $P$ be the set of partitions of $M$ and let $Q$ be the set of partitions of $N$. Given (a) a partition $P \in P$ of size $|P| = m + n - h$, (b) a partition $Q \in Q$ and (c) a bijection $\alpha : P \rightarrow Q$, we can construct $\prod_{p \in P} W_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$ distinct graphs $G \in W_{m,n,h}$ by the following steps:

1. Start with $G$ as the graph with vertex set $M \cup N$ and no edges.

2. For each $p \in P$ add one of the subgraphs in $W_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$ on the vertices $p \cup \alpha(p)$.

Since all graphs in $W_{m,n,h}$ can be constructed uniquely by the above steps we have the following theorem.

**Theorem 1** For $m, n, h \in \mathbb{N}$,

$$f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} w_{i,j,h}$$
where

\[ w_{m,n,h} = \sum_{P \in P} \sum_{Q \in Q} \sum_{\alpha : P \to Q} \prod_{p \in P} w|p|,|\alpha(p)|,|p|+|\alpha(p)|-1 \]

where

\[ w|p|,|\alpha(p)|,|p|+|\alpha(p)|-1 = |p|^{|\alpha(p)|-1} |\alpha(p)|^{p-1}. \]

### 4.2 An improved formula for \( f_{m,n,h} \)

We will derive two further equations for \( f_{m,n,h} \) (in Theorems 2 and 3, which are related to Theorem 1). They result in a slightly more complicated algorithm, but will allow us to compute \( f_{m,n,h} \) faster than (2).

For all \( m, n, h \in \mathbb{N} \), define

\[ W_{m,n,h}' = \{ G \in W_{m,n,h} : \text{the vertices in } N \text{ have degree } \geq 2 \}. \]

Given any \( G \in B_{m,n,h} \), we can delete the isolated vertices and the leaves (vertices of degree 1) in \( N \) to obtain a spanning forest in \( W_{i,j,h-k} \) for some \( i \leq m \) and \( j \leq n \) and \( k \in \mathbb{N} \). Conversely, given any \( G \in W_{i,j,h-k}' \), we can add:

- \( m - i \) isolated vertices to \( M \) in \( \binom{m}{m-i} \) ways, so as to increase \(|M|\) to \( m \), then
- \( n - j \) isolated vertices to \( N \) in \( \binom{n}{n-j} \) ways, so as to increase \(|N|\) to \( n \), then
- \( k \) edges to \( G \) in \( (n-j)^k \) ways, so as to increase the number of leaves in \( N \) by \( k \),

thereby obtaining a spanning forest in \( B_{m,n,h} \). Similar to the derivation of (2), we need to relabel the vertices so as to preserve the order of the indices of the non-isolated vertices within \( M \) and the non-isolated non-leaf vertices within \( N \). Hence

\[ f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \min(n-j,h) \binom{m}{i} \binom{n}{j,k,n-j-k} m^k w_{i,j,h-k}'. \]

All of the steps involved for finding the formula (3) for \( w_{m,n,h} \) are still valid for \( w_{m,n,h}' \). Hence (3) remains true if we replace \( w \) with \( w' \). However, we can no longer make use of the simple formula for \( w|p|,|\alpha(p)|,|p|+|\alpha(p)|-1 \). Nevertheless, we will be able to find a formula for \( w|p|,|\alpha(p)|,|p|+|\alpha(p)|-1 \) via Inclusion-Exclusion.

For \( I \subseteq N \), define

\[ A_I = \{ G \in W_{m,n,m+n-1} : \text{vertices in } I \text{ are leaves} \}. \]

Note that spanning forests in \( A_I \) might also have leaves outside of \( I \). Hence \(|A_0| = |W_{m,n,m+n-1}| = m^{n-1} \). By symmetry, if \( I, J \subseteq N \) and \(|I| = |J|\) then \(|A_I| = |A_J|\). So we will assume \( I = \{ n - i + 1, n - i + 2, \ldots, n \} \). We can construct the graphs in \( A_I \) by adding \( i \) leaves to the graphs in \( W_{m,n-|I|,m+n-|I|-1} \) (i.e., the set of spanning trees of \( K_{m,n-|I|} \)), and these leaves can be added in \( m^i \) ways. Hence

\[ |A_I| = m^i \cdot \text{number of spanning trees of } K_{m,n-i} = m^i m^{n-i-1} (n-i)^{m-1} = m^{n-1} (n-i)^{m-1}. \]
when \( m \geq 1 \) and \( n \geq 1 \). Here we take \( 0^0 = 1 \), since we want to account for \( K_{1,0} \in W_{1,0,0} \). Hence, by Inclusion-Exclusion, we find

\[
\begin{align*}
  w'_{m,n,m+n-1} &= |A_0| - \left| \bigcup_{I \subseteq N, I \neq \emptyset} A_I \right| \\
  &= m^{n-1}n^{m-1} - \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} m^{n-1}(n-i)^{m-1} \\
  &= m^{n-1} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^{m-1}
\end{align*}
\]

when \( m \geq 1 \) and \( n \geq 1 \). By definition, we also have \( w'_{m,n,m+n-1} = 0 \) when either \( m = 0 \) or \( n = 0 \) (and \( w'_{m,n,m+n-1} \) is not defined when \( m = 0 \) and \( n = 0 \)).

Before identifying formulae for \( w_{m,n,h} \) and \( w'_{m,n,h} \) in the next section, we make the following observation.

**Lemma 5** For all \( m, n, h \in \mathbb{N} \), we have \( n! \) divides \( w'_{m,n,h} \).

**Proof:** The symmetric group on \( N \) acts on \( W'_{m,n,h} \) by permuting the vertices in \( N \). Suppose a graph \( G \in W'_{m,n,h} \) has a non-trivial stabiliser subgroup under this action. Then there exists a non-identity permutation \( \alpha \) such that \( \alpha G = G \), and distinct vertices \( v, v' \in N \) for which \( \alpha(v) = v' \). Since \( G \in W'_{m,n,h} \), the degree of \( v \) is 2 or more, so assume \( v \) has distinct neighbours \( a, b \in M \). Then, since \( \alpha G = G \), we find that \( v' = \alpha(v) \) has the neighbours \( a \) and \( b \) too. Thus, \( \{v, v', a, b\} \) induces a 4-cycle, giving a contradiction. Hence, all stabiliser subgroups are trivial, and by the Orbit-Stabiliser Theorem, all orbits have size \( n! \). Hence \( n! \) divides \( w'_{m,n,h} \). \( \square \)

4.3 Formulae for \( w_{m,n,h} \) and \( w'_{m,n,h} \)

Now we will simplify (3), noting that our simplifications remain valid when we replace \( w \) with \( w' \).

For \( a \geq 1 \) and \( t \geq 1 \), let \( S_{a,t} \) be the set of (number) partitions of \( a \) into \( t \) non-zero parts. We will interpret elements of \( S_{a,t} \) as multisets; for example \( \{4, 3, 1, 1\} \in S_{12,5} \). We will require \( S_{a,0} = \emptyset \) when \( t < 0 \), and \( S_{0,t} = \{\emptyset\} \) and \( S_{0,t} = \emptyset \) when \( t > 0 \). For \( z \in S_{a,t} \), let \( T[z] \) be the set of partitions of \( \{1, 2, \ldots, a\} \) whose part sizes induce the number partition \( z \). For \( z \in S_{a,t} \), let \( \hat{z} \) denote an arbitrary element of \( T[z] \). If \( P \in T[x] \) and \( Q \in T[z] \), then any bijection \( \alpha : P \to Q \) induces a bijection \( \beta : x \to z \).
Hence

\[ w_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \sum_{P \in T[x]} \sum_{Q \in T[z]} \prod_{\alpha: P \to Q \text{ is a bijection}} u_{|P|,|P|,|P|+|\alpha(p)|-1} \]

\[ = \sum_{x,z \in S_{m,m+n-h}} \sum_{P \in T[x]} \sum_{Q \in T[z]} \prod_{\alpha: z \to z \text{ is a bijection}} u_{|P|,|\alpha(p)|,|P|+|\alpha(p)|-1} \]

\[ = \sum_{x,z \in S_{m,m+n-h}} |T[x]| \cdot |T[z]| \sum_{\beta: x \to z} \prod_{r \in \beta} w_{r, \beta(r), r+\beta(r)-1}. \]

If \( z \in S_{a,t} \), then

\[ |T[z]| = \frac{a!}{\prod_{i \geq 1} i^{s_i(z)} s_i(z)!} \]

where \( s_i(z) \) denotes the number of parts \( i \) in \( z \) (see e.g. [1] Theorem 13.2]). Hence we have the following theorems.

**Theorem 2** For \( m, n, h \in \mathbb{N} \),

\[ f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} w_{i,j,h} \]

where

\[ w_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \sum_{\beta: x \to z} \prod_{r \in \beta} \frac{m!}{\prod_{i \geq 1} i^{s_i(z)} s_i(z)!} \frac{n!}{\prod_{i \geq 1} i^{s_i(z)} s_i(z)!} \prod_{r \in \beta} r^{\beta(r)-1} (r+\beta(r)-1). \] (4)

**Theorem 3** For \( m, n, h \in \mathbb{N} \),

\[ f_{m,n,h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \min(n-j,h) \binom{m}{i} \binom{n}{j} \frac{m!}{\prod_{i \geq 1} i^{s_i(z)} s_i(z)!} \frac{n!}{\prod_{i \geq 1} i^{s_i(z)} s_i(z)!} \prod_{r \in \beta} r^{\beta(r)-1} (r+\beta(r)-1) \]

where

\[ w'_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \sum_{\beta: x \to z} \prod_{r \in \beta} \frac{m!}{\prod_{i \geq 1} i^{s_i(z)} s_i(z)!} \frac{n!}{\prod_{i \geq 1} i^{s_i(z)} s_i(z)!} \prod_{r \in \beta} w'_{r, \beta(r), r+\beta(r)-1} \]

where

\[ w'_{m,n,m+n-1} = m^{n-1} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^{m-1}. \]

The main advantage of Theorems 2 and 3 over Theorem 1 is the summation is over number partitions (rather than set partitions).
5 Implementation
The author has implemented the two formulae for \( f_{m,n,h} \) described in the preceding section, along with a simple backtracking algorithm. The results between all three implementations concur, giving confidence in the accuracy of the code. The C source code is available as supplementary material to this document. The GMP library was used for arbitrary precision arithmetic \([6]\).

The formula involving \( w'_{m,n,h} \) instead of \( w_{m,n,h} \) was (unsurprisingly) much faster, largely because many values of \( w'_{m,n,h} \) equal 0, such as when \( m \leq n \), which can be used to drastically reduce the search tree.

5.1 Symmetry breaking
The run-time of the program was also improved through the use of symmetry breaking, which we will now describe. For \( x \in S_{m,m+n-h} \) and \( z \in S_{n,m+n-h} \), there are often many bijections \( \beta : x \rightarrow z \) which map the same elements to the same elements (since both \( x \) and \( z \) are multisets). In these instances, a naïve algorithm would repeat the same computation unnecessarily. Let \( x \) and \( z \) be the multisets \( x = \{x_1, x_2, \ldots, x_{m+n-h}\} \) and \( z = \{z_1, z_2, \ldots, z_{m+n-h}\} \). We define the condition:

\[
\text{Symmetry breaking condition 1: We say } \beta : x \rightarrow z \text{ is half-canonical if } \beta^{-1}(z_{i-1}) < \beta^{-1}(z_i) \text{ whenever } z_{i-1} = z_i.
\]

We need to add a multiplicative factor to adjust for the restriction to half-canonical bijections. Hence

\[
w_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \sum_{\substack{\beta : x \rightarrow z \\beta \text{ is a bijection} \\beta \text{ is half-canonical}}} \left( \prod_{i \geq 1} i^{s_i(z)} \right) \frac{m!}{\prod_{i \geq 1} i^{s_i(x)} s_i(x)!} \frac{n!}{\prod_{i \geq 1} i^{s_i(z)} s_i(z)!} \prod_{r \in x} w_{r,\beta(r),r+\beta(r)-1}
\]

and similarly with \( w' \) in place of \( w \). Using this assumption, we reduce the number of bijections by a factor of \( \prod_{i \geq 1} i^{s_i(z)} \), which results in a substantial time saving.

Instead of the single symmetry breaking assumption, it is possible to utilise symmetry breaking using an additional condition:

\[
\text{Symmetry breaking condition 2: We say } \beta : x \rightarrow z \text{ is canonical if it is half-canonical and } \beta(x_{i-1}) < \beta(x_i) \text{ whenever } x_i = x_{i-1}.
\]

Again, we find

\[
w_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \sum_{\substack{\beta : x \rightarrow z \\beta \text{ is a bijection} \\beta \text{ is canonical}}} \Gamma_{x,z,\beta} \left( \prod_{i \geq 1} i^{s_i(x)} s_i(x)! \right) \left( \prod_{i \geq 1} i^{s_i(z)} s_i(z)! \right) \prod_{r \in x} w_{r,\beta(r),r+\beta(r)-1},
\]

and similarly with \( w' \) in place of \( w \), where \( \Gamma_{x,z,\beta} \) is the number of bijections between the multisets \( x \) and \( z \) which map the same elements to the same elements (as \( \beta \)). A formula for \( \Gamma_{x,z,\beta} \) was given by e.g. \([3]\), namely

\[
\Gamma_{x,z,\beta} = \frac{\prod_{i \geq 1} s_i(x)! \prod_{i \geq 1} s_i(z)!}{\prod_{i \geq 1} \prod_{j \geq 1} s_{i,j}(x,\beta)!},
\]

where
The number of $h$-edge spanning forests in $K_{m,n}$

- $s_i(z)$ denotes the number of parts $i$ in $z$ (as before), and
- $s_{i,j}(x,\beta)$ is the number of elements $(i,j)$ in the multiset $\{(r,\beta(r)) : r \in x\}$.

The author has implemented both of these symmetry breaking schemes in order to compare their performance (see Section 5.3).

5.2 Pseudo-code

Algorithm 1 gives a pseudo-code version of the C code used to implement the algorithm described by Theorem 3 using the half-canonical symmetry breaking condition. The partitions of $m$ and $n$ into $k$ parts were computed whenever needed and stored in memory. Iterating through the half-canonical bijections was performed “on the fly” using a backtracking algorithm.

While Theorems 2 and 3 are valid for all $m,n,h \in \mathbb{N}$, we need to set $f_{m,n,0} = 1$ separately in the C code.

5.3 Performance

The C code was run on a $2 \times 2.66$ GHz processor (although the code itself is not parallelised). The following table gives the run-times (in seconds) for the two algorithms under the two symmetry breaking schemes when computing all non-zero $f_{m,n,h}$ with $m,n \leq 19$.

<table>
<thead>
<tr>
<th></th>
<th>half-canonical</th>
<th>canonical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 2</td>
<td>27.1</td>
<td>6.8</td>
</tr>
<tr>
<td>Theorem 3</td>
<td>4.1</td>
<td>4.5</td>
</tr>
</tbody>
</table>

The fastest version is Theorem 3 using half-canonical symmetry breaking. Under these conditions, the code had the following run-times to find all non-zero values of $f_{m,n,h}$ with $m,n \leq t$:

<table>
<thead>
<tr>
<th>$t$</th>
<th>time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>20.5</td>
</tr>
<tr>
<td>26</td>
<td>27.7</td>
</tr>
<tr>
<td>27</td>
<td>35.4</td>
</tr>
<tr>
<td>28</td>
<td>46.0</td>
</tr>
<tr>
<td>29</td>
<td>62.2</td>
</tr>
<tr>
<td>30</td>
<td>83.0</td>
</tr>
</tbody>
</table>

This table indicates the scalability of the program, which is not overwhelming for these values of $t$. The author also ran the program to compute $f_{m,n,h}$ with $m,n \leq 20$ where $t = 50$, which took under 2 days (to be precise, it took 1 day 15 hours and 17 minutes). The largest number encountered was $f_{50,50,101}$, which has 167 digits, and is equal to the number of spanning trees of $K_{50,50}$, which is $50^{50-1} \cdot 50^{50-1} = 50^{98}$.

5.4 Complexity

In the worst case, $x = z = \{1,2,3,\ldots,t\}$, where $\min(m,n) = 1 + 2 + \cdots + t = \frac{1}{2}t(t + 1)$ (so $t = O(\sqrt{\min(m,n)})$, in which case the program must iterate through all $t!$ bijections from $x$ to $z$. So, for a worst case analysis, we assume $m = n$ and $m + n - h = \lceil \sqrt{m} \rceil$. 


Algorithm 1 Implementation of Theorem 3 using half-canonical symmetry breaking

\( f_{m,n,m+n-1} := 0; \) for all \( m, n, h \)
\( w'_{m,n,m+n-1} := 0; \) for all \( m, n, h \)
\( w'_{0,0,0} := 1; \)

for \( m = 0, 1, \ldots, \text{MAX}(m) \) do
  for \( n = 0, 1, \ldots, \text{MAX}(n) \) do
    if \( m > n \) and \( m > 0 \) and \( n > 0 \) then // Other values of \( w'_{m,n,h} \) are 0
      for \( i = 0, 1, \ldots, n \) do
        // Using the Inclusion-Exclusion formula
        \( w'_{m,n,m+n-1} := w'_{m,n,m+n-1} + (-1)^i \binom{n}{i} (n - i)^{m-1}; \)
      end for
    end if
  end for
  for \( h = 0, 1, \ldots, m + n - 2 \) do
    \( w'_{m,n,h} := 0; \)
    Compute all partitions of \( m \) into \( m + n - h \) parts and store in memory.
    Compute all partitions of \( n \) into \( m + n - h \) parts and store in memory.
    for partitions \( x \) of \( m \) into \( m + n - h \) parts do
      for partitions \( z \) of \( n \) into \( m + n - h \) parts do
        for half-canonical bijections \( \beta : x \rightarrow z \) do // via backtracking algorithm
          \( \Gamma : = \prod_{i \geq 1} i!^{s_i(x)}; \)
          \( w'_{m,n,h} := w'_{m,n,h} + \Gamma \cdot \frac{m!}{\prod_{i \geq 1} i!^{s_i(x)}} \cdot \frac{n!}{\prod_{i \geq 1} i!^{s_i(z)}} \cdot \prod_{r \in x} \frac{1}{w'_{r,\beta(r),r+h-1}}; \)
        end for
      end for
    end for
  end for
end for
end for

\( f_{m,n,0} = 1; \)
for \( h = 1, 2, \ldots, m + n - 1 \) do
  for \( i = 0, 1, \ldots, m \) do
    for \( j = 0, 1, \ldots, n \) do
      for \( k = 0, 1, \ldots, \min(n - j, h) \) do
        \( f_{m,n,h} := f_{m,n,h} + \binom{m}{i} \binom{n}{j,k,n-j-k} m^k w'_{i,j,h-k}; \)
      end for
    end for
  end for
end for
for \( h = 0, 1, \ldots, m + n - 1 \) do
  Print \( f_{m,n,h}; \)
end for
end for
The number of $h$-edge spanning forests in $K_{m,n}$

Hence, given $x, z \in S_{m,m+n-h}$, there can be $O(\sqrt{m}!)$ canonical bijections from $x$ to $z$. Hardy and Ramanujan’s asymptotic formula for the number of partitions of $m$, namely

$$\frac{1}{4m\sqrt{3}}e^{\pi\sqrt{2m/3}},$$

gives a crude asymptotic upper bound on $|S_{m,\sqrt{m}}|^2$, specifically

$$|S_{m,\sqrt{m}}|^2 = O\left(\frac{e^{\text{const}m}}{m^2}\right)$$

as the number of ways of choosing $x$ and $z$. Hence, $[4]$ has

$$O\left(e^{\text{const}\sqrt{m}m^{1.5}}\right)$$

terms, by Stirling’s Approximation. In contrast, a backtracking algorithm would need to generate and check around $\left(\frac{m^2}{\sqrt{m}}\right) \leq \frac{1}{\sqrt{m}}! m^{2\sqrt{m}}$ graphs, which, when $m = n$ and $h = 2m - \sqrt{m}$, is

$$O(e^{\sqrt{m}m^{1.5}}+\text{const})$$

iterations, by Stirling’s Approximation. Of course, when implementing these algorithms, we use pruning whenever possible to reduce the search space, which makes a drastic difference not accounted for in these approximations.

5.4.1 When $h$ is fixed

We conclude this paper with the observation that, when $h$ is fixed, computing $f_{m,n,h}$ is asymptotically “easy”. The underlying reason is that, for sufficiently large $m$ or $n$, we must have isolated vertices in graphs in $B_{m,n,h}$. Thus, $[2]$ only contains a finite number of non-zero terms.

**Theorem 4** For fixed $h$, computing $f_{m,n,h}$ can be performed in time $O\left(\log(mn)\right)$.

**Proof:** If $i > h$ or $j > h$, then $w_{i,j,h} = 0$ (since any graph in $B_{i,j,h}$ must have an isolated vertex). Hence, $[2]$ is equivalent to

$$f_{m,n,h} = \sum_{i=0}^{h} \sum_{j=0}^{h} \binom{m}{i} \binom{n}{j} w_{i,j,h}.$$ 

For fixed $h$, there is a finite number of terms in this sum. Thus, for fixed $h$, we could write a program in which:

- we store a list of the pairs $(i, j)$ for which $w_{i,j,h}$ is non-zero, along with the value of $w_{i,j,h}$,
- we iterate through this list, computing $\binom{m}{i} \binom{n}{j} w_{i,j,h}$, and add it to a running total.

We can compute $\binom{m}{i} = \frac{1}{i} m(m-1) \cdots (m-i+1)$ using $O(h)$ multiplications (since $i \leq h$), each of which takes time $O(\log m)$, and one division. Hence, $\binom{n}{j}$ can be computed in time $O(\log m)$ time (since $h$ is fixed). Similarly $\binom{m}{i}$ can be computed in $O(\log n)$ time. We conclude that the whole summation can be performed in time $O\left(\log(mn)\right)$. \qed
Acknowledgements

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