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# Computing the number of $h$ -edge spanning forests in complete bipartite graphs

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Let  $f_{m,n,h}$  be the number of spanning forests with  $h$  edges in the complete bipartite graph  $K_{m,n}$ . Kirchhoff's Matrix Tree Theorem implies  $f_{m,n,m+n-1} = m^{n-1}n^{m-1}$  when  $m \geq 1$  and  $n \geq 1$ , since  $f_{m,n,m+n-1}$  is the number of spanning trees in  $K_{m,n}$ . In this paper, we give an algorithm for computing  $f_{m,n,h}$  for general  $m, n, h$ . We implement this algorithm and use it to compute all non-zero  $f_{m,n,h}$  when  $m \leq 50$  and  $n \leq 50$  in under 2 days.

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**Keywords:** complete bipartite graph, spanning forest

## 1 Introduction

A consequence of Kirchhoff's Matrix Tree Theorem is that the number of spanning trees in the complete bipartite graph  $K_{m,n}$  is  $m^{n-1}n^{m-1}$  when  $m \geq 1$  and  $n \geq 1$ . In this paper, we consider a generalisation of this counting problem: the number  $f_{m,n,h}$  of  $h$ -edge spanning forests in  $K_{m,n}$ . The number of spanning trees in  $K_{m,n}$  is  $f_{m,n,m+n-1}$  when  $m \geq 1$  and  $n \geq 1$ , since spanning trees of  $K_{m,n}$  have  $m + n - 1$  edges.

The numbers  $f_{m,n,h}$  arise in relation to the Tutte polynomial of  $K_{m,n}$ . For a general graph  $G = (V, E)$ , the Tutte polynomial is

$$T_G(x, y) = \sum_{A \subseteq E} (x - 1)^{k(A) - k(E)} (y - 1)^{k(A) + |A| - |V|}$$

where  $k(A)$  is the number of connected components in the graph  $(V, A)$ . Importantly, for the Tutte polynomial, we take  $0^0 = 1$ . Thus, the number of spanning forests of  $G$  is given by

$$\begin{aligned} T_G(2, 1) &= \sum_{A \subseteq E} 0^{k(A) + |A| - |V|} \\ &= \#\{A \subseteq E : k(A) + |A| = |V|\} \\ &= \text{number of spanning forests of } G, \end{aligned} \tag{1}$$

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since  $k(A) + |A| = |V|$  if and only if  $A$  is a spanning forest.

In the present paper, we are interested in the case of when  $G = K_{m,n}$ , and will look for  $h$ -edge spanning forests. In this situation,

$$T_{K_{m,n}}(2, 1) = \sum_{h \geq 0} f_{m,n,h}.$$

For general graphs  $G = (V, E)$ , Myrvold [10] gave an algorithm for computing the number of  $k$ -component spanning forests of  $G$ . Bjöklund et al. [4] described an algorithm that could compute  $T_G(2, 1)$  in time  $2^{|V|}|V|^{O(1)}$  by using Kirchhoff's Matrix Tree Theorem for each of the  $2^{|V|}$  subsets of  $V$ , then combining the results together with Inclusion-Exclusion. Other relevant work for the enumeration of spanning forests includes [9]. Porter [11] gave an algorithm for generating the spanning trees of  $K_{m,n}$ . Farr and McDiarmid [5] showed that computing the number of circuits in  $G$  is #P-complete<sup>(i)</sup>.

Bounds for the number of spanning forests in graphs were given by Teranishi [13], from which we can deduce

$$T_{K_{m,n}}(2, 1) \geq \sum_{k \geq 0} \left( \frac{\min(m, n)}{2} \right)^{m+n-k} \binom{m+n}{k}$$

for all  $m \geq 0$  and  $n \geq 0$ . Jin and Liu [8] gave a simple formula for the number of rooted spanning forests in  $K_{m,n}$  (see also [7] and [12]).

This paper instead heads in a different direction to previous work: we derive algorithms for enumerating  $h$ -edge spanning forests in  $K_{m,n}$ .

## 2 Basic results

To be clear, all forests in this paper will be labelled. Let  $\mathbb{N} = \{0, 1, 2, \dots\}$ . The following lemma gives the boundary conditions on  $f_{m,n,h}$ .

**Lemma 1** *Suppose  $m, n, h \in \mathbb{N}$ . Then*

- $f_{m,n,0} = 1$ ,
- if  $h \geq 1$ , then  $f_{0,n,h} = f_{m,0,h} = 0$ ,
- if  $m \geq 1$  and  $n \geq 1$ , then  $f_{m,n,h} > 0$  if and only if  $0 \leq h \leq m + n - 1$ ,
- if  $m \geq 1$  and  $n \geq 1$ , then  $f_{m,n,m+n-1} = m^{n-1}n^{m-1}$ .

**Proof:** If  $h = 0$ , then there is exactly one subgraph of  $K_{m,n}$  with no edges (and it is a spanning forest), so  $f_{m,n,0} = 1$ . If  $m = 0$  or  $n = 0$ , then  $K_{m,n}$  has no edges, and thus we obtain the second bulleted item.

Now assume  $m \geq 1$  and  $n \geq 1$ . Since  $K_{m,n}$  is connected, it has a spanning tree, which must have exactly  $m + n - 1$  edges. By deleting edges from this spanning tree, we find  $h$ -edge forests in  $K_{m,n}$  for all  $0 \leq h \leq m + n - 1$ . Hence  $f_{m,n,h} > 0$  when  $0 \leq h \leq m + n - 1$ .

Now assume  $A$  is an  $h$ -subset of the edges in  $K_{m,n}$ , where  $m \geq 1$  and  $n \geq 1$  and  $h \geq m + n$ . To be a spanning forest, we need  $k(A) + h = m + n$  as per (1), but if  $h \geq m + n$ , then  $k(A) \leq 0$ , giving a contradiction.

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<sup>(i)</sup> #P is the set of counting problems which ask for the number of "yes" instances for decision problems in NP. A #P problem is in #P-complete whenever any other problem in #P can be reduced to it by a polynomial-time counting reduction.

$m$	$n$	$h = 0$	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1										
1	2	1	2	1									
1	3	1	3	3	1								
1	4	1	4	6	4	1							
1	5	1	5	10	10	5	1						
1	6	1	6	15	20	15	6	1					
2	2	1	4	6	4								
2	3	1	6	15	20	12							
2	4	1	8	28	56	64	32						
2	5	1	10	45	120	200	192	80					
2	6	1	12	66	220	480	672	544	192				
3	3	1	9	36	84	117	81						
3	4	1	12	66	220	477	648	432					
3	5	1	15	105	455	1335	2673	3375	2025				
3	6	1	18	153	816	3015	7938	14499	16524	8748			
4	4	1	16	120	560	1784	3936	5632	4096				
4	5	1	20	190	1140	4785	14544	31520	44800	32000			
4	6	1	24	276	2024	10536	40704	117376	244224	331776	221184		
5	5	1	25	300	2300	12550	51030	155900	347500	515625	390625		
5	6	1	30	435	4060	27255	138606	544525	1641000	3645000	5400000	4050000	
6	6	1	36	630	7140	58680	369792	1834992	7210080	22083840	50388480	77262336	60466176

Fig. 1: Small non-zero values of  $f_{m,n,h}$ .

The fourth bulleted item is from Kirchhoff’s Matrix Tree Theorem (already mentioned). □

The following lemma is an important (but basic) consequence of the symmetry of the problem.

**Lemma 2** For all  $m, n, h \in \mathbb{N}$ , we have  $f_{m,n,h} = f_{n,m,h}$ .

In Figure 1 we list the non-zero values of  $f_{m,n,h}$  when  $0 \leq m \leq n \leq 6$ . These values were generated using a straightforward backtracking algorithm.

### 3 Combinatorial equivalence

In this section, we will describe some combinatorial objects that are equivalent to  $h$ -edge spanning forests of  $K_{m,n}$ .

Let  $M$  be an  $m \times n$   $(0, 1)$ -matrix. We define a *cycle* in  $M$  to be a set  $\{e_1, e_2, \dots, e_{2t}\}$  of entries of  $M$ , such that:

- each  $e_i$  contains the symbol 1, and
- for  $i \in \{1, 2, \dots, t\}$ , we have that (a)  $e_{2i-1}$  belongs to the same row as  $e_{2i}$  and (b)  $e_{2i}$  belongs to the same column as  $e_{2i+1}$  (where we take  $e_{2t+1} = e_1$ ).

For example, the matrix

0	Ⓛ	0	1	Ⓛ
0	0	1	0	1
Ⓛ	0	0	0	Ⓛ
Ⓛ	Ⓛ	0	0	0

has a cycle (consisting of the circled 1's), whereas the matrices

$$\begin{array}{|c|c|c|} \hline 0 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

do not have cycles.

Let  $A_{m,n,h}$  be the set of  $(0,1)$ -matrices with exactly  $h$  elements equal to 1. Let  $p_{m,n,h}$  be the probability that an element of  $A_{m,n,h}$  chosen uniformly at random contains a cycle. An anonymous user of [math.stackexchange.com](http://math.stackexchange.com) asked<sup>(ii)</sup> for a formula for  $p_{m,n,h}$ . In fact, this was the original motivation for the author to study this problem.

**Lemma 3** For all  $m, n, h \in \mathbb{N}$ , we have  $f_{m,n,h}$  is the number of  $m \times n$   $(0,1)$ -matrices with  $h$  1's that do not contain a cycle, and hence

$$f_{m,n,h} = (1 - p_{m,n,h}) \binom{mn}{h}.$$

**Proof:** A matrix  $M \in A_{m,n,h}$  can be interpreted as the biadjacency matrix<sup>(iii)</sup> of an  $h$ -edge subgraph  $G = G(M)$  of  $K_{m,n}$ . A cycle in  $M \in A_{m,n,h}$  corresponds to a cycle in  $G$ . Hence

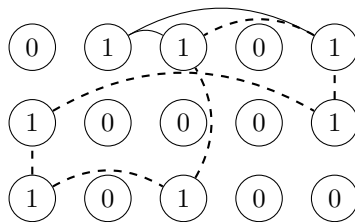
$$\begin{aligned} f_{m,n,h} &= (1 - p_{m,n,h}) |A_{m,n,h}| \\ &= (1 - p_{m,n,h}) \binom{mn}{h}. \end{aligned}$$

□

A problem related to the  $(0,1)$ -matrix problem above comes from the study of  $(0,1)$ -matrices that do not have a submatrix which is the incidence matrix of any cycle of length at least 3; these are called “totally balanced matrices” (see e.g. [2]).

Another interpretation of matrices in  $A_{m,n,h}$  is as induced subgraphs of  $K_m \square K_n$ , where  $\square$  represents the Cartesian product of graphs. The graph  $K_m \square K_n$  is sometimes called the “rook’s graph”, since the edges represent the legal moves of a rook on an  $m \times n$  chess board. The vertices in  $K_m \square K_n$  are  $\{(i, j) : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$  and there is an edge between distinct vertices  $(i, j)$  and  $(i', j')$  whenever  $i = i'$  or  $j = j'$ . The graph  $K_m \square K_n$  is also the line graph of  $K_{m,n}$ .

There is a bijection between induced subgraphs  $H = H(M)$  of  $K_m \square K_n$  and  $(0,1)$ -matrices  $M = (m_{ij}) \in A_{m,n,h}$ : we include the vertex  $(i, j)$  in  $H$  if and only if  $m_{ij} = 1$ . For example, if we ignore the vertices marked 0 in



<sup>(ii)</sup> Full URL: <http://math.stackexchange.com/q/24800>

<sup>(iii)</sup> The biadjacency matrix of a bipartite graph  $G$  on the vertex set  $\{u_i\}_{1 \leq i \leq m} \cup \{v_j\}_{1 \leq j \leq n}$  is the  $m \times n$   $(0,1)$ -matrix  $M = (m_{ij})$  with  $m_{ij} = 1$  if and only if  $u_i v_j$  is an edge in  $G$ .

we obtain an induced subgraph  $H(M)$  of  $K_3 \square K_5$ , corresponding to the matrix

$$M = \begin{array}{|ccccc|} \hline 0 & 1 & \textcircled{1} & 0 & \textcircled{1} \\ \hline \textcircled{1} & 0 & 0 & 0 & \textcircled{1} \\ \hline \textcircled{1} & 0 & \textcircled{1} & 0 & 0 \\ \hline \end{array}$$

in  $A_{3,5,6}$ . We see that cycles in  $M$  map from cycles  $C_k$  in  $H$  for some  $k \in \{4, 6, 8, \dots\}$  (an example of a 6-cycle is highlighted). There are other cycles in  $H$  (e.g.  $C_3$  in the above example; in fact, if  $M$  above had a row of 1's, then  $H$  would have a  $K_5$  subgraph). However, induced cycles  $C_k$  in  $H(M)$  of length  $k \in \{4, 6, 8, \dots\}$  are in one-to-one correspondence with cycles in  $M$  via the above bijection.

**Lemma 4** For all  $m, n, h \in \mathbb{N}$ , we have that  $f_{m,n,h}$  is the number of  $h$ -vertex induced subgraphs of  $K_m \square K_n$  that do not contain an induced  $C_k$  for any  $k \in \{4, 6, 8, \dots\}$ .

## 4 Simplifying the equation

### 4.1 A formula for $f_{m,n,h}$

We will look at two related ways of simplifying  $f_{m,n,h}$ . Let  $K_{m,n}$  have the vertex bipartition  $M \cup N$ , where  $M = \{u_1, u_2, \dots, u_m\}$  and  $N = \{v_1, v_2, \dots, v_n\}$ . Let  $B_{m,n,h}$  be the set of  $h$ -edge spanning forests of  $K_{m,n}$ . Hence  $f_{m,n,h} = |B_{m,n,h}|$ . Define

$$W_{m,n,h} = \{G \in B_{m,n,h} : G \text{ has no isolated vertices}\}$$

and  $w_{m,n,h} = |W_{m,n,h}|$ . We observe the following:

- Given any spanning forest in  $B_{m,n,h}$  we can delete the isolated vertices to obtain a spanning forest in  $W_{i,j,h}$  for some  $i \leq m$  and  $j \leq n$ .
- Conversely, given any spanning forest in  $W_{i,j,h}$ , we can add isolated vertices to it in  $\binom{m}{m-i} \binom{n}{n-j}$  ways to obtain a spanning forest in  $B_{m,n,h}$ .

In both of the operations above, we need to relabel the vertices, but we preserve the order of the indices of the non-isolated vertices within  $M$  and  $N$ .

Hence

$$f_{m,n,h} = \sum_{i=0}^m \sum_{j=0}^n \binom{m}{i} \binom{n}{j} w_{i,j,h}. \tag{2}$$

This formula eliminates the need to further account for isolated vertices.

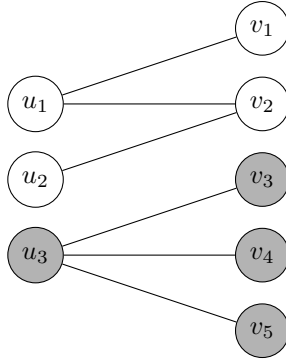
We can decompose any  $G \in W_{m,n,h}$  into disjoint components, each of which belong to  $W_{i,j,k}$  for some  $i \leq m$  and  $j \leq n$  and  $k \leq h$ . Hence any  $G \in W_{m,n,h}$  decomposes into the following.

- A partition  $P$  of  $M$  where  $x, y \in M$  belong to the same part if there is a path from  $x$  to  $y$  in  $G$ .
- A partition  $Q$  of  $N$  where  $x', y' \in N$  belong to the same part if there is a path from  $x'$  to  $y'$  in  $G$ .
- A bijection  $\alpha : P \rightarrow Q$  such that every edge  $ab$  in  $G$  has  $a \in p$  and  $b \in \alpha(p)$  for some  $p \in P$ . Hence we have  $|P| = |Q|$ .

- For each  $p \in P$ , a subgraph in  $W_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$  induced by the vertices in  $p \cup \alpha(p)$ .

Regarding the final bulleted item, we observe that the subgraph induced by  $p \cup \alpha(p)$  for some  $p \in P$  must be a spanning tree on  $K_{|p|,|\alpha(p)|}$ . Any more edges would cause a cycle, while fewer edges would cause disjoint components (in which case  $G$  could be further decomposed).

We give an example of a graph  $G$  in  $W_{3,5,6}$  below.



Here, the partitions  $P$  and  $Q$  are given by  $P = \{\{u_1, u_2\}, \{u_3\}\}$  and  $Q = \{\{v_1, v_2\}, \{v_3, v_4, v_5\}\}$  and we have the bijection  $\alpha$  such that

$$\begin{aligned} \{u_1, u_2\} &\mapsto \{v_1, v_2\} \\ \{u_3\} &\mapsto \{v_3, v_4, v_5\}. \end{aligned}$$

We see that  $G$  decomposes into a graph in  $W_{2,2,3}$  and a graph in  $W_{1,3,3}$ .

In general, for any  $p \in P$ , we must have exactly  $|p| + |\alpha(p)| - 1$  edges in  $p \cup \alpha(p)$ . Summing this over all  $p \in P$  gives  $h = m + n - |P|$ . We conclude that

$$|P| = m + n - h$$

(and thus  $|Q| = m + n - h$ ).

We will now describe how to construct any graph in  $W_{m,n,h}$  via its decomposition. Let  $\mathcal{P}$  be the set of partitions of  $M$  and let  $\mathcal{Q}$  be the set of partitions of  $N$ . Given (a) a partition  $P \in \mathcal{P}$  of size  $|P| = m + n - h$ , (b) a partition  $Q \in \mathcal{Q}$  and (c) a bijection  $\alpha : P \rightarrow Q$ , we can construct  $\prod_{p \in P} w_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$  distinct graphs  $G \in W_{m,n,h}$  by the following steps:

1. Start with  $G$  as the graph with vertex set  $M \cup N$  and no edges.
2. For each  $p \in P$  add one of the subgraphs in  $W_{|p|,|\alpha(p)|,|p|+|\alpha(p)|-1}$  on the vertices  $p \cup \alpha(p)$ .

Since all graphs in  $W_{m,n,h}$  can be constructed uniquely by the above steps we have the following theorem.

**Theorem 1** For  $m, n, h \in \mathbb{N}$ ,

$$f_{m,n,h} = \sum_{i=0}^m \sum_{j=0}^n \binom{m}{i} \binom{n}{j} w_{i,j,h}$$

where

$$w_{m,n,h} = \sum_{\substack{P \in \mathcal{P} \\ |P|=m+n-h}} \sum_{\substack{Q \in \mathcal{Q} \\ |Q|=m+n-h}} \sum_{\substack{\alpha: P \rightarrow Q \\ \alpha \text{ is a bijection}}} \prod_{p \in P} w_{|p|, |\alpha(p)|, |p|+|\alpha(p)|-1} \quad (3)$$

where

$$w_{|p|, |\alpha(p)|, |p|+|\alpha(p)|-1} = |p|^{|\alpha(p)|-1} |\alpha(p)|^{|p|-1}.$$

#### 4.2 An improved formula for $f_{m,n,h}$

We will derive two further equations for  $f_{m,n,h}$  (in Theorems 2 and 3), which are related to Theorem 1. They result in a slightly more complicated algorithm, but will allow us to compute  $f_{m,n,h}$  faster than (2).

For all  $m, n, h \in \mathbb{N}$ , define

$$W'_{m,n,h} = \{G \in W_{m,n,h} : \text{the vertices in } N \text{ have degree } \geq 2\}.$$

Given any  $G \in B_{m,n,h}$  we can delete the isolated vertices and the leaves (vertices of degree 1) in  $N$  to obtain a spanning forest in  $W'_{i,j,h-k}$  for some  $i \leq m$  and  $j \leq n$  and  $k \in \mathbb{N}$ . Conversely, given any  $G \in W'_{i,j,h-k}$ , we can add:

- $m - i$  isolated vertices to  $M$  in  $\binom{m}{m-i}$  ways, so as to increase  $|M|$  to  $m$ , then
- $n - j$  isolated vertices to  $N$  in  $\binom{n}{n-j}$  ways, so as to increase  $|N|$  to  $n$ , then
- $k$  edges to  $G$  in  $\binom{n-j}{k} m^k$  ways, so as to increase the number of leaves in  $N$  by  $k$ ,

thereby obtaining a spanning forest in  $B_{m,n,h}$ . Similar to the derivation of (2), we need to relabel the vertices so as to preserve the order of the indices of the non-isolated vertices within  $M$  and the non-isolated non-leaf vertices within  $N$ . Hence

$$f_{m,n,h} = \sum_{i=0}^m \sum_{j=0}^n \sum_{k=0}^{\min(n-j,h)} \binom{m}{i} \binom{n}{j, k, n-j-k} m^k w'_{i,j,h-k}.$$

All of the steps involved for finding the formula (3) for  $w_{m,n,h}$  are still valid for  $w'_{m,n,h}$ . Hence (3) remains true if we replace  $w$  with  $w'$ . However, we can no longer make use of the simple formula for  $w_{|p|, |\alpha(p)|, |p|+|\alpha(p)|-1}$ . Nevertheless, we will be able to find a formula for  $w'_{|p|, |\alpha(p)|, |p|+|\alpha(p)|-1}$  via Inclusion-Exclusion.

For  $I \subseteq N$ , define

$$A_I = \{G \in W_{m,n,m+n-1} : \text{vertices in } I \text{ are leaves}\}.$$

Note that spanning forests in  $A_I$  might also have leaves outside of  $I$ . Hence  $|A_\emptyset| = |W_{m,n,m+n-1}| = m^{n-1} n^{m-1}$ . By symmetry, if  $I, J \subseteq N$  and  $|I| = |J|$  then  $|A_I| = |A_J|$ . So we will assume  $I = \{n - i + 1, n - i + 2, \dots, n\}$ . We can construct the graphs in  $A_I$  by adding  $i$  leaves to the graphs in  $W_{m,n-|I|, m+n-|I|-1}$  (i.e., the set of spanning trees of  $K_{m,n-|I|}$ ), and these leaves can be added in  $m^i$  ways. Hence

$$\begin{aligned} |A_I| &= m^i \cdot \text{number of spanning trees of } K_{m,n-i} \\ &= m^i m^{n-i-1} (n-i)^{m-1} \\ &= m^{n-1} (n-i)^{m-1} \end{aligned}$$



when  $m \geq 1$  and  $n \geq 1$ . Here we take  $0^0 = 1$ , since we want to account for  $K_{1,0} \in W_{1,0,0}$ . Hence, by Inclusion-Exclusion, we find

$$\begin{aligned} w'_{m,n,m+n-1} &= |A_\emptyset| - \left| \bigcup_{\substack{I \subseteq N \\ I \neq \emptyset}} A_I \right| \\ &= m^{n-1} n^{m-1} - \sum_{i=1}^n (-1)^{i-1} \binom{n}{i} m^{n-1} (n-i)^{m-1} \\ &= m^{n-1} \sum_{i=0}^n (-1)^i \binom{n}{i} (n-i)^{m-1} \end{aligned}$$

when  $m \geq 1$  and  $n \geq 1$ . By definition, we also have  $w'_{m,n,m+n-1} = 0$  when either  $m = 0$  or  $n = 0$  (and  $w'_{m,n,m+n-1}$  is not defined when  $m = 0$  and  $n = 0$ ).

Before identifying formulae for  $w_{m,n,h}$  and  $w'_{m,n,h}$  in the next section, we make the following observation.

**Lemma 5** *For all  $m, n, h \in \mathbb{N}$ , we have  $n!$  divides  $w'_{m,n,h}$ .*

**Proof:** The symmetric group on  $N$  acts on  $W'_{m,n,h}$  by permuting the vertices in  $N$ . Suppose a graph  $G \in W'_{m,n,h}$  has a non-trivial stabiliser subgroup under this action. Then there exists a non-identity permutation  $\alpha$  such that  $\alpha G = G$ , and distinct vertices  $v, v' \in N$  for which  $\alpha(v) = v'$ . Since  $G \in W'_{m,n,h}$ , the degree of  $v$  is 2 or more, so assume  $v$  has distinct neighbours  $a, b \in M$ . Then, since  $\alpha G = G$ , we find that  $v' = \alpha(v)$  has the neighbours  $a$  and  $b$  too. Thus,  $\{v, v', a, b\}$  induces a 4-cycle, giving a contradiction. Hence, all stabiliser subgroups are trivial, and by the Orbit-Stabiliser Theorem, all orbits have size  $n!$ . Hence  $n!$  divides  $w'_{m,n,h}$ .  $\square$

### 4.3 Formulae for $w_{m,n,h}$ and $w'_{m,n,h}$

Now we will simplify (3), noting that our simplifications remain valid when we replace  $w$  with  $w'$ .

For  $a \geq 1$  and  $t \geq 1$ , let  $S_{a,t}$  be the set of (number) partitions of  $a$  into  $t$  non-zero parts. We will interpret elements of  $S_{a,t}$  as multisets; for example  $\{4, 3, 3, 1, 1\} \in S_{12,5}$ . We will require  $S_{a,t} = \emptyset$  when  $t < 0$ , and  $S_{0,0} = \{\emptyset\}$  and  $S_{0,t} = \emptyset$  when  $t > 0$ . For  $z \in S_{a,t}$ , let  $T[z]$  be the set of partitions of  $\{1, 2, \dots, a\}$  whose part sizes induce the number partition  $z$ . For  $z \in S_{a,t}$ , let  $\hat{z}$  denote an arbitrary element of  $T[z]$ . If  $P \in T[x]$  and  $Q \in T[z]$ , then any bijection  $\alpha : P \rightarrow Q$  induces a bijection  $\beta : x \rightarrow z$ .

Hence

$$\begin{aligned}
 w_{m,n,h} &= \sum_{x,z \in S_{m,m+n-h}} \sum_{P \in T[x]} \sum_{Q \in T[z]} \sum_{\substack{\alpha: P \rightarrow Q \\ \alpha \text{ is a bijection}}} \prod_{p \in P} w_{|p|, |\alpha(p)|, |p| + |\alpha(p)| - 1} \\
 &= \sum_{x,z \in S_{m,m+n-h}} \sum_{P \in T[x]} \sum_{Q \in T[z]} \sum_{\substack{\alpha: \hat{x} \rightarrow \hat{z} \\ \alpha \text{ is a bijection}}} \prod_{p \in \hat{x}} w_{|p|, |\alpha(p)|, |p| + |\alpha(p)| - 1} \\
 &= \sum_{x,z \in S_{m,m+n-h}} |T[x]| \cdot |T[z]| \sum_{\substack{\alpha: \hat{x} \rightarrow \hat{z} \\ \alpha \text{ is a bijection}}} \prod_{p \in \hat{x}} w_{|p|, |\alpha(p)|, |p| + |\alpha(p)| - 1} \\
 &= \sum_{x,z \in S_{m,m+n-h}} |T[x]| \cdot |T[z]| \sum_{\substack{\beta: x \rightarrow z \\ \beta \text{ is a bijection}}} \prod_{r \in x} w_{r, \beta(r), r + \beta(r) - 1}.
 \end{aligned}$$

If  $z \in S_{a,t}$ , then

$$|T[z]| = \frac{a!}{\prod_{i \geq 1} i! s_i(z) s_i(z)!}$$

where  $s_i(z)$  denotes the number of parts  $i$  in  $z$  (see e.g. [1, Theorem 13.2]). Hence we have the following theorems.

**Theorem 2** For  $m, n, h \in \mathbb{N}$ ,

$$f_{m,n,h} = \sum_{i=0}^m \sum_{j=0}^n \binom{m}{i} \binom{n}{j} w_{i,j,h}$$

where

$$w_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \sum_{\substack{\beta: x \rightarrow z \\ \beta \text{ is a bijection}}} \frac{m!}{\prod_{i \geq 1} i! s_i(x) s_i(x)!} \frac{n!}{\prod_{i \geq 1} i! s_i(z) s_i(z)!} \prod_{r \in x} r^{\beta(r)-1} \beta(r)^{r-1}. \quad (4)$$

**Theorem 3** For  $m, n, h \in \mathbb{N}$ ,

$$f_{m,n,h} = \sum_{i=0}^m \sum_{j=0}^n \sum_{k=0}^{\min(n-j,h)} \binom{m}{i} \binom{n}{j, k, n-j-k} m^k w'_{i,j,h-k}.$$

where

$$w'_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \sum_{\substack{\beta: x \rightarrow z \\ \beta \text{ is a bijection}}} \frac{m!}{\prod_{i \geq 1} i! s_i(x) s_i(x)!} \frac{n!}{\prod_{i \geq 1} i! s_i(z) s_i(z)!} \prod_{r \in x} w'_{r, \beta(r), r + \beta(r) - 1}$$

where

$$w'_{m,n,m+n-1} = m^{n-1} \sum_{i=0}^n (-1)^i \binom{n}{i} (n-i)^{m-1}.$$

The main advantage of Theorems 2 and 3 over Theorem 1 is the summation is over number partitions (rather than set partitions).

## 5 Implementation

The author has implemented the two formulae for  $f_{m,n,h}$  described in the preceding section, along with a simple backtracking algorithm. The results between all three implementations concur, giving confidence in the accuracy of the code. The C source code is available as supplementary material to this document. The GMP library was used for arbitrary precision arithmetic [6].

The formula involving  $w'_{m,n,h}$  instead of  $w_{m,n,h}$  was (unsurprisingly) much faster, largely because many values of  $w'_{m,n,h}$  equal 0, such as when  $m \leq n$ , which can be used to drastically reduce the search tree.

### 5.1 Symmetry breaking

The run-time of the program was also improved through the use of symmetry breaking, which we will now describe. For  $x \in S_{m,m+n-h}$  and  $z \in S_{n,m+n-h}$ , there are often many bijections  $\beta : x \rightarrow z$  which map the same elements to the same elements (since both  $x$  and  $z$  are multisets). In these instances, a naïve algorithm would repeat the same computation unnecessarily. Let  $x$  and  $z$  be the multisets  $x = \{x_1, x_2, \dots, x_{m+n-h}\}$  and  $z = \{z_1, z_2, \dots, z_{m+n-h}\}$ . We define the condition:

*Symmetry breaking condition 1:* We say  $\beta : x \rightarrow z$  is *half-canonical* if  $\beta^{-1}(z_{i-1}) < \beta^{-1}(z_i)$  whenever  $z_{i-1} = z_i$ .

We need to add a multiplicative factor to adjust for the restriction to half-canonical bijections. Hence

$$w_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \sum_{\substack{\beta: x \rightarrow z \\ \beta \text{ is a bijection} \\ \beta \text{ is half-canonical}} \left( \prod_{i \geq 1} i^{s_i(z)} \right) \frac{m!}{\prod_{i \geq 1} i^{s_i(x)} s_i(x)!} \frac{n!}{\prod_{i \geq 1} i^{s_i(z)} s_i(z)!} \prod_{r \in x} w_{r, \beta(r), r + \beta(r) - 1}$$

and similarly with  $w'$  in place of  $w$ . Using this assumption, we reduce the number of bijections by a factor of  $\prod_{i \geq 1} i^{s_i(z)}$ , which results in a substantial time saving.

Instead of the single symmetry breaking assumption, it is possible to utilise symmetry breaking using an additional condition:

*Symmetry breaking condition 2:* We say  $\beta : x \rightarrow z$  is *canonical* if it is half-canonical and  $\beta(x_{i-1}) < \beta(x_i)$  whenever  $x_i = x_{i-1}$ .

Again, we find

$$w_{m,n,h} = \sum_{x,z \in S_{m,m+n-h}} \sum_{\substack{\beta: x \rightarrow z \\ \beta \text{ is a bijection} \\ \beta \text{ is canonical}}} \Gamma_{x,z,\beta} \frac{m!}{\prod_{i \geq 1} i^{s_i(x)} s_i(x)!} \frac{n!}{\prod_{i \geq 1} i^{s_i(z)} s_i(z)!} \prod_{r \in x} w_{r, \beta(r), r + \beta(r) - 1},$$

and similarly with  $w'$  in place of  $w$ , where  $\Gamma_{x,z,\beta}$  is the number of bijections between the multisets  $x$  and  $z$  which map the same elements to the same elements (as  $\beta$ ). A formula for  $\Gamma_{x,z,\beta}$  was given by e.g. [3], namely

$$\Gamma_{x,z,\beta} = \frac{\prod_{i \geq 1} s_i(x)! \prod_{i \geq 1} s_i(z)!}{\prod_{i \geq 1} \prod_{j \geq 1} s_{i,j}(x, \beta)!}$$

where

- $s_i(z)$  denotes the number of parts  $i$  in  $z$  (as before), and
- $s_{i,j}(x, \beta)$  is the number of elements  $(i, j)$  in the multiset  $\{(r, \beta(r)) : r \in x\}$ .

The author has implemented both of these symmetry breaking schemes in order to compare their performance (see Section 5.3).

### 5.2 Pseudo-code

Algorithm 1 gives a pseudo-code version of the C code used to implement the algorithm described by Theorem 3 using the half-canonical symmetry breaking condition. The partitions of  $m$  and  $n$  into  $k$  parts were computed whenever needed and stored in memory. Iterating through the half-canonical bijections was performed “on the fly” using a backtracking algorithm.

While Theorems 2 and 3 are valid for all  $m, n, h \in \mathbb{N}$ , we need to set  $f_{m,n,0} = 1$  separately in the C code.

### 5.3 Performance

The C code was run on a  $2 \times 2.66$  GHz processor (although the code itself is not parallelised). The following table gives the run-times (in seconds) for the two algorithms under the two symmetry breaking schemes when computing all non-zero  $f_{m,n,h}$  with  $m, n \leq 19$ .

	half-canonical	canonical
Theorem 2	27.1	6.8
Theorem 3	4.1	4.5

The fastest version is Theorem 3 using half-canonical symmetry breaking. Under these conditions, the code had the following run-times to find all non-zero values of  $f_{m,n,h}$  with  $m, n \leq t$ :

$t$	time (sec)
25	20.5
26	27.7
27	35.4
28	46.0
29	62.2
30	83.0

This table indicates the scalability of the program, which is not overwhelming for these values of  $t$ . The author also ran the program to compute  $f_{m,n,h}$  with  $m, n \leq t$  where  $t = 50$ , which took under 2 days (to be precise, it took 1 day 15 hours and 17 minutes). The largest number encountered was  $f_{50,50,101}$ , which has 167 digits, and is equal to the number of spanning trees of  $K_{50,50}$ , which is  $50^{50-1} \cdot 50^{50-1} = 50^{98}$ .

### 5.4 Complexity

In the worst case,  $x = z = \{1, 2, 3, \dots, t\}$ , where  $\min(m, n) = 1 + 2 + \dots + t = \frac{1}{2}t(t + 1)$  (so  $t = O(\sqrt{\min(m, n)})$ ), in which case the program must iterate through all  $t!$  bijections from  $x$  to  $z$ . So, for a worst case analysis, we assume  $m = n$  and  $m + n - h = \lfloor \sqrt{m} \rfloor$ .

**Algorithm 1** Implementation of Theorem 3 using half-canonical symmetry breaking

---

```

 $f_{m,n,m+n-1} := 0$ ; for all  $m, n, h$ 
 $w'_{m,n,m+n-1} := 0$ ; for all  $m, n, h$ 
 $w'_{0,0,0} := 1$ ;
for  $m = 0, 1, \dots, MAX(m)$  do
  for  $n = 0, 1, \dots, MAX(n)$  do
    if  $m > n$  and  $m > 0$  and  $n > 0$  then // Other values of  $w'_{m,n,h}$  are 0
      for  $i = 0, 1, \dots, n$  do
        // Using the Inclusion-Exclusion formula
         $w'_{m,n,m+n-1} := w'_{m,n,m+n-1} + (-1)^i \binom{n}{i} (n-i)^{m-1}$ ;
      end for
      for  $h = 0, 1, \dots, m+n-2$  do
         $w'_{m,n,h} := 0$ ;
        Compute all partitions of  $m$  into  $m+n-h$  parts and store in memory.
        Compute all partitions of  $n$  into  $m+n-h$  parts and store in memory.
        for partitions  $x$  of  $m$  into  $m+n-h$  parts do
          for partitions  $z$  of  $n$  into  $m+n-h$  parts do
            for half-canonical bijections  $\beta : x \rightarrow z$  do // via backtracking algorithm
               $\Gamma := \prod_{i \geq 1} i^{s_i(z)}$ ;
               $w'_{m,n,h} := w'_{m,n,h} + \Gamma \cdot \frac{m!}{\prod_{i \geq 1} i^{s_i(x)} s_i(x)!} \frac{n!}{\prod_{i \geq 1} i^{s_i(z)} s_i(z)!} \prod_{r \in x} w'_{r, \beta(r), r + \beta(r) - 1}$ ;
            end for
          end for
        end for
      end for
    end if
     $f_{m,n,0} = 1$ ;
    for  $h = 1, 2, \dots, m+n-1$  do
      for  $i = 0, 1, \dots, m$  do
        for  $j = 0, 1, \dots, n$  do
          for  $k = 0, 1, \dots, \min(n-j, h)$  do
             $f_{m,n,h} := f_{m,n,h} + \binom{m}{i} \binom{n}{j, k, n-j-k} m^k w'_{i,j,h-k}$ ;
          end for
        end for
      end for
    end for
  end for
end for

```

---

Hence, given  $x, z \in S_{m,m+n-h}$ , there can be  $O(\lfloor \sqrt{m} \rfloor!)$  canonical bijections from  $x$  to  $z$ . Hardy and Ramanujan's asymptotic formula for the number of partitions of  $m$ , namely

$$\frac{1}{4m\sqrt{3}} e^{\pi\sqrt{2m/3}},$$

gives a crude asymptotic upper bound on  $|S_{m,\lfloor \sqrt{m} \rfloor}|^2$ , specifically

$$|S_{m,\lfloor \sqrt{m} \rfloor}|^2 = O\left(\frac{e^{\text{const}\cdot m}}{m^2}\right)$$

as the number of ways of choosing  $x$  and  $z$ . Hence, (4) has

$$O\left(e^{\text{const}\cdot\sqrt{m}} m^{\sqrt{m}/2+\text{const}}\right)$$

terms, by Stirling's Approximation. In contrast, a backtracking algorithm would need to generate and check around  $\binom{m^2}{\lfloor \sqrt{m} \rfloor} \leq \frac{1}{\lfloor \sqrt{m} \rfloor!} m^{2\lfloor \sqrt{m} \rfloor}$  graphs, which, when  $m = n$  and  $h = 2m - \sqrt{m}$ , is

$$O(e^{\sqrt{m}} m^{1.5\sqrt{m}+\text{const}})$$

iterations, by Stirling's Approximation. Of course, when implementing these algorithms, we use pruning whenever possible to reduce the search space, which makes a drastic difference not accounted for in these approximations.

#### 5.4.1 When $h$ is fixed

We conclude this paper with the observation that, when  $h$  is fixed, computing  $f_{m,n,h}$  is asymptotically "easy". The underlying reason is that, for sufficiently large  $m$  or  $n$ , we must have isolated vertices in graphs in  $B_{m,n,h}$ . Thus, (2) only contains a finite number of non-zero terms.

**Theorem 4** For fixed  $h$ , computing  $f_{m,n,h}$  can be performed in time  $O(\log(mn))$ .

**Proof:** If  $i > h$  or  $j > h$ , then  $w_{i,j,h} = 0$  (since any graph in  $B_{i,j,h}$  must have an isolated vertex). Hence, (2) is equivalent to

$$f_{m,n,h} = \sum_{i=0}^h \sum_{j=0}^h \binom{m}{i} \binom{n}{j} w_{i,j,h}.$$

For fixed  $h$ , there is a finite number of terms in this sum. Thus, for fixed  $h$ , we could write a program in which:

- we store a list of the pairs  $(i, j)$  for which  $w_{i,j,h}$  is non-zero, along with the value of  $w_{i,j,h}$ ,
- we iterate through this list, computing  $\binom{m}{i} \binom{n}{j} w_{i,j,h}$ , and add it to a running total.

We can compute  $\binom{m}{i} = \frac{1}{i!} m(m-1)\cdots(m-i+1)$  using  $O(h)$  multiplications (since  $i \leq h$ ), each of which takes time  $O(\log m)$ , and one division. Hence,  $\binom{m}{i}$  can be computed in time  $O(\log m)$  time (since  $h$  is fixed). Similarly  $\binom{n}{j}$  can be computed in  $O(\log n)$  time. We conclude that the whole summation can be performed in time  $O(\log(mn))$ .  $\square$

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