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Graphs where every k-subset of vertices is an identifying set

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Let $G = (V, E)$ be an undirected graph without loops and multiple edges. A subset $C \subseteq V$ is called *identifying* if for every vertex $x \in V$ the intersection of C and the closed neighbourhood of x is nonempty, and these intersections are different for different vertices x. Let k be a positive integer. We will consider graphs where *every* k-subset is identifying. We prove that for every $k > 1$ the maximal order of such a graph is at most $2k-2$. Constructions attaining the maximal order are given for infinitely many values of k. The corresponding problem of k-subsets identifying any at most ℓ vertices is considered as well.

Keywords: identifying code, extremal graph, strongly regular graph, Plotkin bound

1 Introduction

Karpovsky *et al.* introduced identifying sets in [9] for locating faulty processors in multiprocessor systems. Since then identifying sets have been considered in many different graphs (see numerous references in [14]) and they find their motivations, for example, in sensor networks and environmental monitoring [10]. For recent developments see for instance [1, 2].

Let $G = (V, E)$ be a simple undirected graph where V is the set of vertices and E is the set of edges. The adjacency between vertices x and y is denoted by $x \sim y$, and an edge between x and y is denoted by $\{x, y\}$ or xy. Suppose $x, y \in V$. The *(graphical) distance* between x and y is the number of edges in any shortest path between these vertices and it is denoted by $d(x, y)$. If there is no such path, then $d(x, y) = \infty$. We denote by $N(x)$ the set of vertices adjacent to x (*neighbourhood*) and the *closed neighbourhood* of a vertex x is $N[x] = \{x\} \cup N(x)$. The closed neighbourhood within radius r centered at x is denoted by $N_r[x] = \{y \in V \mid d(x, y) \leq r\}$. We denote further $S_r(x) = \{y \in V \mid d(x, y) = r\}$. Moreover, for $X \subseteq V$, $N_r[X] = \cup_{x \in X} N_r[x]$. For $C \subseteq V$, $X \subseteq V$, and $x \in V$ we denote

$$I_r(C; x) = I_r(x) = N_r[x] \cap C$$

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and

$$I_r(C;X) = I_r(X) = N_r[X] \cap C = \bigcup_{x \in X} I_r(C;x).$$

If $r = 1$, we drop it from the notations. When necessary, we add a subscript G. We also write, for example, $N[x,y]$ and $I(C;x,y)$ for $N[\{x,y\}]$ and $I(C;\{x,y\})$. The *symmetric difference* of two sets is

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

The cardinality of a set X is denoted by $|X|$; we will also write $|G|$ for the order $|V|$ of a graph $G = (V,E)$. The *degree* of a vertex x is $\deg(x) = |N(x)|$. Moreover, $\delta_G = \delta = \min_{x \in V} \deg(x)$ and $\Delta_G = \Delta = \max_{x \in V} \deg(x)$. The *diameter* of a graph $G = (V,E)$ is $\operatorname{diam}(G) = \max\{d(x,y) \mid x,y \in V\}$.

We say that a vertex $x \in V$ *dominates* a vertex $y \in V$ if and only if $y \in N[x]$. As well we can say that a vertex y is *dominated* by x (or vice versa). A subset C of vertices V is called a *dominating set* (or *dominating*) if $\cup_{c \in C} N[c] = V$.

Definition 1 *A subset C of vertices of a graph $G = (V,E)$ is called* $(r, \leq \ell)$-identifying *(or an* $(r, \leq \ell)$-identifying set*) if for all $X, Y \subseteq V$ with $|X| \leq \ell$, $|Y| \leq \ell$, $X \neq Y$ we have*

$$I_r(C;X) \neq I_r(C;Y).$$

If $r = 1$ and $\ell = 1$, then we speak about an identifying set*.*

The idea behind identification is that we can uniquely determine the subset X of vertices of a graph $G = (V,E)$ by knowing only $I_r(C;X)$ — provided that $|X| \leq \ell$ and $C \subseteq V$ is an $(r, \leq \ell)$-identifying set.

Definition 2 *Let, for $n \geq k \geq 1$ and $\ell \geq 1$, $\mathfrak{Gr}(n,k,\ell)$ be the set of graphs on n vertices such that every k-element set of vertices is $(1, \leq \ell)$-identifying. Moreover, we denote $\mathfrak{Gr}(n,k,1) = \mathfrak{Gr}(n,k)$ and $\mathfrak{Gr}(k) = \bigcup_{n \geq k} \mathfrak{Gr}(n,k)$.*

In other words, in a sensor network which is modeled by a graph in the class $\mathfrak{Gr}(n,k,\ell)$ we can choose freely k sensors [10] i.e. vertices to locate any ℓ objects in vertices.

Example 3 (i) *For every $\ell \geq 1$, an empty graph $E_n = (\{1,\ldots,n\}, \emptyset)$ belongs to $\mathfrak{Gr}(n,k,\ell)$ if and only if $k = n$.*

(ii) *A cycle C_n ($n \geq 4$) belongs to $\mathfrak{Gr}(n,k)$ if and only if $n-1 \leq k \leq n$. A cycle C_n with $n \geq 7$ is in $\mathfrak{Gr}(n,n,2)$.*

(iii) *A path P_n of n vertices ($n \geq 3$) belongs to $\mathfrak{Gr}(n,k)$ if and only if $k = n$.*

(iv) *A complete bipartite graph $K_{n,m}$ ($n+m \geq 4$) is in $\mathfrak{Gr}(n+m,k)$ if and only $n+m-1 \leq k \leq n+m$.*

(v) *In particular, a star $S_n = K_{1,n-1}$ ($n \geq 4$) is in $\mathfrak{Gr}(n,k)$ if and only if $n-1 \leq k \leq n$.*

(vi) *The complete graph K_n ($n \geq 2$) is not in $\mathfrak{Gr}(n,k)$ for any k.*

We are interested in the maximum number n of vertices which can be reached by a given k. We study mainly the case $\ell = 1$ and define

$$\Xi(k) = \max\{n : \mathfrak{Gr}(n,k) \neq \emptyset\}. \tag{1}$$

Conversely, the question is for a given graph on n vertices what is the smallest number k such that every k-subset of vertices is an identifying set (or a $(1, \leq \ell)$-identifying set). (Note that even if we take $k = n$, there are graphs on n vertices that do not belong to $\mathfrak{Gr}(n,n)$, for example the complete graph K_n, $n \geq 2$.) The ratio n/k is called the *rate*.

In particular, we are interested in the asymptotics as $k \to \infty$. Combining Theorem 17 and Corollary 26, we obtain the following, which in particular shows that the rate is always less than 2.

Theorem 4 *$\Xi(k) \leq 2k - 2$ for all $k \geq 2$, and $\lim_{k\to\infty} \frac{\Xi(k)}{k} = 2$.*

We will see in Section 4 that $\Xi(k) = 2k - 2$ for infinitely many k.

We give some basic results in Section 2 and study small k in Section 2.1 where we give a complete description of the sets $\mathfrak{Gr}(k)$ for $k \leq 4$. In Section 3 we give an upper bound, which bases on a relation with error-correcting codes. We consider strongly regular graphs and some modifications of them in Section 4; this provides us with examples (e.g., Paley graphs) that attain or almost attain the upper bound in Theorem 4. In Section 5 we give results for the case $\ell \geq 2$.

2 Basic results

We begin with some simple consequences of the definition. We omit the simple proofs.

Lemma 5 *If $G = (V,E) \in \mathfrak{Gr}(n,k,\ell)$, then every induced subgraph $G[A]$, where $A \subseteq V$, of order $|A| = m \geq k$ belongs to $\mathfrak{Gr}(m,k,\ell)$.*

Lemma 6 *If G has connected components G_i, $i = 1,\ldots,m$, with $|G| = n$ and $|G_i| = n_i$, then $G \in \mathfrak{Gr}(n,k,\ell)$ if and only if $G_i \in \mathfrak{Gr}(n_i, k + n_i - n, \ell)$ for every i. In other words, $G_i \in \mathfrak{Gr}(n_i,k_i,\ell)$ with $n_i - k_i = n - k$.*

A graph G belongs to $\mathfrak{Gr}(n,k,\ell)$ if and only if every k-subset intersects every symmetric difference of the neighbourhoods of two sets that are of size at most ℓ. Equivalently, $G \in \mathfrak{Gr}(n,k,\ell)$ if and only if the complement of every such symmetric difference of two neighbourhoods contains less than k vertices. We state this as a theorem.

Theorem 7 *Let $G = (V,E)$ and $|V| = n$. A graph G belongs to $\mathfrak{Gr}(n,k,\ell)$ if and only if*

$$n - \min_{\substack{X,Y \subseteq V \\ X \neq Y \\ |X|,|Y| \leq \ell}} \{|N[X] \triangle N[Y]|\} \leq k - 1. \tag{2}$$

Now take $\ell = 1$, and consider $\mathfrak{Gr}(n,k)$. The characterization in Theorem 7 can be written as follows, since X and Y either are empty or singletons.

Corollary 8 *Let $G = (V,E)$ and $|V| = n$. A graph G belongs to $\mathfrak{Gr}(n,k)$ if and only if*

(i) *$\delta_G \geq n - k$, and*

(ii) $\max_{x,y\in V,\, x\neq y}\{|N[x]\cap N[y]|+|V\setminus(N[x]\cup N[y])|\}\leq k-1$.

In particular, if $G\in\mathfrak{Gr}(n,k)$ then every vertex is dominated by every choice of a k-subset, and for all distinct $x,y\in V$ we have $|N[x]\cap N[y]|\leq k-1$.

Example 9 *Let G be the 3-dimensional cube, with 8 vertices. Then $|N[x]|=4$ for every vertex x, and $|N[x]\triangle N[y]|$ is 4 when $d(x,y)=1$, 4 when $d(x,y)=2$, and 8 when $d(x,y)=3$. Hence, Theorem 7 shows that $G\in\mathfrak{Gr}(8,5)$.*

Lemma 10 *Let $G_0=(V_0,E_0)\in\mathfrak{Gr}(n_0,k_0)$ and let $G=(V_0\cup\{a\},E_0\cup\{\{a,x\}\mid x\in V_0\})$ for a new vertex $a\notin V_0$. In words, we add a vertex and connect it to all other vertices. Then $G\in\mathfrak{Gr}(n_0+1,k_0+1)$ if (and only if) $|N_{G_0}[x]|\leq k_0-1$ for every $x\in V_0$, or, equivalently, $\Delta_{G_0}\leq k_0-2$.*

Proof: An immediate consequence of Theorem 7 (or Corollary 8). □

Example 11 *If G_0 is the 3-dimensional cube in Example 9, which belongs to $\mathfrak{Gr}(8,5)$ and is regular with degree $3=5-2$, then Lemma 10 yields a graph $G\in\mathfrak{Gr}(9,6)$. G can be regarded as a cube with centre.*

2.1 Small k

Example 12 *For $k=1$, it is easily seen that $\mathfrak{Gr}(n,1)=\emptyset$ for $n\geq 2$, and thus $\mathfrak{Gr}(1)=\{K_1\}$ and $\Xi(1)=1$.*

Example 13 *Let $k=2$. If $G\in\mathfrak{Gr}(2)$, then G cannot contain any edge xy, since then $N[x]\cap\{x,y\}=\{x,y\}=N[y]\cap\{x,y\}$, so $\{x,y\}$ does not separate $\{x\}$ and $\{y\}$. Consequently, G has to be an empty graph E_n, and then $\delta_G=0$ and Corollary 8*(i) *(or Example 3*(i)*) shows that $n=k=2$. Thus $\mathfrak{Gr}(2)=\{E_2\}$ and $\Xi(2)=2$.*

Example 14 *Let $k=3$. First, assume $n=|G|=3$. There are only four graphs G with $|G|=3$, and it is easily checked that $E_3,P_3\in\mathfrak{Gr}(3,3)$ (Example 3*(i)(iii)*), while $C_3=K_3\notin\mathfrak{Gr}(3,3)$ (Example 3*(vi)*) and a disjoint union $K_1\cup K_2\notin\mathfrak{Gr}(3,3)$, for example by Lemma 6 since $K_2\notin\mathfrak{Gr}(2,2)$. Hence $\mathfrak{Gr}(3,3)=\{E_3,P_3\}$.*

Next, assume $n\geq 4$. Since there are no graphs in $\mathfrak{Gr}(n_1,k_1)$ if $n_1>k_1$ and $k_1\leq 2$, it follows from Lemma 6 that there are no disconnected graphs in $\mathfrak{Gr}(n,3)$ for $n\geq 4$. Furthermore, if $G\in\mathfrak{Gr}(n,3)$, then every induced subgraph with 3 vertices is in $\mathfrak{Gr}(3,3)$ and is thus E_3 or P_3; in particular, G contains no triangle.

If $G\in\mathfrak{Gr}(4,3)$, it follows easily that G must be C_4 or S_4, and indeed these belong to $\mathfrak{Gr}(4,3)$ by Example 3(ii)(v)*. Hence $\mathfrak{Gr}(4,3)=\{C_4,S_4\}$.*

Next, assume $G\in\mathfrak{Gr}(5,3)$. Then every induced subgraph with 4 vertices is in $\mathfrak{Gr}(4,3)$ and is thus C_4 or S_4. Moreover, by Corollary 8, $\delta_G\geq 5-3=2$. However, if we add a vertex to C_4 or S_4 such that the degree condition $\delta_G\geq 2$ is satisfied and we do not create a triangle we get $K_{2,3}$ – a complete bipartite graph, and we know already $K_{2,3}\notin\mathfrak{Gr}(5,3)$ (Example 3(iv)*). Consequently $\mathfrak{Gr}(5,3)=\emptyset$, and thus $\mathfrak{Gr}(n,3)=\emptyset$ for all $n\geq 5$.*

Consequently, $\mathfrak{Gr}(3)=\mathfrak{Gr}(3,3)\cup\mathfrak{Gr}(4,3)=\{E_3,P_3,S_4,C_4\}$ and $\Xi(3)=4$.

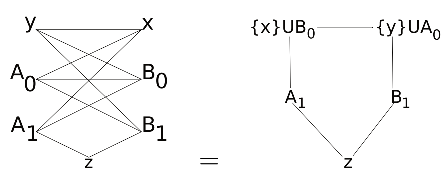
Example 15 *Let $k=4$. First, it follows easily from Lemma 6 and the descriptions of $\mathfrak{Gr}(j)$ for $j\leq 3$ above that the only disconnected graphs in $\mathfrak{Gr}(4)$ are E_4 and the disjoint union $P_3\cup K_1$; in particular, every graph in $\mathfrak{Gr}(n,4)$ with $n\geq 5$ is connected.*

Next, if $G \in \mathfrak{Gr}(n, 4)$, there cannot be a triangle in G because otherwise if a 4-subset includes the vertices of a triangle, one more vertex cannot separate the vertices of the triangle from each other. (Cf. Lemma 19.)

For $n = 4$, the only connected graphs of order 4 that do not contain a triangle are C_4, P_4 and S_4, and these belong to $\mathfrak{Gr}(4, 4)$ by Example 3(ii)(iii)(v). *Hence $\mathfrak{Gr}(4, 4) = \{C_4, P_4, S_4, E_4, P_3 \cup K_1\}$.*

Now assume that $G \in \mathfrak{Gr}(n, 4)$ with $n \geq 5$.

(i) Suppose first that a graph $K_1 \cup K_2 = (\{x, y, z\}, \{\{x, y\}\})$ is an induced subgraph of G. Then all the other vertices of G are adjacent to either x or y but not both, since otherwise there would be an induced triangle or an induced $E_2 \cup K_2$ or $K_2 \cup K_2$, and these do not belong to $\mathfrak{Gr}(4, 4)$. Let $A = N(x) \setminus \{y\}$ and $B = N(y) \setminus \{x\}$, so we have a partition of the vertex set as $\{x, y, z\} \cup A \cup B$. There can be further edges between A and B, z and A, z and B but not inside A and B. Let $A = A_0 \cup A_1$ and $B = B_0 \cup B_1$, where $A_1 = \{a \in A \mid a \sim z\}$, $A_0 = A \setminus A_1$ and $B_1 = \{b \in B \mid b \sim z\}$, $B_0 = B \setminus B_1$. If $a \in A_0$ and $b \in B$, then the 4-subset $\{a, b, x, z\}$ does not distinguish a and x unless $a \sim b$. Similarly, if $a \in A$ and $b \in B_0$, then $a \sim b$. On the other hand, if $a \in A_1$ and $b \in B_1$, then $a \not\sim b$, since otherwise abz would be a triangle. Thus, we have, where one or more of the sets A_0, A_1, B_0, B_1 might be empty, where an edge



is a complete bipartite graph on sets incident to it, and there are no edges inside these sets.

If $n \geq 6$, then there are at least two elements in one of the sets $\{x\} \cup B_0$, $\{y\} \cup A_0$, A_1 or B_1. However, these two vertices have the same neighbourhood and hence they cannot be separated by the other $n - 2 \geq 4$ vertices. Thus, $n = 5$.

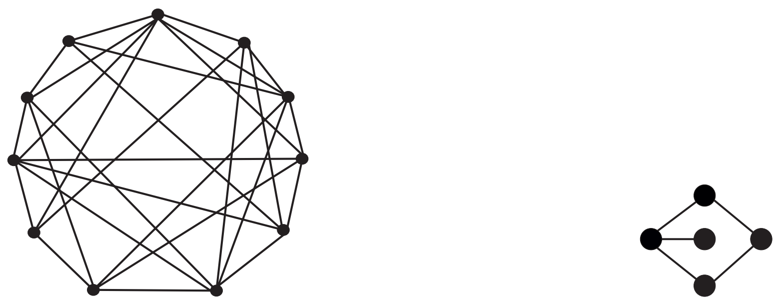
If $n = 5$, and both A_1 and B_1 are non-empty, we must have $A_0 = B_0 = \emptyset$ and $G = C_5$, which is in $\mathfrak{Gr}(5, 4)$ by Example 3(ii).

Finally, assume $n = 5$ and $A_1 = \emptyset$ (the case $B_1 = \emptyset$ is the same after relabelling). Then B_1 is non-empty, since G is connected. If B_0 is non-empty, let $b_0 \in B_0$ and $b_1 \in B_1$, and observe that $\{x, b_0, b_1, z\}$ does not separate z and b_1. Hence $B_0 = \emptyset$. We thus have either $|A_0| = 1$ and $|B_1| = 1$, or $|A_0| = 0$ and $|B_1| = 2$, and both cases yield the graph in Figure 1(b) which easily is seen to be in $\mathfrak{Gr}(5, 4)$.

(ii) Suppose that there is no induced subgraph $K_1 \cup K_2$. Since G is connected, we can find an edge $x \sim y$. Let, as above, $A = N(x) \setminus \{y\}$ and $B = N(y) \setminus \{x\}$. If $a \in A$ and $b \in B$ and $a \not\sim b$, then $(\{a, x, b\}, \{\{a, x\}\})$ is an induced subgraph and we are back in case (i). Hence, all edges between sets A and B exist and thus, recalling that G has no triangles, G is the complete bipartite graph with bipartition $(A \cup \{y\}, B \cup \{x\})$. By Example 3(iv), *then $n \leq 5$. If $n = 5$, we get $G = K_{2,3}$ or $G = K_{1,4} = S_5$, which both belong to $\mathfrak{Gr}(5, 4)$ by Example 3*(iv).

We summarize the result in a theorem.

Theorem 16 *We have $\Xi(4) = 5$. More precisely, $\mathfrak{Gr}(4) = \mathfrak{Gr}(4, 4) \cup \mathfrak{Gr}(5, 4)$, where $\mathfrak{Gr}(4, 4) = \{C_4, P_4, S_4, E_4, P_3 \cup K_1\}$ and $\mathfrak{Gr}(5, 4) = \{S_5, C_5, K_{2,3}, G_5\}$ where G_5 is the graph in Figure 1(b).*



(a) A graph in $\mathfrak{Gr}(11, 7)$ found by a computer search. (b) Graph G_5 in $\mathfrak{Gr}(5, 4)$

Fig. 1: Examples in $\mathfrak{Gr}(11, 7)$ and $\mathfrak{Gr}(5, 4)$.

Upper and lower bounds for $\Xi(k)$ for $1 \leq k \leq 20$ are given in Table 1. Note that we have determined $\Xi(k)$ exactly for $k \leq 6$ and for 9, 19, but not for other values of k when $k \leq 20$.

3 Upper estimates on the order

In the next theorem we give an upper on bound on $\Xi(k)$, which is obtained using knowledge on error-correcting codes.

Theorem 17 *If $k \geq 2$, then $\Xi(k) \leq 2k - 2$.*

Proof: We begin by giving a construction from a graph in $\mathfrak{Gr}(n, k)$ to error-correcting codes. A non-existence result of error-correcting codes then yields the non-existence of $\mathfrak{Gr}(n, k)$ graphs of certain parameters. Let $G = (V, E) \in \mathfrak{Gr}(n, k)$, where $V = \{x_1, x_2, \ldots, x_n\}$. We construct $n + 1$ binary strings $\mathbf{y}_i = (y_{i1}, \ldots, y_{in})$ of length n, for $i = 0, \ldots, n$, from the sets $\emptyset = N[\emptyset]$ and $N[x_i]$ for $i = 1, \ldots, n$ by defining $y_{0j} = 0$ for all j and

$$y_{ij} = \begin{cases} 0 & \text{if } x_j \notin N[x_i] \\ 1 & \text{if } x_j \in N[x_i] \end{cases}, \qquad 1 \leq i \leq n.$$

Let C denote the code which consists of these binary strings as codewords. Because $G \in \mathfrak{Gr}(n, k)$, the symmetric difference of two closed neighbourhoods $N[x_i]$ and $N[x_j]$, or of one neigbourhood $N[x_i]$ and $\emptyset$, is at least $n - k + 1$ by (2); in other words, the minimum Hamming distance $d(C)$ of the code C is at least $n - k + 1$.

We first give a simple proof that $\Xi(k) \leq 2k - 1$. Thus, suppose that there is a $G \in \mathfrak{Gr}(n, k)$ such that $n = 2k$. In the corresponding error-correcting code C, the minimum distance is at least $d = n - k + 1 = k + 1 > n/2$. Let the maximum cardinality of the error-correcting codes of length n and minimum distance at least d be denoted by $A(n, d)$. We can apply the Plotkin bound (see for example [15, Chapter 2, §2]), which says $A(n, d) \leq 2\lfloor d/(2d - n)\rfloor$, when $2d > n$. Thus, we have

$$A(n, d) \leq 2 \left\lfloor \frac{k+1}{2} \right\rfloor \leq k + 1.$$

Because $k+1 < 2k = n < |C|$, this contradicts the existence of C. Hence, there cannot exist a graph $G \in \mathfrak{Gr}(2k, k)$, and thus $\mathfrak{Gr}(n, k) = \emptyset$ when $n \geq 2k$.

The Plotkin bound is not strong enough to imply $\Xi(k) \leq 2k-2$ in general, but we obtain this from the proof of the Plotkin bound as follows. (In fact, for odd k, $\Xi(k) \leq 2k-2$ follows from the Plotkin bound for an odd minimum distance. We leave this to the reader since the argument below is more general.)

Suppose that $G = (V, E) \in \mathfrak{Gr}(n, k)$ with $n = 2k-1$. We thus have a corresponding error-correcting code C with $|C| = n+1 = 2k$ and minimum Hamming distance at least $n-k+1 = k$. Hence, letting d denote the Hamming distance,

$$\sum_{0 \leq i < j \leq n} d(y_i, y_j) \geq \binom{n+1}{2} k = \frac{2k(2k-1)}{2} k = (2k-1)k^2. \tag{3}$$

On the other hand, if there are s_m strings y_i with $y_{im} = 1$, and thus $|C| - s_m = 2k - s_m$ strings with $y_{im} = 0$, then the number of ordered pairs (i, j) such that $y_{im} \neq y_{jm}$ is $2s_m(2k - s_m)$ and this parabola gives $2s_m(2k - s_m) \leq 2k^2$. Hence each bit contributes at most k^2 to the sum in (3), and summing over m we find

$$\sum_{0 \leq i < j \leq n} d(y_i, y_j) \leq nk^2 = (2k-1)k^2. \tag{4}$$

Consequently, we have equality in (3) and (4), and thus $d(y_i, y_j) = k$ for all pairs (i, j) with $i \neq j$.

In particular, $|N[x_i]| = d(y_i, y_0) = k$ for $i = 1, \ldots, n$, and thus every vertex in G has degree $k-1$, i.e., G is $(k-1)$-regular. Hence, $2|E| = n(k-1) = (2k-1)(k-1)$, and k must be odd.

Further, if $i \neq j$, then $|N[x_i] \triangle N[x_j]| = d(y_i, y_j) = k$, and since $N[x_i] \setminus N[x_j]$ and $N[x_j] \setminus N[x_i]$ have the same size $k - |N[x_i] \cap N[x_j]|$, they have both the size $k/2$ and k must be even.

This contradiction shows that $\mathfrak{Gr}(2k-1, k) = \emptyset$, and thus $\Xi(k) \leq 2k-2$. □

The next theorem (which does not use Theorem 17) will lead to another upper bound in Theorem 20. It can be seen as an improvement for the extreme case $\mathfrak{Gr}(2k-2, k)$ of Mantel's [16] theorem on existence of triangles in a graph. Note that this result fails for $k = 5$ by Example 9.

Theorem 18 *Suppose $G \in \mathfrak{Gr}(n, k)$ and $k \geq 6$. If $n \geq 2k-2$, then there is a triangle in G.*

Proof: Let $G = (V, E) \in \mathfrak{Gr}(n, k)$. Suppose to the contrary that there are no triangles in G. If there is a vertex $x \in V$ such that $\deg(x) \geq k+1$, then we select in $N(x)$ a k-set X and a vertex y outside it; since X has to dominate y, it is clear that there exists a triangle xyz. Hence $\deg(x) \leq k$ for every x. On the other hand, we know by Corollary 8(i) that for all $x \in V$ $\deg(x) \geq n-k \geq k-2$.

Let $x \in V$ be a vertex whose degree is minimum. We denote $V \setminus N[x] = B$ and we use the fact that $|B| \leq k-1$.

1) Suppose first $\deg(x) = k$. Because $\deg(x)$ is minimum we know that for all $a \in N(x)$, $\deg(a) = k$. This is possible if and only if $|B| = k-1$ and for all $a \in N(x)$ we have $B \cap N(a) = B$. But then in the k-subset $C = \{x\} \cup B$ we have $I(C; a) = I(C; b)$ for all $a, b \in N(x)$. This is impossible.

2) Suppose then $\deg(x) = k-1$. If now $|B| \leq k-2$ the graph is impossible as in the first case (choose $C = N[x]$). Hence, $|B| = k-1$. For every $a \in N(x)$ there are at least $k-2$ adjacent vertices in B, and thus at most 1 non-adjacent. This implies that for all $a, b \in N(x)$, $a \neq b$, we have $|N(a) \cap N(b) \cap B| \geq k-3 \geq 2$, when $k \geq 5$. Hence, by choosing $a, b \in N(x)$, $a \neq b$, we have

the k-subset $C = \{x\} \cup (N(x) \setminus \{a, b\}) \cup \{c_1, c_2\}$, where $c_1, c_2 \in N(a) \cap N(b) \cap B$. In this k-subset $I(C; a) = I(C; b)$, which is impossible.

3) Suppose finally $\deg(x) = k-2$. Now $|B| = k-1$, otherwise we cannot have $n \geq 2k-2$. If there is $b \in B$ such that $|N(b) \cap N(x)| = k-2$, then because $\deg(b) \leq k$ we have $|N(b) \cap B| \geq 2$ and $|B \setminus N[b]| \geq k-4 \geq 2$, when $k \geq 6$. Hence, there are $c_1, c_2 \in B \setminus N[b]$, $c_1 \neq c_2$, and in the k-subset $C = N(x) \cup \{c_1, c_2\}$ we have $I(C; x) = I(C; b)$ which is impossible.

Thus, for all $b \in B$ we have $|N(b) \cap N(x)| \leq k-3$. On the other hand, each of the $k-2$ vertices in $N(x)$ has at least $k-3$ adjacent vertices in B, so the vertices in B have on the average at least $(k-2)(k-3)/(k-1) > k-4$ adjacent vertices in the set $N(x)$. Hence, we can find $b \in B$ such that $|N(b) \cap N(x)| = k-3$. Because $\deg(b) \geq k-2$ we have at least one $b_0 \in B$ such that $d(b, b_0) = 1$. Because there are no triangles, each of the $k-3$ neighbours of b in $N(x)$ is not adjacent with b_0, and therefore adjacent to at least $k-3$ of the $k-2$ vertices in $B \setminus \{b_0\}$. Hence, for all $a_1, a_2 \in N(x) \cap N(b)$, $a_1 \neq a_2$, we have $|N(a_1) \cap N(a_2) \cap B| \geq k-4 \geq 2$ when $k \geq 6$. In the k-subset $C = \{x, b, c_1, c_2\} \cup (N(x) \setminus \{a_1, a_2\})$, where $c_1, c_2 \in N(a_1) \cap N(a_2) \cap B$, we have $I(C; a_1) = I(C; a_2)$, which is impossible. $\square$

Lemma 19 *If there is a graph $G \in \mathfrak{Gr}(n,k)$ that contains a triangle, then $n \leq 3k-9$. (In particular, $k \geq 5$.)*

Proof: Suppose that $G = (V, E) \in \mathfrak{Gr}(n,k)$ and that there is a triangle $\{x, y, z\}$ in G. Let, for $v, w \in V$, $J_w(v)$ denote the indicator function given by $J_w(v) = 1$ if $v \in N[w]$ and $J_w(v) = 0$ if $v \notin N[w]$. Define the set $M_{xy} = \{v \in V : J_x(v) = J_y(v)\}$, and $M'_{xy} = M_{xy} \setminus \{x, y, z\}$. Since M_{xy} does not distinguish x and y, we have $|M_{xy}| \leq k-1$. Further, $\{x, y, z\} \subseteq M_{xy}$, and thus $|M'_{xy}| \leq k-4$. Define similarly M_{xz}, M_{yz}, M'_{xz}, M'_{yz}; the same conclusion holds for these.

Since the indicator functions take only two values, M_{xy}, M_{xz} and M_{yz} cover V, and thus

$$n = |V| = |M'_{xy} \cup M'_{xz} \cup M'_{yz} \cup \{x, y, z\}| \leq 3(k-4) + 3 = 3k-9.$$

Since $n \geq k$, this entails $3k-9 \geq k$ and thus $k \geq 5$. $\square$

The following upper bound is generally weaker than Theorem 17, but it gives the optimal result for $k = 6$.

Theorem 20 *Suppose $k \geq 6$. Then $\Xi(k) \leq 3k-9$.*

Proof: Suppose that $G \in \mathfrak{Gr}(n,k)$. If G does not contain any triangle, then Theorem 18 yields $n \leq 2k-3 \leq 3k-9$. If G does contain a triangle, then Lemma 19 yields $n \leq 3k-9$. $\square$

4 Strongly regular graphs

A graph $G = (V, E)$ is called *strongly regular* with parameters (n, t, λ, μ) if $|V| = n$, $\deg(x) = t$ for all $x \in V$, any two adjacent vertices have exactly λ common neighbours, and any two nonadjacent vertices have exactly μ common neighbours; we then say that G is an (n, t, λ, μ)-SRG. See [3] for more information. By [3, Proposition 1.4.1] we know that if G is an (n, t, λ, μ)-SRG, then $n = t + 1 + t(t - 1 - \lambda)/\mu$.

We give two examples of strongly regular graphs that will be used below.

Example 21 *The well-known Paley graph $P(q)$, where q is a prime power with $q \equiv 1 \pmod 4$, is a $(q,\, (q-1)/2,\, (q-5)/4,\, (q-1)/4)$-SRG, see for example [3]. The vertices of $P(q)$ are the elements of the finite field F_q, with an edge ij if and only if $i-j$ is a non-zero square in the field; when q is a prime, this means that the vertices are $\{1,\dots,q\}$ with edges ij when $i-j$ is a quadratic residue mod q.*

Example 22 *Another construction of strongly regular graphs uses a regular symmetric Hadamard matrix with constant diagonal (RSHCD) [6], [4], [5]. In particular, in the case (denoted RSHCD+) of a regular symmetric $n \times n$ Hadamard matrix $H = (h_{ij})$ with diagonal entries $+1$ and constant positive row sums $2m$ (necessarily even when $n > 1$), then $n = (2m)^2 = 4m^2$ and the graph G with vertex set $\{1,\dots,n\}$ and an edge ij (for $i \neq j$) if and only if $h_{ij} = +1$ is a $(4m^2,\, 2m^2+m-1,\, m^2+m-2,\, m^2+m)$-SRG [4, §8D].*

It is not known for which m such RSHCD+ exist (it has been conjectured that any $m \geq 1$ is possible) but constructions for many m are known, see [6], [17, V.3] and [5, IV.24.2]. For example, starting with the 4×4 RSHCD+

$$H_4 = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

its tensor power $H_4^{\otimes r}$ is an RSHCD+ with $n = 4^r$, and thus $m = 2^{r-1}$, for any $r \geq 1$. This yields a $(2^{2r},\, 2^{2r-1}+2^{r-1}-1,\, 2^{2r-2}+2^{r-1}-2,\, 2^{2r-2}+2^{r-1})$-SRG with vertex set $\{1,2,3,4\}^r$, where two different vertices $(i_1,\dots,i_r)$ and $(j_1,\dots,j_r)$ are adjacent if and only if the number of coordinates ν such that $i_\nu + j_\nu = 5$ is even.

Theorem 23 *A strongly regular graph $G = (V,E)$ with parameters (n,t,λ,μ) belongs to $\mathfrak{Gr}(n,k)$ if and only if*

$$k \geq \max\bigl\{n-t,\, n-2t+2\lambda+3,\, n-2t+2\mu-1\bigr\},$$

or, equivalently, $t \geq n-k$ and $2\max\{\lambda+1, \mu-1\} \leq k+2t-n-1$.

Proof: An immediate consequence of Theorem 7, since $|N[x]| = t+1$ for every vertex x and $|N[x] \triangle N[y]|$ equals $2(t-\lambda-1)$ when $x \sim y$ and $2(t+1-\mu)$ when $x \not\sim y$, $x \neq y$. □

We can extend this construction to other values of n by modifying the strongly regular graph.

Theorem 24 *If there exists a strongly regular graph with parameters (n_0,t,λ,μ), then for every $i = 0,\dots,n_0+1$ there exists a graph in $\mathfrak{Gr}(n_0+i, k_0+i)$, where*

$$k_0 = \max\bigl\{n_0-t,\, t,\, n_0-2t+2\lambda+3,\, n_0-2t+2\mu-1,\, 2t-2\lambda-1,\, 2t-2\mu+2\bigr\},$$

provided $k_0 \leq n_0$.

Proof: For $i = 0$, this is a weaker form of Theorem 23. For $i \geq 1$, we suppose that $G_0 = (V_0,E_0)$ is (n_0,t,λ,μ)-SRG and build a graph G_i in $\mathfrak{Gr}(n_0+i, k_0+i)$ from G_0 by adding suitable new vertices and edges.

If $1 \leq i \leq n_0$, choose i different vertices $x_1,x_2,\dots,x_i$ in V_0. Construct a new graph $G_i = (V_i,E_i)$ by taking G_0 and adding to it new vertices $x'_1,x'_2,\dots,x'_i$ and new edges $x'_j y$ for $j \leq i$ and all $y \notin N_{G_0}(x_j)$.

First, $\deg_{G_i}(x) \geq \deg_{G_0}(x) = t$ for $x \in V_0$ and $\deg_{G_i}(x') = n_0 - t$ for $x' \in V_i' = V_i \setminus V_0$. We proceed to investigate $N[x] \triangle N[y]$, and separate several cases.

(i) If $x, y \in V_0$, with $x \neq y$, then

$$\left|N[x] \triangle N[y]\right| \geq \left|(N[x] \triangle N[y]) \cap V_0\right| = \left|N_{G_0}[x] \triangle N_{G_0}[y]\right|,$$

which equals $2(t - \lambda - 1)$ if $x \sim y$ and $2(t - \mu + 1)$ if $x \not\sim y$.

(ii) If $x \in V_0$, $y' \in V_i'$, then, since $\triangle$ is associative and commutative,

$$\left|(N[x] \triangle N[y']) \cap V_0\right| = \left|(N_{G_0}[x] \triangle (V_0 \triangle N_{G_0}(y))\right| = n_0 - \left|N_{G_0}[x] \triangle N_{G_0}(y)\right|,$$

which equals $n_0 - 1$ if $x = y$, $n_0 - (2t - 2\lambda - 1)$ if $x \sim y$, and $n_0 - (2t - 2\mu + 1)$ if $x \not\sim y$ and $x \neq y$. If $x \sim y$, further, $\left|(N[x] \triangle N[y']) \cap V_i'\right| \geq 1$, since $y' \notin N[x]$.

(iii) If $x', y' \in V_i'$, with $x' \neq y'$, then

$$\left|(N[x'] \triangle N[y']) \cap V_0\right| = \left|(V_0 \setminus N_{G_0}(x)) \triangle (V_0 \setminus N_{G_0}(y))\right| = \left|N_{G_0}(x) \triangle N_{G_0}(y)\right|,$$

which equals $2(t - \lambda)$ if $x \sim y$ and $2(t - \mu)$ if $x \not\sim y$. Further, $\left|(N[x'] \triangle N[y']) \cap V_i'\right| = |\{x', y'\}| = 2$.

Collecting these estimates, we see that $G_i \in \mathfrak{Gr}(n_0 + i, k_0 + i)$ by Theorem 7 (or Corollary 8) with our choice of k_0. Note that $2k_0 \geq (n_0 - 2t + 2\lambda + 3) + (2t - 2\lambda - 1) = n_0 + 2 \geq 3$, so $k_0 \geq 2$.

Finally, for $i = n_0 + 1$, we construct G_{n_0+1} by adding a new vertex to G_{n_0} and connecting it to all other vertices. The graph G_{n_0} has by construction maximum degree $\Delta_{G_{n_0}} = n_0 \leq k_0 + n_0 - 2$. Hence, Lemma 10 shows that $G_{n_0+1} \in \mathfrak{Gr}(2n_0 + 1, k_0 + n_0 + 1)$. □

We specialize to the Paley graphs, and obtain from Example 21 and Theorems 23–24 the following.

Theorem 25 *Let q be an odd prime power such that $q \equiv 1 \pmod 4$.*

(i) *The Paley graph $P(q) \in \mathfrak{Gr}(q, (q + 3)/2)$.*

(ii) *There exists a graph in $\mathfrak{Gr}(q + i, (q + 3)/2 + i)$ for all $i = 0, 1, \ldots, q + 1$.*

Note that the rate $2q/(q + 3)$ for the Paley graphs approaches 2 as $q \to \infty$; in fact, with $n = q$ and $k = (q+3)/2$ we have $n = 2k - 3$, almost attaining the bound $2k - 2$ in Theorem 17. (The Paley graphs thus almost attain the bound in Theorem 17, but never attain it exactly.)

Corollary 26 $\Xi(k) \geq 2k - o(k)$ *as $k \to \infty$.*

Proof: Let $q = p^2$ where (for $k \geq 6$) p is the largest prime such that $p \leq \sqrt{2k - 3}$. It follows from the prime number theorem that $p/\sqrt{2k - 3} \to 1$ as $k \to \infty$, and thus $q = 2k - o(k)$. Hence, if k is large enough, then $k \leq q \leq 2k - 3$, and Theorem 25 shows that $P(q) \in \mathfrak{Gr}(q, (q + 3)/2) \subseteq \mathfrak{Gr}(q, k)$, so $\Xi(k) \geq q = 2k - o(k)$. (Alternatively, we may let q be the largest prime such that $q \leq 2k - 3$ and $q \equiv 1 \pmod 4$ and use the prime number theorem for arithmetic progressions [8, Chapter 17] to see that then $q = 2k - o(k)$.) □

We turn to the strongly regular graphs constructed in Example 22 and find from Theorem 23 that they are in $\mathfrak{Gr}(4m^2, 2m^2 + 1)$, thus attaining the bound in Theorem 17. We state that as a theorem.

Theorem 27 *The strongly regular graph constructed in Example 22 from an $n \times n$ RSHCD+ belongs to $\mathfrak{Gr}(n, n/2+1)$.*

Corollary 28 *There exist infinitely many integers k such that $\Xi(k) = 2k - 2$.*

Proof: If $k = n/2 + 1$ for an even n such that there exists an $n \times n$ RSHCD+, then $\Xi(k) \geq n = 2k - 2$ by Theorem 27. The opposite inequality is given by Theorem 17. By Example 22, this holds at least for $k = 2^{2r-1} + 1$ for any $r \geq 1$. □

5 On $\mathfrak{Gr}(n, k, \ell)$

In this section we consider $\mathfrak{Gr}(n, k, \ell)$ for $\ell \geq 2$. Let us denote

$$\Xi(k, \ell) = \max\{n : \mathfrak{Gr}(n, k, \ell) \neq \emptyset\}.$$

Trivially, the empty graph $E_k \in \mathfrak{Gr}(k, k, \ell)$ for any $\ell \geq 1$; thus $\Xi(k, \ell) \geq$k.

Note that a graph $G = (V, E)$ with $|V| = n$ admits a $(1, \leq\ell)$-identifying set $\iff$ V is $(1, \leq\ell)$-identifying $\iff$ $G \in \mathfrak{Gr}(n, n, \ell)$.

Theorem 29 *Suppose that $G = (V, E) \in \mathfrak{Gr}(n, k, \ell)$, where $n > k$ and $\ell \geq 2$. Then the following conditions hold:*

- (i) *For all $x \in V$ we have $\ell + 1 < n - k + \ell + 1 \leq |N[x]| \leq k - \ell$. In other words, $\delta_G \geq n - k + \ell$ and $\Delta_G \leq k - \ell - 1$.*
- (ii) *For all $x, y \in V$, $x \neq y$, $|N[x] \cap N[y]| \leq k - 2\ell + 1$.*
- (iii) *$n \leq 2k - 2\ell - 1$ and $k \geq 2\ell + 2$.*

Proof: (i) Suppose first that there is a vertex $x \in V$ such that $|N[x]| \leq n - k + \ell$. By removing $n - k$ vertices from V, starting in $N[x]$, we find a k-subset C with $I(C; x) = \{c_1, \dots, c_m\}$ for some $m \leq \ell$. If $m = 0$, then $I(C; x) = I(C; \emptyset)$, which is impossible. If $1 \leq m < \ell$, we can arrange (by removing x first) so that $x \notin C$, and thus $x \notin Y = \{c_1, \dots, c_m\}$. Then $I(C; \{x\} \cup Y) = I(C; Y)$, a contradiction. If $m = \ell \geq 2$, we can conversely arrange so that $x \in C$, and thus $x \in I(C; x)$, say $c_1 = x$. Then $I(C; c_2, \dots, c_m) = I(C; c_1, \dots, c_m)$, another contradiction. Consequently, $|N[x]| \geq n - k + \ell + 1$.

Suppose then $|N[x]| \geq k - \ell + 1$. If $|N[x]| \geq k$, we can choose a k-subset C of $N[x]$; then $I(C; x) = C = I(C; x, y)$ for any y, which is impossible. If $k > |N[x]| \geq k - \ell + 1$, we can choose a k-subset $C = N[x] \cup \{c_1, \dots, c_{k-|N[x]|}\}$. Choose also $a \in N(c_1)$ (which is possible because $\deg(c_1) \geq 1$ by (i)). Now $I(C; x, c_1, \dots, c_{k-|N[x]|}) = C = I(C; x, a, c_2, \dots, c_{k-|N[x]|})$, which is impossible.

(ii) Suppose to the contrary that there are $x, y \in V$, $x \neq y$, such that $|N[x] \cap N[y]| \geq k - 2\ell + 2$. Let $A = N(y) \setminus N[x]$. Then, according to (i), $|A| \leq |N[y] \setminus N[x]| = |N[y]| - |N[x] \cap N[y]| \leq k - \ell - (k - 2\ell + 2) = \ell - 2$. Since $k > \ell - 2$ by (i), there is a k-subset $C \subseteq V \setminus \{y\}$ such that $A \subset C$. Then $I(C; A \cup \{x, y\}) = I(C; A \cup \{x\})$, a contradiction.

(iii) An immediate consequence of (i), which implies $n - k + \ell + 1 \leq k - \ell$ and $\ell + 1 < k - \ell$. □

Theorem 30 *For $\ell \geq 2$, $\Xi(k, \ell) \leq \max\{\frac{\ell}{\ell-1}(k-2), k\}$.*

Proof: If $\Xi(k,\ell) = k$, there is nothing to prove. Assume then that there exists a graph $G = (V,E) \in \mathfrak{Gr}(n,k,\ell)$, where $n > k$. By Theorem 29(iii), $\ell < k/2 < n$. Let us consider any set of vertices $Z = \{z_1, z_2, \dots, z_\ell\}$ of size ℓ. We will estimate $|N[Z]|$ as follows. By Theorem 29(i) we know $|N[z_1]| \geq n-k+\ell+1$. Now $N[z_1,z_2]$ must contain at least $n-k+1$ vertices, which *do not* belong to $N[z_1]$ due to Theorem 7 which says that $|N[X] \triangle N[Y]| \geq n-k+1$, where we take $X = \{z_1\}$ and $Y = \{z_1, z_2\}$. Analogously, each set $N[z_1,\dots,z_i]$ $(i = 2,\dots,\ell)$ must contain at least $n-k+1$ vertices which are not in $N[z_1,\dots,z_{i-1}]$. Hence, for the set Z we have $|N[Z]| \geq n-k+\ell+1+(\ell-1)(n-k+1) = \ell(n-k+2)$. Since trivially $|N[Z]| \leq n$, we have $(\ell-1)n \leq \ell(k-2)$, and the claim follows. □

Corollary 31 *For $\ell \geq 2$, we have $\frac{\Xi(k,\ell)}{k} \leq 1 + \frac{1}{\ell-1}$.*

The next results improve the result of Theorem 30 for $\ell = 2$.

Lemma 32 *Assume that $n > k$. Let $G = (V,E)$ belong to $\mathfrak{Gr}(n,k,2)$. Then*

$$n + \frac{n-k+2}{n-1}(n-k+3) \leq 2k-3$$

Proof: Suppose $x \in V$. Let

$$f(n,k) = \frac{n-k+2}{n-1}(n-k+3).$$

Our aim is first to show that there exists a vertex in $N(x)$ or in $S_2(x)$ which dominates at least $f(n,k)$ vertices of $N[x]$. Let

$$\lambda_x = \max\{|N[x] \cap N[a]| \mid a \in N(x)\}.$$

If $\lambda_x \geq f(n,k)$, we are already done. But if $\lambda_x < f(n,k)$, then we show that there is a vertex in $S_2(x)$ that dominates at least $f(n,k)$ vertices of $N[x]$. Let us estimate the number of edges between the vertices in $N(x)$ and in $S_2(x)$ — we denote this number by M. By Theorem 29(i), every vertex $y \in N(x)$ yields at least $|N[y]| - \lambda_x \geq n-k+3-\lambda_x$ such edges and there are at least $n-k+2$ vertices in $N(x)$. Consequently, $M \geq (n-k+2)(n-k+3-\lambda_x)$. On the other hand, again by Theorem 29(i), $|S_2(x)| \leq n - |N[x]| \leq k-3$. Hence, there must exist a vertex in $S_2(x)$ incident with at least $M/(k-3)$ edges whose other endpoint is in $N(x)$. Now, if $\lambda_x < f(n,k)$, then

$$\frac{M}{k-3} > \frac{(n-k+2)(n-k+3-f(n,k))}{k-3} = f(n,k).$$

Hence there exists in this case a vertex in $S_2(x)$ that is incident to at least $f(n,k)$ such edges, i.e., it dominates at least $f(n,k)$ vertices in $N(x)$.

In any case there thus exists $z \neq x$ such that $|N[x] \cap N[z]| \geq f(n,k)$. Let $C = (N[x] \cap N[z]) \cup (V \setminus N[x])$. Then $I(C;x,z) = I(C;z)$, so C is not $(1,\leq 2)$-identifying and thus $|C| < k$. Hence, using Theorem 29(i),

$$k-1 \geq |C| \geq f(n,k) + n - |N[x]| \geq f(n,k) + n - (k-2),$$

and thus $n + f(n,k) \leq 2k-3$ as asserted. □

Theorem 33 *If $k \le 5$, then $\Xi(k,2) = k$. If $k \ge 6$, then*

$$\Xi(k,2) < \Big(1 + \frac{1}{\sqrt{2}}\Big)(k-2) + \frac{1}{4}.$$

Proof: Let $n = \Xi(k,2)$, and let $m = k-2$. If $n > k$, then $k \ge 6$ by Theorem 29(iii); hence $n = k$ when $k \le 5$. Further, still assuming $n > k$, Lemma 32 yields

$$n + \frac{(n-m)(n-m+1)}{n-1} \le 2m+1$$

or

$$0 \ge n(n-1) + (n-m)^2 + n - m - (2m+1)(n-1) = 2\big(n - (m + \tfrac{1}{4})\big)^2 - m^2 + \tfrac{7}{8}.$$

Hence, $n - (m + \frac{1}{4}) < m/\sqrt{2}$. □

Corollary 34 *For $\ell = 2$, we have $\Xi(k,2)/k \le 1 + \frac{1}{\sqrt{2}}$.*

Problem 35 *What is $\limsup_{k\to\infty} \Xi(k,\ell)/k$ for $\ell \ge 2$? In particular, is $\limsup_{k\to\infty} \Xi(k,\ell)/k > 1$?*

The following theorem implies that for any $\ell \ge 2$ there exist graphs in $\mathfrak{Gr}(n,k,\ell)$ for $n \approx k + \log_2 k$. In particular, we have such graphs with $n > k$.

Theorem 36 *Let $\ell \ge 2$ and $m \ge \max\{2\ell - 2, 4\}$. A binary hypercube of dimension m belongs to $\mathfrak{Gr}(2^m, 2^m - m + 2\ell - 2, \ell)$.*

Proof: Suppose first $\ell \ge 3$. By [11, Theorem 2] we know that then a set in a binary hypercube is $(1, \le \ell)$-identifying if and only if every vertex is dominated by at least $2\ell - 1$ different vertices belonging to the set. Hence, we can remove any $m + 1 - (2\ell - 1)$ vertices from the set of vertices, and there will still be a big enough multiple domination to assure that the remaining set is $(1, \le \ell)$-identifying.

Suppose then that $\ell = 2$ and $G = (V,E)$ is the binary m-dimensional hypercube. Let us denote by $C \subseteq V$ a $(2^m - m + 2)-$subset. Every vertex is dominated by at least $m + 1 - (m-2) = 3$ vertices of C. For all $x, y \in V$, $x \ne y$ we have $|N[x] \cap N[y]| = 2$ if and only if $1 \le d(x,y) \le 2$ and otherwise $|N[x] \cap N[y]| = 0$. Hence, for all $x, y, z \in V$ with $x \ne y$, $I(y) = N[y] \cap C$ contains at least 3 vertices, and these cannot all be dominated by x; thus, we have $I(x) \ne I(y)$ and $I(x) \ne I(y,z)$.

We still need to show that $I(x,y) \ne I(z,w)$ for all $x,y,z,w \in V$, $x \ne y$, $z \ne w$, $\{x,y\} \ne \{z,w\}$. By symmetry we may assume that $x \notin \{z,w\}$. Suppose $I(x,y) = I(z,w)$.

If $|I(x)| \ge 5$, then any two vertices $z, w \ne x$ cannot dominate $I(x)$, a contradiction.

If $|I(x)| = 4$, then $|I(z) \cap I(x)| = |I(w) \cap I(x)| = 2$ and $I(x) \cap I(z) \cap I(w) = \emptyset$. It follows that $3 \le d(z,w) \le 4$ which implies $I(z) \cap I(w) = \emptyset$. Since $|N[x] \setminus C| = |N[x]| - |I(x)| = m - 3$, all except one vertex, say v, of $V \setminus C$ belong to $N[x]$, so $V \setminus N[x] \subseteq C \cup \{v\}$; the vertex v cannot belong to both $N[z]$ and $N[w]$ since these are disjoint, so we may (w.l.o.g.) assume that $v \notin N[z]$, and thus $N[z] \setminus N[x] \subseteq C$, whence $N[z] \setminus N[x] \subseteq I(z) \setminus I(x)$. Hence, $|I(z) \cap I(y)| \ge |I(z) \setminus I(x)| \ge |N[z] \setminus N[x]| = |N[z]| - |N[z] \cap N[x]| = m + 1 - 2 \ge 3$. Thus $y = z$; however, then $I(y) \cap I(w) = I(z) \cap I(w) = \emptyset$ and since $I(w) \not\subseteq I(x)$, we have $I(w) \not\subseteq I(x,y)$.

Suppose finally that $|I(x)| = 3$; w.l.o.g. we may assume $|I(z) \cap I(x)| = 2$. Now $|N[x] \setminus C| = |N[x]| - |I(x)| = m - 2 = |V \setminus C|$, and thus $V \setminus C = N[x] \setminus C \subseteq N[x]$; hence, $V \setminus N[x] \subseteq C$ and thus

$N[z]\setminus N[x] \subseteq I(z)\setminus I(x)$. Consequently, $|I(z)\cap I(y)| \geq |I(z)\setminus I(x)| \geq |N[z]\setminus N[x]| \geq m+1-2 \geq 3$, and thus $z = y$. But similarly $N[w] \setminus N[x] \subseteq I(w) \setminus I(x)$ and the same argument shows $w = y$, and thus $w = z$, a contradiction. □

We finally consider graphs without isolated vertices (i.e., no vertices with degree zero), and in particular connected graphs.

By [13, Theorem 8] a graph with no isolated vertices admitting a $(1, \leq \ell)$-identifying set has minimum degree at least ℓ. Hence, always $n \geq \ell + 1$.

In [7] and [12] it has been proven that there exist connected graphs which admit $(1, \leq \ell)$-identifying sets. For example, the smallest known connected graph admitting a $(1, \leq 3)$-identifying set has 16 vertices [12]. It is unknown whether there are such graphs with smaller order. In the next theorem we solve the case of graphs admitting $(1, \leq 2)$-identifying sets.

Theorem 37 *The smallest $n \geq 3$ such that there exists a connected graph (or a graph without isolated vertices) in $\mathfrak{Gr}(n, n, 2)$ is $n = 7$.*

(If we allow isolated vertices, we can trivially take the empty graph E_n for any $n \geq 2$.)

Proof: The cycle $C_n \in \mathfrak{Gr}(n, n, 2)$ for $n \geq 7$ by Example 3(ii) (see also [12]).

Assume that $G = (V, E) \in \mathfrak{Gr}(n, n, 2)$ is a graph of order $n \leq 6$ without isolated vertices; we will show that this leads to a contradiction. By [13], we know that $\deg(v) \geq 2$ for all $v \in V$. We will use this fact frequently in the sequel.

If G is disconnected, the only possibility is that $n = 6$ and that G consists of two disjoint triangles, but this graph is not even in $\mathfrak{Gr}(n, n, 1)$.

Hence, G is connected. Let $x, y \in V$ be such that $d(x, y) = \text{diam}(G)$.

(i) Suppose that $\text{diam}(G) = 1$, or more generally that there exists a dominating vertex x. Then $N[x, y] = N[x]$ for any $y \in V$, which is a contradiction.

(ii) Suppose next $\text{diam}(G) = 2$. Moreover, by the previous case we can assume that for any $v \in V$ there is $w \in V$ such that $d(v, w) = 2$.

Assume first $|N(x)| = 4$. Then $S_2(x) = \{y\}$. Since $\deg(y) \geq 2$, there exist two vertices $w_1, w_2 \in N(y) \cap N(x)$, but then $N[x, w_1] = N[x, w_2]$.

Assume next $|N(x)| = 3$, say $N(x) = \{u_1, u_2, u_3\}$. Then $|S_2(x)| = n - |N[x]| \leq 2$. Since the four sets $N[x]$ and $N[x, u_i]$, $i = 1, 2, 3$, must be distinct, we can assume without loss of generality that $|S_2(x)| = 2$, say $S_2(x) = \{y, w\}$, and that the only edges between the elements in $S_2(x)$ and $N(x)$ are u_1y, u_2w, u_3y and u_3w. Then $N[x, u_3] = N[y, u_2]$.

Assume finally that $|N(x)| = 2$. By the previous discussion we may assume that $|N(v)| = 2$ for all $v \in V$. Then G must be a cycle C_n, but it can easily be seen that $C_n \notin \mathfrak{Gr}(n, n, 2)$ for $3 \leq n \leq 6$.

(iii) Suppose that $\text{diam}(G) = 3$. Clearly $|N(x)| \geq 2$ and $|S_2(x)| \geq 1$. If $|S_2(x)| = 1$, say $S_2(x) = \{w\}$, then $N[w, y] = N[w]$, which is not allowed. Since $n \leq 6$, we thus have $|N(x)| = 2$ and $|S_2(x)| = 2$, say $N(x) = \{u_1, u_2\}$ and $S_2(x) = \{w_1, w_2\}$. We can assume without loss of generality that $u_1w_1 \in E$. If $w_2u_2 \in E$, then $N[w_1, u_2] = N[x, y]$. If $w_2u_2 \notin E$, then $N[w_1, w_2] = N[w_1]$.

(iv) Suppose that $\text{diam}(x, y) \geq 4$. Then G contains an induced path P_5. There is at most one additional vertex, but it is impossible to add it to P_5 and obtain $\delta_G \geq 2$ and $\text{diam}(G) \geq 4$.

This completes the proof. □

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Tab. 1: Lower and upper bounds for $\Xi(k)$ for $1 \leq k \leq 20$. The lower bounds come from the examples given in the last column; for $n \geq 8$ using Theorem 23, 25 or 27 or Lemma 10. The strongly regular graphs used here can be found from [5]. The upper bounds for $k \geq 7$ come from Theorem 17.

k	lower bound	upper bound	example
1	1	1 (Ex. 12)	E_1
2	2	2 (Ex. 13)	E_2
3	4	4 (Ex. 14, Th.17)	C_4, S_4
4	5	5 (Th. 16)	Figure 1(b)
5	8	8 (Th. 17)	Example 9
6	9	9 (Th. 20)	Example 11, $P(9)$
7	11	12 (Th. 17, Th. 20)	Figure 1(a)
8	13	14	$P(13)$
9	16	16	RSHCD+
10	17	18	$P(17)$
11	18	20	Th. 25(ii)
12	21	22	(21,10,3,6)-SRG
13	22	24	Lemma 10
14	25	26	$P(25)$
15	26	28	(26,15,8,9)-SRG
16	29	30	$P(29)$
17	30	32	Th. 25(ii)
18	31	34	Th. 25(ii)
19	36	36	RSHCD+
20	37	38	$P(37)$

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