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Proving exact values for the 2-limited broadcast domination number on grid graphs

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We establish exact values for the 2-limited broadcast domination number of various grid graphs, in particular $C_m \square C_n$ for $3 \leq m \leq 6$ and all $n \geq m$, $P_m \square C_3$ for all $m \geq 3$, and $P_m \square C_n$ for $4 \leq m \leq 5$ and all $n \geq m$. We also produce periodically optimal values for $P_m \square C_4$ and $P_m \square C_6$ for $m \geq 3$, $P_4 \square P_n$ for $n \geq 4$, and $P_5 \square P_n$ for $n \geq 5$. Our method completes an exhaustive case analysis and eliminates cases by combining tools from linear programming with various mathematical proof techniques.

Keywords: Graph theory, broadcast domination, limited broadcast domination, linear programming, integer linear programming, grid graphs

1 Introduction

Suppose there is a transmitter located at each vertex of a graph G. A *k-limited broadcast* f on G is a function $f : V(G) \mapsto \{0, 1, \ldots, k\}$. The integer $f(v)$ represents the *strength* of the broadcast from v, where $f(v) = 0$ means the transmitter at v is not broadcasting. A broadcast of positive strength $f(v)$ from v is *heard* by all vertices u such that $d(u, v) \leq f(v)$, where $d(u, v)$ is the distance between the u and v in G. A broadcast f is *dominating* if each vertex of G hears the broadcast from some vertex. The *cost* of a broadcast f is $\sum_{v \in V(G)} f(v)$. The *k-limited broadcast domination number* $\gamma_{b,k}(G)$ of a graph G is the minimum cost of a k-limited dominating broadcast on G.

The k-limited broadcast domination number can be seen to be the optimum solution to ILP 1.1 shown below. Let G be a graph and fix $1 \leq k \leq rad(G)$, where $rad(G)$ is the *radius* of G. For each vertex

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$i \in V(G)$ and $\ell \in \{1, 2, \ldots, k\}$, let $x_{i,\ell} = 1$ if vertex i is broadcasting at strength ℓ and 0 otherwise.

$$\begin{aligned}
&\text{Minimize:} && \sum_{\ell=1}^{k} \sum_{i \in V(G)} \ell \cdot x_{i,\ell} \\
&\text{Subject to:} && (1) \quad \sum_{\ell=1}^{k} \sum_{\substack{i \,\in\, V(G) \text{ s.t.} \\ d(i,j) \,\leq\, \ell}} x_{i,\ell} \geq 1, \quad \text{for each vertex } j \in V(G), \\
& && (2) \quad x_{i,\ell} \in \{0,1\} \quad \text{for each vertex } i \in V(G) \text{ and } \ell \in \{1, 2, \ldots, k\}.
\end{aligned} \tag{ILP 1.1}$$

The k-limited broadcast domination number of a graph was first defined in Erwin (2001) (also see Erwin (2004)). The first major results for k-limited broadcast domination were given in 2018 and are specific to 2-limited dominating broadcasts in trees Cáceres et al. (2018a). These results were generalized to k-limited dominating broadcasts to give a best possible upper bound of $\gamma_{b,k}(T) \leq \left\lceil \frac{k+2}{k+1} \cdot \frac{n}{3} \right\rceil$, where T is a tree on n vertices Cáceres et al. (2018b). Specific to 2-limited dominating broadcasts, if G is a connected graph of order n, then $\gamma_{b,2}(G) \leq \left\lceil \frac{4n}{9} \right\rceil$ and if G is a graph of order n that contains a dominating path, then $\gamma_{b,2}(G) \leq \left\lceil \frac{2n}{5} \right\rceil$ Cáceres et al. (2018a). Yang showed that if G is a cubic $(C_4, C_6)-$free graph of order n, then $\gamma_{b,2}(G) \leq \frac{n}{3}$ Yang (2019); Henning et al. (2021). Park recently extended this result to cubic C_4-free graph of order n Park (2023).

For each fixed positive integer k, the problem of deciding whether there exists a k-limited dominating broadcast of cost at most a given integer B is NP-complete Cáceres et al. (2018b); Yang (2019). The results of Yang (2019), respectively, establish $O(n^3)$, $O(n^2)$, $O(n^2)$, and $O(n^3)$ time algorithms for the k-limited broadcast domination number of strongly chordal graphs, interval graphs, circular arc graphs, and proper interval bigraphs. The algorithm for k-limited broadcast on strongly chordal graphs in Yang (2019) is a specialization of the $O(n^3)$ time algorithm for (general) broadcast domination on strongly chordal graphs in Yang (2015); Brewster et al. (2019).

The k-limited broadcast domination problem is a restriction of the broadcast domination problem in which vertices can broadcast with strength up to $rad(G)$. The *broadcast domination number* $\gamma_b(G)$ of a graph G is optimum solution to the ILP obtained from ILP 1.1 by setting $k = rad(G)$. The broadcast domination number of a graph was introduced in Erwin (2001). Erwin proved that, for every non-trivial connected graph,

$$\left\lceil \frac{diam(G)+1}{3} \right\rceil \leq \gamma_b(G) \leq \min\{rad(G), \gamma(G)\}.$$

It immediately follows that $\gamma_b(P_n) = \left\lceil \frac{n}{3} \right\rceil$. Broadcast domination in trees was first explored in Herke (2007) (also see Cockayne et al. (2011); Herke and Mynhardt (2009)). This work establishes $\gamma_b(T) \leq \left\lceil \frac{n}{3} \right\rceil$, where T is a tree of order n. The broadcast domination number is known for the Cartesian products of two paths Dunbar et al. (2006), two cycles Dunbar et al. (2006), and strong grids Brešar and Špacapan (2009). Note that, as $\gamma_b\left(C_m \Box C_n\right) \leq \gamma_b\left(P_m \Box C_n\right) \leq \gamma_b\left(P_m \Box P_n\right)$, the previously stated results provide bounds for $\gamma_b\left(P_m \Box C_n\right)$. A survey of results on broadcast domination can be found in Henning et al. (2021).

This paper presents lower bounds for the 2-limited broadcast domination number of the Cartesian products of two paths, a path and a cycle, and two cycles. Our computational approach completes an exhaustive search of all possible small induced sub-broadcasts of given costs on a graph. Cases which provably cannot be part of an optimal broadcast are then eliminated. This approach can likely be extended to other graphs as well as general k-limited broadcast domination.

Some intuition for our method is provided by example in Section 2. Section 3 describes and proves the correctness of the six schemes used to eliminate cases in the exhaustive search. Section 3 concludes with the statement of our main algorithm (Algorithm 2) to prove lower bounds and the proof of its correctness. Our results are summarized in Section 4. These include exact values for the 2-limited broadcast domination number of $C_m \square C_n$ for $3 \leq m \leq 6$ and all $n \geq m$, $P_m \square C_3$ for all $m \geq 3$, and $P_m \square C_n$ for $4 \leq m \leq 5$ and all $n \geq m$, and periodically optimal values for $P_m \square C_4$ and $P_m \square C_6$ for $m \geq 3$, $P_4 \square P_n$ for $n \geq 4$, and $P_5 \square P_n$ for $n \geq 5$. These results improve upon the bounds in Slobodin's M.Sc. thesis Slobodin (2021).

2 Intuition and Definitions

This section includes a high-level overview of our method, an example specific to $P_5 \square C_n$, and relevant definitions.

Definition 2.1. Let f be a 2-limited broadcast on the graph G and let $X \subseteq V(G)$. Define the *sub-broadcast g induced by X* by

$$g(x) = \begin{cases} f(x) & \text{if } x \in X \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Throughout this paper, we consider a class of graphs $G_{m,n}$ equal to $P_m \square C_n$ or $C_m \square C_n$ for a fixed number of rows m. In this way, the vertex in the ith row and jth column can be denoted by (i, j). The goal is to prove that $\gamma_{b,2}(G_{m,n})$ is greater than or equal to a function $B(m, n)$. We proceed by induction on n. After checking the appropriate base cases computationally, we assume the bound holds for all $n < n_0$ for some integer n_0. Let f be a 2-limited dominating broadcast of G_{m,n_0}. We choose values r and s such that, if the minimum cost (with respect to f) of a sub-broadcast induced by r consecutive columns of G_{m,n_0} is strictly greater than s, then $B(m, n_0) \leq \gamma_{b,2}(G_{m,n_0})$. We then exhaustively enumerate (computationally) all possible sub-broadcast induced by r consecutive columns of G_{m,n_0} of cost less than or equal to s. If it is possible to conclude that, for each possible sub-broadcast g, either g cannot be a sub-broadcast of an optimal 2-limited dominating broadcast on G_{m,n_0} or g forces $B(m, n_0) \leq \gamma_{b,2}(G_{m,n_0})$, then the desired bounds follows. See Example 2.2.

Example 2.2. Consider the following result.

Proposition 2.3. *(Slobodin et al., 2023, Theorem 3) For $n \geq 3$,*

$$\gamma_{b,2}(P_5 \square C_n) \leq n + \begin{cases} 0 & \text{for } n \equiv 0 \pmod 2 \text{ and} \\ 1 & \text{for } n \equiv 1 \pmod 2. \end{cases}$$

Suppose we wish to obtain optimal values for $\gamma_{b,2}(P_5 \square C_n)$ when $n \equiv 0 \pmod 2$ by proving that $n \leq \gamma_{b,2}(P_5 \square C_n)$. We have that $n \leq \gamma_{b,2}(P_5 \square C_n)$ for $3 \leq n \leq 16$ by computation. Suppose the bound holds for all $n < n_0$ for some $n_0 > 16$. Let f be a 2-limited dominating broadcast of $P_5 \square C_{n_0}$. Let C be the subgraph of $P_5 \square C_{n_0}$ induced by the vertices appearing in a minimum cost (with respect to f) set of eight consecutive columns (here $r = 8$). If the sub-broadcast induced by $V(C)$ has cost at least 8 (here $s = 7$), then $cost(f) \geq n_0$. It is therefore sufficient to consider, for each integer $x \leq 7$, all possible sub-broadcasts of cost x induced by $V(C)$. If it is possible to conclude that, for each such sub-broadcast g, either g cannot be a sub-broadcast of an optimal 2-limited dominating broadcast on $P_5 \square C_{n_0}$ or g forces $n_0 \leq \gamma_{b,2}(P_5 \square C_{n_0})$, then the desired bounds follows.

We conclude this section with definitions used throughout the rest of the paper.

Definition 2.4. Given two 2-limited broadcasts f and g on a graph G, for each $x \in V(G)$, define

$$(f \oplus g)(x) = \max\{f(x), g(x)\}$$

and

$$(f \ominus g)(x) = \begin{cases} 0 & \text{if } g(x) > 0 \text{ and} \\ f(x) & \text{otherwise.} \end{cases}$$

Definition 2.5. If f is a broadcast on G, then we say f *dominates* $y \in V(G)$ if there exists a vertex x such that $f(x) \geq d(x, y)$. Further, we say that f *dominates* $X \subseteq V(G)$ if it dominates every vertex $x \in X$.

Definition 2.6. Let f be a broadcast on G. The *broadcast range* of f is the set of vertices which hear a broadcast under f.

3 Eliminating Possible Induced Sub-broadcasts of Fixed Cost

Fix m and suppose we wish to establish the function $B(n)$ as a lower bound for the 2-limited broadcast domination number of $G_{m,n}$. A positive fixed number $r = r(m) \geq 5$ of columns is chosen. In the inductive step, we consider $n_0 > r + 10$ such that $B(n)$ is a lower bound for $\gamma_{b,2}(G_{m,n})$ for all $n < n_0$. Let C be the subgraph of G_{m,n_0} induced by the vertices of r consecutive columns. We complete an exhaustive search of all possible sub-broadcasts g induced by $V(C)$ and subject each such g to a series of tests in the hope of excluding g or concluding that g forces $B(n_0) \leq \gamma_{b,2}(G_{m,n_0})$. Given C, four columns are added to both the left and right of C in order to ensure that the subgraph considered is large enough to include all vertices dominated by any vertex that could potentially dominate some vertex in C.

In summary, our algorithm takes as an input, $H_{m,k} = (P_m \text{ or } C_m) \square P_k$, where $k = r + 8$, with columns labelled $c_1, c_2, \ldots, c_k$, where the vertices of C are in columns $c_5, c_6, \ldots, c_{k-4}$. Note that $(P_m \text{ or } C_m) \square P_k$ is understood to mean $P_m \square P_k$ or $C_m \square P_k$ dependent upon $G_{m,n}$. Observe that $k \geq 13$ and $n_0 \geq k + 3$. The assumptions defined previously are used in Sections 3.1 through 3.6 which describe and prove the correctness of the six schemes we use to eliminate sub-broadcasts. These schemes appear in the same order as in Algorithm 2 in Section 3.7.

3.1 Domination Requirement

Since we are looking for a lower bound for the cost of an optimal 2-limited dominating broadcast f on G_{m,n_0}, any induced sub-broadcast that forces vertices of G_{m,n_0} to not be dominated, can be eliminated.

Observation 1. *If the sub-broadcast g induced by $V(C)$ does not dominate the vertices of columns c_7, c_8, ..., c_{k-6}, then g cannot be a sub-broadcast of a dominating broadcast.*

See Figure 1. The region containing $V(C)$ is depicted by the thick black rectangle. The black circles with a black inner fill indicate vertices broadcasting at a non-zero strength. The thick red dotted lines indicate the broadcast ranges of the broadcasting vertices at their centers. In this example, there is one vertex broadcasting at strength 2 in columns c_4 and c_{k-3} and one vertex broadcasting at strength 1 in column c_{k-3}. Let $DoesNotDominate(H_{m,k}, g)$ return true if g does not dominate the vertices of columns $c_7, c_8, \ldots, c_{k-6}$ and false otherwise.

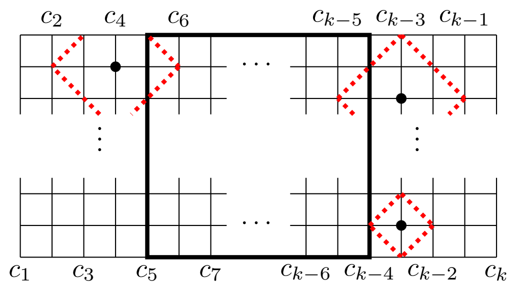

Fig. 1: The graph $H_{m,k}$ with columns labelled $c_1, c_2, \ldots, c_k$, C indicated by the thick black rectangle, and possible broadcast vertices exterior to C which can only dominate vertices in columns c_5, c_6, c_{k-5}, and c_{k-4}.

3.2 Forbidden Broadcasts

To improve the speed of our computations, we have identified four simple forbidden broadcast structures. Without loss of generality, a sub-broadcast g cannot contain any of the forbidden broadcasts shown in

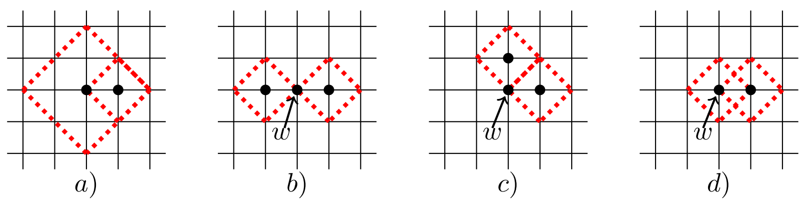

Fig. 2: Forbidden broadcasts.

Figure 2 because the broadcast in $a)$ cannot be found in an optimal 2-limited broadcast, and the broadcasts in $b)$, $c)$, and $d)$ can be replaced by a broadcast of strength 2 from w while preserving the cost of g and extending the range of g. Let $ForbiddenBroadcast(H_{m,k}, g)$ return true if g exhibits $a)$, $b)$, $c)$, or $d)$ and false otherwise.

3.3 Optimality Requirement

As we are attempting to prove a lower bound for the cost of an optimal 2-limited dominating broadcast on G_{m,n_0}, any possible sub-broadcast that is not optimal can be eliminated.

Observation 2. *If the broadcast range R of a possible sub-broadcast g induced by $V(C)$ can be dominated by a broadcast h on $H_{m,k}$ of cost strictly less than $cost(g)$, then g cannot be a sub-broadcast of an optimal dominating broadcast.*

Let $HasBroadcast(H_{m,k}, R, x)$ return true if R can be dominated with cost less than or equal to x on $H_{m,k}$ and false otherwise.

3.4 Proof by Induction

Recall that, after checking the appropriate base cases, we assume $B(n)$ is a lower bound of $\gamma_{b,2}(G_{m,n})$ for all $n < n_0$ where $n_0 \geq k{+}3$. Additionally, the values $r = k{-}8$ and s are chosen such that, if the minimum cost of a sub-broadcast induced by r consecutive columns of G_{m,n_0} is strictly greater than s, then $B(n_0) \leq \gamma_{b,2}(G_{m,n_0})$. As such, we may be able to conclude (via the inductive assumption) that possible sub-broadcasts g of cost less than or equal to s induced by r consecutive columns of G_{m,n_0} imply the bound we hope to prove. This can be done by deleting $i \leq k$ columns and "patching" the graph back together to obtain a 2-limited dominating broadcast on G_{m,n_0-i}, the cost of which we assumed to be greater than or equal to $B(n_0 - i)$. In general, given some possible induced sub-broadcast g whose broadcast range R is contained within k consecutive columns, we delete a particular selection of i columns, for each i from 1 to k. To this end, define

$$m_i = \begin{cases} \max_{n \geq n_0} \{B(n) - B(n-i)\} & \text{should it exist and} \\ \infty & \text{otherwise.} \end{cases}$$

Algorithm 1 and Lemma 3.1 formalize our approach. See Example 3.2 for an illustration of this test on $P_5 \square C_n$.

Algorithm 1: Routine to determine if assumed sub-broadcast implies bound.

function InductiveArgument $(H_{m,k}, R, x, m_1, m_2, \ldots, m_k)$;
Input : A graph $H_{m,k} = (P_m \text{ or } C_m) \square P_k$ with columns labelled from left to right by $c_1, c_2, \ldots, c_k$, and rows from 1 to m, a set of vertices $R \subseteq V(H_{m,k})$ labelled according to their row number and column label, the cost x used to dominate R by some 2-limited broadcast g whose broadcast range lies entirely within $H_{m,k}$, and m_i (for each $i = 1$ to k) as defined in Section 3.4.
Output: Value of the truth statement: "sub-broadcast implies bound."
Create a sorted list L of the columns that contain at least one vertex of R so that they are first ordered from maximum to minimum according to the number of vertices that are in R. Resolve ties by sorting so that c_i comes before c_j if $i < j$;
for i **from** 1 **to** $length(L)$ **do**
    Let S be the set of columns contained in the first i entries of list L;
    Let $H_{m,k-|S|} = (P_m \text{ or } C_m) \square P_{k-|S|}$;
    Set $R' = \{v \in R : v \in H_{m,k-|S|}\}$;
    if $HasBroadcast(H_{m,k-|S|}, R', x - m_i)$ **then return** True;
end
return False;

Lemma 3.1. *Let $k \geq 13$, $B(n)$ be a lower bound of $\gamma_{b,2}(G_{m,n})$ for all $n < n_0$ for some $n_0 \geq k+3$, and let g be a broadcast of cost x whose broadcast range R is contained in some k-column induced subgraph $H_{m,k}$ of G_{m,n_0}. If g is a sub-broadcast of an optimal broadcast on G_{m,n_0} and $InductiveArgument(H_{m,k},\ R,\ x,\ m_1,\ m_2,\ \ldots,\ m_k)$ is true, then $B(n_0) \leq \gamma_{b,2}(G_{m,n_0})$. Here m_i (for each $i = 1$ to k) is defined as in Section 3.4.*

Proof: Assume the conditions of Lemma 3.1. Let f be an optimal 2-limited dominating broadcast of G_{m,n_0} which contains g as an induced sub-broadcast. Let $f' = f \ominus g$ and let S, $H_{m,k-|S|}$, and R' be defined as in Algorithm 1. Let $G_{m,n_0-|S|}$ be constructed by removing the set of columns S from G_{m,n_0} which resulted in $InductiveArgument$ returning true on line 7 and adding edges in the natural way such that the resulting graph is isomorphic to $(P_m$ or $C_m) \square C_{n_0-|S|}$.

Observation 3. *As $n_0 \geq k+3$ and $|S| \leq k$, we have that $n_0 - |S| \geq 3$. Thus, $G_{m,n_0-|S|}$ is well-defined.*

Let f'' be the broadcast formed by restricting f' to $G_{m,n_0-|S|}$. That is, let $f''(v) = f'(v)$ for all $v \in V(G_{m,n_0-|S|})$. For each vertex $v \in V(G_{m,n_0}) \setminus V(G_{m,n_0-|S|})$ (i.e. all vertices v in the set of columns S) broadcasting with non-zero strength under f', pick a vertex u in the same row as v and in an undeleted column nearest to v and let $f''(u) = \max\{f'(u), f'(v)\}$. Note that $cost(f'') \leq cost(f')$.

Observation 4. *The broadcast f'' dominates $V(G_{m,n_0-|S|})$ with the possible exception of the vertices of R'.*

Proof: Fix $v \in V(G_{m,n_0-|S|}) \setminus R'$. As $v \notin R'$, v is not dominated by g on G_{m,n_0}. There are two cases:

Case I, v hears a broadcast from a vertex u under f' and u is not in the set of columns S. As $f''(u) \geq f'(u)$, since v hears a broadcast from u under f' on G_{m,n_0}, v hears a broadcast from u under f'' on $G_{m,n_0-|S|}$.

Case II, v hears a broadcast from a vertex u under f' and u is in the set of columns S. As $u \notin V(G_{m,n_0-|S|})$, there exists a vertex $u' \in V(G_{m,n_0-|S|})$ in the same row as u and in a nearest column undeleted to u such that $f''(u') \geq f'(u)$. Note that, by Observation 3, such a vertex exists. It suffices to check that v hears a broadcast from u' under f'' on $G_{m,n_0-|S|}$. Since $f'(u) \leq 2$ and $G_{m,n_0} = (P_m$ or $C_m) \square C_{n_0}$, there are two subcases:

Subcase II.a), u' is on the same column as v or u' is on the column between v and u on G_{m,n_0}. Since u' is in the same row as u, $d(u', v) \leq d(u, v)$ in G_{m,n_0}. As $f''(u') \geq f'(u)$, since v hears a broadcast from u under f' on G_{m,n_0}, v hears a broadcast from u' under f'' on $G_{m,n_0-|S|}$.

Subcase II.b), u is on a column between v and u' on G_{m,n_0}. As u' is on a nearest column to u that is undeleted, the column of u' on $G_{m,n_0-|S|}$ is at distance at most $d_{G_{m,n_0}}(u, v)$ (the distance between u and v in G_{m,n_0}) from the column containing v. As u' is in the same row as u and $f''(u') \geq f'(u)$, since v hears a broadcast from u under f' on G_{m,n_0}, v hears a broadcast from u' under f'' on $G_{m,n_0-|S|}$. □

By Observation 4, to dominate $G_{m,n_0-|S|}$, it suffices to find a 2-limited broadcast which dominates R'. As the function call on line 7 returns true, R' can be dominated by a 2-limited broadcast h with cost less than or equal to $x - m_{|S|}$ where $x = cost(g)$. Note that, for this to be true, $m_{|S|} \neq \infty$. Let $f''' = f'' \oplus h$. The broadcast f''' is a 2-limited dominating broadcast on $G_{m,n_0-|S|}$. By the inductive assumption,

$$B(n_0 - |S|) \leq \gamma_{b,2}(G_{m,n_0-|S|}) \leq cost(f'''). \tag{2}$$

Observe that

$$\begin{aligned} cost(f''') = cost(f'' \oplus h) = cost(f'') + cost(h) &\leq cost(f') + cost(h) \\ &\leq cost(f') + x - m_{|S|} \\ &= cost(f \ominus g) + x - m_{|S|} \\ &= cost(f) - cost(g) + x - m_{|S|} \\ &= cost(f) - m_{|S|}. \end{aligned} \tag{3}$$

As $m_{|S|} \geq B(n_0) - B(n_0 - |S|)$, when combined with Equations 2 and 3, we have that

$$B(n_0) \leq B(n_0 - |S|) + m_{|S|} \leq cost(f''') + m_{|S|} \leq cost(f) = \gamma_{b,2}(G_{m,n_0})$$

as desired.

□

Example 3.2. Suppose we wish to establish $n \leq \gamma_{b,2}(P_5 \square C_n)$; doing so will yield periodically optimal values for $\gamma_{b,2}(P_5 \square C_n)$. By computation, we have that $n \leq \gamma_{b,2}(P_5 \square C_n)$ for $3 \leq n \leq 16$. Suppose the bound holds for all $n < n_0$ where $n_0 > 16$. Let f be an optimal 2-limited dominating broadcast of $P_5 \square C_{n_0}$ and let C be the subgraph of $P_5 \square C_{n_0}$ induced by the vertices appearing in a minimum cost set of eight consecutive columns with respect to f. Suppose $V(C)$ induces the sub-broadcast g shown in Figure 3.

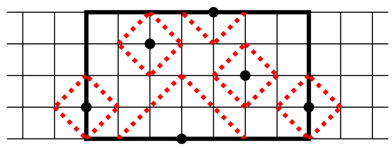

Fig. 3: Assumed sub-broadcast g induced by $V(C)$ of cost 7.

Let $f' = f \ominus g$ and let R be the range of g. The broadcast f' dominates $V(P_5 \square C_{n_0})$ with the possible exception of the vertices of R. Suppose we delete the four columns indicated in Figure 4 (Left) from the grid and add edges in the natural way such that the resulting graph $G_{5,n_0-4} = P_5 \square C_{n_0-4}$. Let f'' be

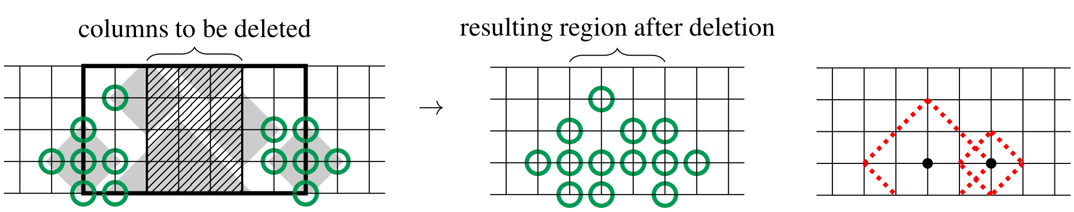

Fig. 4: (Left & Middle) Procedure which reduces G_{m,n_0} to G_{m,n_0-4}. (Right) Broadcast of cost 3 which dominates R'.

the broadcast formed by restricting f' to G_{m,n_0-4}. That is, let $f''(v) = f'(v)$ for all $v \in V(G_{m,n_0-4})$. Let $R' = \{v \in V(G_{m,n_0-4}) : v \in R\}$. The vertices of R' are indicated by the green circles in Figures 4 (Middle). The broadcast f'' dominates $V(G_{m,n_0-4})$ with the possible exception of the vertices of R'. However, R' can be dominated by the broadcast h of cost 3 as shown in Figure 4 (Right). Let $f''' = f'' \oplus h$. The broadcast f''' is a 2-limited dominating broadcast on $G_{5,n_0-4} = P_5 \square C_{n_0-4}$. By assumption, $n_0 - 4 \leq \gamma_{b,2}(P_5 \square C_{n_0-4}) \leq cost(f''')$. As

$$cost(f''') \leq cost(f'') + 3 = cost(f \ominus g) + 3 \leq cost(f) - 4,$$

we conclude that $n_0 \leq cost(f) = \gamma_{b,2}(P_5 \square C_{n_0})$ as desired.

3.5 Necessary Broadcasts

Given some possible sub-broadcast g, the broadcast structure of g may imply the existence of a larger sub-broadcast g'. If this broadcast g' does not pass the optimality requirement (see Section 3.3), then g can be eliminated. If this broadcast g' allows for the induction argument (see Section 3.4), then g implies the bound we hope to prove.

Observation 5. *As g is induced by $V(C)$, for each vertex in column c_6 or c_{k-5} not dominated by g, any dominating broadcast of G_{m,n_0} must have a vertex broadcasting at strength 2, in the same row and in column c_4 or c_{k-3}.*

See Figure 5 (Left). Let g' be the broadcast constructed from g by adding these necessary broadcasts. Let R' be the broadcast range of g'. Let $NecessaryBroadcast(H_{m,k}, g, m_1, m_2, \ldots, m_k)$ return true if either $HasBroadcast(H_{m,k}, R', cost(g') - 1)$ or $InductiveArgument(H_{m,k}, R', cost(g'), m_1, m_2, \ldots, m_k)$ returns true and false otherwise.

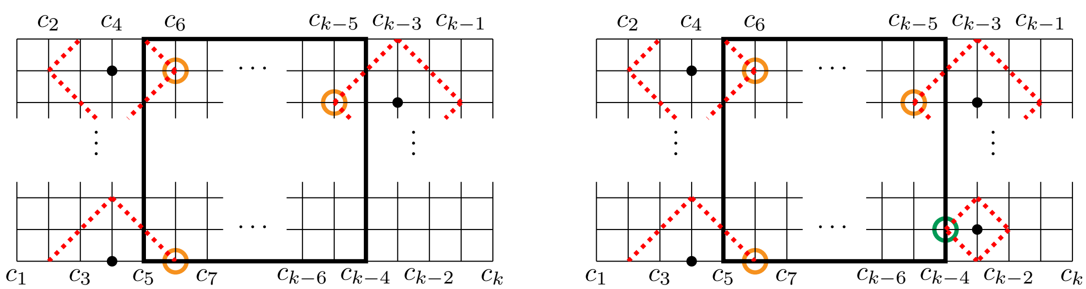

Fig. 5: The graph $H_{m,k}$ with columns labelled $c_1, c_2, \ldots, c_k$ and C indicated by the thick black rectangle. (Left) Resulting necessary broadcasts of strength 2 in columns c_4 and c_{k-3} forced by vertices undominated in columns c_6 and c_{k-5}. (Right) Necessary broadcast g' as described in Section 3.5, the vertex undominated by g' indicated by the green circle, and one of the five possible sub-broadcasts g'' which extend g' to dominate the vertex undominated by g' in column c_{k-4}.

3.6 Considering All Possible Sub-Cases

When considering all possible sub-broadcasts g and the necessary sub-broadcasts g' they imply (see Section 3.5), some sub-broadcasts may require that we consider all possible induced sub-broadcasts g'' which extend g' to dominate $V(C)$.

Let g' be the necessary broadcast implied by g as described in Section 3.5. Note g' dominates $V(C)$ with the possible exception of vertices in columns c_5 and c_{k-4}. There are many possible ways a dominating 2-limited broadcast could dominate the vertices in columns c_5 and c_{k-4} undominated by g'. Let $\mathcal{C}'$ be the collection of broadcasts formed by extending g' to include vertices from $V(G_{m,k}) \setminus [V(c_5) \cup V(c_6) \cup \cdots \cup V(c_{k-4})]$ broadcasting at strength 0, 1, or strength 2 until every vertex in columns c_5 and c_{k-4} which do not hear a broadcast under g' is dominated. See Figure 5 (Right). Let $AllSubcases(H_{m,k}, g, m_1, m_2, \ldots, m_k)$ return true if either $HasBroadcast(H_{m,k}, R', cost(g') - 1)$ or $InductiveArgument(H_{m,k}, R'', cost(g''), m_1, m_2, \ldots, m_k)$ returns true for every $g'' \in \mathcal{C}'$ (where R'' is the broadcast range of g'') and false otherwise.

Without loss of generality, we assume that, for each broadcast $g'' \in \mathcal{C}'$, $ForbiddenBroadcast(G_{m,k}, g'')$ is false. Additionally, for each $g'' \in \mathcal{C}'$, if the vertices u and v are added to the broadcast to dominate a vertex in column c_5, we may assume that the set of vertices dominated in column c_5 by u is not a subset of the set of vertices in column c_5 dominated by v, else u is redundant in terms of dominating the vertices of column c_5. The same argument applies to the vertices dominating the vertices in column c_{k-4}.

3.7 Algorithm to Prove Lower Bounds

Algorithm 2: *ProvedLowerBound* implements the battery of tests described in Section 3.1 through 3.6 to prove lower bounds for $C_m \square C_n$ and $P_m \square C_n$. The correctness of Algorithm 2 is proven by Theorem 3.3.

Algorithm 2: Routine to prove lower bound.

1 function ProvedLowerBound $(H_{m,k}, s, t, m_1, m_2, \ldots, m_k)$;
  Input : A graph $H_{m,k} = (P_m \text{ or } C_m)\square P_k$, where $k \geq 13$, with columns labelled from left to right by $c_1, c_2, \ldots, c_k$, the minimum s and maximum t possible costs of a sub-broadcast g of $H_{m,k}$ induced by columns $c_5, c_6, \ldots, c_{k-4}$, and m_i (for each $i = 1$ to k) as defined in Section 3.4.
  Output: Value of the truth statement: "lower bound proven."
2 Let $\mathcal{C}$ be all possible sub-broadcasts g of costs s to t induced by the vertices of columns $c_5, c_6, \ldots, c_{k-4}$;
3 **foreach** $g \in \mathcal{C}$ **do**
4   **if** $DoesNotDominate(H_{m,k}, g)$ **then goto** line 12;
5   **if** $ForbiddenBroadcast(H_{m,k}, g)$ **then goto** line 12;
6   Let R be the broadcast range of g;
7   **if** $HasBroadcast(H_{m,k}, R, cost(g) - 1)$ **then goto** line 12;
8   **if** $InductiveArgument(H_{m,k}, R, cost(g), m_1, m_2, \ldots, m_k)$ **then goto** line 12;
9   **if** $NecessaryBroadcast(H_{m,k}, g, m_1, m_2, \ldots, m_k)$ **then goto** line 12;
10   **if** $AllSubcases(H_{m,k}, g, m_1, m_2, \ldots, m_k)$ **then goto** line 12;
11   **return** False;
12
13 **end**
14 **return** True;

Theorem 3.3. *Let $k \geq 13$ and $B(n)$ be a lower bound of $\gamma_{b,2}(G_{m,n})$ for all $n < n_0$ for some $n_0 \geq k+3$. Let $s = \gamma_{b,2}(H_{m,k-12})$ and $t = max\left\{\ell : \exists n \geq n_0 \text{ s.t. } \frac{n\ell}{k-8} < B(n)\right\}$. If $ProvedLowerBound(H_{m,k}, s, t, m_1, m_2, \ldots, m_k)$ is true, then $B(n) \leq \gamma_{b,2}(G_{m,n})$ for all n. Here m_i (for each $i = 1$ to k) is defined as in Section 3.4.*

Proof: Assume the conditions of Theorem 3.3 and suppose $ProvedLowerBound(H_{m,k}, s, t, m_1, m_2, \ldots, m_k)$ is true. Let f be an optimal 2-limited dominating broadcast of G_{m,n_0}. Let C be the subgraph of G_{m,n_0} induced by the vertices appearing in a minimum cost set (with respect to f) of $r = k-8$ consecutive columns of G_{m,n_0}. Let $H_{m,k}$ be the subgraph of G_{m,n_0} centred on C with columns labelled $c_1, c_2, \ldots, c_k$ and let C' be the subgraph of $H_{m,k}$ induced by columns $c_7, c_8, \ldots, c_{k-6}$. Note the vertices of C are in columns $c_5, c_6, \ldots, c_{k-4}$ of $H_{m,k}$ and $C' = H_{m,k-12}$. The sub-broadcast g of f induced by $V(C)$ must dominate C' (Observation 1). As $s = \gamma_{b,2}(C')$, we have that $cost(g) \geq s$. By our choice of C,

$$cost(f) \geq \frac{n \cdot cost(g)}{r} = \frac{n \cdot cost(g)}{k-8}.$$

If $cost(g) > t$ then, by the definition of t, $cost(f) \geq B(n)$ for all $n \geq n_0$. Thus, $cost(g) \leq t$. As $\mathcal{C}$ defined on line 2 is the set of all possible sub-broadcasts of cost between s and t, inclusive, induced by the vertices of columns $c_5, c_6, \ldots, c_{k-4}$, $g \in \mathcal{C}$. As $ProvedLowerBound$ returned true, one of the function calls on lines 4 through 10 returned true for g. From the results in Sections 3.1 through 3.6, this proves the claim. □

4 Results

Our implementation includes a canonicity test for $P_m \square P_n$ so as to only consider the set $\mathcal{C}$ of all possible sub-broadcasts induced by the vertices of some set of r consecutive columns with costs between s and t, inclusive, up to isomorphism. That is, for each pair of broadcasts $g, g^* \in \mathcal{C}$, there is no group action on $P_m \square P_n$ which defines an automorphism between g and g^*. This test was done by checking that each broadcast (when expressed as a sequence) was the maximum lexicographically when compared to all broadcasts isomorphic to it. When adapting the code to work on $C_m \square C_n$, we did not update the canonicity test to reduce the number of cases up to isomorphism on $C_m \square P_n$ from $P_m \square P_n$. Fortunately, this redundancy was acceptable in terms of run time. The number of induced sub-broadcasts (i.e. $|\mathcal{C}|$) have been verified by Pólya's Theorem (see (Brualdi, 2010, Theorem 14.3.3)).

Our implementation of $ProvedLowerBound$ has allowed us to prove Theorems 4.1 through 4.11, and their respective corollaries. For each theorem, we include a table which summarizes the number of broadcasts rejected at each step of the algorithm per considered cost. Steps with zero cases are omitted. Additionally, as $AllSubcases$ considers all possible induced sub-broadcasts of a given case, the total number of cases considered will be at least $|\mathcal{C}|$.

Our implementation is written in C++ and available here Slobodin (2022). All ILP calls are run with a Gurobi solver Gurobi Optimization (2021). All computations in this section were run on Slobodin's 2021 16GB MacBook Pro with an Apple M1 Pro processor.

Theorem 4.1. *For $n \geq 3$, $\gamma_{b,2}(C_3 \square C_n) = \lceil \frac{2n}{3} \rceil$.*

Proof: Theorem 6 of Slobodin et al. (2023) proves that $\gamma_{b,2}(C_3 \square C_n) \leq \lceil 2n/3 \rceil$. Fix $r = 6$ and let $k = 14 = r + 8$. By computation, we know that the upper bound is optimal for all $3 \leq n \leq 16 = k + 2$. Given the upper bound, for $1 \leq i \leq 14 = k$, m_i is defined as follows:

$$(m_1, m_2, \ldots, m_{14}) = (1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, 9, 10).$$

As $r - 4 = 2$ and $\gamma_{b,2}(C_3 \square P_2) = 2$, set $s = 2$. Set $t = 3$. Observe that, for $n = 17$,

$$\left\lceil \frac{3n}{6} \right\rceil = 9 < 12 = \left\lceil \frac{2n}{3} \right\rceil = B(n),$$

thus $t \geq 3$. If $t > 3$, then there exists an $n > 16$ such that

$$\left\lceil \frac{4n}{6} \right\rceil \leq \frac{tn}{6} < B(n) = \left\lceil \frac{4n}{6} \right\rceil,$$

which is a contradiction. As $ProvedLowerBound\,(C_3 \square P_{14}, 2, 3, m_1, m_2, \ldots, m_{14})$ is true, the result follows. $\square$

Running *ProvedLowerBound* for the above values took less than one second.

	Cost 2	Cost 3
$\lvert\mathcal{C}\rvert$	54	302
$DoesNotDominate$	48	231
$ForbiddenBroadcast$	4	45
$InductiveArgument$	0	12
$NecessaryBroadcast + HasBroadcast$	0	8
$NecessaryBroadcast + InductiveArgument$	2	3
$AllSubcases + HasBroadcast$	0	45
$AllSubcases + InductiveArgument$	0	63

Tab. 1: Cases considered in the proof of Theorem 4.1.

Corollary 4.2. *For* $m \geq 3$, $\gamma_{b,2}\,(P_m \square C_3) = \lceil \frac{2m}{3} \rceil$.

Proof: The bound is easily verified by computation for $3 \leq m \leq 22$. Theorem 5 of Slobodin et al. (2023) proves that $\gamma_{b,2}\,(P_m \square C_3) \leq \lceil 2n/3 \rceil$ for all $m \geq 23$. As any 2-limited dominating broadcast on $P_m \square C_3$ is a 2-limited dominating broadcast on $C_m \square C_3$, $\gamma_{b,2}\,(C_3 \square C_m) \leq \gamma_{b,2}\,(P_m \square C_3)$. The result follows from Theorem 4.1. $\square$

Theorem 4.3. *For* $n \geq 4$, $\gamma_{b,2}\,(C_4 \square C_n) = 4\left\lfloor \frac{n}{6} \right\rfloor + \begin{cases} 0 & \text{for } n \equiv 0 \pmod 6, \\ 2 & \text{for } n \equiv 1 \text{ or } 2 \pmod 6, \\ 3 & \text{for } n \equiv 3 \text{ or } 4 \pmod 6, \text{ and} \\ 4 & \text{for } n \equiv 5 \pmod 6. \end{cases}$

Proof: Theorem 6 of Slobodin et al. (2023) proves that $\gamma_{b,2}\,(C_4 \square C_n)$ is less than or equal to the stated value. Fix $r = 6$ and let $k = 14 = r + 8$. By computation, we know that the stated value is optimal for all $3 \leq n \leq 16 = k + 2$. Given the upper bound, for $1 \leq i \leq 14 = k$, m_i is defined as follows:

$$(m_1, m_2, \ldots, m_{14}) = (2, 2, 3, 3, 4, 4, 6, 6, 7, 7, 8, 8, 10, 10).$$

As $r - 4 = 2$ and $\gamma_{b,2}\,(C_4 \square P_2) = 2$, set $s = 2$. Let n_6 be the least residue of n modulo 6 and let $c(n_6)$ be the constant in the upper bound dependent upon n_6. Set $t = 4$. Observe that, for $n = 19$,

$$\left\lceil \frac{4n}{6} \right\rceil = 13 < 14 = 4\left\lfloor \frac{n}{6} \right\rfloor + 2 = 4\left\lfloor \frac{n}{6} \right\rfloor + n_6 = B(n),$$

thus $t \geq 4$. If $t > 4$, then there exists an $n > 16$ such that

$$\frac{5n}{6} \leq \frac{tn}{6} < B(n) = 4\left\lfloor \frac{n}{6} \right\rfloor + c(n_6) = \frac{4n}{6} - \frac{4n_6}{6} + c(n_6) \leq \frac{4n}{6} + \frac{4}{3} \Rightarrow n < 8,$$

	Cost 2	Cost 3	Cost 4
$\lvert\mathcal{C}\rvert$	84	644	4,302
$DoesNotDominate$	83	610	3,770
$ForbiddenBroadcast$	0	16	378
$InductiveArgument$	0	0	98
$NecessaryBroadcast + HasBroadcast$	1	16	33
$NecessaryBroadcast + InductiveArgument$	0	2	20
$AllSubcases + HasBroadcast$	0	0	2
$AllSubcases + InductiveArgument$	0	0	30

Tab. 2: Cases considered in the proof of Theorem 4.3.

which is a contradiction. As $ProvedLowerBound\,(C_4\Box P_{14}, 2, 4, m_1, m_2, \ldots, m_{14})$ is true, the result follows. $\square$

Running *ProvedLowerBound* for the above values took less than one second.

Corollary 4.4. *For* $m \geq 3$, $\gamma_{b,2}\,(P_m\Box C_4) = 4\left\lfloor \frac{m}{6}\right\rfloor + \begin{cases} 0 \text{ or } 1 & \text{for } m \equiv 0 \pmod 6, \\ 2 & \text{for } m \equiv 1 \text{ or } 2 \pmod 6, \\ 3 & \text{for } m \equiv 3 \pmod 6, \\ 3 \text{ or } 4 & \text{for } m \equiv 4 \pmod 6, \text{ and} \\ 4 & \text{for } m \equiv 5 \pmod 6. \end{cases}$

Proof: The bound is easily verified by computation for $3 \leq m \leq 22$. Theorem 5 of Slobodin et al. (2023) proves that $\gamma_{b,2}\,(P_m\Box C_4)$ is less than or equal to the bound in the corollary statement for $m \geq 23$. As any 2-limited dominating broadcast on $P_m\Box C_4$ is a 2-limited dominating broadcast on $C_m\Box C_4$, $\gamma_{b,2}\,(C_4\Box C_m) \leq \gamma_{b,2}\,(P_m\Box C_4)$. The result follows from Theorem 4.3. $\square$

Theorem 4.5. *For* $n \geq 5$, $\gamma_{b,2}\,(C_5\Box C_n) = n$.

Proof: Theorem 6 of Slobodin et al. (2023) proves that $\gamma_{b,2}\,(C_5\Box C_n) \leq n$. Fix $r = 8$ and let $k = 16 = r + 8$. By computation, we know that the upper bound is optimal for all $3 \leq n \leq 18 = k + 2$. Given the upper bound, for $1 \leq i \leq 16 = k$, m_i is defined as follows:

$$(m_1, m_2, \ldots, m_{16}) = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16).$$

As $r - 4 = 4$ and $\gamma_{b,2}\,(C_5\Box P_4) = 5$, set $s = 5$. Set $t = 7$. Observe that, for $n = 19$,

$$\left\lceil \frac{7n}{8} \right\rceil = 17 < 19 = n = B(n),$$

thus $t \geq 7$. If $t > 7$, then there exists an $n > 18$ such that

$$\frac{8n}{8} \leq \frac{tn}{8} < B(n) = n,$$

which is a contradiction. As $ProvedLowerBound\,(C_5\Box P_{16}, 5, 7, m_1, m_2, \ldots, m_{16})$ is true, the result follows. $\square$

Running *ProvedLowerBound* for the above values took less than one minute.

	Cost 5	Cost 6	Cost 7
$\lvert\mathcal{C}\rvert$	264,148	1,925,104	12,162,548
$DoesNotDominate$	264,115	1,922,880	12,103,722
$ForbiddenBroadcast$	8	1,423	48,899
$HasBroadcast$	0	161	5,198
$InductiveArgument$	25	632	4,696
$NecessaryBroadcast$ $+InductiveArgument$	0	5	27
$AllSubcases + HasBroadcast$	0	27	0
$AllSubcases$ $+InductiveArgument$	0	48	30

Tab. 3: Cases considered in the proof of Theorem 4.5.

Corollary 4.6. *For* $m \geq 3$*,* $m \leq \gamma_{b,2}\left(P_m \square C_5\right) \leq m+1$.

Proof: Theorem 5 of Slobodin et al. (2023) proves that $\gamma_{b,2}(P_m \square C_5) \leq m+1$. The lower bound is easily verified by computation for $3 \leq m \leq 4$. As any 2-limited dominating broadcast on $P_m \square C_5$ is a 2-limited dominating broadcast on $C_m \square C_5$, $\gamma_{b,2}\left(C_5 \square C_m\right) \leq \gamma_{b,2}\left(P_m \square C_5\right)$. The result follows from Theorem 4.5. □

Theorem 4.7. *For* $n \geq 6$*,* $\gamma_{b,2}\left(C_6 \square C_n\right) = n + \begin{cases} 0 & \text{for } n \equiv 0 \pmod 4 \text{ and} \\ 1 & \text{for } n \equiv 1, 2, \text{ or } 3 \pmod 4. \end{cases}$

Proof: Theorem 6 of Slobodin et al. (2023) proves that $\gamma_{b,2}\left(C_6 \square C_n\right)$ is less than or equal to the stated value. Fix $r=8$ and let $k=16=r+8$. By computation, we know that the stated value is optimal for all $3 \leq n \leq 18 = k+2$. Given the upper bound, for $1 \leq i \leq 16 = k$, m_i is defined as follows:

$$(m_1, m_2, \ldots, m_{16}) = (2, 3, 4, 4, 6, 7, 8, 8, 10, 11, 12, 12, 14, 15, 16, 16).$$

As $r-4=4$ and $\gamma_{b,2}\left(C_6 \square P_4\right) = 5$, set $s=5$. Let n_4 be the least residue of n modulo 4 and let $c(n_4)$ be the constant in the upper bound dependent upon n_4. Set $t=8$. Observe that, for $n=19$,

$$\left\lceil \frac{8n}{8} \right\rceil = 19 < 20 = n+1 = n + c(n_4) = B(n),$$

thus $t \geq 8$. If $t > 8$, then there exists an $n > 18$ such that

$$\frac{9n}{8} \leq \frac{tn}{8} < B(n) = n + c(n_4) \leq n+1 \Rightarrow n < 8$$

which is a contradiction. As $ProvedLowerBound\left(C_6 \square P_{16}, 5, 8, m_1, m_2, \ldots, m_{16}\right)$ is true, the result follows. □

Running *ProvedLowerBound* for the above values took less than seven minutes.

Corollary 4.8. *For* $m \geq 3$*,* $\gamma_{b,2}\left(P_m \square C_6\right) = m + \begin{cases} 0 \text{ or } 1 & \text{for } m \equiv 0 \pmod 4 \text{ and} \\ 1 & \text{for } m \equiv 1, 2, \text{ or } 3 \pmod 4. \end{cases}$

	Cost 5	Cost 6	Cost 7	Cost 8
$\lvert\mathcal{C}\rvert$	635,628	5,506,384	41,289,876	273,548,430
$DoesNotDominate$	635,625	5,506,080	41,277,225	273,227,125
$ForbiddenBroadcast$	0	138	9,204	278,760
$HasBroadcast$	0	21	1,368	31,477
$InductiveArgument$	0	67	1,698	10,361
$NecessaryBroadcast$ $+ HasBroadcast$	3	78	330	563
$NecessaryBroadcast$ $+InductiveArgument$	0	0	39	102
$AllSubcases$ $+HasBroadcast$	0	0	1,262	5914
$AllSubcases$ $+InductiveArgument$	0	0	78	204

Tab. 4: Cases considered in the proof of Theorem 4.7.

Proof: Theorem 5 of Slobodin et al. (2023) proves that $\gamma_{b,2}(P_m \square C_6) \leq m+1$. The lower bound is easily verified by computation for $3 \leq m \leq 5$. As any 2-limited dominating broadcast on $P_m \square C_6$ is a 2-limited dominating broadcast on $C_m \square C_6$, $\gamma_{b,2}(C_6 \square C_m) \leq \gamma_{b,2}(P_m \square C_6)$. The result follows from Theorem 4.7. □

Theorem 4.9. *For $n \geq 3$, $\gamma_{b,2}(P_4 \square C_n) = 8\left\lfloor \frac{n}{10} \right\rfloor + c(n_{10})$ where $c(n_{10})$ is dependent upon the least residue n_{10} of n modulo 10 and given in Table 5.*

	Least residue n_{10} of n modulo 10									
n_{10}	0	1	2	3	4	5	6	7	8	9
$c(n_{10})$:	0	2	2	3	4	5	5	6	7	8

Tab. 5: Values of $c(n_{10})$ for $\gamma_{b,2}(P_4 \square C_n)$ stated in Theorem 4.9.

Proof: Theorem 3 of Slobodin et al. (2023) proves that $\gamma_{b,2}(P_4 \square C_n)$ is less than or equal to the stated value. Fix $r = 10$ and let $k = 18 = r + 8$. By computation, we know that the stated value is optimal for all $3 \leq n \leq 20 = k + 2$. Given the upper bound, for $1 \leq i \leq 18 = k$, m_i is defined as follows:

$$(m_1, m_2, \ldots, m_{18}) = (2, 2, 3, 4, 5, 5, 6, 7, 8, 8, 10, 10, 11, 12, 13, 13, 14, 15).$$

As $r - 4 = 6$ and $\gamma_{b,2}(P_4 \square P_6) = 6$, set $s = 6$. Let n_{10} be the least residue of n modulo 10 and let $c(n_{10})$ be the constant in the upper bound dependent upon n_{10}. Set $t = 8$. Observe that, for $n = 21$,

$$\left\lceil \frac{8n}{10} \right\rceil = 17 < 18 = 8\left\lfloor \frac{n}{10} \right\rfloor + 2 = 8\left\lfloor \frac{n}{10} \right\rfloor + c(n_{10}) = B(n),$$

thus $t \geq 8$. If $t > 8$, then there exists an $n > 20$ such that

$$\frac{9n}{10} \leq \frac{tn}{10} < B(n) = 8\left\lfloor \frac{n}{10} \right\rfloor + c(n_{10}) = \frac{8n}{10} - \frac{8n_{10}}{10} + c(n_{10}) \leq \frac{8n}{10} + \frac{12}{10} \Rightarrow n < 12,$$

which is a contradiction. As $ProvedLowerBound\left(P_4\Box P_{18}, 6, 8, m_1, m_2, \ldots, m_{18}\right)$ is true, the result follows. □

Running *ProvedLowerBound* for the above values took less than 30 seconds.

	Cost 6	Cost 7	Cost 8
$\lvert\mathcal{C}\rvert$	1,922,800	12,154,870	67,920,535
$DoesNotDominate$	1,922,790	12,153,957	67,886,561
$ForbiddenBroadcast$	1	546	27,525
$HasBroadcast$	0	136	4,944
$InductiveArgument$	7	215	1,472
$NecessaryBroadcast + HasBroadcast$	1	8	16
$NecessaryBroadcast$ $+InductiveArgument$	0	6	12
$AllSubcases + HasBroadcast$	3	8	61
$AllSubcases$ $+InductiveArgument$	13	28	43

Tab. 6: Cases considered in the proof of Theorem 4.9.

Corollary 4.10. *(a) For $n \geq 4$, $8\left\lfloor \frac{n}{10} \right\rfloor + c(n_{10}) \leq \gamma_{b,2}\left(P_4\Box P_n\right)$ where $c(n_{10})$ is dependent upon the least residue n_{10} of n modulo 10 and given in Table 5. (b) The lower bound stated in (a) gives optimal values for $\gamma_{b,2}(P_4\Box P_n)$ for all $n \equiv 1, 4, 5,$ and $9 \pmod{10}$*

Proof: (a) As any 2-limited dominating broadcast on $P_4\Box P_n$ is a 2-limited dominating broadcast on $P_4\Box C_n$, $\gamma_{b,2}\left(P_4\Box C_n\right) \leq \gamma_{b,2}\left(P_4\Box P_n\right)$. The lower bound follows from Theorem 4.9. (b) Theorem 1 of Slobodin et al. (2023) proves that $\gamma_{b,2}\left(P_4\Box P_n\right) \leq 8\left\lfloor \frac{n}{10} \right\rfloor + d(n_{10})$ where $d(n_{10})$ is dependent upon the least residue n_{10} of n modulo 10 and given in Table 7. The result follows. □

	Least residue n_{10} of n modulo 10									
n_{10}	0	1	2	3	4	5	6	7	8	9
$d(n_{10})$	1	2	3	4	4	5	6	7	8	8

Tab. 7: Values of $d(n_{10})$ for $\gamma_{b,2}\left(P_4\Box P_n\right)$ stated in Theorem 3.1 of Slobodin et al. (2023).

Theorem 4.11. *For $n \geq 3$,* $\gamma_{b,2}\left(P_5\Box C_n\right) = n + \begin{cases} 0 & \text{for } n \equiv 0 \pmod{2} \text{ and} \\ 1 & \text{for } n \equiv 1 \pmod{2}. \end{cases}$

Proof: Theorem 3 of Slobodin et al. (2023) proves that $\gamma_{b,2}\left(P_5\Box C_n\right)$ is less than or equal to stated value. Fix $r = 11$ and let $k = 19 = r + 8$. By computation, we know that the stated value is optimal for all $3 \leq n \leq 21 = k + 2$. Given the upper bound, for $1 \leq i \leq 19 = k$, m_i is defined as follows:

$$(m_1, m_2, \ldots, m_{19}) = (2, 2, 4, 4, 6, 6, 8, 8, 10, 10, 12, 12, 14, 14, 16, 16, 18, 18, 20).$$

As $r - 4 = 7$ and $\gamma_{b,2}(P_5 \square P_7) = 8$, set $s = 8$. Let n_2 be the least residue of n modulo 2 and let $c(n_2)$ be the constant in the upper bound dependent upon n_2. Set $t = 11$. Observe that, for $n = 23$,

$$\left\lceil \frac{11n}{11} \right\rceil = 23 < 24 = n + 1 = n + c(n_2) = B(n),$$

thus $t \geq 11$. If $t > 11$, then there exists an $n > 21$ such that

$$\frac{12n}{11} \leq \frac{tn}{11} < B(n) = n + c(n_2) \leq n + 1 \Rightarrow n < 11,$$

which is a contradiction. As $ProvedLowerBound\,(P_5 \square P_{19}, 8, 11, m_1, m_2, \ldots, m_{19})$ is true, the result follows. $\square$

Running *ProvedLowerBound* for the above values took less than 30 minutes.

	Cost 8	Cost 9	Cost 10	Cost 11
$\lvert\mathcal{C}\rvert$	777,158,275	5,239,827,968	32,027,967,253	179,128,860,188
$DoesNotDominate$	777,158,269	5,239,826,944	32,027,887,652	179,125,748,233
$ForbiddenBroadcast$	0	514	61,253	2,782,915
$HasBroadcast$	0	213	14,112	311,874
$InductiveArgument$	5	287	4,208	17,127
$NecessaryBroadcast$ + $HasBroadcast$	1	7	14	21
$NecessaryBroadcast$ +$InductiveArgument$	0	3	12	14
$AllSubcases$ +$HasBroadcast$	0	0	1	25
$AllSubcases$ +$InductiveArgument$	0	0	15	26

Tab. 8: Cases considered in the proof of Theorem 4.11.

Corollary 4.12. *For* $n \geq 5$, $\gamma_{b,2}(P_5 \square P_n) = n + \begin{cases} 0 \text{ or } 1 & \text{for } n \equiv 0 \pmod 2 \text{ and} \\ 1 & \text{for } n \equiv 1 \pmod 2. \end{cases}$

Proof: Theorem 1 of Slobodin et al. (2023) proves that $\gamma_{b,2}(P_5 \square P_n) \leq n+1$. As any 2-limited dominating broadcast on $P_5 \square P_n$ is a 2-limited dominating broadcast on $P_5 \square C_n$, $\gamma_{b,2}(P_5 \square C_n) \leq \gamma_{b,2}(P_5 \square P_n)$. The result follows from Theorem 4.11. $\square$

5 Future Work

This paper presents a method for computationally proving lower bounds for the 2-limited broadcast domination of the Cartesian product of two paths, a path and a cycle, and two cycles. Exact values for the 2-limited broadcast domination number of $C_m \square C_n$ for $3 \leq m \leq 6$ and all $n \geq m$, $P_m \square C_3$ for all

$m \geq 3$, and $P_m \square C_n$ for $4 \leq m \leq 5$ and all $n \geq m$ have been found, as have periodically optimal values for $P_m \square C_4$ and $P_m \square C_6$ for $m \geq 3$, $P_4 \square P_n$ for $n \geq 4$, and $P_5 \square P_n$ for $n \geq 5$. Our method can likely be extended to other graphs and k-limited broadcast domination for $k > 2$. We note the follow rather natural questions.

Problem 5.1. *Can this method be optimized further to prove bounds on larger graphs or graph other than the Cartesian product of two paths, a path and a cycle, and two cycles?*

Problem 5.2. *Can this method be altered to prove bounds for the k-limited broadcast domination number on the Cartesian product of two paths, a path and a cycle, and two cycles?*

Note: Using the methods described in this paper, and an improved backtracking technique, we have also proven, for $n \geq 8$,

$$\gamma_{b,2}(C_8 \square C_n) = 8 \left\lfloor \frac{n}{6} \right\rfloor + \begin{cases} 0 & \text{for } n \equiv 0 \pmod 6, \\ 2 & \text{for } n \equiv 1 \pmod 6, \\ 4 & \text{for } n \equiv 2 \pmod 6, \\ 6 & \text{for } n \equiv 3 \text{ or } 4 \pmod 6, \text{ and} \\ 8 & \text{for } n \equiv 5 \pmod 6. \end{cases}$$

This computation took one year and considered over 223 trillion cases. For each proof in this paper, we used a backtracking algorithm to construct the set $\mathcal{C}$ of all possible sub-broadcasts. In our improved backtracking algorithm, we forbid the addition of any ForbiddenBroadcast (see Section 3.2). The number of cases produced by this improved backtrack cannot be verified by Pólya's Theorem (see (Brualdi, 2010, Theorem 14.3.3)). As such, these results will be reported elsewhere with an updated methodology and justification.

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