Discrete Mathematics and Theoretical Computer Science ....
.... **vol. 25:3 special issue ICGT'22 #6 (2023)**

Exactly Hittable Interval Graphs

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revisions 3rd Jan. 2023, 12th Sep. 2023; accepted 31st Oct. 2023.

Given a set system (also well-known as a hypergraph) $H = \{\mathcal{U}, \mathcal{X}\}$, where $\mathcal{U}$ is a set of elements and $\mathcal{X}$ is a set of subsets of $\mathcal{U}$, an exact hitting set S is a subset of $\mathcal{U}$ such that each subset in $\mathcal{X}$ contains exactly one element in S. We refer to a set system as *exactly hittable* if it has an exact hitting set. In this paper, we study interval graphs which have intersection models that are exactly hittable. We refer to these interval graphs as *Exactly Hittable Interval Graphs* (EHIG). We present a forbidden structure characterization for EHIG. We also show that the class of proper interval graphs is a strict subclass of EHIG. Finally, we give an algorithm that runs in polynomial time to recognize graphs belonging to the class of EHIG.

Keywords: Exact Hitting Sets, Interval Graphs, Forbidden structure characterization

1 Introduction

We study classes of simple graphs which are intersection graphs of set systems that have exact hitting sets. In particular, we introduce a class of interval graphs which are intersection graphs of intervals that have exact hitting sets. We refer to this class as *Exactly Hittable Interval Graphs* (EHIG). We also present an infinite family of forbidden structures for EHIG. In the following, we introduce a setting of exact hitting sets and intersection graphs, before presenting our results.

Exact Hitting Sets: Set systems are synonymous with hypergraphs. A *hitting set* of a hypergraph H is a subset T of the vertex set of H such that T has at least one vertex from every hyperedge. If every hyperedge has exactly one element from T, then T is called an *exact hitting set*. The EXACT HITTING SET problem is a well-studied decision problem that aims to find if a given hypergraph has an exact hitting set. It finds applications in combinatorial cryptosystems (Downey and Fellows (2013)) and computational biology among many others. The EXACT HITTING SET problem is the dual of the EXACT COVER problem which, in turn, seeks to find a set cover that covers all vertices of a hypergraph such that the number of occurrences each vertex has in the cover is exactly one. Some famous examples of the EXACT COVER problem are sudoku, tiling dominoes, and the n-queens problem. The EXACT COVER problem is a special case of the MINIMUM MEMBERSHIP SET COVER problem (MMSC) (Karp (1972)). While the classic SET COVER problem seeks to find a set cover of minimum cardinality, MMSC aims to find a set cover that minimizes the maximum number of occurrences each vertex has in the cover. MMSC is known to be NP-complete on arbitrary set systems (Kuhn et al. (2005)). However, for interval hypergraphs, MMSC was shown to be solvable in polynomial time by Dom et al. (2006). If a hypergraph H has an exact hitting set, we refer to H as an *exactly hittable hypergraph*. Dhannya and Narayanaswamy (2020) have shown that a conflict-free coloring of a set of intervals is exactly a partition into sets of intervals, such that each set has an exact hitting set. This motivates the question of characterizing those sets of intervals which have an exact hitting set. A natural characterization is obtained by writing the hitting set linear program with one constraint per interval. This system is totally unimodular and thus defines an integer polytope (Dom et al. (2006)). Thus, the intervals have an exact hitting set if and only if the polytope defined by the exact hitting set linear program is non-empty. Further, it is possible to find if the interval hypergraph is exactly hittable in

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ISSN 1365–8050

polynomial time (Dom et al. (2006)). In this work, we consider a related graph theoretic version of this question - can we characterize the class of interval graphs that are the *intersection graphs* (defined in Section 1.1) of a set of intervals that have an exact hitting set? We refer to this class as the class of *Exactly Hittable Interval Graphs* (EHIG).

Intersection Graphs: The theory of graphs and hypergraphs are connected by a very well-studied notion of intersection graphs (Erdős et al. (1966)). It is well-known that every graph G is an intersection graph of some hypergraph H (Harary (1969)). H is referred to as an *intersection model* or a *set representation* of G (Golumbic (2004); Harary (1969)). Interestingly, certain special classes of graphs are characterized by the structure of their intersection models. For instance, Gavril (1974) has shown that the class of chordal graphs are the intersection graphs of subtrees of a tree . When the hyperedges are restricted to be paths on a tree, the resulting intersection graph class is that of path chordal graphs, which is a proper subclass of the class of chordal graphs (Chalopin and Gonçalves (2009); Gavril (1978); Lévêque et al. (2009); Monma and Wei (1986)). These characterizations result in recognition algorithms which are very well studied. The book by Golumbic (2004) can be considered a pilgrimage for anyone interested in the characterization and recognition of different natural sub-classes of perfect graphs. The recognition problem in the class of perfect graphs itself remained a fascinating open problem with a long history of results (survey in the classic book by Grötschel et al. (2012)) till the Strong Perfect Graph Conjecture was proven by Chudnovsky et al. (2003). While many classes have efficient recognition algorithms, there are those for which the recognition problem is NP-complete. Tolerance graphs are a sub-class of interval graphs, and the recognition problem for this class has been shown to be NP-Complete by Mertzios et al. (2010). The thinness of a graph, on the other hand, is a width parameter that generalizes certain properties of interval graphs. Interval graphs are exactly the graphs of thinness one. In their work, Bonomo-Braberman and Brito (2023) have presented characterizations of 2-thin and proper 2-thin graphs as intersection graphs of rectangles in the plane, as vertex intersection graphs of paths on a grid, and by forbidden ordered patterns. Forbidden induced subgraph characterization for restricted cases of known graph classes are well-studied. For instance, even though a structural characterization by minimal forbidden induced subgraphs for the entire class of circle graphs is not known, Bonomo-Braberman et al. (2022) have given a characterization by minimal forbidden induced subgraphs of circle graphs, restricted to split graphs. Rectangle intersection graphs are the intersection graphs of axis-parallel rectangles in the plane. A graph is said to be a k-stabbable rectangle intersection graph (k-SRIG), if it has a rectangle intersection representation in which k horizontal lines can be placed such that each rectangle intersects at least one of them. Chakraborty et al. (2021) have introduced some natural subclasses of 2-SRIG, and have shown that one of these subclasses can be recognized in linear-time if the input graphs are restricted to be triangle-free. Earlier, Chakraborty and Francis (2020) had developed a forbidden structure characterization for block graphs that are 2-ESRIG (in the case when each rectangle intersects exactly one of the k horizontal lines) and trees that are 3-ESRIG, which lead to polynomial-time recognition algorithms for these two classes of graphs. These forbidden structures are natural generalizations of asteroidal triples.

A result which has the flavour of Exact Hitting Set is in a recent paper by Bhyravarapu et al. (2021). They consider the problem of coloring the vertex set of a graph with k non-zero colors and one zero colour such that for each vertex v, there is a vertex u in $N(v)$ which has a non-zero colour different from all the other vertices in $N(v)$. This is called the $CFON^*$ colouring problem, and the goal is to find the minimum value of k for which the graph has a $CFON^*$ colouring. For $k = 1$, this problem is the Exact Hitting Set problem of a set system in which the sets are the set of neighbours of each vertex. For unit disk graphs, they show that testing if there is a $CFON^*$ coloring with one non-zero colour is NP-complete.

Forbidden Structure Characterizations: While a graph G may be identified as an intersection graph of a structured hypergraph, characterization of G based on forbidden structures has also been equally well-studied. For instance, the class of chordal graphs are characterized by the absence of induced cycles of size 4 or more (Golumbic (2004)). Similarly, by the celebrated theorem of Kuratowski (West (2000)), the class of planar graphs must not have subgraphs that are subdivisions of K_5 and $K_{3,3}$. Interval graphs are known to be the class of chordal graphs without an asteroidal triple as induced subgraph Lekkerkerker and Boland (1962). Recall that an asteroidal triple of a graph G is a set of three independent vertices such that there is path between each pair of these vertices that does not contain any vertex of the neighborhood of the third. The class of proper interval graphs is a subclass of interval graphs that do not have a $K_{1,3}$ as an induced subgraph (Roberts (1978)). Refer to Table 1 for a summary of these examples. Clearly, characterization of simple graphs based on their intersection models and forbidden structures are extremely well-studied notions in defining graph classes.

Graph Class	*Intersection Model*	*Forbidden Structures*
Simple	An exactly hittable hypergraph	NIL
Planar	Segments on a plane	Subdivisions of K_5 and $K_{3,3}$ West (2000)
Chordal	Subtrees of a tree	C_k, for $k \geq 4$ Golumbic (2004)
Path chordal	Paths on a tree	*List given in Lévêque et al. (2009)*
Interval	Subpaths on a path	C_k, for $k \geq 4$ and asteroidal triple Lekkerkerker and Boland (1962)
Proper interval	Sets of intervals not properly contained in each other	C_k, for $k \geq 4$, asteroidal triple and $K_{1,3}$ Roberts (1978)
Exactly Hittable Interval Graphs (New graph class)	Exactly hittable sets of intervals	C_k, for $k \geq 4$, asteroidal triple and induced path P_k which has, in its open neighbourhood, an independent set of $k+3$ vertices

Tab. 1: Intersection models and forbidden structures for well-known graph classes

Our results

1. We begin our set of results with a simple extension to a well-known theorem by Harary (1969) that every graph G is the intersection graph of some hypergraph H.

   Observation 1 *Every simple undirected graph is the intersection graph of an exactly hittable hypergraph. Further, if G is a chordal graph, then it is the intersection graph of an exactly hittable set of subtrees of a tree.*

   We present proof of this observation in Section 2. Further to this observation, we look at a subclass of chordal graphs, namely interval graphs, which are intersection graphs of subpaths on a path. We ask if there is an exactly hittable intersection model for every interval graph, where the intersection model consists of subpaths on a path. Interestingly, the answer is no.

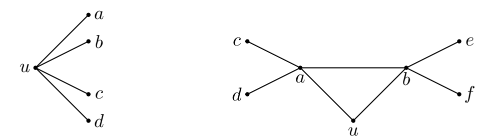
2. We introduce the class of *Exactly Hittable Interval Graphs* (EHIG), which is the set of interval graphs that have an exactly hittable interval representation. A given set of intervals defines a unique interval graph, but an interval graph can have many interval representations. We say that an interval graph is an exactly hittable interval graph if and only if it has at least one exactly hittable interval representation.

   Definition 1 (Exactly Hittable Interval Graphs) *The class of exactly hittable interval graphs is the class of interval graphs which are intersection graphs of intervals that have exact hitting sets.*

   We present a forbidden structure characterization for EHIG. First, we define a family $\mathcal{F}$ of simple graphs as follows:

   Definition 2 *For each $k \geq 1$, $\mathcal{F}_k$ denotes the set of connected interval graphs whose vertex set can be partitioned into an induced path P consisting of k vertices and the open neighbourhood of P (consisting of only those vertices which are not in P) which is an independent set of size $k+3$. Further, $\mathcal{F}$ is defined to be $\bigcup_{k\geq 1} \mathcal{F}_k$.*

Our main contribution in this paper is to prove that every graph in $\mathcal{F}$ is a forbidden structure for EHIG. See Fig.1 for examples of forbidden structures. In Fig.1(i), u is the induced path P consisting of one vertex with an independent set of four vertices a, b, c, d in its neighbourhood. Similarly, in Fig.1(ii), a-b is the induced path P consisting of two vertices and $\{c, d, u, e, f\}$ is an independent set of five vertices in the neighbourhood of P.



(i) (ii)

Fig. 1: Two simple instances of forbidden structures

By Definition 2, for any k, the set $\mathcal{F}_k$ may contain more than one graph which are forbidden structures. For example, both the graphs in Fig. 2 belong to $\mathcal{F}_2$.

(i) (ii)

Fig. 2: Two instances of forbidden structures in $\mathcal{F}_2$

It may, however, be noted that, Fig. 2(ii) contains $K_{1,4}$ as an induced subgraph, which itself is a forbidden structure in $\mathcal{F}_1$.

Theorem 1 *An interval graph G is exactly hittable if and only if it does not contain any graph from the set $\mathcal{F}$ as an induced subgraph.*

This theorem is proved in Section 3. We believe that this result is an interesting addition to the existing graph characterizations, primarily because we could not find such an equivalence elsewhere in the literature, including graph classes repositories like graphclasses.org.

3. In Section 2, we introduce, what we refer to as, a *canonical interval representation* for an interval graph. Given an interval graph G, a canonical interval representation H_G is an **interval hypergraph** given by $H_G = ([n], \mathcal{I})$, where $[n] = \{1, \ldots, n\}$ and $\mathcal{I} \subseteq \{\{i, i+1, \ldots, j\} \mid i \leq j, i, j \in [n]\}$, and all intervals have distinct left endpoints and distinct right endpoints. Further, for each $v \in G$, $I_v \in \mathcal{I}$ denotes the corresponding interval. For construction of H_G, we start with the well known linear ordering of maximal cliques associated with an interval graph (Gilmore and Hoffman (2011); Golumbic (2004)). An interval representation is constructed from the ordering such that the intersection graph of this representation is isomorphic to G. By construction, there exists exactly one canonical interval representation for every interval graph. While the canonical representation may be of independent interest, this representation is crucial in proving Theorem 1 in this paper.

   In Section 2, we prove the following theorem.

   Theorem 2 *Let G be an interval graph. Let H_G be its canonical interval representation constructed as described in Section 2. Then, G is exactly hittable if and only if H_G is exactly hittable.*

4. Given an interval graph G and its canonical interval representation H_G, we show that the algorithm by Dom et al. (2006) to solve the MMSC problem in interval hypergraphs can be used to recognize EHIG. We present the details in Section 3.2.

5. We show that the class EHIG is positioned between the class of proper interval graphs and the class of interval graphs in the containment hierarchy of graph classes.

Theorem 3 *Proper interval graphs $\subset$ EHIG $\subset$ Interval Graphs.*

The proof of the second part of the above theorem follows from the definition of EHIG. We prove the first part of the containment relationship in Section 3.3. Interestingly, the smallest forbidden structure of EHIG is $K_{1,4}$ whereas that of the class of proper interval graphs is $K_{1,3}$.

1.1 Preliminaries

Definition 3 (Intersection Graphs) *Given a set system $\mathcal{X} = (\mathcal{U}, \mathcal{S})$, the intersection graph $G(\mathcal{X})$ of sets in $\mathcal{X}$ is the simple graph obtained as follows. For every set $S \in \mathcal{S}$, there exists a vertex $v_S \in G$. An edge (v_{S_i}, v_{S_j}) occurs in G if and only if there exists two sets $S_i, S_j \in \mathcal{F}$ such that $S_i \cap S_j \neq \emptyset$. The family $\mathcal{S}$ is called a set representation of the graph G. A set representation is also referred to as an intersection model (Golumbic (2004), Harary (1969)).*

A hypergraph $H = (\mathcal{V}, \mathcal{E})$ is a graph theoretic representation of a set system $\mathcal{X} = (\mathcal{U}, \mathcal{S})$, where the set $\mathcal{V}$ corresponds to $\mathcal{U}$ and the set $\mathcal{E}$ corresponds to $\mathcal{S}$. The set $\mathcal{V}$ contains *vertices* of hypergraph H and the set $\mathcal{E}$ contains *hyperedges*. In the intersection graph G, for every hyperedge $E \in \mathcal{E}$, there exists a vertex $v_E \in G$. An edge (v_{E_i}, v_{E_j}) occurs in G if and only if the hyperedges E_i and E_j have a non-empty intersection.

Definition 4 (Interval Graphs) *A graph $G = (V, E)$ is an interval graph if there exists an assignment of intervals on the real line to each vertex $v \in V(G)$ such that for each edge (u, v) in G, the associated intervals $I(u)$ and $I(v)$ have a non-empty intersection. The set of intervals $\{I(v)\}_{v \in V(G)}$ is an interval representation or intersection model of G.*

Open and Closed neighborhoods: For a vertex v in a graph $G = (V, E)$, the open neighborhood of v in G, denoted by $N(v)$, is the set $\{u \in V \mid \{u, v\} \in E\}$ and the closed neighborhood of v in G, denoted by $N[v]$, is the set $N(v) \cup \{v\}$.

Definition 5 *Cheilaris and Smorodinsky (2012) An* ***interval hypergraph*** *is any hypergraph $H = ([n], \mathcal{I})$, where $[n] = \{1, \ldots, n\}$ and $\mathcal{I} \subseteq \{\{i, i+1, \ldots, j\} \mid i \leq j, i, j \in [n]\}$.*

Each hyperedge in $\mathcal{I}$ is a set of consecutive integers, which we call an *interval*. In an interval $I = \{i, i + 1, \ldots, j\}$, i and j are the *left* and *right endpoints* of I respectively, which we denote by $l(I)$ and $r(I)$, respectively. We use $\mathcal{V}(H)$ (or simply $\mathcal{V}$) and $\mathcal{I}(H)$ (or simply $\mathcal{I}$) to denote the vertex set and the hyperedge set, respectively, of an interval hypergraph H. An interval hypergraph is said to be *proper* if no interval is contained in another interval. If, for an interval graph G, there exists an interval representation in which no interval is properly contained inside another interval, then G is a *proper interval graph*.

An interval graph is characterized by the existence of a linear ordering of its maximal cliques. In Section 3, we use the following characterization to obtain an exactly hittable interval representation for an interval graph, if such a representation exists.

Theorem 4 (Gilmore and Hoffman (2011)) *The maximal cliques of an interval graph G can be linearly ordered such that, for every vertex x of G, the maximal cliques containing x occur consecutively.*

The class of interval graphs is a subfamily of the class of *chordal graphs*, which, in turn, is a subfamily of the class of *perfect graphs*. A chordal graph is a simple graph that does not contain any induced cycle of size ≥ 4 (Golumbic (2004)). Chordal graphs are known to be intersection graphs of subtrees of a tree (Gavril (1974)). A clique tree T of a graph G is a tree with the maximal cliques of G as nodes, such that for every vertex v of G, the maximal cliques containing v induce a subtree $T(v)$ in T. In fact, chordal graphs are exactly the graphs that admit a clique tree (McKee and McMorris (1999)). A clique tree is also known as a *tree decomposition* of a graph.

Note: We draw the reader's attention to the distinction between *interval hypergraphs* and *interval graphs*, and *proper interval hypergraphs* and *proper interval graphs*, as these are used extensively throughout the paper. Furthermore, recall that an interval graph is an Exactly Hittable Interval Graph if it has an intersection model, made of intervals, that has an exact hitting set. On the other hand, an Exactly Hittable Interval Hypergraph is one that has an exact hitting set.

Observation 2 *Since our goal is to characterize interval graphs that have an exactly hittable interval representation, we assume without loss of generality that, in the graph G, for every sequence of consecutive maximal cliques in a linear ordering, there is at most one vertex which starts and ends in this sequence.*

Indeed, if a given graph violates this property and there are two or more vertices that start at the same clique and end at the same clique in a sequence, then we retain only one of those vertices. The justification for this assertion is that if the resulting graph has an exactly hittable interval representation, so does the original graph.

Notations: All other definitions and notations on simple graphs, used throughout this paper, have been taken from West (2000).

2 A Canonical Interval Representation

In this section, we obtain *a canonical interval representation* H_G of a given interval graph G. The canonical interval representation is nothing but a special intersection model of G. Consequently, the intersection graph of intervals in H_G is isomorphic to G. The construction follows a well-defined set of steps with the result that every interval graph has a unique canonical interval representation. The canonical representation H_G is obtained by *stretching* intervals so that all intervals have distinct left endpoints and distinct right endpoints. In other words, no pair of intervals start at the same point or end at the same point. The canonical interval representation is crucial to the proof of our main result in Section 3.

Outline: The starting point of this construction is to use the well known linear ordering of maximal cliques associated with an interval graph (Golumbic (2004)) (refer Theorem 4). Fig. 3 gives an illustration of how to obtain the canonical interval representation of an interval graph. Let $G = (V, E)$ be the given interval graph. Let $\mathcal{O} = \{Q_1, Q_2 \dots Q_t\}$ be a linear ordering of maximal cliques in G. For each $v \in V(G)$, let the interval representation of G obtained from $\mathcal{O}$ be $I(v) = [l(v), r(v)]$, where $l(v)$ is the index of the leftmost clique in $\mathcal{O}$ that contains v, and $r(v)$ is the index of the rightmost clique containing v. Let $\mathcal{I}' = \{I(v) \mid v \in V(G)\}$. To construct the canonical interval representation, we associate a gadget D_i with maximal clique Q_i, for $1 \leq i \leq t$. For every maximal clique Q_i, we look at D_i and stretch those intervals in $\mathcal{I}'$ that either start at i or end at i. Intuitively, we can think of $I(v)$ as being *stretched to the left* if $l(v) = i$ and as being *stretched to the right* if $r(v) = i$. Inside gadget D_i, there is a point, which we denote by z_i, with the following property: any interval for which $l(v) = i$, starts at z_i or to the left of z_i and any interval for which $r(v) = i$, ends at z_i or to the right of z_i. We refer to z_i as the *zero-point* of gadget D_i. The exact construction of stretched intervals is detailed in the subsequent paragraphs.

The gadgets $D_1, D_2 \dots, D_t$ are arranged in the same order as that of the maximal cliques in $\mathcal{O}$. Further, for each $v \in V(G)$, the stretched interval associated with $I(v)$ has $D_{l(v)}$ as its left-most gadget and $D_{r(v)}$ as its rightmost gadget. To complete the construction, between each pair of consecutive gadgets, we add an additional point, and we refer to these points as intermediate points. These points play a crucial role in our characterization of EHIGs in Section 3.1. The stretched interval of $I(v)$ contains all these additional points between consecutive gadgets in the ordered set $\{D_{l(v)}, D_{l(v)+1}, \dots, D_{r(v)}\}$. Let $H_G = (\mathcal{V}, \mathcal{I})$ denote the canonical interval hypergraph thus obtained. $\mathcal{V}$ is the set of all points internal to the gadgets (defined below) and the $t-1$ additional points between consecutive gadgets (as described above). The intervals in $\mathcal{I}$ are the stretched intervals corresponding to each interval in $\mathcal{I}'$. We now describe the gadget D_i associated with maximal clique Q_i, $1 \leq i \leq t$.

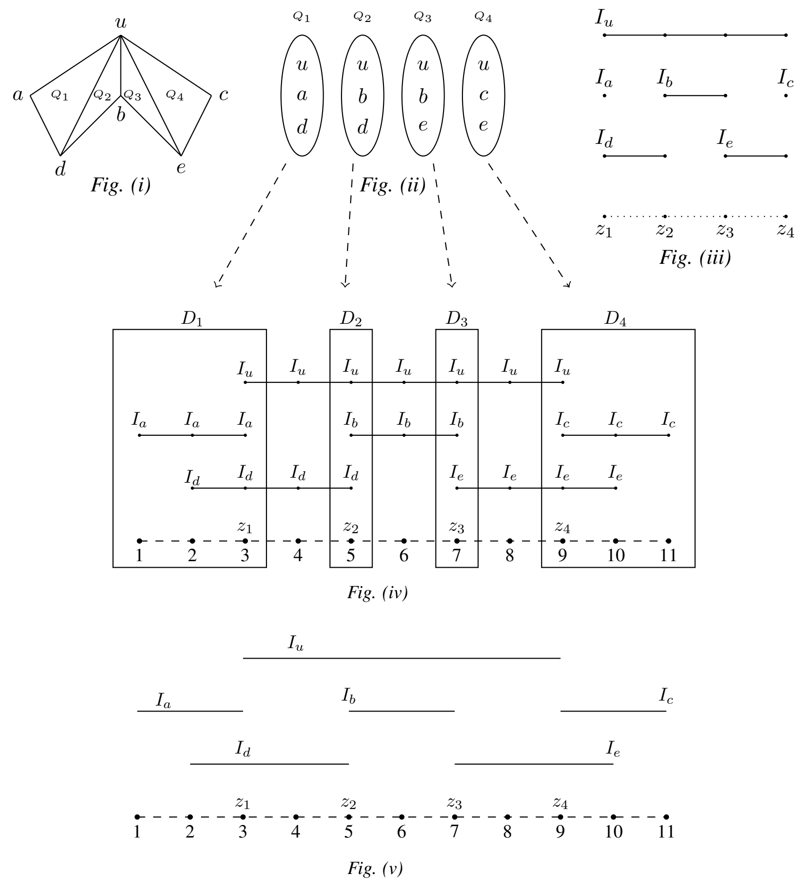


Fig. 3: Construction of Canonical Interval Representation *(i)* Interval Graph G with its maximal cliques Q_1, Q_2, Q_3, Q_4 *(ii)* Linear ordering of maximal cliques $\mathcal{O} = \{Q_1, Q_2, Q_3, Q_4\}$ *(iii)* Interval representation of G obtained from $\mathcal{O}$ *(iv)* Gadgets D_1 to D_4 *(v)* Canonical interval representation for G

Construction of the gadget D_i for maximal clique Q_i: Let $\{I(v_1), I(v_2), \ldots, I(v_m)\}$ be the ordered set of intervals such that for each $1 \leq k \leq m$, $l(v_k) = i$ and $r(v_k) > r(v_j)$ whenever $1 \leq k < j \leq m$. In other words, the ordered set considers the intervals whose left endpoint is i in descending order of their right endpoints. Then, for each $1 \leq k \leq m$, the left endpoint of the interval of $I(v_k)$ is stretched $k-1$ points to the left. By this, $l(v_1)$ is kept at z_i itself, as no stretching is done on it (see Fig. 4).

On the integer line, the left end point of $I(v_k) \in D_1$, which is the most left stretched interval in D_1, is taken as point 1. We next consider those intervals $I(v)$ such that $r(v) = i$. Let $\{I(v_1), I(v_2), \ldots, I(v_m)\}$ be the ordered set of intervals such that for each $1 \leq k \leq t$, $r(v_k) = i$ and $l(v_k) < l(v_j)$ whenever $1 \leq k < j \leq m$. In other words, the ordered set considers the intervals whose right endpoint is i in ascending order of their left endpoints. Then, for each $1 \leq k \leq m$, the right endpoint of the interval of $I(v_k)$ is stretched $k-1$ points to the right. On the integer line, the right endpoint of the stretched interval of $I(v_k)$ would be $z_i + k - 1$. This completes the description of the gadget D_i. Note that for $I(v)$ in $\mathcal{I}$, the stretched interval is stretched to the left only in the leftmost gadget in which it is present, and it is stretched to the right in the rightmost gadget in which it is present. By construction, no two intervals share the same left endpoint and the same right endpoint.

Lemma 1 *Let H_G be the canonical interval representation of graph G as constructed using the above procedure. Then, G is isomorphic to the intersection graph of intervals in H_G.*

Proof: The gadgets $D_1, \ldots, D_t$ are arranged in the same order as the maximal cliques in the ordered set $\mathcal{O} = \{Q_1, Q_2 \ldots Q_t\}$. For each $v \in G$, the starting gadget (and the ending gadget) of interval $I(v)$ and the starting maximal clique (and the ending maximal clique) of vertex v in $\mathcal{O}$ are the same by construction. Further,

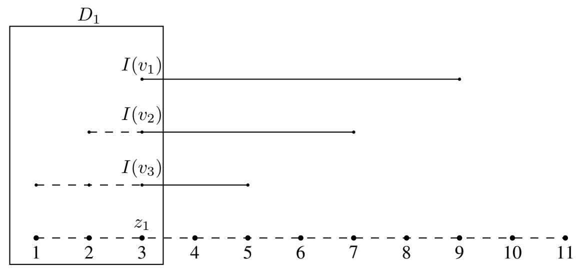


Fig. 4: *Stretching* intervals to the left

$I(v)$ contains all the points in the intervening gadgets between the starting and ending gadgets of $I(v)$ just as v occurs in all the intervening maximal cliques between the starting and ending maximal cliques to which v belongs to. It follows that $I(u)$ and $I(v)$ intersect if and only if the corresponding stretched intervals have a non-empty intersection. Thus the intersection graph of intervals in H_G is isomorphic to G. □

3 Exactly Hittable Interval Graphs

Characterizing simple graphs as intersection graphs is a well-pursued line of study in graph theory. Harary (1969) had presented results on this problem in his book. We address the question of when a simple graph is the intersection graph of an exactly hittable hypergraph. We modify the proof given by Harary to answer this question. In addition, we present similar results for the class of chordal graphs (refer to Section 1.1 for definition). We recall and prove Observation 1 about arbitrary graphs and arbitrary chordal graphs.

Observation 1 *Every simple undirected graph is the intersection graph of an exactly hittable hypergraph. Further, if G is a chordal graph, then it is the intersection graph of an exactly hittable set of subtrees of a tree.*

Proof: The proof of the first statement is based on a slight modification to the intersection model constructed from G in Theorem 2.5 in the book by Harary (1969). Let $H = (\mathcal{V}, \mathcal{E})$ be the intersection model constructed as follows. The universe $\mathcal{V}$ of the hypergraph is $V(G) \cup E(G)$. The set $\mathcal{E}$ contains a hyperedge E_v for each vertex $v \in V(G)$, and E_v contains all the edges incident on v and the element v. Clearly, the intersection graph of H is isomorphic to G and $V(G)$ is an exact hitting set of H.
The proof of the second statement, which is for a chordal graph G, is similar and is as follows. Since G is a chordal graph let it be isomorphic to the intersection graph of some subtrees of a tree T. In particular, let T be the clique tree of the chordal graph G (Golumbic (2004)). Let $\{T_v \mid v \in V(G)\}$ be the set of subtrees in T, where T_v is the subtree associated with v and the tree nodes in T_v correspond to those maximal cliques in G which contain the vertex v. We modify T to get T' by adding $n = |V(G)|$ new nodes, each corresponding to a vertex in $V(G)$. For each $v \in V(G)$, the new node corresponding to v is made adjacent in T to some node in T_v. The resulting tree is T' and T'_v is the subtree of T' consisting of T_v and the new node corresponding to v. Clearly, the newly added nodes form an exact hitting set of the set $\{T'_v \mid v \in V(G)\}$ in T', and the intersection graph of the subtrees $\{T'_v \mid v \in G\}$ is the same as G. □

Interestingly, not every interval graph has an exactly hittable interval representation. In this paper, we present a forbidden structure characterization for the class of interval graphs that have an exactly hittable interval representation. In this section, we prove that every graph in $\mathcal{F}$ (see Definition 1) is a forbidden structure for EHIG. First, we state and prove one direction of Theorem 1.

We use the following notations throughout the section. H' denotes an interval representation of G. We denote the open neighbourhood of vertex v by $N(v)$. $N(P)$ denotes open neighbourhood of all vertices in path P, excluding the vertices in P. $\mathcal{I}(P)$ denotes the set of intervals in H' corresponding to vertices in path P, $X_{N(P)}$ denotes the set of independent vertices in $N(P)$ and $\mathcal{I}(X_{N(P)})$ denotes set of intervals in H' corresponding to $X_{N(P)}$.

Lemma 2 *Let G be an interval graph. Let $F \in \mathcal{F}$ be any forbidden structure. If G contains F as an induced subgraph, then G is not an Exactly Hittable Interval Graph.*

Proof: Our proof is by contradiction. Let H' be any exactly hittable interval representation of G and let G contain $F \in \mathcal{F}$ as an induced subgraph. Let F contain P, an induced path of length k in G that has an independent set of at least $k+3$ vertices in its neighbourhood. Let S be an exact hitting set of H'. Recall that $\mathcal{I}(P)$ denotes the set of intervals in H' corresponding to vertices in path P. By our assumption that G contains F, the number of intervals in $\mathcal{I}(X_{N(P)})$ is at least $k+3$. Hence $|\mathcal{I}(X_{N(P)}) \cap S| \geq k+3$. Since $X_{N(P)}$ is an independent set, there can be at most two intervals in $\mathcal{I}(X_{N(P)})$ that have at least one endpoint each outside the union of intervals in $\mathcal{I}(P)$ - one on either side of P. Therefore, even if these two intervals in $\mathcal{I}(X_{N(P)})$ are hit outside the intervals in $\mathcal{I}(P)$ at either ends, the remaining $k+1$ independent intervals have to be hit inside the union of intervals in $\mathcal{I}(P)$. Hence $|\mathcal{I}(P) \cap S| \geq k+1$. But there are only k intervals inside $\mathcal{I}(P)$. Therefore, by the pigeonhole principle, at least one interval among the intervals in $\mathcal{I}(P)$ has to be hit more than once. Thus S cannot be an exact hitting set of H'. We have arrived at a contradiction to the assumption that H' is exactly hittable. Since we started with an arbitrary exactly hittable representation and arrived at a contradiction, we conclude that G is not exactly hittable. □

Now, we prove the other direction of Theorem 1, i.e, an interval graph G which contains no graph from the set $\mathcal{F}$ as an induced subgraph is exactly hittable. Let $\mathcal{O} = \{Q_1, Q_2 \dots Q_t\}$ be a linear ordering of maximal cliques in G (refer Theorem 4 and Section 2). Let H_G be the canonical interval representation of G obtained from $\mathcal{O}$. We use the following notations in this section. We denote a minimum clique cover of the neighbourhood of a vertex v, which is formed by the minimum number of maximal cliques in $\mathcal{O}$, by $C(N[v])$. Recall that a clique cover for a vertex set S is a set of cliques such that each vertex in S appears in at least one clique. Note that such a clique cover exists.
We prove a simple observation here.

Observation 3 *If $Q_i \dots Q_j,\ i,j \in [1,t], i \leq j$ denote the maximal cliques containing vertex $v \in V$, then $Q_j \in C(N[v])$.*

Proof: We prove this by contradiction. Let us assume that $Q_j \notin C(N[v])$. As $Q_j \neq Q_{j-1}$, there exists a vertex u in Q_j which is not in Q_{j-1}. It follows that u is not contained in any maximal cliques that occur before Q_{j-1} in $\mathcal{O}$ since the maximal cliques containing a vertex occur consecutively in the linear ordering of maximal cliques of an interval graph. Therefore, if $Q_j \notin C(N[v])$, then u is not covered. It contradicts the fact that $C(N[v])$ is a clique cover of $N[v]$. It follows that $Q_j \in C(N[v])$. □

From now on, when we refer to a minimum clique cover of the input graph, we mean a minimum clique cover formed by the minimum number of maximal cliques in $\mathcal{O}$ unless specified otherwise. Let $|C(N[v])|$ denote the number of cliques in $C(N[v])$. Similarly, we denote a minimum clique cover of vertices in the maximal cliques Q_i to Q_j in the ordering $\mathcal{O}$, $i < j$, by $C(Q_i, \dots, Q_j)$.

Our proof is based on the structural properties of a path P in G, the construction of which is presented in Algorithm 1. The structural properties of path P are proved as lemmas later in the section.

Outline of Algorithm 1:

We construct an induced path P which contains a minimal set of vertices from graph G. The vertices in path P are selected such that every maximal clique in $\mathcal{O}$ has a non-empty intersection with path P. Further, we incrementally construct a clique cover of G by taking the clique cover of the closed neighbourhood of each of the individual vertices in P.

Let $\{v_1, v_2, \dots, v_p\}$ be the ordered set of vertices in the constructed path P with respect to the linear ordering $\mathcal{O}$. Let $v_i, \dots, v_j, 1 \leq i \leq j \leq p$ be any subset of vertices in path P. We use $CC(N[v_i, v_{i+1}, \dots, v_j]),\ i \leq j$ to denote a clique cover of $(N[v_i] \cup N[v_{i+1}] \cup \dots \cup N[v_j])$ and $|CC(N[v_i, v_{i+1}, \dots, v_j])|$ to denote the number of cliques in $CC(N[v_i, v_{i+1}, \dots, v_j])$. Note that $CC(N[v_i, \dots, v_j])$ is a clique cover of graph G when $i = 1, j = p$. Thus obtained clique cover of G, $CC(N[v_1, \dots, v_p])$, is stored in $\mathcal{K}$. We denote the maximal cliques which constitute $\mathcal{K}$ in the order in which they appear in $CC(N[v_1, \dots, v_p])$, by $K_1, K_2, \dots, K_{\alpha'}$.

Algorithm 1: Construction of path P and computation of clique cover
Input: An interval graph G with a linear ordering of maximal cliques $\mathcal{O} = \{Q_1, Q_2 \dots Q_t\}$
Output: Path P

$i = 1$
$v_1 \leftarrow$ Interval in Q_1 with largest right endpoint
$P \leftarrow v_1$
$Q_r^1 =$ Maximal clique in which v_1 ends
$C(N[v_1]) =$ Minimum clique cover of $N[v_1]$
$Q_{r'}^1 =$ Maximal clique immediately preceding Q_r^1 in $C(N[v_1])$
$CC(N[v_1]) = C(N[v_1])$
while $Q_r^i \neq Q_t$ **do**
  $i = i + 1$
  $v_i =$ Interval $I \in Q_r^{i-1} \setminus Q_{r'}^{i-1}$ which has largest right endpoint; If there are more than one such vertex, then the one with the smallest left endpoint is chosen
  $P \leftarrow P \cup v_i$
  Q_r^i = Maximal clique in which v_i ends
  $CC(N[v_1, \dots, v_i]) = CC(N[v_1, \dots, v_{i-1}]) \cup C(Q_{r+1}^{i-1}, \dots, Q_r^i)$, where Q_{r+1}^{i-1} is the maximal clique immediately succeeding Q_r^{i-1} in $\mathcal{O}$
  $Q_{r'}^i$ = Maximal clique immediately preceding Q_r^i in $CC(N[v_1, \dots, v_i])$
end while
$\mathcal{K} = CC(N[v_1, \dots, v_i])$
return P

Here the notation α' is used to indicate that $\mathcal{K}$ is a minimal clique cover of G rather than a minimum clique cover.

In any perfect graph, the size of a minimum clique cover equals the size of a maximum independent set. Based on this and the fact that interval graphs are perfect graphs, we state an observation which we use in proving some important properties of the constructed clique cover of the neighbourhood of vertices in path P.

Observation 4 *In any perfect graph G', for each maximal clique K in a minimum clique cover $\mathcal{K}$ of G', there exists a vertex $u \in K$ such that u does not belong to any other maximal clique in $\mathcal{K}$.*

Lemma 3 *For $1 \leq i \leq p$, $|C(N[v_i])| \leq 3$.*

Proof: The proof is by contradiction. Let $| C(N[v_i]) | > 3$. By definition, $C(N[v_i])$ contains only the maximal cliques from the linear ordering $\mathcal{O}$. From Observation 4, it follows that for each maximal clique $Q \in C(N[v_i])$, there exists a vertex w which is unique to Q. Since $|C(N[v_i])| > 3$, there exists at least 4 such vertices each belonging to different maximal cliques in $C(N[v_i])$. Let those vertices be denoted as w_1, w_2, w_3, w_4. We can easily see that the vertices w_1, w_2, w_3, w_4 form an independent set, since each of them belong only to their respective maximal cliques. It follows that v_i together with w_1, w_2, w_3, w_4 form a forbidden structure $K_{1,4}$ (refer Fig. 1 (i)). This is a contradiction to our premise that G does not contain any forbidden structure. □

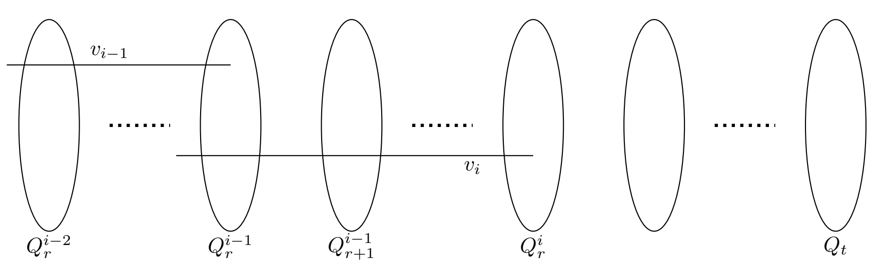


Fig. 5: Construction of path P

Lemma 4 *In the path P, if for any vertex v_i, $1 \leq i < p$, $|C(N[v_i])| = 3$, then $|C(N[v_{i+1}])| \leq 2$.*

Proof: The proof is by contradiction. Assume that there exists a vertex $v_i \in P$, $1 \leq i \leq p$ for which $|C(N[v_i])| = 3$ and $|C(N[v_{i+1}])| \geq 3$. Also note that by Lemma 3, $|C(N[v_{i+1}])|$ cannot exceed 3. Vertices v_i

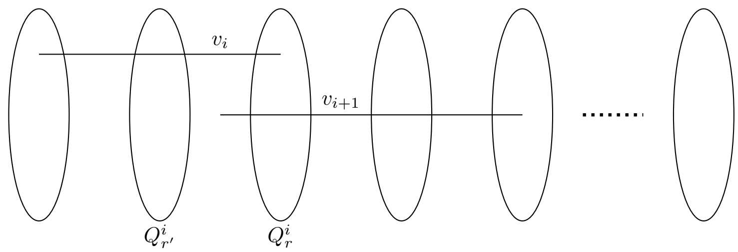


Fig. 6: Forbidden structure formation

and v_{i+1}, being adjacent, form an edge in the path P. We consider the following cases based on the cardinality of $C(N[v_i]) \cup C(N[v_{i+1}])$.

Case when $| \ C(N[v_i]) \cup \ C(N[v_{i+1}]) \ |= 4$: Recall from Algorithm 1 that Q^{i-1}_r and Q^i_r are the maximal cliques in the ordering $\mathcal{O}$ which contains the right endpoints of the intervals corresponding to v_{i-1} and v_i respectively. For any vertex $v_i \in P$, $Q^{i-1}_r \in C(N[v_i])$ and $Q^i_r \in C(N[v_i])$. By our choice of v_{i+1} in the construction of path P, $v_{i+1} \in \{Q^i_r \setminus Q^i_{r'}\}$. Thus v_{i+1} is covered by Q^i_r in $C(N[v_i])$. As per our assumption, $| \ C(N[v_i]) \cup C(N[v_{i+1}]) \ | \ = 4$, and by our premise, $| \ C(N[v_i]) \ | \ = 3$. Therefore those vertices of $N[v_{i+1}]$ which are not covered by Q^i_r have to be covered by exactly one more clique. It follows that $| \ C(N[v_{i+1}]) \ | \ = 2$, which is a contradiction to our assumption that $| \ C(N[v_{i+1}]) \ | \ = 3$. This, in turn, is a contradiction to our initial premise that $| \ C(N[v_i]) \cup \ C(N[v_{i+1}]) \ |= 4$. Thus the only possibility is, $| \ C(N[v_i]) \cup C(N[v_{i+1}]) \ |= 5$, which we discuss in the next case. Observe that $| \ C(N[v_i]) \cup C(N[v_{i+1}]) \ |$ cannot be greater than 5 since $v_{i+1} \in Q^i_r$ and $| \ C(N[v_{i+1}]) \ | \ = 3$. Therefore, if $| \ C(N[v_i]) \cup C(N[v_{i+1}]) \ |$ is greater than 5, then those vertices in $N[v_{i+1}]$ which are not covered in Q^i_r will be covered by 3 maximal cliques, which would make $| \ C(N[v_{i+1}]) \ |= 4$. This is again a contradiction to the assumption that $| \ C(N[v_{i+1}]) \ | \ = 3$.

Case when $| \ C(N[v_i]) \cup C(N[v_{i+1}]) \ |= 5$: The proof is by contradiction to our premise that G does not contain any forbidden structure. We first show that $C(N[v_i]) \cup C(N[v_{i+1}])$ is indeed a minimum clique cover of $N[v_i] \cup N[v_{i+1}]$. Then, using Observation 4, we show that there exists a forbidden structure. By definition, $C(N[v_i])$ is a minimum clique cover of $N[v_i]$. Therefore, each of the three maximal cliques in $C(N[v_i])$ has atleast one unique vertex which does not belong to any other maximal clique. Since $v_{i+1} \in Q^i_r$ and $Q^i_r \in C(N[v_i])$, $Q^i_r \in C(N[v_{i+1}])$. Let the other two maximal cliques in $C(N[v_{i+1}]$ be Q_j and Q_k. By Observation 4, Q_j and Q_k contain a unique vertex each. It follows that any minimum clique cover of $N[v_i] \cup N[v_{i+1}]$ contains all three maximal cliques of $C(N[v_i])$ along with Q_j and Q_k. Hence $C(N[v_i]) \cup C(N[v_{i+1}])$ is a minimum clique cover of $N[v_i] \cup N[v_{i+1}]$. By Observation 4, on $C(N[v_i]) \cup \ C(N[v_{i+1}])$, there is a set V' of 5 vertices in $C(N[v_i]) \cup \ C(N[v_{i+1}])$ that are mutually disjoint and form an independent set of size five. The edge (v_i, v_{i+1}), together with V' form a forbidden structure (see Definition 2). Thus we have arrived at a contradiction. Therefore, if $| \ C(N[v_i]) \ | \ = 3$, then $| \ C(N[v_{i+1}]) \ | \ \leq 2$. $\square$

Observation 5 *For each vertex $v \in P \setminus v_p$, $| \ C(N[v]) \ |\geq 2$.*

Proof: By construction of path P,

$$CC(N[v_1, \ldots, v_i]) = CC(N[v_1, \ldots, v_{i-1}]) \cup C(Q^{i-1}_{r+1}, \ldots, Q^i_r)$$

Q^{i-1}_r is the rightmost maximal clique in $CC(N[v_1, \ldots, v_{i-1}])$ and it covers v_i, since $v_i \in Q^{i-1}_r$. Q^i_r is the rightmost maximal clique which v_i belongs to, in the ordering $\mathcal{O}$. Note that $C(N[v_i]) = Q^{i-1}_r \cup C(Q^{i-1}_{r+1}, \ldots, Q^i_r)$.

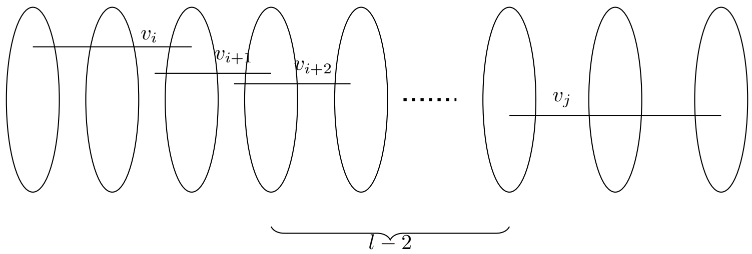


Fig. 7: Vertices v_i and v_j belong to three consecutive cliques in the clique cover

Let $A = N[v_i] \cap (Q^{i-1}_{r+1} \cup \cdots \cup Q^i_r)$. Since $v_i \neq v_p$, we know that there is $v_{i+1} \in P$ which is chosen such that $v_{i+1} \notin Q^{i-1}_r$ and $v_{i+1} \in N[v_i]$. It follows that $A \neq \emptyset$. By choice of v_i, it has the rightmost right endpoint among all vertices in $Q^{i-1}_r \setminus Q^{i-1}_{r'}$ and $v_i \neq v_p$. Hence $\exists u \in A$ which is not covered by Q^{i-1}_r. Therefore, there exists at least one $Q \in C(Q^{i-1}_{r+1}, \ldots, Q^i_r)$ that covers all the vertices in A,. In other words, $C(Q^{i-1}_{r+1}, \ldots, Q^i_r) \neq \emptyset$. It follows that $C(N[v_i])$ is of size at least 2. □

From Lemma 4 and Observation 5, we present the following claim.

Lemma 5 *In path P, there is at most one vertex v where $\mid C(N[v]) \mid \; = 3$.*

Proof: The proof of this claim is again by contradiction. Assume that there is more than one vertex with size of minimum clique cover equal to 3 in path P. Let v_i be the first such vertex in P in the increasing order of left endpoints. By Lemma 4, we know that the minimum clique cover of v_{i+1} is of size less than 3. By our assumption, $\exists j > i+1$ such that minimum clique cover of v_j is of size 3. Let the number of vertices in the subpath of P from v_i to v_j (including both v_i and v_j) be l. It follows from Observation 5 that for each vertex $v_k \in P$, $k \in [i{+}1, j{-}1]$, the minimum clique cover is of size 2. We compute $CC(N[v_i, \ldots, v_k])$ with respect to $CC(N[v_i, \ldots, v_{k-1}])$. Note that v_k is already covered in $CC(N[v_i, \ldots, v_{k-1}])$ and $CC(N[v_i, \ldots, v_k])$ additionally covers $N[v_k] \setminus N[v_{k-1}]$. Since $|C(N[v_k])| = 2$, it adds just 1 to the number of cliques in $CC(N[v_i, \ldots, v_{k-1}])$. That is,

$$\mid CC(N[v_i, \ldots, v_k]) \mid \; = \; \mid CC(N[v_i, \ldots, v_{k-1}]) \mid + 1$$

It follows that each of the $l-2$ vertices in $\{v_{i+1}, \ldots, v_{j-1}\}$ increments the size of the clique cover by 1. That is, $\mid CC(N[v_{i+1}, \ldots, v_{j-1}]) \mid = l-2$. Thus

$$\begin{aligned} \mid CC(N[v_i, \ldots, v_j]) \mid &= \; \mid C(N[v_i]) \mid + \mid CC(N[v_{i+1}, \ldots, v_{j-1}]) \mid + (\mid C(N[v_j]) \mid -1) \\ &= 3 + (l-2) + 3 - 1 \\ &= l + 3 \end{aligned}$$

Note that we deduct 1 from $\mid C(N[v_j]) \mid$ since v_j is already covered by $CC(N[v_{i+1}], \ldots, N[v_{j-1}])$. We can see that the vertices from v_i to v_j form a path of length l which has an independent set of size $l+3$ in its neighbourhood. The vertices from v_i to v_j, together with the independent set of size $l+3$ in its neighbourhood forms a forbidden structure. We have arrived at a contradiction to our premise that G does not contain any forbidden structures. Therefore it is proved that in path P, there is at most one vertex which has a minimum clique cover of size 3. □

3.1 Computing the exact hitting set from $\mathcal{K}$

We now complete the characterization of EHIGs. We first prove the characterization when p is at most 3 and then complete the proof by an inductive argument. From Algorithm 1, we use the minimal clique cover, $\mathcal{K} = \{K_1, K_2 \ldots K_{\alpha'}\}$, which consists of maximal cliques in $\mathcal{O}$ and the path P consisting of vertices $v_1, \ldots, v_p$ from left to right, in the arguments below. Further, for each $1 \leq i \leq \alpha'$, we use D_i to denote the gadget corresponding to K_i. For each $1 \leq i \leq p$, let L_i and R_i denote the leftmost and rightmost, respectively, maximal cliques in $\mathcal{O}$ which contains v_i. DL_i and DR_i denote the gadgets in H_G corresponding to L_i and R_i, respectively. Recall that H_G denotes the canonical interval representation of G. It is useful to note that DL_1 and D_1 both denote the gadget associated with K_1, and DR_p and $D_{\alpha'}$ both denote the gadget associated with $K_{\alpha'}$. The inductive argument below refers to two interval graphs G and G', and the gadgets we refer to are in H_G or $H_{G'}$, and will be clear from the context. In all our arguments that follow, we use the maximal clique ordering $\mathcal{O}$ to reason about the position of a maximal clique with respect to another maximal clique. We thus use the words and phrases *before, after, at or before, at or after* with respect to $\mathcal{O}$. These relationships are also represented by $<, >, \leq, \geq$, whenever it is more convenient to use these symbols.

We now prove the characterization based on the different cases for p. For $p = 1$, we explicitly present the hitting set, and when $2 \leq p \leq 3$, we present the exact hitting sets which satisfy some additional properties. These additional properties are useful in the inductive argument for $p \geq 3$. In all the following lemmas in this section, we assume that G is an interval graph which does not have an $F \in \mathcal{F}$ as an induced subgraph. Further, all the statements are based on the value of p which is the number of vertices in the path P computed by Algorithm 1, and $\mathcal{K}$.

Lemma 6 *Let v_1 be the only vertex in P. Then the canonical interval representation H_G satisfies one of the following two statements.*

- *Let $N[v_1]$ have a minimum clique cover of size 2, which is $\{K_1, K_2\}$. Then $\{z_1, z_2 + 1\}$ and $\{z_1 - 1, z_2\}$ are two exact hitting sets of H_G.*
- *Let $N[v_1]$ have a minimum clique cover of size 3, which is $\{K_1, K_2, K_3\}$. Let $w_1 = |K_1 \cap K_2|$ and let $w_3 = |K_2 \cap K_3|$. Then $\{z_1 - w_1, z_2, z_3 + w_3\}$ is an exact hitting set of H_G.*

Proof: In the first case, K_1 and K_2 are the first and last cliques of $\mathcal{O}$, respectively. Let D_1 and D_2 be the gadgets corresponding to K_1 and K_2, respectively, in G. By the definition of H_G, the interval associated with v_1 has z_1 as the left endpoint and z_2 as the right endpoint. Consider the set $\{z_1, z_2 + 1\}$. z_1 hits all the intervals whose left endpoint is in D_1, and $z_2 + 1$ hits all the intervals whose left endpoint is to the right of D_1, and right endpoint is to the right of z_2 in D_2. Further, each interval is hit, and no interval is hit twice. By a symmetric argument, $\{z_1 - 1, z_2\}$ is also an exact hitting set.

In the second case, consider the cliques $B_1 = K_1 \setminus K_2$, $B_3 = K_3 \setminus K_2$, and $B_2 = K_2$. Let $w_1 = |K_1 \cap K_2|$ and $w_3 = |K_3 \cap K_2|$. We show that the set $\{z_1 - w_1, z_2, z_3 + w_3\}$ is an exact hitting set of H_G. First, we observe that the point z_2 hits all the intervals in D_2. Further, from the construction of H_G it follows that, if I_1 is an interval in $D_1 \setminus D_2$ and I_2 is an interval in $D_1 \cap D_2$, then in D_1 the left endpoint of I_1 is smaller than the left endpoint of I_2. Thus, among the w_1 intervals in $D_1 \cap D_2$, the longest interval has the left endpoint in z_1 and the remaining $w_1 - 1$ intervals start at different points in $\{(z_1 - 1), \ldots, (z_1 - (w_1 - 1))\}$. Therefore, the point $z_1 - w_1$ does not hit any of the w_1 intervals that belong to $D_1 \cap D_2$. Further, it hits all the intervals in $D_1 \setminus D_2$. A symmetric argument shows that $z_3 + w_3$ hits all the intervals in $D_3 \setminus D_2$ and does not hit any interval in $D_3 \cap D_2$. The remaining intervals are all in D_2, and they are not hit by either $z_1 - w_1$ or $z_3 + w_3$, and they all contain z_2 which is the zero-point of gadget D_2. Thus, this base case is proved. □

Lemma 7 *Let P consist of only v_1 and v_2. Then, the following statements hold for H_G.*

- *Let $N[v_1]$ and $N[v_2]$ have minimum clique cover of size 2. In $\mathcal{K} = \{K_1, K_2, K_3\}$, let $w_1 = |K_1 \cap L_2|$ and let $w_2 = |K_3 \cap K_2|$. Then $\{z_1 - w_1, z_2, z_3 + w_3\}$ is an exact hitting set of H_G.*
- *Let $N[v_1]$ have minimum clique cover of size 2 and $N[v_2]$ have minimum clique cover of size 3. In $\mathcal{K} = \{K_1, K_2, K_3, K_4\}$, let $w_1 = |K_1 \cap L_2|$ and $w_4 = |K_4 \cap K_3|$. Then H_G has an exact hitting set which contains $\{z_1 - w_1, z_4 + w_4\}$.*
- *Let $N[v_1]$ have minimum clique cover of size 3 and $N[v_2]$ have minimum clique cover of size 2. In $\mathcal{K} = \{K_1, K_2, K_3, K_4\}$, let $w_1 = |K_1 \cap K_2|$ and $w_4 = |K_4 \cap K_3|$. Then H_G has an exact hitting set which contains $\{z_1 - w_1, z_4 + w_4\}$.*

Proof: If $p = 2$, then $\mathcal{K}$ either has 3 maximal cliques or 4 maximal cliques. Thus α' which is the index of the last clique in $\mathcal{K}$ is in $\{3, 4\}$. Further, in all the cases, $z_1 - w_1$ is in D_1 and for $\alpha' \in \{3, 4\}$, $z_{\alpha'} + w_{\alpha'}$ is in

$D_{\alpha'}$. Consider the point $z_1 - w_1$ which is in D_1. In the first two cases, $z_1 - w_1$ hits the intervals associated with vertices in $K_1 \setminus L_2$ and does not hit any interval associated with vertices in $K_1 \cap L_2$. Consider the interval graph obtained by removing $K_1 \setminus L_2$. Since $N[v_1]$ has a clique cover of size 2, it follows that $L_2 \setminus K_1 \subset K_2$. Consider the maximal cliques from K_2 to K_3. v_2 is common to all of these cliques. We now consider the first case and the second case separately.

In the first case, the minimum clique cover of $N[v_2]$ is of size 2. Let w_3 be the number of vertices in $K_2 \cap K_3$. Consider the exact hitting set $\{z_2, z_3 + w_3\}$. Clearly, z_2 hits all the intervals in D_2, and $z_3 + w_3$ hits all the intervals in $D_3 \setminus D_2$ and does not hit anything in D_2; this is because the right endpoints of the intervals in $D_2 \cap D_3$ are smaller than $z_3 + w_3$. Thus, for the first case $\{z_1 - w_1, z_2, z_3 + w_3\}$ is an exact hitting set of H_G.

In the second case, the minimum clique cover of $N[v_2]$ is of size 3. Since the first clique containing v_2 is L_2, the minimum clique cover of $N[v_2]$ is L_2, K_3, K_4. Consider G' to be the interval graph induced by $N[v_2]$ and consider $H_{G'}$. L_2 is the first maximal clique in G' and in this case let z denote the zero-point of the gadget of DL_2 in H_G. By using the second case of Lemma 6, consider the exact hitting set $\{z - |L_2 \cap K_3|, z_3, z_4 + |K_3 \cap K_4|\}$ of $H_{G'}$. From this set, we construct an exact hitting set of H_G based on the following two subcases. The first subcase is that there are vertices in $L_2 \setminus K_1$ whose leftmost clique is L_2, and whose rightmost clique is before K_3. Then, the interval in $H_{G'}$ of such a vertex is hit by $z - |L_2 \cap K_3|$ and this also hits all the intervals associated with vertices in $L_2 \setminus K_3$. Thus, in this subcase, we get an exact hitting set for H_G consisting of $\{z_1 - w_1, z - |L_2 \cap K_3|, z_3, z_4 + |K_3 \cap K_4|\}$. In the second subcase, for all the vertices whose leftmost clique is L_2, the rightmost clique is at or after K_3. Let h be the intermediate point just preceding DL_2, which is the gadget associated with L_2 in H_G. Consider the set $\{z_1 - w_1, h, z_3, z_4 + |K_3 \cap K_4|\}$. $z_1 - w_1$ hits all intervals associated with vertices in $K_1 \setminus L_2$, h hits all intervals associated with vertices in $L_2 \setminus (L_2 \cap K_3)$, and this set includes $K_1 \cap L_2$. z_3 hits all intervals associated with vertices in $(L_2 \cap K_3) \cup (K_3 \cap K_4)$, and $z_4 + |K_3 \cap K_4|$ hits all intervals associated with vertices in $K_4 \setminus K_3$ (that is, the intervals in $D_4 \setminus D_3$). Further, it is clear that it is an exact hitting set. Thus, the first two cases are proved. The third case is symmetric to the second case[(i)], and thus it is also true. Hence the lemma. □

Remark: The approach of obtaining an exact hitting set for H_G from an exact hitting set for $H_{G'}$, by using a preceding or following intermediate point, is a template that repeats in the following proofs. However, this template is used in different contexts, and it seems that the repeated case-by-case usage of this template is unavoidable.

Lemma 8 *Let P consist of only v_1, v_2, v_3, and let $N[v_2]$ have a minimum clique cover of size 3. Further, let $w_1 = |K_1 \cap L_2|$ and let $w_{\alpha'} = |K_{\alpha'} \cap K_{\alpha'-1}|$. Then H_G has an exact hitting set containing $\{z_1 - w_1, z_{\alpha'} + w_{\alpha'}\}$.*

Proof: Consider the interval graph G' obtained by removing $K_1 \setminus L_2$. Let $H_{G'}$ be the canonical interval representation. The path obtained from Algorithm 1 on G', using the cliques from L_2 to $K_{\alpha'}$ is the path consisting of v_2 and v_3 only. The first maximal clique in $H_{G'}$ is L_2 which is the leftmost clique containing v_2. Further, the rightmost clique containing v_2 is $K_{\alpha'-1}$, and the minimum clique cover of $N[v_2]$ is 3 and the minimum clique cover of $N[v_3]$ is 2 (from Lemma 5). Let S be an exact hitting set of $H_{G'}$ obtained from using Lemma 7. Let z denote the zero-point of the gadget DL_2. For some $w > 0$, let $z - w$ be in S. Also, in the construction of S it is ensured that the interval associated with v_2 is not hit by $z - w$. From the structure of S defined in Lemma 7, $\{z - w, z_{\alpha'-1}, z_{\alpha'} + w_{\alpha'}\} \subseteq S$. We get an exact hitting set for H_G from S based on two cases.

The first case is that there are vertices in $L_2 \setminus K_1$ for which the leftmost clique is L_2 and the interval associated with them are hit by $z - w$ in $H_{G'}$. Such vertices will also occur in the rightmost clique containing v_1, since $N[v_1]$ has a clique cover of size 2 in G. Thus in this case, $S \cup \{z_1 - w_1\}$ is an exact hitting set for H_G. The main reason is that the intervals associated with the vertices in $L_2 \setminus K_1$ whose leftmost clique is at or before L_2 will all be hit by $z - w$. Further, $z_1 - w_1$ hits all the intervals associated with vertices in $K_1 \setminus L_2$.

In the second case, all the intervals in $L_2 \setminus K_1$ whose leftmost clique is L_2 are not hit by $z - w$; they are hit by another element of S. Thus, $z - w$ is in the hitting set of $H_{G'}$ only to hit intervals whose left endpoint is to the left of DL_2 in H_G. Let h be the intermediate point just to preceding the gadget associated with DL_2 in H_G. Consider the set $S \setminus \{z - w\} \cup \{z_1 - w_1, h\}$. This is an exact hitting set of H_G since S is an exact hitting

(i) A detailed description of the symmetry is presented in the inductive argument in Lemma 9 set of $H_{G'}$, the intervals associated with the vertices in $L_2 \setminus K_1$ whose leftmost clique is before L_2 will all be hit by h, and $z_1 - w_1$ hits all the intervals associated with vertices in $K_1 \setminus L_2$. Hence the lemma. □

Lemma 9 *Let G be an interval graph which does not have an $F \in \mathcal{F}$ as an induced subgraph. Then the canonical interval representation H_G has an exact hitting set. Further, if $p \geq 2$, there is an exact hitting satisfying the following two additional properties:*

- *The intervals corresponding to the vertices in $K_1 \setminus N[v_2]$ are hit by a point in D_1 to the left of z_1 and $D_1 \cap N[v_2]$ is hit by a point in DL_2 which is to the left of the zero-point of DL_2 or the intermediate point just before DL_2.*
- *The intervals corresponding to vertices in $K_{\alpha'} \setminus N[v_{p-1}]$ are hit by a point in $D_{\alpha'}$ to the right of $z_{\alpha'}$ and $D_{\alpha'} \cap N[v_{p-1}]$ is hit by a point in $D_{\alpha'-1}$ to the right of the zero-point of $D_{\alpha'-1}$ or the intermediate point just after $D_{\alpha'-1}$.*

Proof: The proof when $p = 1$ follows from Lemma 6. The proof is by induction on $p \geq 2$. The base case is for $p = 2$, and from Lemma 7 we know H_G has a hitting set which satisfies the additional properties. We now prove the claim for all $p \geq 3$.

First, we consider the case in which minimum clique cover of either $N[v_1]$ or $N[v_2]$ is of size 3 and the minimum clique cover of either $N[v_{p-1}]$ or $N[v_p]$ is of size 3. From Lemma 5, we know that there is at most one vertex in P which has a minimum clique cover of size 3. Therefore, in this case, it follows that $p = 3$, and $N[v_2]$ has a minimum clique cover of size 3. From Lemma 8, we know that H_G has an exact hitting set which satisfies the additional properties.

Next we consider the case in which either the minimum clique cover of $N[v_1]$ and $N[v_2]$ are both of size 2 and if not, the minimum clique cover of $N[v_{p-1}]$ and $N[v_p]$ are both of size 2. The induction hypothesis is that the claim is true for all natural numbers in the set $\{2, \ldots, p-1\}$. Given this, we prove that the claim is true for p.

Let us first consider the case when $N[v_1]$ and $N[v_2]$ both have a minimum clique cover of size 2. By construction, in Algorithm 1, we know that K_2 is the rightmost clique in $\mathcal{O}$ which contains v_1. From Observation 3, we know that K_2 is in a minimum clique cover of $N[v_1]$. Any vertex in $N[v_1]$ which is not present in K_2 will be present in K_1, since minimum clique cover of $N[v_1]$ is 2. Let L_2 be the leftmost clique in $\mathcal{O}$ which contains v_2. We know that $K_1 < L_2 \leq K_2$. That is, L_2 is a clique to the right of K_1 and is not later than K_2 in $\mathcal{O}$.

Consider the partition of K_1 into $K_1 \setminus L_2$ and $K_1 \cap L_2$. Let w_1 be the number of vertices in $K_1 \cap L_2$. Let $B_1 = K_1 \setminus L_2 = K_1 \setminus N[v_2]$. Consider the point $z_1 - w_1$ in the gadget D_1 in H_G. By the nature of the construction of H_G, this point is to the left of the starting point of the intervals associated with vertices in $K_1 \cap L_2$. Consider the interval graph G' obtained by removing B_1 from G. Since the elements in $N[v_1]$ are in the cliques K_1 and K_2, the first clique in the maximal clique ordering of G' is L_2. Further, among all the vertices in L_2, v_2 is the vertex in L_2 for which the rightmost clique containing it has the largest index in $\mathcal{O}$. Thus, Algorithm 1 on G' will compute the path $P \setminus v_1$ which has the $p-1$ vertices $v_2, v_3, \ldots v_p$. Let L_3 denote the leftmost maximal clique in G' which contains v_3. The additional properties satisfied by G' are as follows:

- For each vertex v in G' for which the leftmost clique containing it (in $\mathcal{O}$) is after K_1 and before L_2, the rightmost clique containing v is before K_3. Otherwise, the choice for v_2 by the algorithm is violated.
- For each vertex v in $K_1 \cap L_2$, the rightmost clique (in $\mathcal{O}$) containing v is at or before K_2.
- For each vertex u in $N[v_2]$ for which the leftmost clique containing it is to the right of L_2, u is also present in K_3. Otherwise, let there exist such a vertex u for which the rightmost clique containing it is before K_3. Then u along with one vertex each in L_2 and K_3 form an independent set of size 3. Thus, minimum clique cover of $N[v_2]$ is 3. This contradicts our premise that for both $N[v_1]$ and $N[v_2]$ have a minimum clique cover of size 2.
- L_3 is after K_2 and not later to K_3 in $\mathcal{O}$.

By the induction hypothesis applied to G', $H_{G'}$ has an exact hitting set such that the intervals corresponding to the vertices in $L_2 \setminus N[v_3]$ are hit at a point to the left of the zero-point of the gadget DL_2 in $H_{G'}$ and the intervals corresponding to vertices in $L_2 \cap N[v_3]$ are hit by a point to the left of the zero-point of the gadget DL_3 or by the intermediate point just before DL_3. In particular, v_2 which is in $L_2 \cap N[v_3]$ is hit by a point to the left of the zero-point in the gadget DL_3 or by the intermediate point just before DL_3. Note that the intermediate point just before DL_3 exists since DL_3 is to the right of D_2. Let this hitting set be denoted by S. Let $h^* \in S$ be the point in the gadget DL_2 in $H_{G'}$. Note that DL_2 is the leftmost gadget in $H_{G'}$.

We first show that h^* hits the intervals in $H_{G'}$ corresponding to the vertices in $K_1 \cap L_2$. This is because, by the construction of P, for each vertex in K_1, the rightmost clique containing it is at or before K_2, and L_3 is after K_2. Thus, the point in $H_{G'}$ that hits any interval corresponding to a vertex in $K_1 \cap L_2$ has to be in a gadget to the left of D_2. If this point is different from h^*, then it would also hit v_2. This contradicts the fact that in the exact hitting set S, v_2 is hit by a point to the left of the zero-point in the gadget DL_3 or to the intermediate point just before DL_3. Clearly, both these points are different from h^* which is a point in the gadget DL_2 and we know that DL_2 is different from DL_3.

To construct the exact hitting set in H_G from S, we define a point h in H_G as follows based on two cases:

- The first case is when there is a vertex such that the leftmost clique containing it is L_2 and the rightmost clique containing it is before K_3 (in $\mathcal{O}$). Among all such vertices, let u be the vertex such that the rightmost clique containing it has the largest index in $\mathcal{O}$. By construction of H_G the left endpoint of the interval associated with u would have been the largest among all such vertices. We take h to be the left endpoint of the interval assigned to u in H_G.

- In the second case there is no vertex such that the leftmost clique containing it is L_2 and the rightmost clique containing it is before K_3 (in $\mathcal{O}$). In this case, h is the intermediate point between DL_2 and the preceding gadget in H_G. Such a point exists, since L_2 is different from the first clique K_1, and by construction of H_G, the gadget for every clique in $\mathcal{O}$ except K_1 has a point to its left.

Then $S \setminus \{h^*\} \cup \{z_1 - w_1, h\}$ is a hitting set of H_G, where $z_1 - w_1$ hits all the intervals in B_1. h hits all the intervals associated with vertices in L_2 whose rightmost endpoint is before K_3, and these are not hit by any other point in S, due to the induction hypothesis, and they are not hit by $z_1 - w_1$. Further, the vertices in B_1 are not elements of L_2, and thus are hit only by $z_1 - w_1$. Finally, the intervals associated with all the other vertices are hit exactly once by S in gadgets different from the gadgets associated with L_2 and K_1, and these gadgets have the same structure in $H_{G'}$ as in H_G. Therefore, $S \setminus \{h^*\} \cup \{z_1 - w_1, h\}$ is an exact hitting set of H_G, and it also satisfies the two additional properties.

Next, we consider the case when $N[v_1]$ or $N[v_2]$ has a minimum clique cover of size 3. Then, since $p \geq 4$, we know that both $N[v_{p-1}]$ and $N[v_p]$ has a minimum clique cover of two cliques. Further, v_{p-1} and v_p are distinct from v_1 and v_2. From Observation 3, we know that $K_{\alpha'}$ is in the minimum clique cover of $N[v_p]$. Consider v_{p-1} in P and we know that $D_{\alpha'-1}$ denotes the rightmost clique in $\mathcal{O}$ which contains v_{p-1}. To prove the claim in this case, we consider the reversed maximal clique ordering $\mathcal{O}$, and the path obtained by executing Algorithm 1 on this ordering. Due to the symmetry of the vertices chosen during the algorithm to be added to the path, it is clear that the path computed will be the reversal of P. Also, we know that $L_{p-1} \leq K_{\alpha'-2} < L_p \leq K_{\alpha'-1} < K_{\alpha'}$. We consider the argument for the previous case by using the reversal of $\mathcal{O}$, the canonical representation constructed using the reversal of $\mathcal{O}$, and the reversal of P. We compare the execution of Algorithm 1 using the reversal of $\mathcal{O}$ and its execution using $\mathcal{O}$: $K_{\alpha'}$ will be in the place of K_1, $K_{\alpha'-1}$ in the place of L_2, L_p in the place of K_2, $K_{\alpha'-2}$ in the place of L_3, and L_{p-1} in place of K_3. By an argument symmetric with the argument using the induction hypothesis for the previous case, using the canonical representation constructed using the reversal of $\mathcal{O}$, it follows that $S \setminus \{h^*\} \cup \{z_{\alpha'} + w_{\alpha'}, h\}$ is an exact hitting set of H_G, and it satisfies the two additional properties. Hence the lemma. □

We complete the proof of the forbidden structure characterization for EHIGs.

Theorem 1 *An interval graph G is exactly hittable if and only if it does not contain any graph from the set $\mathcal{F}$ as an induced subgraph.*

Proof: From Section 2, we know that every interval graph G has a unique canonical interval representation, which we denote by H_G. Furthermore, if G does not have an $F \in \mathcal{F}$ as an induced subgraph, then by Lemma 9, H_G has an exact hitting set. We have shown in Lemma 2 that if G has an exactly hittable interval representation, then G does not have any $F \in \mathcal{F}$ as an induced subgraph. This proves the theorem. □

Using the above theorem, we prove Theorem 2.

Theorem 2 *Let G be an interval graph. Let H_G be its canonical interval representation constructed as described in Section 2. Then, G is exactly hittable if and only if H_G is exactly hittable.*

Proof: Let the hypergraph H_G constructed from interval graph G has an exact hitting set. By Lemma 1, the intersection graph G' of H_G is isomorphic to G. It follows that if G' has an exactly hittable interval representation, then G also has one. Thus, G is exactly hittable.

To show the other direction, let G be an EHIG. By Theorem 1, if G is exactly hittable, then G does not have any forbidden structure. Then, it follows from Lemma 9 that the canonical representation H_G of G has an exact hitting set. □

3.2 Algorithm to recognize exactly hittable interval graphs

In this section, we present an algorithm to recognize an exactly hittable interval graph. This algorithm makes use of the canonical interval representation in Section 2 and the result by Dom et al. (2006) for MMSC problem (described in Section 1) on interval hypergraphs. In their paper, Dom et al. showed that an integer linear programming (ILP) formulation, say $\mathcal{L}$, for MMSC problem on interval hypergraphs can be solved in polynomial time. The coefficients of inequalities in $\mathcal{L}$ results in a totally unimodular matrix and the polyhedron corresponding to $\mathcal{L}$ is an integer polyhedron. If the input instance to ILP is an exactly hittable instance, then the solution returned is 1. We use this algorithm below to test if a given interval hypergraph instance is exactly hittable.

Algorithm `isEHIG`: Given an interval graph G, construct the canonical interval representation as described in Section 2. Let H_G be the resulting interval representation. Run MMSC algorithm by Dom et al. (2006) on H_G as input. If the algorithm returns value 1, then return `yes`. Else return `no`.

Lemma 10 *Algorithm `isEHIG`(G) outputs yes if and only if G is exactly hittable in polynomial time.*

Proof: The proof follows from Lemma 1, Theorem 2 and the correctness of algorithm for MMSC problem on interval hypergraphs. □

It is also clear that the inductive argument in Lemma 9 can be converted into a polynomial time combinatorial algorithm to check if H_G has an exact hitting set. This leverages the fact that a minimum clique cover of a perfect graph can be computed in polynomial time.

3.3 Proper Interval Graphs is a subclass of EHIG

We now recall and complete the proof of Theorem 3.

Theorem 3 *Proper interval graphs $\subset$ EHIG $\subset$ Interval Graphs.*

Proof: Let G be a proper interval graph and let it be the intersection graph of the interval hypergraph $H = (\mathcal{V}, \mathcal{I})$ in which no interval properly contains another. Since H is a proper interval hypergraph, no two intervals in $\mathcal{I}$ can have the same left endpoint. Hence order intervals in $\mathcal{I}$ according to increasing order of their left endpoints. Let this ordering be $I_1 < I_2 < \ldots < I_m$. Add $r(I_1)$ (which is the smallest right endpoint among all intervals) to set S. Remove all intervals hit by $r(I_1)$. Recurse on the remaining set of intervals until all the intervals are hit by S. Clearly, S is an exact hitting set.

To show the strict containment, we show that the graph $K_{1,3}$ which is a forbidden structure (Roberts (1978)) for Proper Interval Graphs has an exactly hittable interval representation. Let the vertices of the $K_{1,3}$ be $\{u, a, b, c\}$ and edges be $\{(u, a), (u, b), (u, c)\}$. The intervals assigned to the vertices a, b, c and u are shown in Fig. 8. Hence the lemma. □

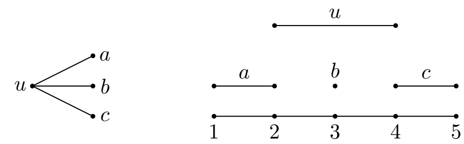


Fig. 8: Exactly hittable interval representation of $K_{1,3}$ (here, $\{1,3,5\}$ is an exact hitting set)

4 Discussion

Our results indicate that there is an interesting hierarchy among the class of interval graphs based on the number of times an interval is hit by a hitting set. We have shown that proper interval graphs have an exactly hittable interval representation. Further it is a strict subclass of the set of EHIGs. The natural question is to characterize interval graphs which have a representation such that each interval is hit at most k times by a hitting set. We also believe that the recognition problem for such graphs is fundamental and interesting.

Acknowledgements

We thank the anonymous reviewers for providing insightful comments on one of the gaps in the crucial proof of characterization. We would also like to thank the organizers of ICGT 2022 for their support during the review process.

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