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On Hamiltonian Paths and Cycles in Sufficiently Large Distance Graphs

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For a positive integer $n \in \mathbb{N}$ and a set $D \subseteq \mathbb{N}$, the distance graph G_n^D has vertex set $\{0, 1, \ldots, n-1\}$ and two vertices i and j of G_n^D are adjacent exactly if $|j - i| \in D$. The condition $\gcd(D) = 1$ is necessary for a distance graph G_n^D being connected. Let $D = \{d_1, d_2\} \subseteq \mathbb{N}$ be such that $d_1 > d_2$ and $\gcd(d_1, d_2) = 1$. We prove the following results.

- If n is sufficiently large in terms of D, then G_n^D has a Hamiltonian path with endvertices 0 and $n-1$.
- If $d_1 d_2$ is odd, n is even and sufficiently large in terms of D, then G_n^D has a Hamiltonian cycle.
- If $d_1 d_2$ is even and n is sufficiently large in terms of D, then G_n^D has a Hamiltonian cycle.

Keywords: Distance graph; Toeplitz graph; circulant graph; Hamiltonian path; Hamiltonian cycle; traceability

1 Introduction

For a finite set of positive integers $D \subseteq \mathbb{N}$, the *infinite distance graph* G^D has vertex set $V(G^D) = \mathbb{Z}$ and two vertices u and v of G^D are adjacent exactly if $|u - v| \in D$. For a graph G and a subset $U \subseteq V(G)$ of the vertex set, we denote by $G[U]$ the subgraph of G induced by U. For $i, j \in \mathbb{Z}$, $i \leq j$, we denote by $[i, j] = \{k \in \mathbb{Z} \mid i \leq k \leq j\}$. For a positive integer $n \in \mathbb{N}$, the *distance graph* (also called *Toeplitz graph* in many papers) $G_n^D = G^D[[0, n-1]]$ is the subgraph of G^D induced by the vertices in $[0, n-1]$.

Infinite distance graphs and especially their colourings were first studied by Eggleton, Erdős, and Skilton [10, 11]. Most of the research on distance graphs focused on their colourings [6, 8, 9, 14, 18, 19, 28]. Distance graphs generalize the very well-studied class of *circulant graphs* [2, 16, 17, 26]. In fact, circulant graphs coincide exactly with the regular distance graphs [23]. Circulant graphs have been proposed for numerous network applications and many of their properties such as connectedness and diameter [4, 2, 16, 17], cycle and path structure [1, 3, 5], and isomorphism testing and recognition [12, 22] have

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been studied in great detail. Several fundamental results concerning circulant graphs were extended to the more general class of distance graphs in [7, 23, 24, 25]. The complexity of the connectedness problem for distance graphs was recently settled by Gómez et al. [13]. In [25, 27, 15] the existence of long paths and cycles in distance graphs is studied. The following main result from [21] confirmed a conjecture from Penso et al. [25]. [20] gives an overview on Hamiltonian cycles and paths in vertex-transitive graphs.

Theorem 1 (Löwenstein et al. [21]) *For a finite set $D \subseteq \mathbb{N}$ with $|D| \geq 2$ and $\gcd(D) = 1$, there are infinitely many $n \in \mathbb{N}$ such that G_n^D has a Hamiltonian cycle and G_{n+1}^D has a Hamiltonian path with endvertices 0 and n.*

We conjecture that the conclusion of the last theorem holds

- for all n that are sufficiently large in terms of D if not all elements of D are odd and
- for all even n that are sufficiently large in terms of D if all elements of D are odd.

The purpose of the present paper is to confirm this conjecture in the case that D contains just two elements. In Section 2 we introduce suitable terminology and collect some properties of distance graphs. In Section 3 we confirm our conjecture proving the existence of Hamiltonian paths. Finally, in Section 4 we provide similar results for Hamiltonian cycles.

2 The structure of G^D

Let $D = \{d_1, d_2\}$ for two positive integers d_1 and d_2 such that $\gcd(d_1, d_2) = 1$ and $d_1 > d_2$.

We define coordinates $(x, y) \in (\mathbb{Z}/(d_1 + d_2)\mathbb{Z}) \times \mathbb{Z}$ for the vertices of the distance graph G^D by

$$(x, y) := y(d_1 + d_2) + a_x,$$

where $a_x = xd_1 \pmod{d_1 + d_2}$. Note that this bidimensional relabelling of the vertices of G^D is a bijection. A vertex (x, y) satisfying $0 \leq xd_1 \pmod{d_1 + d_2} < d_2$ is called *lower*. A vertex (x, y) satisfying $d_2 \leq xd_1 \pmod{d_1 + d_2} < d_1$ is called *middle*. A vertex (x, y) satisfying $d_1 \leq xd_1 \pmod{d_1 + d_2} < d_1 + d_2$ is called *upper*.

For a lower vertex (x, y), we have

$$\begin{aligned}
(x, y) + d_1 &= (x + 1, y),\\
(x, y) + d_2 &= (x - 1, y),\\
(x, y) - d_1 &= (x - 1, y - 1),\\
(x, y) - d_2 &= (x + 1, y - 1),
\end{aligned}$$

which implies that a lower vertex (x, y) is adjacent to the vertices $(x + 1, y)$, $(x - 1, y)$, $(x + 1, y - 1)$, and $(x - 1, y - 1)$.

Similarly, for a middle vertex (x, y), we have

$$\begin{aligned}
(x, y) + d_1 &= (x + 1, y + 1),\\
(x, y) + d_2 &= (x - 1, y),\\
(x, y) - d_1 &= (x - 1, y - 1),\\
(x, y) - d_2 &= (x + 1, y),
\end{aligned}$$

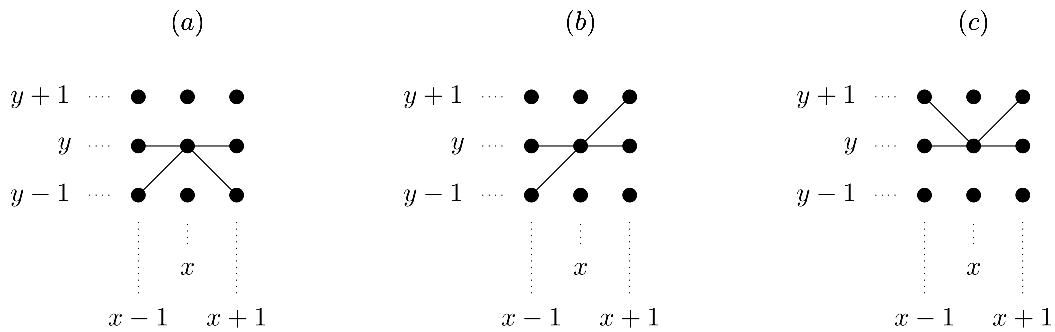

Fig. 1: Neighborhood of (a) a lower, (b) a middle, and (c) an upper vertex.

which implies that a middle vertex (x, y) is adjacent to the vertices $(x+1, y)$, $(x-1, y)$, $(x+1, y+1)$, and $(x-1, y-1)$.

Finally, for an upper vertex (x, y), we have

$$\begin{aligned}
(x, y) + d_1 &= (x+1, y+1), \\
(x, y) + d_2 &= (x-1, y+1), \\
(x, y) - d_1 &= (x-1, y), \\
(x, y) - d_2 &= (x+1, y),
\end{aligned}$$

which implies that an upper vertex (x, y) is adjacent to the vertices $(x+1, y)$, $(x-1, y)$, $(x+1, y+1)$, and $(x-1, y+1)$.

See Figure 1 for an illustration of these observations.

For $c \in \mathbb{Z}/(d_1+d_2)\mathbb{Z}$, all vertices (x, y) of G^D with $x = c$ form the *column* c. Similarly, for $r \in \mathbb{Z}$, all vertices (x, y) satisfying $y = r$ form the *row* r. Note that the vertices in a column are either all lower, or all middle, or all upper. A column that consists of lower (middle, upper) vertices is called *lower (middle, upper)*. See Figure 2 for an illustration.

Lemma 2 *(i) For $c \in \mathbb{Z}/(d_1+d_2)\mathbb{Z}$, the column c is lower if and only if the column $c+1$ is upper.*

(ii) Column 0 is lower.

(iii) Column 1 is upper.

Proof: For $x \in \mathbb{Z}/(d_1+d_2)\mathbb{Z}$, we have $0 \leq xd_1 \pmod{d_1+d_2} < d_2$ if and only if $d_1 \leq (x+1)d_1 \pmod{d_1+d_2} < d_1+d_2$, which proves (i). (ii) follows, because $0 \leq 0 = 0d_1 \pmod{d_1+d_2} < d_2$. Finally, (i) and (ii) imply (iii). □

The columns $x, x+1, \ldots, x+l-1$ form a *block of length* l, if column x is lower, column $x+l$ is lower, and none of the columns $x+1, \ldots, x+l-1$ is lower. The block that contains column 0 is denoted by B_0. Let l be the length of block B_i and let column x be the unique lower column that belongs to block

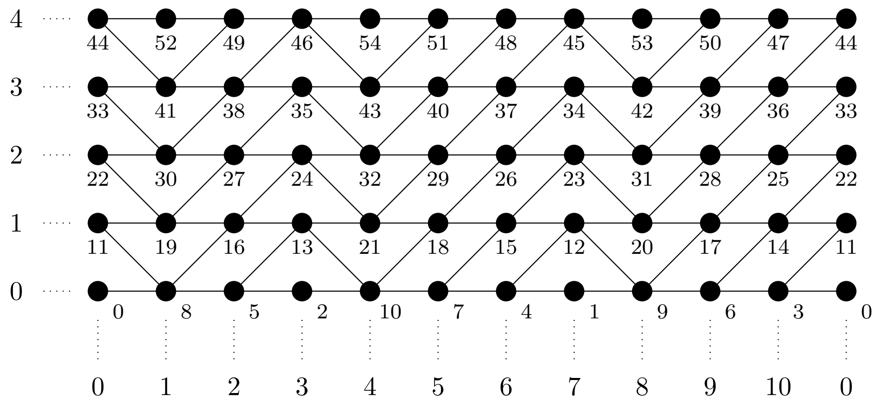

Fig. 2: The distance graph $G_{55}^{\{8,3\}}$. Note that the vertices of column 0 are drawn twice. In order to simplify the drawing, we adopt the convention that such a vertex is adjacent to the union of the neighbors of the two copies, i.e. vertex 22 is adjacent to the vertices 19, 30, 14, and 25.

B_i, then the block that contains column $x+l$ is denoted by B_{i+1}. Note that the indices of the blocks are elements of $\mathbb{Z}/d_2\mathbb{Z}$. For $i \in \mathbb{Z}/d_2\mathbb{Z}$, let x_i denote the unique lower column in block B_i. Figure 3 shows the blocks of $G_{85}^{\{12,5\}}$.

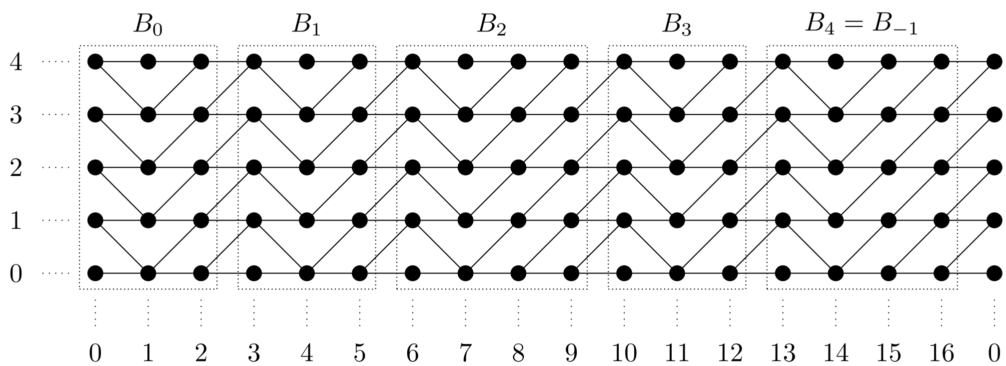

Fig. 3: Blocks of $G_{85}^{\{12,5\}}$. Note that 4 equals -1 in $\mathbb{Z}/5\mathbb{Z}$, that is, $B_4 = B_{-1}$.

Lemma 3 *(i) The length of a block is either* $\left\lfloor \frac{d_1}{d_2} \right\rfloor + 1$ *or* $\left\lceil \frac{d_1}{d_2} \right\rceil + 1$.

(ii) The length of B_0 *is* $\left\lfloor \frac{d_1}{d_2} \right\rfloor + 1$.

(iii) The length of B_{-1} *is* $\left\lceil \frac{d_1}{d_2} \right\rceil + 1$.

(iv) The number of blocks is d_2.

Proof: Let $x, x+1, \dots, x+l-1$ be the columns of a block B of length l. By definition and Lemma 2 (i), x is the unique lower column of block B, $x+1$ is the unique upper column of block B, and $x+l$ is a lower column. Hence, for all $y \in \mathbb{Z}$ and $x+1 \leq k \leq x+l-1$, we have $(k,y)-(k+1,y)=d_2$ and therefore $(x+1,y)-(x+l,y)=d_2(l-1)$. Since column $x+1$ is upper and column $x+l$ is lower, we have $d_1-d_2+1 \leq (x+1,y)-(x+l,y) \leq d_1+d_2-1$, which implies (i).

If $B=B_0$, then $x=0$ and $(x+1,y) \equiv d_1 \pmod{d_1+d_2}$ for all $y \in \mathbb{Z}$. Hence $(x+1,y)-(x+l,y) \leq d_1$. Together with $(x+1,y)-(x+l,y)=d_2(l-1)$ this implies (ii).

If $B=B_{-1}$, then $x+l=0$ and $(x+l,y) \equiv 0 \pmod{d_1+d_2}$ for all $y \in \mathbb{Z}$. Since column $x+1$ is upper, we have $(x+1,y)-(x+l,y) \geq d_1$. Together with $(x+1,y)-(x+l,y)=d_2(l-1)$ this implies (iii).

Since the function $f:\{0,\dots,d_1+d_2-1\} \to \{0,\dots,d_1+d_2-1\}$ with $f(x)=xd_1 \pmod{d_1+d_2}$ is bijective for $\gcd(d_1,d_2)=1$, there are exactly d_2 lower columns and therefore d_2 blocks, which proves (iv). □

3 Hamiltonian paths of G_n^D

The main result of this section is the following.

Theorem 4 *For every $D=\{d_1,d_2\} \subseteq \mathbb{N}$ with $d_1>d_2$ and $\gcd(d_1,d_2)=1$, there is some $n_0 \in \mathbb{N}$ such that for all integers n with $n \geq n_0$, the distance graph G_n^D has a Hamiltonian path with endvertices 0 and $n-1$.*

As before let $D=\{d_1,d_2\}$ for two positive integers d_1 and d_2 such that $\gcd(d_1,d_2)=1$ and $d_1>d_2$. For two lower vertices (x,y) and (x',y') with $x \neq x'$ and $y<y'$ in the distance graph G^D, a path in G^D with endvertices (x,y) and (x',y') whose vertex set consists of all vertices in the rows $y, y+1, \dots, y'-1$ and the vertex (x',y') is called an (x,y)-(x',y')-*climbing path* of G^D. See Figure 5 for an illustration.

Before we proceed to the proof of Theorem 4, we establish a series of lemmas concerning the existence of climbing paths.

Lemma 5 *If B_i is a block of even length in G^D, then G^D has an (x_i,y)-$(x_{i+1},y+2)$-climbing path for all y.*

Proof: Let

$$\begin{aligned} P \quad : \quad & (x_{i+1}-1,y),(x_{i+1},y+1),(x_{i+1}-1,y+1),(x_{i+1}-2,y), \\ & (x_{i+1}-3,y),(x_{i+1}-2,y+1),(x_{i+1}-3,y+1),(x_{i+1}-4,y), \\ & \dots,(x_i+3,y),(x_i+4,y+1),(x_i+3,y+1),(x_i+2,y). \end{aligned}$$

The sequence

$$\begin{aligned} & (x_i,y),(x_i-1,y),\dots,(x_{i+1},y), \\ & P,(x_i+1,y), \\ & (x_i+2,y+1),(x_i+1,y+1),\dots,(x_{i+1}+1,y+1), \\ & (x_{i+1},y+2) \end{aligned}$$

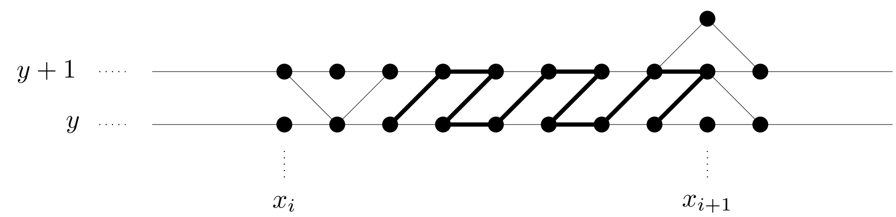

Fig. 4: P for a block B_i of length 8.

defines an (x_i, y)-$(x_{i+1}, y+2)$-climbing path in G^D. See Figures 4 and 5 for an illustration. □

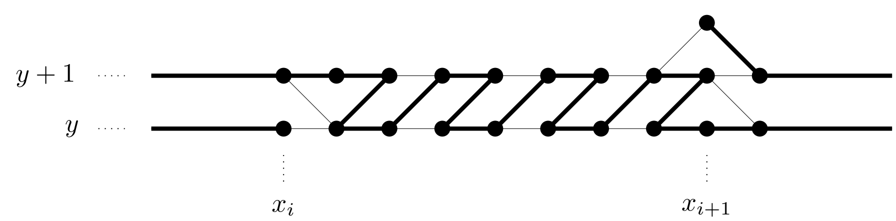

Fig. 5: An (x_i, y)-$(x_{i+1}, y+2)$-climbing path for block B_i of length 8.

Lemma 6 *If B_{i-1} is a block of even length in G^D, then G^D has an (x_i, y)-$(x_{i-1}, y+2)$-climbing path for all y.*

Proof: Let

$$\begin{aligned} P \quad : \quad & (x_{i-1}+3, y+1), (x_{i-1}+2, y), (x_{i-1}+3, y), (x_{i-1}+4, y+1), \\ & (x_{i-1}+5, y+1), (x_{i-1}+4, y), (x_{i-1}+5, y), (x_{i-1}+6, y+1), \\ & \ldots, (x_i-1, y+1), (x_i-2, y), (x_i-1, y), (x_i, y+1). \end{aligned}$$

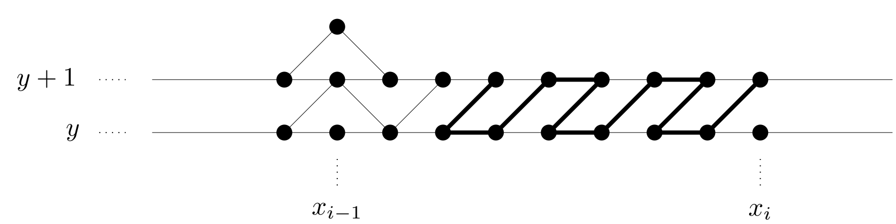

Fig. 6: P for a block B_{i-1} of length 8.

The sequence

$$
\begin{aligned}
&(x_i, y), (x_i + 1, y), \ldots, (x_{i-1} + 1, y), \\
&(x_{i-1}, y + 1), (x_{i-1} + 1, y + 1), (x_{i-1} + 2, y + 1), P, \\
&(x_i + 1, y + 1), (x_i + 2, y + 1), \ldots, (x_{i-1} - 1, y + 1), \\
&(x_{i-1}, y + 2)
\end{aligned}
$$

defines an (x_i, y)-$(x_{i-1}, y + 2)$-climbing path of G^D. See Figures 6 and 7 for an illustration. $\square$

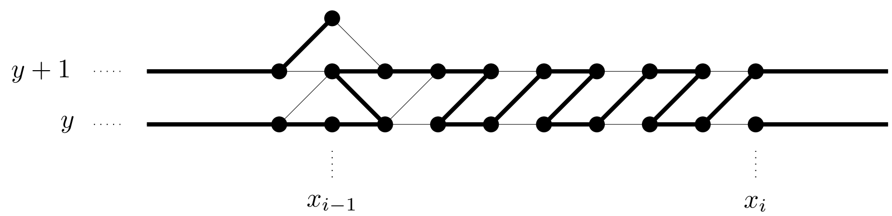

Fig. 7: An (x_i, y)-$(x_{i-1}, y + 2)$-climbing path for block B_{i-1} of length 8.

Lemma 7 *If G^D has at least $j + 2$ blocks for some $j \geq 1$ and for some $i \in \mathbb{Z}/d_2\mathbb{Z}$, the blocks $B_i, B_{i+1}, \ldots, B_{i+j}$ of G^D are such that B_i and B_{i+j} are of odd length and $B_{i+1}, \ldots, B_{i+j-1}$ are of even length at least 4, then G^D has an (x_i, y)-$(x_{i+j+1}, y + 3)$-climbing path for all y.*

Proof: By Lemma 3, the blocks B_i and B_{i+j} are of length at least 3.

Let

$$
\begin{aligned}
P_{i+j} \quad : \quad & (x_{i+j+1} - 1, y), (x_{i+j+1}, y + 1), (x_{i+j+1} - 1, y + 1), (x_{i+j+1} - 2, y), \\
& (x_{i+j+1} - 3, y), (x_{i+j+1} - 2, y + 1), (x_{i+j+1} - 3, y + 1), (x_{i+j+1} - 4, y), \\
& \ldots, (x_{i+j} + 2, y), (x_{i+j} + 3, y + 1), (x_{i+j} + 2, y + 1), (x_{i+j} + 1, y).
\end{aligned}
$$

For $1 \leq q \leq j - 1$, let

$$
\begin{aligned}
P_{i+q} \quad : \quad & (x_{i+q} + 3, y + 2), (x_{i+q} + 2, y + 1), (x_{i+q} + 3, y + 1), (x_{i+q} + 4, y + 2), \\
& (x_{i+q} + 5, y + 2), (x_{i+q} + 4, y + 1), (x_{i+q} + 5, y + 1), (x_{i+q} + 6, y + 2), \\
& \ldots, (x_{i+q+1} - 3, y + 2), (x_{i+q+1} - 4, y + 1), (x_{i+q+1} - 3, y + 1), (x_{i+q+1} - 2, y + 2)
\end{aligned}
$$

and let

$$
\begin{aligned}
P'_{i+q} \quad : \quad & P_{i+q}, (x_{i+q+1} - 1, y + 2), (x_{i+q+1} - 2, y + 1), (x_{i+q+1} - 1, y + 1), (x_{i+q+1}, y + 1), \\
& (x_{i+q+1} + 1, y + 1), (x_{i+q+1}, y + 2), (x_{i+q+1} + 1, y + 2), (x_{i+q+1} + 2, y + 2).
\end{aligned}
$$

Note that P_{i+q} is empty if B_{i+q} is of length 4. Furthermore, let

$$
\begin{aligned}
P_i \quad : \quad & (x_{i+1} - 2, y + 2), (x_{i+1} - 3, y + 1), (x_{i+1} - 4, y + 1), (x_{i+1} - 3, y + 2), \\
& (x_{i+1} - 4, y + 2), (x_{i+1} - 5, y + 1), (x_{i+1} - 6, y + 1), (x_{i+1} - 5, y + 2), \\
& \ldots, (x_i + 3, y + 2), (x_i + 2, y + 1), (x_i + 1, y + 1), (x_i + 2, y + 2).
\end{aligned}
$$

Note that P_i is empty if B_i is of length 3.

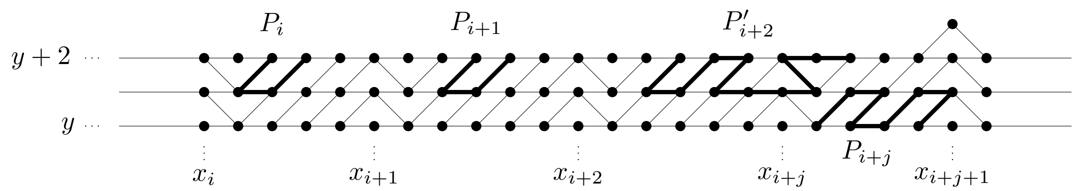

Fig. 8: P_i, P_{i+1}, P'_{i+2}, and P_{i+j} for $j = 3$.

Now, the sequence

$$
\begin{aligned}
&(x_i, y), (x_i - 1, y), \ldots, (x_{i+j+1}, y), P_{i+j},\\
&(x_{i+j}, y), (x_{i+j} - 1, y), \ldots, (x_{i+1} - 1, y),\\
&(x_{i+1}, y+1), (x_{i+1}+1, y+1), (x_{i+1}, y+2), (x_{i+1}+1, y+2), (x_{i+1}+2, y+2),\\
&P'_{i+1}, P'_{i+2}, \ldots, P'_{i+j-1},\\
&(x_{i+j}+3, y+2), (x_{i+j}+4, y+2), \ldots, (x_{i+j+1}, y+2),\\
&(x_{i+j+1}+1, y+1), (x_{i+j+1}+2, y+1), \ldots, (x_i, y+1),\\
&(x_i+1, y), (x_i+2, y), \ldots, (x_{i+1}-2, y),\\
&(x_{i+1}-1, y+1), (x_{i+1}-2, y+1), (x_{i+1}-1, y+2), P_i,\\
&(x_i+1, y+2), (x_i, y+2), \ldots, (x_{i+j+1}+1, y+2),\\
&(x_{i+j+1}, y+3)
\end{aligned}
$$

defines an (x_i, y)-$(x_{i+j+1}, y+3)$-climbing path of G^D. See Figures 8 and 9 for an illustration. □

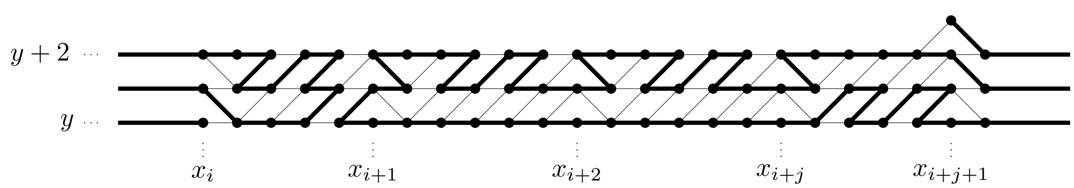

Fig. 9: An (x_i, y)-$(x_{i+j+1}, y+3)$-climbing path for $j = 3$.

Lemma 8 *If G^D has at least $j+2$ blocks for some $j \geq 1$ and for some $i \in \mathbb{Z}/d_2\mathbb{Z}$, the blocks $B_i, B_{i+1}, \ldots, B_{i+j}$ of G^D are such that B_i and B_{i+j} are of length 3 and $B_{i+1}, \ldots, B_{i+j-1}$ are of length 2, then G^D has an (x_i, y)-$(x_{i+j+1}, y+j+2)$-climbing path for all y.*

Proof: Note that $x_{i+j+1} = x_i + 2j + 4$. For $1 \leq q \leq j-1$, let

$$P_q \quad : \quad (x_i + 2j + 2, y + q), (x_i + 2j + 3, y + q + 1), (x_i + 2j + 4, y + q + 2),$$

$(x_i + 2j + 5, y + q + 1), (x_i + 2j + 6, y + q + 1), \ldots, (x_i, y + q + 1),$
$(x_i + 1, y + q), (x_i + 2, y + q), \ldots, (x_i + 2j - 2q + 2, y + q),$
$(x_i + 2j - 2q + 1, y + q + 1), (x_i + 2j - 2q + 2, y + q + 1), \ldots, (x_i + 2j + 1, y + q + 1).$

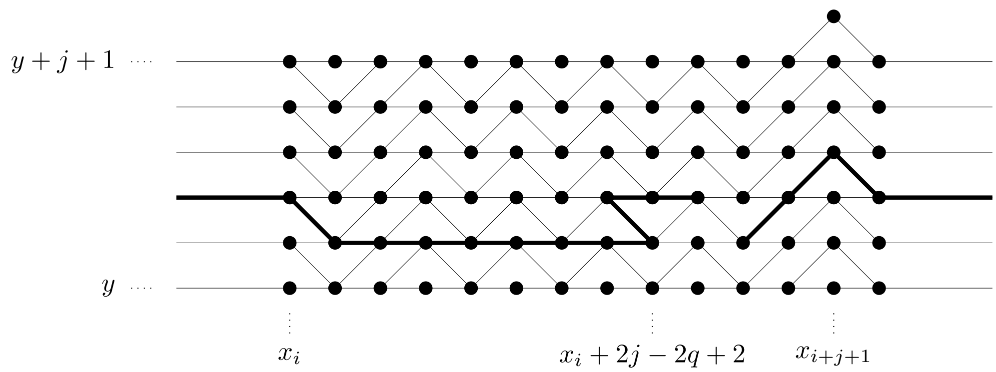

Fig. 10: P_q for $j = 4$ and $q = 1$.

Now, the sequence

$$
\begin{aligned}
&(x_i, y), (x_i - 1, y), \ldots, (x_i + 2j + 4, y),\\
&(x_i + 2j + 3, y), (x_i + 2j + 4, y + 1), (x_i + 2j + 3, y + 1), (x_i + 2j + 4, y + 2),\\
&(x_i + 2j + 5, y + 1), (x_i + 2j + 6, y + 1), \ldots, (x_i, y + 1),\\
&(x_i + 1, y), (x_i + 2, y), \ldots, (x_i + 2j + 2, y),\\
&(x_i + 2j + 1, y + 1),\\
&P_1, P_2, \ldots, P_{j-1},\\
&(x_i + 2j + 2, y + j),\\
&(x_i + 2j + 3, y + j + 1)(x_i + 2j + 2, y + j + 1), \ldots, (x_i + 3, y + j + 1),\\
&(x_i + 2, y + j), (x_i + 1, y + j),\\
&(x_i + 2, y + j + 1)(x_i + 1, y + j + 1), \ldots, (x_i + 2j + 5, y + j + 1),\\
&(x_i + 2j + 4, y + j + 2)
\end{aligned}
$$

defines an (x_i, y)-$(x_{i+j+1}, y + j + 2)$-climbing path of G^D. See Figures 10 and 11 for an illustration. □

We are now in a position to prove the main result of this section. A path P in G^D with $V(P) = [\min(V(P)), \max(V(P))]$ is called *special*, if the endvertices of P are $\min(V(P))$ and $\max(V(P))$.

Proof of Theorem 4: If $d_2 = 1$, then the statement of the theorem is trivial. Hence we assume that $d_2 > 1$. The idea of the proof is to show the existence of two distinct positive integers p_1 and p_2 with

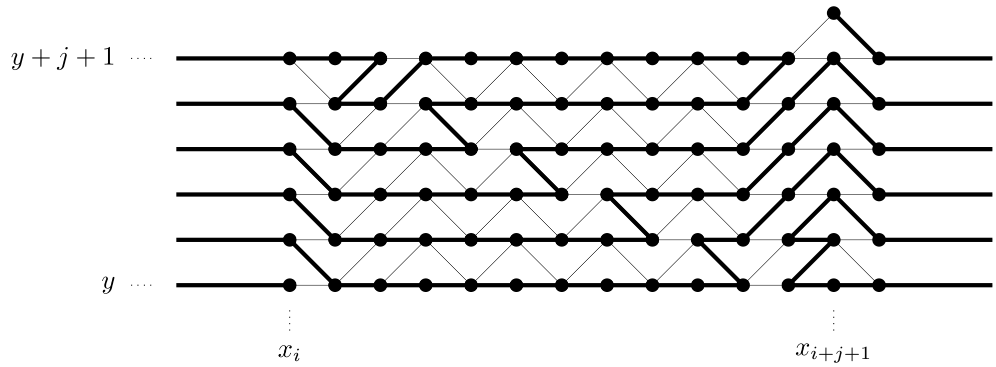

Fig. 11: An (x_i, y)-$(x_{i+j+1}, y+j+2)$-climbing path for $j = 4$.

$\gcd(p_1, p_2) = 1$ such that the distance graph G^D has a special path of length p_1 and a special path of length p_2. Note that we can shift special paths. If $P : v_0, \ldots, v_l$ is a special path, then also $P + h : v_0 + h, \ldots, v_l + h$ is a special path. Furthermore, we can concatenate special paths. If $P : v_0, \ldots, v_l$ is a special path of length l and $Q : v_l, \ldots, v_{l+h}$ is a special path of length h, then $PQ : v_0, \ldots, v_{l+h}$ is a special path of length $l + h$. Since $\gcd(p_1, p_2) = 1$, it follows from the extended Euclidean algorithm that every sufficiently large integer p is a positive integral linear combination of p_1 and p_2. In fact, if $p > (2p_2 - 1)p_1$ and $p = a_1p_1 + a_2p_2$ for integers a_1 and a_2 such that $a_1 \leq 0$, then $a_1 + sp_2 > 0$ and $a_2 - sp_1 > 0$ for $s = \lceil \frac{-a_1}{p_2} \rceil + 1$ and thus $p = (a_1 + sp_2)p_1 + (a_2 - sp_1)p_2$ is a positive integral linear combination of p_1 and p_2. Therefore, the desired result follows by shifting and concatenating copies of the special paths of lengths p_1 and p_2, which we construct now.

It has been observed in [25, 21] that $G^D_{d_1+d_2+1}$ has a Hamiltonian path with endvertices 0 and $d_1 + d_2$. Hence, for p_1, we choose $d_1 + d_2$.

For p_2, we show that there is a positive integer p_2 with $p_2 \equiv -1 \pmod{d_1 + d_2} \equiv -1 \pmod{p_1}$, such that G^D has a special path of length p_2, thus $\gcd(p_1, p_2) = 1$.

Let x' be such that $x'd_1 \equiv -1 \pmod{d_1 + d_2}$. By definition and Lemma 2 (i), column x' is upper and column $x' - 1$ is lower. In order to get a special path with endvertices $(0, 0)$ and (x', y') for some y', we concatenate climbing paths to form a $(0, 0)$-$(x' - 1, y')$-climbing path and append the path $(x' - 2, y'), (x' - 3, y'), \ldots, (x', y')$.

Let k be such that the block B_k contains column x', that is, $x_k = x' - 1$. Since column x' is upper, column $x' - 2$ belongs to block B_{k-1}.

Since $\gcd(d_1, d_2) = 1$, at least one of d_1 and d_2 is odd.

Case 1 *One of d_1 and d_2 is even and G^D has at most 2 blocks of odd length.*

Since $d_1 + d_2$ is odd, the number of blocks of odd length is odd, that is, it equals 1.

We first assume that all blocks $B_0, B_1, \ldots, B_{k-1}$ are of even length. By Lemma 5, there exists an $(x_i, 2i)$-$(x_{i+1}, 2i + 2)$-climbing path P_i for $0 \leq i \leq k - 1$. Since $x' - 1 = x_k$, the concatenation of the paths $P_0, P_1, \ldots, P_{k-1}$ forms a $(0, 0)$-$(x' - 1, y')$-climbing path for $y' = 2k$.

Next, we assume that all blocks $B_k, B_{k+1}, \ldots, B_{-1}$ are of even length. Then, by Lemma 6, there exists an $(x_{i+1}, 2(d_1+d_2-i)-2)$-$(x_i, 2(d_1+d_2-i))$-climbing path P_i for $k \leq i \leq d_1+d_2-1$. Since $x'-1 = x_k$, the concatenation of the paths $P_{d_1+d_2-1}, P_{d_1+d_2-2}, \ldots, P_k$ forms a $(0,0)$-$(x'-1, y')$-climbing path for $y' = 2(d_1+d_2-k)$. This concludes the first case.

Case 2 *One of d_1 and d_2 is even and G^D has at least 3 blocks of odd length.*

Since $d_1 + d_2$ is odd, the number of blocks of odd length is odd. This implies that one of the two sequences $B_0, B_1, \ldots, B_{k-1}$ and $B_0, B_1, \ldots, B_{-1}, B_0, B_1, \ldots, B_{k-1}$ has an even number of blocks with odd length. We call this sequence $\mathcal{S}$. We can partition $\mathcal{S}$ into subsequences $S_1, S_2, \ldots, S_t$, where each subsequence is either a block of even length or a sequence $B_i, B_{i+1}, \ldots, B_{i+j}$ of blocks with $i \in \mathbb{Z}/d_2\mathbb{Z}$ and $j \geq 1$, such that block B_i has odd length, block B_{i+j} has odd length, and blocks $B_{i+1}, \ldots, B_{i+j-1}$ have even length. For a subsequence S_q, $1 \leq q \leq t$, that consists of one block B_i with $i \in \mathbb{Z}/d_2\mathbb{Z}$, Lemma 5 implies that there exists an (x_i, y)-$(x_{i+1}, y+2)$-climbing path $P_{q,y}$ for every y. If $\frac{d_1}{d_2} < 2$, then Lemma 3 implies that the lengths of the blocks are 2 and 3. For a subsequence S_q, $1 \leq q \leq t$, that consists of at least two blocks $B_i, B_{i+1}, \ldots, B_{i+j}$ with $i \in \mathbb{Z}/d_2\mathbb{Z}$ and $j \geq 1$, Lemma 8 implies that there exists an (x_i, y)-$(x_{i+j+1}, y+j+2)$-climbing path $P_{q,y}$ for every y. If $\frac{d_1}{d_2} \geq 2$, then Lemma 3 implies that the lengths of the blocks are at least 3. For a subsequence S_q, $1 \leq q \leq t$, that consists of at least two blocks $B_i, B_{i+1}, \ldots, B_{i+j}$ with $i \in \mathbb{Z}/d_2\mathbb{Z}$ and $j \geq 1$, Lemma 7 implies that there exists an (x_i, y)-$(x_{i+j+1}, y+3)$-climbing path $P_{q,y}$ for every y.

The concatenation of the paths $P_{1,y_1}, P_{2,y_2}, \ldots, P_{t,y_t}$ forms a $(0,0)$-$(x'-1, y')$-climbing path for $y_1 = 0$, suitable y_q's, where $2 \leq q \leq t$, and $y' = y_t$.

This concludes the second case.

If both d_1 and d_2 are odd, then $d_1 + d_2$ is even, which implies that the number of blocks of odd length is even and exactly those vertices are even integers that are in a column with an even index. This implies that x' is odd and $x_k = x' - 1$ is even. Since column 0 and column $x'-1$ are lower, the sequence $B_0, B_1, \ldots, B_{k-1}$ has an even number of blocks with odd length.

Case 3 *Both d_1 and d_2 are odd and G^D has at most 2 blocks of odd length.*

Since $d_2 \geq 2$, G has exactly 2 blocks of odd length. This implies that one of the two sequences $B_0, B_1, \ldots, B_{k-1}$ and $B_k, B_{k+1}, \ldots, B_{-1}$ has only blocks of even length. Now we are in the same situation as in Case 1. Arguing as in Case 1 completes this case.

Case 4 *Both d_1 and d_2 are odd and G^D has at least 4 blocks of odd length.*

Since the sequence $B_0, B_1, \ldots, B_{k-1}$ has an even number of blocks of odd length, we are in the same situation as in Case 2. Arguing as in Case 2 completes this case, which concludes the proof of the theorem. $\square$

4 Hamiltonian cycles of G_n^D

The main results of this section are the following.

Theorem 9 *For every $D = \{d_1, d_2\} \subseteq \mathbb{N}$ with $d_1 > d_2$, $d_1 d_2$ odd, and $\gcd(d_1, d_2) = 1$, there is some $n_0 \in \mathbb{N}$ such that for all even integers n with $n \geq n_0$, the distance graph G_n^D has a Hamiltonian cycle.*

Theorem 10 *For every $D = \{d_1, d_2\} \subseteq \mathbb{N}$ with $d_1 > d_2$, $d_1 d_2$ even, and $\gcd(d_1, d_2) = 1$, there is some $n_0 \in \mathbb{N}$ such that for all integers n with $n \geq n_0$, the distance graph G_n^D has a Hamiltonian cycle.*

Note that the distance graphs considered in Theorem 9 are necessarily bipartite. Therefore, they can only have a Hamiltonian cycle if their order is even.

As in Section 3 we establish several lemmas before proceeding to the proofs of Theorems 9 and 10.

For two lower vertices (x, y) and (x', y') with $x \neq x'$, $0 \notin \{x'+1, x'+2, \ldots, x\}$, and $y < y'$ in the distance graph G^D, a set of vertex disjoint paths $R_y, R_{y+1}, \ldots, R_{y'-1}$ in G^D is called an (x, y)-(x', y')-*path-collection* of G^D, if it satisfies the following conditions:

- for $y \leq i < y'$, P_i has the endvertices $(0, i)$ and $(-1, i+1)$,
- for $y \leq i < y'$, the path $(0, i), (1, i), \ldots, (x', i)$ is a subpath of P_i,
- for $y \leq i < y'$, the path $(x, i+1), (x+1, i+1), \ldots, (-1, i+1)$ is a subpath of P_i,
- the union of the vertex sets of the paths consists of all vertices in the rows $y+1, y+2, \ldots, y'-1$, the vertices $\{(0, y), (1, y), \ldots, (x-1, y)\}$, and the vertices $\{(x', y'), (x'+1, y'), \ldots, (-1, y')\}$, and
- no edge of the form $\{(-1, z), (0, z')\}$ for some $z, z' \in \mathbb{Z}$ is in the union of the edge sets of the paths.

See Figures 13, 15, and 16 for an illustration. Note, that (x, y) does not belong to any path of an (x, y)-(x', y')-path-collection.

Lemma 11 *If for some $i \neq -1$, B_i is a block of even length in G^D, then G^D has an (x_{i+1}, y)-$(x_i, y+1)$ path collection for all y.*

Proof: Let

$$\begin{aligned} P \quad : \quad & (x_i+3, y+1), (x_i+2, y), (x_i+3, y), (x_i+4, y+1), \\ & (x_i+5, y+1), (x_i+4, y), (x_i+5, y), (x_i+6, y+1), \\ & \ldots, (x_{i+1}-1, y+1), (x_{i+1}-2, y), (x_{i+1}-1, y), (x_{i+1}, y+1). \end{aligned}$$

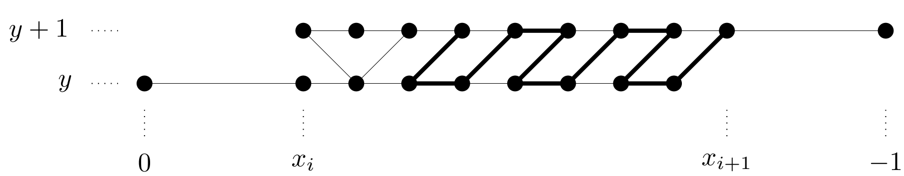

Fig. 12: P for a block B_i of length 8.

The sequence

$$\begin{array}{l}(0,y),(1,y),\ldots,(x_i+1,y),\\ (x_i,y+1),(x_i+1,y+1),(x_i+2,y+1),P,\\ (x_{i+1}+1,y+1),(x_{i+1}+2,y+1),\ldots,(-1,y+1)\end{array}$$

defines an (x_{i+1},y)-$(x_i,y+1)$-path-collection in G^D. See Figures 12 and 13 for an illustration. □

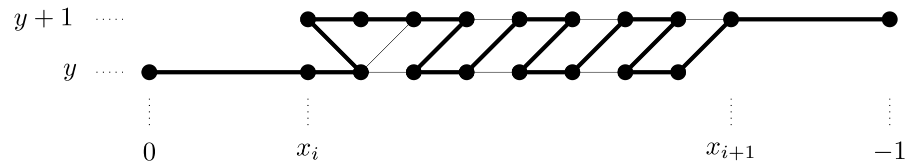

Fig. 13: An (x_{i+1},y)-$(x_i,y+1)$-path-collection for a block B_i of length 8.

Lemma 12 *If for some $i\in\mathbb{Z}/d_2\mathbb{Z}$ and for some $j\geq 1$, the blocks $B_i,B_{i+1},\ldots,B_{i+j}$ of G^D are such that $-1\notin\{i,i+1,\ldots,i+j\}$, B_i and B_{i+j} are of odd length and $B_{i+1},\ldots,B_{i+j-1}$ are of even length at least 4, then G^D has an (x_{i+j+1},y)-$(x_i,y+2)$-path-collection for all y.*

Proof: By Lemma 3, the blocks B_i and B_{i+j} are of length at least 3.

Let

$$\begin{array}{rcl}P_i & : & (x_i+1,y),(x_i+2,y+1),(x_i+3,y+1),(x_i+2,y),\\ & & (x_i+3,y),(x_i+4,y+1),(x_i+5,y+1),(x_i+4,y),\\ & & \ldots,(x_{i+1}-2,y),(x_{i+1}-1,y+1),(x_{i+1},y+1),(x_{i+1}-1,y).\end{array}$$

For $1\leq q\leq j-1$, let

$$\begin{array}{rcl}P_{i+q} & : & (x_{i+q},y),(x_{i+q}+1,y),\\ & & (x_{i+q}+2,y),(x_{i+q}+3,y+1),(x_{i+q}+4,y+1),(x_{i+q}+3,y),\\ & & (x_{i+q}+4,y),(x_{i+q}+5,y+1),(x_{i+q}+6,y+1),(x_{i+q}+5,y),\\ & & \ldots,(x_{i+q+1}-2,y),(x_{i+q+1}-1,y+1),(x_{i+q+1},y+1),(x_{i+q+1}-1,y).\end{array}$$

Furthermore, let

$$\begin{array}{rcl}P_{i+j} & : & (x_{i+j}+2,y),(x_{i+j}+3,y+1),(x_{i+j}+4,y+1),(x_{i+j}+3,y),\\ & & (x_{i+j}+4,y),(x_{i+j}+5,y+1),(x_{i+j}+6,y+1),(x_{i+j}+5,y),\\ & & \ldots,(x_{i+j+1}-3,y),(x_{i+j+1}-2,y+1),(x_{i+j+1}-1,y+1),(x_{i+j+1}-2,y).\end{array}$$

For $1\leq q\leq j$, let

$$\begin{array}{rcl}Q_{i+q} & : & (x_{i+q-1}+4,y+2),(x_{i+q-1}+5,y+2),\ldots,(x_{i+q}+2,y+2),\\ & & (x_{i+q}+1,y+1),(x_{i+q}+2,y+1),(x_{i+q}+3,y+2).\end{array}$$

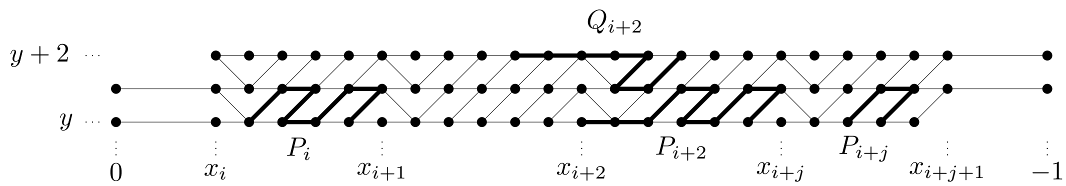

Fig. 14: P_i, P_{i+2}, P_{i+j}, and Q_{i+2} for $j = 3$.

Now, R_y and R_{y+1}, where

$$\begin{aligned} R_y \quad : \quad & (0, y), (1, y), \dots, (x_i, y), \\ & P_i, \\ & P_{i+1}, P_{i+2}, \dots, P_{i+j-1}, \\ & (x_{i+j}, y), (x_{i+j}+1, y), P_{i+j}, (x_{i+j+1}-1, y), \\ & (x_{i+j+1}, y+1), (x_{i+j+1}+1, y+1), \dots, (-1, y+1) \end{aligned}$$

and

$$\begin{aligned} R_{y+1} \quad : \quad & (0, y+1), (1, y+1), \dots, (x_i+1, y+1), \\ & (x_i, y+2), (x_i+1, y+2), (x_i+2, y+2), (x_i+3, y+2), \\ & Q_{i+1}, Q_{i+2}, \dots, Q_{i+j}, \\ & (x_{i+j+1}-1, y+2), (x_{i+j+1}, y+2), \dots, (-1, y+2) \end{aligned}$$

define an (x_{i+j+1}, y)-$(x_i, y+2)$-path-collection of G^D. See Figures 14 and 15 for an illustration. $\square$

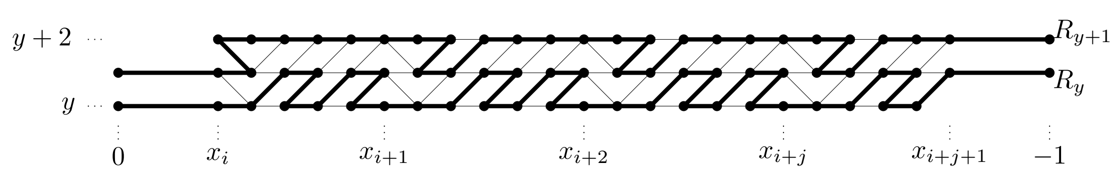

Fig. 15: An (x_{i+j+1}, y)-$(x_i, y+2)$-path-collection for $j = 3$.

Lemma 13 *If for some $i \in \mathbb{Z}/d_2\mathbb{Z}$ and for some $j \geq 1$, the blocks $B_i, B_{i+1}, \dots, B_{i+j}$ of G^D are such that $-1 \notin \{i, i+1, \dots, i+j\}$, B_i and B_{i+j} are of length 3 and $B_{i+1}, \dots, B_{i+j-1}$ are of length 2, then G^D has an (x_{i+j+1}, y)-$(x_i, y+j+1)$-path-collection for all y.*

Proof: Note that $x_{i+j+1} = x_i + 2j + 4$. For $0 \leq q \leq j-1$, let

$$R_{i+q} \quad : \quad (0, y+q), (1, y+q), \dots, (x_i+1, y+q),$$

$$(x_i+2, y+q+1), (x_i+3, y+q+1), \ldots, (x_i+2q+3, y+q+1),$$
$$(x_i+2q+2, y+q), (x_i+2q+3, y+q), \ldots, (x_i+2j+3, y+q),$$
$$(x_i+2j+4, y+q+1), (x_i+2j+5, y+q+1), \ldots, (-1, y+q+1).$$

and let

$$\begin{aligned} R_{i+j} \quad : \quad & (0, y+j), (1, y+j), \ldots, (x_i+1, y+j), \\ & (x_i, y+j+1), (x_i+1, y+j+1), \ldots, (x_i+2j+3, y+j+1), \\ & (x_i+2j+2, y+j), (x_i+2j+3, y+j), \\ & (x_i+2j+4, y+j+1), (x_i+2j+5, y+j+1), \ldots, (-1, y+j+1). \end{aligned}$$

Now, $R_i, R_{i+1}, \ldots, R_{i+q}$ define an (x_{i+j+1}, y)-$(x_i, y+j+1)$-path-collection of G^D. See Figure 16 for an illustration. □

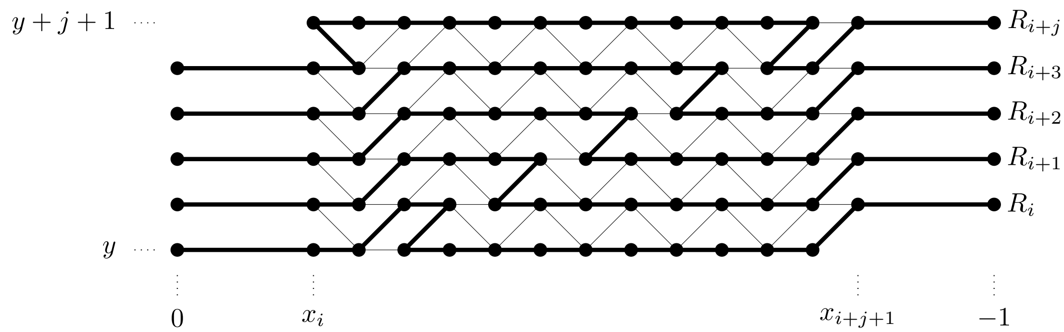

Fig. 16: An (x_{i+j+1}, y)-$(x_i, y+j+1)$-path-collection for $j = 4$.

Lemma 14 *If for some $i \in \mathbb{Z}/d_2\mathbb{Z}$ and for some $j \geq 0$, the sequence $\mathcal{S} = B_i, B_{i+1}, \ldots, B_{i+j}$ of blocks of G^D is such that $-1 \notin \{i, i+1, \ldots, i+j\}$ and the number of blocks of odd length among $B_i, B_{i+1}, \ldots, B_{i+j}$ is even, then G^D has an (x_{i+j+1}, y)-$(x_i, y+\Delta y)$-path-collection for some Δy and for all y.*

Proof: By definition, the union of suitable path-collections is a path-collection: If for some x, x', x'', y, y', y'', G^D has an (x, y)-(x', y')-path-collection and an (x', y')-(x'', y'')-path-collection, then G^D has an (x, y)-(x'', y'')-path-collection. We can partition $\mathcal{S}$ into subsequences, where each subsequence is either a block of even length or a sequence $B_k, B_{k+1}, \ldots, B_{k+l}$ of blocks with $k \in \mathbb{Z}/d_2\mathbb{Z}$ and $l \geq 1$, such that blocks B_k and B_{k+l} have odd length and blocks $B_{k+1}, \ldots, B_{k+l-1}$ have even length. For a subsequence that consists of one even block B_k with $k \in \mathbb{Z}/d_2\mathbb{Z}$, Lemma 11 implies that there exists a (x_{k+1}, y)-$(x_k, y+1)$ path collection for every y. If $\frac{d_1}{d_2} < 2$, then Lemma 3 implies that the lengths of the blocks are 2 and 3. For a subsequence that consists of at least two blocks $B_k, B_{k+1}, \ldots, B_{k+l}$ with $k \in \mathbb{Z}/d_2\mathbb{Z}$ and $l \geq 1$, Lemma 13 implies that there exists an (x_{k+l+1}, y)-$(x_k, y+l+1)$-path-collection for every y. If $\frac{d_1}{d_2} \geq 2$, then Lemma 3 implies that the lengths of the blocks are at least 3. For a subsequence that consists of at least two blocks $B_k, B_{k+1}, \ldots, B_{k+l}$ with $k \in \mathbb{Z}/d_2\mathbb{Z}$ and $l \geq 1$, Lemma 12 implies that there exists an (x_{k+l+1}, y)-$(x_k, y+2)$-path-collection for every y. Hence, a suitable union of path-collections forms an (x_{i+j+1}, y)-$(x_i, y+\Delta y)$-path-collection for a suitable Δy and all y. $\square$

Lemma 15 *If for some $-i \in \mathbb{Z}/d_2\mathbb{Z}$, the blocks $B_{-i}, B_{-i+1}, \ldots, B_{-1}$ of G^D are such that B_{-i} is of odd length and $B_{-i+1}, \ldots, B_{-1}$ are of even length at least 4, then for all y, G^D has a path with endvertices $(-1, y+1)$ and $(-1, y+2)$ that consists of all vertices of rows y and $y+1$ and the vertices $(x_{-i}, y+2), (x_{-i}+1, y+2), \ldots, (-1, y+2)$.*

Proof: For $1 \leq q \leq i-1$, let

$$\begin{aligned} Q_{-q} \quad : \quad & (x_{-q+1}-3, y), (x_{-q+1}-4, y), \ldots, (x_{-q}, y), \\ & (x_{-q}-1, y), (x_{-q}, y+1), (x_{-q}-1, y+1), (x_{-q}-2, y) \end{aligned}$$

and let

$$\begin{aligned} Q_{-i} \quad : \quad & (x_{-i+1}-3, y), (x_{-i+1}-2, y+1), (x_{-i+1}-3, y+1), (x_{-i+1}-4, y), \\ & (x_{-i+1}-5, y), (x_{-i+1}-4, y+1), (x_{-i+1}-5, y+1), (x_{-i+1}-6, y), \\ & \ldots, (x_{-i}+2, y), (x_{-i}+3, y+1), (x_{-i}+2, y+1), (x_{-i}+1, y). \end{aligned}$$

Furthermore, let for $1 \leq q \leq i-1$

$$\begin{aligned} P_{-q} \quad : \quad & (x_{-q}, y+2), (x_{-q}+1, y+2), \\ & (x_{-q}+2, y+2), (x_{-q}+1, y+1), (x_{-q}+2, y+1), (x_{-q}+3, y+2), \\ & (x_{-q}+4, y+2), (x_{-q}+3, y+1), (x_{-q}+4, y+1), (x_{-q}+5, y+2), \\ & \ldots, (x_{-q+1}-2, y+2), (x_{-q+1}-3, y+1), (x_{-q+1}-2, y+1), (x_{-q+1}-1, y+2). \end{aligned}$$

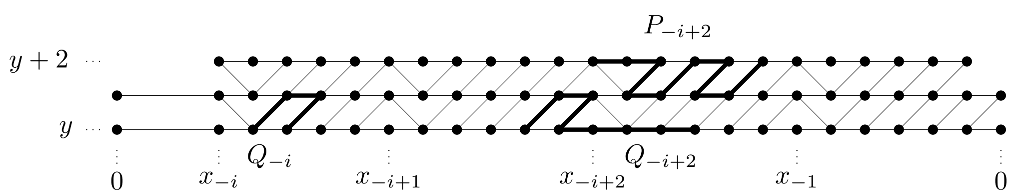

Fig. 17: Q_{-i}, Q_{-i+2}, and P_{-i+2} for $i = 4$.

Now, the sequence

$$\begin{aligned} & (-1, y+1), (-2, y), \\ & Q_{-1}, Q_{-2}, \ldots, Q_{-i}, \\ & (x_{-i}, y), (x_{-i}-1, y), \ldots, (-1, y), \\ & (0, y+1), (1, y+1), \ldots, (x_{-i}+1, y+1), \\ & (x_{-i}, y+2), (x_{-i}+1, y+2), \ldots, (x_{-i+1}-1, y+2), \\ & P_{-i+1}, P_{-i+2}, \ldots, P_{-1} \end{aligned}$$

defines a path that satisfies the conditions of the lemma. See Figures 17 and 18 for an illustration. □

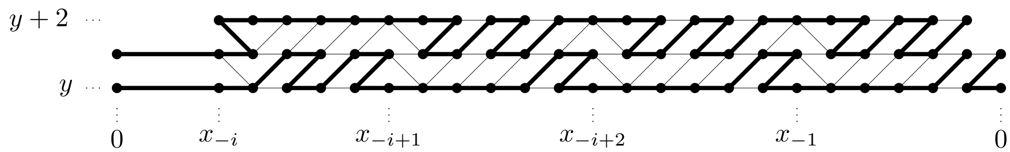

Fig. 18: A path for $i = 4$.

A cycle C in G^D is called *special*, if $V(C) = [\min(V(C)), \max(V(C))]$.

Lemma 16 *For every $D = \{d_1, d_2\} \subseteq \mathbb{N}$ with $d_1 > d_2$, $d_1 d_2$ even, and $\gcd(d_1, d_2) = 1$, there is some $n \in \mathbb{N}$ with $n \equiv 0 \pmod{d_1 + d_2}$ such that G^D has a special cycle C of order $n+1$ with $V(C) = [0, n]$.*

Proof: Clearly, vertex n is in column 0. Since $d_1 d_2$ is even and $\gcd(d_1, d_2) = 1$, we obtain that $d_1 + d_2$ is odd and hence the number of blocks of odd length is odd, i.e. at least 1. Let $i \in \mathbb{Z}/d_2\mathbb{Z}$, such that block B_i is of odd length and the blocks $B_{i+1}, \ldots, B_{-1}$ are of even length. Clearly, by Lemma 3, the length of the blocks $B_{i+1}, \ldots, B_{-1}$ are at least 4. By Lemma 15, G^D has a path Q with endvertices $(-1, 1)$ and $(-1, 2)$ that consists of all vertices of rows 0 and 1 and the vertices $(x_i, 2), (x_i + 1, 2), \ldots, (-1, 2)$. Since the number of blocks of $B_0, \ldots, B_{i-1}$ of odd length is even, by Lemma 14, G^D has an $(x_i, 2)$-$(0, y')$-path-collection $\mathcal{R}$ for some y'. Note, that if G^D has only one block of odd length, then $\mathcal{R} = \emptyset$. In this case we define $y' = 2$. Let

$$P = \left(Q \cup \bigcup_{R \in \mathcal{R}} R \right) + \bigcup_{y=1}^{y'-2} \{\{(-1, y), (0, y+1)\}\}.$$

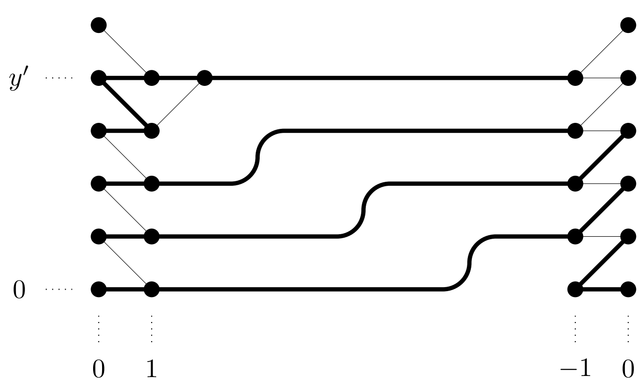

Fig. 19: The path P.

By construction, P is a path with endvertices $(-1, y'-1)$ and $(-1, y')$ that consists of all vertices of rows $0, 1, \ldots, y'$. The vertex $(0, y')$ has the neighbors $(1, y'-1)$ and $(1, y')$ in P. Since the vertex $(1, y')$ is an upper vertex, $(1, y')$ has the neighbors $(0, y')$ and $(2, y')$ in P and $\{(1, y'-1), (2, y')\} \in E(G^D)$. Now,

$$\begin{aligned} C &= P \\ &\quad + \{\{(1, y'-1), (2, y')\}, \{(-1, y'-1), (0, y')\}, \{(-1, y'), (0, y'+1)\}, \{(0, y'+1), (1, y')\}\} \\ &\quad - \{\{(1, y'-1), (0, y')\}, \{(1, y'), (2, y')\}\} \end{aligned}$$

is a special cycle of G^D of order $n+1$ with $n = (y'+1)(d_1+d_2)$ and $V(C) = [0, n]$. See Figures 19 and 20 for an illustration. $\square$

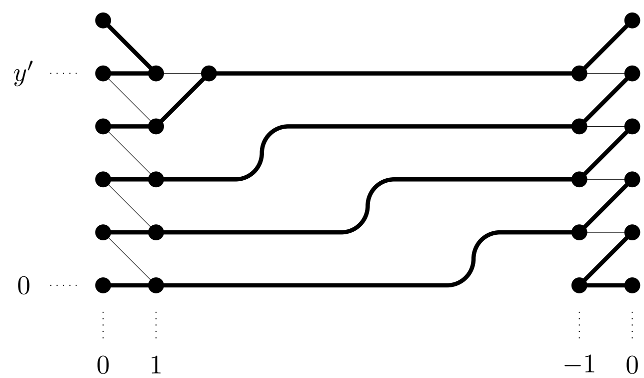

Fig. 20: The cycle C in the proof of Lemma 16.

Lemma 17 *For every $D = \{d_1, d_2\} \subseteq \mathbb{N}$ with $d_1 > d_2$, $d_1 d_2$ odd, and $\gcd(d_1, d_2) = 1$, there is some $n \in \mathbb{N}$ with $n \equiv 0 \pmod{d_1 + d_2}$ such that G^D has a special cycle C of order $n+2$ with $V(C) = [0, n+1]$.*

Proof: Clearly, vertex n is in column 0. First we assume that $d_2 = 1$. In that case, G^D has only one block and the vertex $n+1$ is in column -1. Let $P = \emptyset$ for $d_1 = 3$, otherwise let

$$\begin{aligned} P \quad : \quad & (1,0), (2,1), (3,1), (2,0), \\ & (3,0), (4,1), (5,1), (4,0), \\ & \ldots, (-5,0), (-4,1), (-3,1), (-4,0). \end{aligned}$$

The sequence

$$C \quad : \quad (0,0), P, (-3,0), (-2,0), (-1,1), (-2,1), (-1,2), (0,2), (1,1), (0,1), (-1,0), (0,0)$$

defines a special cycle of G^D of order $2(d_1+d_2)+2$ with $V(C) = [0, 2(d_1+d_2)+1]$. See Figure 21 for an illustration.

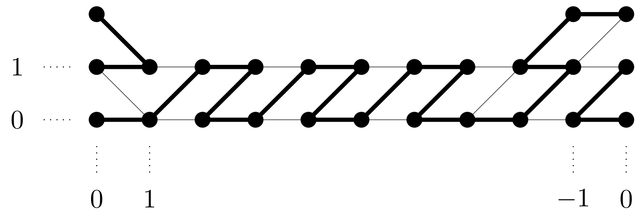

Fig. 21: The special cycle C for $d_1 = 9$ and $d_2 = 1$.

Now we assume that $d_2 > 1$. Hence, by Lemma 3, G^D has more than one block. This implies that vertex $n+1$ is lower. Let $k \in \mathbb{Z}/d_2\mathbb{Z}$, such that vertex $n+1$ belongs to block B_k. Since $d_1 + d_2$ is even, exactly those vertices are even integers that are in a column with an even index. Since vertex $n+1$ is lower and an odd integer, the number of blocks among $B_k, B_{k+1}, \ldots, B_{-1}$ of odd length is odd, i.e. at least one. Let $i \in \mathbb{Z}/d_2\mathbb{Z}$ be such that block B_i is of odd length and the blocks $B_{i+1}, B_{i+2}, \ldots, B_{-1}$ are of even length. Clearly, by Lemma 3, the length of the blocks $B_{i+1}, B_{i+2}, \ldots, B_{-1}$ are at least 4. By Lemma 15, G^D has a path Q_1 with endvertices $(-1,1)$ and $(-1,2)$ that consists of all vertices of rows 0 and 1 and the vertices $(x_i,2),(x_i+1,2),\ldots,(-1,2)$. Since the number of blocks of $B_k, B_{k+1}, \ldots, B_{i-1}$ of odd length is even, by Lemma 14, G^D has an $(x_i,2)$-(x_k,y')-path-collection $\mathcal{R}_1$ for some y'. Note, that if $i = k$, then $\mathcal{R}_1 = \emptyset$. In this case we define $y' = 2$. By the same arguments, G^D has a path Q_2 with endvertices $(-1,y'+2)$ and $(-1,y'+3)$ that consists of all vertices of rows $y'+1$ and $y'+2$ and the vertices $(x_i,y'+3),(x_{i+1},y'+3),\ldots,(-1,y'+3)$ and G^D has an $(x_i,y'+3)$-$(x_k,2y'+1)$-path-collection $\mathcal{R}_2$.

By definition, for every $y'+1 \le y \le 2y'$, the edges $\{(0,y),(1,y)\}$ and $\{(x_k-1,y),(x_k,y)\}$ belong to Q_2 or a path in $\mathcal{R}_2$. Furthermore, the path

$$P_0 : (x_k+1,2y'),(x_k,2y'+1),(x_k+1,2y'+1),(x_k+2,2y'+1)$$

is a subpath of a path in $\{Q_2\} \cup \mathcal{R}_2$. Let

$$P_1 : (0,y'),(1,y'),\ldots,(x_k-1,y')$$

and let

$$\begin{aligned} P_2 \quad : \quad & (x_k,2y'+1),(x_k+1,2y'+1),(x_k,2y'+2), \\ & (x_k-1,2y'+1),(x_k-2,2y'+1),\ldots,(1,2y'+1). \end{aligned}$$

Now,

$$\begin{aligned} C \quad = \quad & (Q_1 \cup Q_2 \cup \mathcal{R}_1 \cup \mathcal{R}_2) \\ & - \left(E(P_0) \cup \bigcup_{y=y'+1}^{2y'} \{\{(0,y),(1,y)\}\} \cup \bigcup_{y=y'+1}^{2y'} \{\{(x_k-1,y),(x_k,y)\}\} \right) \\ & + E(P_1) \cup E(P_2) \\ & + \bigcup_{y=1}^{y'} \{\{(-1,y),(0,y+1)\}\} \end{aligned}$$

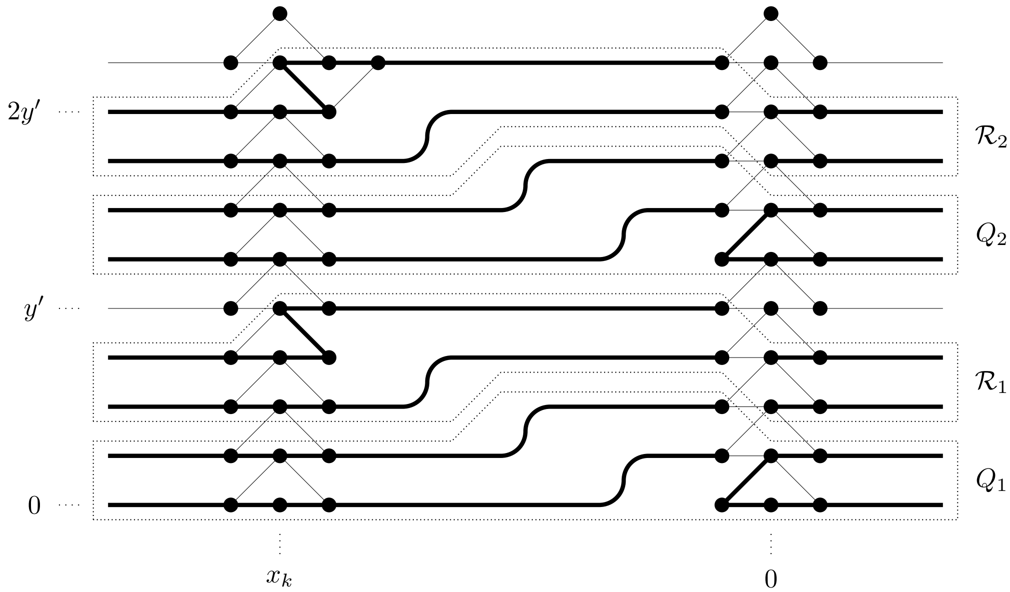

Fig. 22: $Q_1 \cup Q_2 \cup \mathcal{R}_1 \cup \mathcal{R}_2$.

$$
\begin{aligned}
&+\bigcup_{y=y'}^{2y'}\{\{(x_k-1,y),(x_k,y+1)\}\}\\
&+\bigcup_{y=y'+2}^{2y'+1}\{\{(-1,y),(0,y+1)\}\}\\
&+\bigcup_{y=y'+1}^{2y'+1}\{\{(0,y+1),(1,y)\}\}\\
&+\{\{(x_k+1,2y'),(x_k+2,2y'+1)\}\}
\end{aligned}
$$

defines a special cycle of G^D of order $n+2$ with $n=(2y'+2)(d_1+d_2)+2$ and $V(C)=[0,n+1]$. See Figures 22 and 23 for an illustration. $\square$

Let C be a special cycle of G^D and let $n'=\max(V(C))$. If for all $a,b\in V(C)$ with $n'-d_1+1\leq a<b\leq n'$, $\{a,b\}\neq\{n'-2d_2,n'-d_2\}$, and $|a-b|\in D$, we have $\{a,b\}\in E(C)$, then we call C *good*.

We are now in a position to prove the main results of this section.

Proof of Theorem 9: If $D=\{1,3\}$, then the result follows by induction on n. $C:0,1,2,3,0$ is a Hamiltonian cycle of G_4^D. Let C_n be a Hamiltonian cycle of G_n^D. Since the vertex $n-1$ has degree 2 in

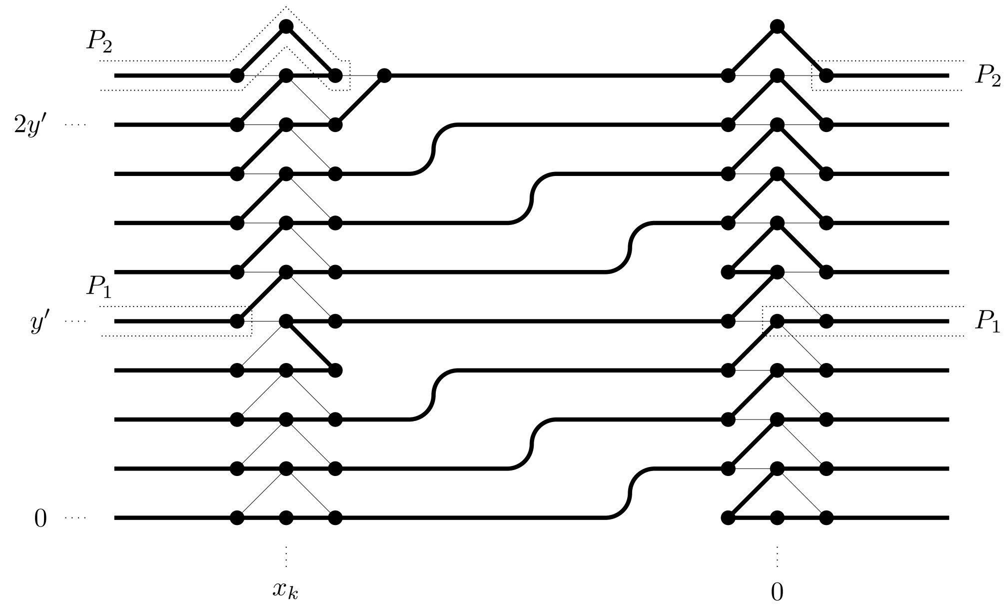

Fig. 23: The cycle C in the proof of Lemma 17.

G_n^D, $\{n-2, n-1\} \in E(C_n)$. Hence,

$$C_{n+2} \quad = \quad C_n + \{\{n-2, n+1\}, \{n+1, n\}, \{n, n-1\}\} - \{\{n-2, n-1\}\}$$

is a Hamiltonian cycle of G_{n+2}^D.

Hence we can assume that $D \neq \{1, 3\}$. Note that we can shift special cycles: If $C : v_0, \ldots, v_l, v_0$ is a special cycle in G^D, then also $C + h : v_0 + h, \ldots, v_l + h, v_0 + h$ is a special cycle in G^D. Furthermore, we can merge special cycles: If C_1 and C_2 are special cycles with $\min(V(C_2)) = \max(V(C_1)) + 1$, $\{a, b\} \in E(C_1)$, $\{c, d\} \in E(C_2)$, and $\{a, c\}, \{b, d\} \in E(G^D)$, then

$$(C_1 \cup C_2) + \{\{a, c\}, \{b, d\}\} - \{\{a, b\}, \{c, d\}\}$$

is a special cycle with vertex set $[\min(V(C_1)), \max(V(C_2))]$. If for $i \leq a < b \leq j$, $\{a, b\}$ is an edge of G^D and at least one of a, b has degree 2 in $G^D[[i, j]]$, then the edge $\{a, b\}$ belongs to every special cycles C of G^D with $V(C) = [i, j]$.

Claim 1 *If C_1 and C_2 are good special cycles of G^D with $\min(V(C_2)) = \max(V(C_1)) + 1$ and $D \neq \{1, 3\}$, then there is a good special cycle C with $V(C) = [\min(V(C_1)), \max(V(C_2))]$.*

Proof of Claim 1: Let $n' = \max(V(C_1))$.

Case 1 $d_1 \neq 2d_2 + 1$.

Since $d_1 \neq 2d_2+1$ and C_1 is good, $e_1 = \{n'-d_1+1, n'-d_1+d_2+1\} \in E(C_1)$. Clearly, $e_2 = \{n'+1, n'+d_2+1\} \in E(C_2)$. Hence,

$$C = (C_1 \cup C_2) + \{\{n'-d_1+1, n'+1\}, \{n'-d_1+d_2+1, n'+d_2+1\}\} - \{e_1, e_2\}$$

is a good special cycle with $V(C) = [\min(V(C_1)), \max(V(C_2))]$. This concludes the first case.

Case 2 $d_1 = 2d_2 + 1$.

Since $D \neq \{1,3\}$, we have $d_2 > 1$. Since $d_1 = 2d_2+1$, and C_1 is good, $e_1 = \{n'-d_1+2, n'-d_1+d_2+2\} \in E(C_1)$. Since $d_2 > 1$, $e_2 = \{n'+2, n'+d_2+2\} \in E(C_2)$. Hence,

$$C = (C_1 \cup C_2) + \{\{n'-d_1+2, n'+2\}, \{n'-d_1+d_2+2, n'+d_2+2\}\} - \{e_1, e_2\}$$

is a good special cycle with $V(C) = [\min(V(C_1)), \max(V(C_2))]$. This concludes the second case and the proof of Claim 1. $\square$

Claim 2 *G^D has a good special cycle of order* $2 \pmod{d_1+d_2}$.

Proof of Claim 2: By Lemma 17, G^D has a special cycle of order $2 \pmod{d_1+d_2}$. Let C_1 be a special cycle of G^D of order $2 \pmod{d_1+d_2}$ and let $n' = \max(V(C_1))$. It follows from [25, 21] that G^D has a special cycle of order d_1+d_2. Note that every vertex in $\{j, j+1, \ldots, j+d_1+d_2-1\}$ has degree 2 in $G^D[[j, j+d_1+d_2-1]]$, for $j \in \mathbb{Z}$ and hence a special cycle of order d_1+d_2 is good. Let C_2 be a special cycle of G^D of order d_1+d_2 with $\min(V(C_2)) = n'+1$. Since vertex n' has degree 2 in $G^D[V(C_1)]$, $\{n'-d_2, n'\} \in E(C_1)$ and since vertex $n'+d_1$ has degree 2 in $G^D[V(C_2)]$, $\{n'+d_1-d_2, n'+d_1\} \in E(C_2)$. Hence,

$$(C_1 \cup C_2) + \{\{n'-d_2, n'+d_1-d_2\}, \{n', n'+d_1\}\} - \{\{n'-d_2, n'\}, \{n'+d_1-d_2, n'+d_1\}\}$$

is a good special cycle of G^D. This concludes the proof of Claim 2. $\square$

Let p_1 with $p_1 \equiv 2 \pmod{d_1+d_2}$, such that G^D has a good special cycle of order p_1. By Claim 2, such a p_1 exists. As said before, G^D has a good special cycle of order $p_2 = d_1+d_2$. Since $\gcd(p_1, p_2) = 2$, it follows from the extended Euclidean algorithm that every sufficiently large even integer is a positive integral linear combination of p_1 and p_2. Therefore and by Claim 1, the desired result follows by shifting and merging copies of good special cycles of order p_1 and p_2. $\square$

Proof of Theorem 10:: The proof is analogous to the proof of Theorem 9. Instead of using Lemma 17 we use Lemma 16. Proceeding as in the proof of Theorem 9 we obtain p_1 with $p_1 \equiv 1 \pmod{d_1+d_2}$ and hence $\gcd(p_1, p_2) = 1$. This clearly allows to establish the theorem for all sufficiently large n and not just for sufficiently large even n. $\square$

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