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On Hamiltonian Paths and Cycles in Sufficiently Large Distance Graphs

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For a positive integer $n \in \mathbb{N}$ and a set $D \subseteq \mathbb{N}$, the distance graph $G_n^D$ has vertex set $\{0, 1, \ldots, n-1\}$ and two vertices $i$ and $j$ of $G_n^D$ are adjacent exactly if $|j-i| \in D$. The condition $\gcd(D) = 1$ is necessary for a distance graph $G_n^D$ being connected. Let $D = \{d_1, d_2\} \subseteq \mathbb{N}$ be such that $d_1 > d_2$ and $\gcd(d_1, d_2) = 1$. We prove the following results.

- If $n$ is sufficiently large in terms of $D$, then $G_n^D$ has a Hamiltonian path with endvertices 0 and $n-1$.
- If $d_1d_2$ is odd, $n$ is even and sufficiently large in terms of $D$, then $G_n^D$ has a Hamiltonian cycle.
- If $d_1d_2$ is even and $n$ is sufficiently large in terms of $D$, then $G_n^D$ has a Hamiltonian cycle.

Keywords: Distance graph; Toeplitz graph; circulant graph; Hamiltonian path; Hamiltonian cycle; traceability

1 Introduction

For a finite set of positive integers $D \subseteq \mathbb{N}$, the infinite distance graph $G^D$ has vertex set $V(G^D) = \mathbb{Z}$ and two vertices $u$ and $v$ of $G^D$ are adjacent exactly if $|u-v| \in D$. For a graph $G$ and a subset $U \subseteq V(G)$ of the vertex set, we denote by $G[U]$ the subgraph of $G$ induced by $U$. For $i, j \in \mathbb{Z}$, $i \leq j$, we denote by $[i, j] = \{k \in \mathbb{Z} \mid i \leq k \leq j\}$. For a positive integer $n \in \mathbb{N}$, the distance graph (also called Toeplitz graph in many papers) $G_n^D = G^D[[0, n-1]]$ is the subgraph of $G^D$ induced by the vertices in $[0, n-1]$.

Infinite distance graphs and especially their colourings were first studied by Eggleton, Erdős, and Skilton [10,11]. Most of the research on distance graphs focused on their colourings [6,8,9,14,18,19,28]. Distance graphs generalize the very well-studied class of circulant graphs [2,16,17,26]. In fact, circulant graphs coincide exactly with the regular distance graphs [23]. Circulant graphs have been proposed for numerous network applications and many of their properties such as connectedness and diameter [4,2,16,17], cycle and path structure [1,3,5], and isomorphism testing and recognition [12,22] have

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been studied in great detail. Several fundamental results concerning circulant graphs were extended to the more general class of distance graphs in [7, 23, 24, 25]. The complexity of the connectedness problem for distance graphs was recently settled by Gómez et al. [13]. In [25, 27, 15] the existence of long paths and cycles in distance graphs is studied. The following main result from [21] confirmed a conjecture from Penso et al. [25]. [20] gives an overview on Hamiltonian cycles and paths in vertex-transitive graphs.

**Theorem 1 (Löwenstein et al. [21])** For a finite set \( D \subseteq \mathbb{N} \) with \(|D| \geq 2\) and \( \gcd(D) = 1 \), there are infinitely many \( n \in \mathbb{N} \) such that \( G^D_n \) has a Hamiltonian cycle and \( G^D_{n+1} \) has a Hamiltonian path with endvertices 0 and \( n \).

We conjecture that the conclusion of the last theorem holds
- for all \( n \) that are sufficiently large in terms of \( D \) if not all elements of \( D \) are odd and
- for all even \( n \) that are sufficiently large in terms of \( D \) if all elements of \( D \) are odd.

The purpose of the present paper is to confirm this conjecture in the case that \( D \) contains just two elements. In Section 2 we introduce suitable terminology and collect some properties of distance graphs. In Section 3 we confirm our conjecture proving the existence of Hamiltonian paths. Finally, in Section 4 we provide similar results for Hamiltonian cycles.

## 2 The structure of \( G^D \)

Let \( D = \{d_1, d_2\} \) for two positive integers \( d_1 \) and \( d_2 \) such that \( \gcd(d_1, d_2) = 1 \) and \( d_1 > d_2 \).

We define coordinates \((x, y) \in (\mathbb{Z}/(d_1 + d_2)\mathbb{Z}) \times \mathbb{Z}\) for the vertices of the distance graph \( G^D \) by
\[
(x, y) := y(d_1 + d_2) + a_x,
\]
where \( a_x = xd_1 \mod (d_1 + d_2) \). Note that this bidimensional relabelling of the vertices of \( G^D \) is a bijection. A vertex \((x, y)\) satisfying \( 0 \leq xd_1 \mod (d_1 + d_2) < d_2 \) is called lower. A vertex \((x, y)\) satisfying \( d_2 \leq xd_1 \mod (d_1 + d_2) < d_1 \) is called middle. A vertex \((x, y)\) satisfying \( d_1 \leq xd_1 \mod (d_1 + d_2) < d_1 + d_2 \) is called upper.

For a lower vertex \((x, y)\), we have
\[
\begin{align*}
(x, y) + d_1 & = (x + 1, y), \\
(x, y) + d_2 & = (x - 1, y), \\
(x, y) - d_1 & = (x - 1, y - 1), \\
(x, y) - d_2 & = (x + 1, y - 1),
\end{align*}
\]
which implies that a lower vertex \((x, y)\) is adjacent to the vertices \((x + 1, y), (x - 1, y), (x + 1, y - 1), \) and \((x - 1, y - 1)\).

Similarly, for a middle vertex \((x, y)\), we have
\[
\begin{align*}
(x, y) + d_1 & = (x + 1, y + 1), \\
(x, y) + d_2 & = (x - 1, y), \\
(x, y) - d_1 & = (x - 1, y - 1), \\
(x, y) - d_2 & = (x + 1, y),
\end{align*}
\]
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which implies that a middle vertex \((x, y)\) is adjacent to the vertices \((x + 1, y)\), \((x - 1, y)\), \((x + 1, y + 1)\), and \((x - 1, y - 1)\).

Finally, for an upper vertex \((x, y)\), we have

\[
\begin{align*}
(x, y) + d_1 &= (x + 1, y + 1), \\
(x, y) + d_2 &= (x - 1, y + 1), \\
(x, y) - d_1 &= (x - 1, y), \\
(x, y) - d_2 &= (x + 1, y),
\end{align*}
\]

which implies that an upper vertex \((x, y)\) is adjacent to the vertices \((x + 1, y)\), \((x - 1, y)\), \((x + 1, y + 1)\), and \((x - 1, y - 1)\).

See Figure 1 for an illustration of these observations.

For \(c \in \mathbb{Z}/(d_1 + d_2)\mathbb{Z}\), all vertices \((x, y)\) of \(G^D\) with \(x = c\) form the column \(c\). Similarly, for \(r \in \mathbb{Z}\), all vertices \((x, y)\) satisfying \(y = r\) form the row \(r\). Note that the vertices in a column are either all lower, or all middle, or all upper. A column that consists of lower (middle, upper) vertices is called lower (middle, upper). See Figure 2 for an illustration.

**Lemma 2**

(i) For \(c \in \mathbb{Z}/(d_1 + d_2)\mathbb{Z}\), the column \(c\) is lower if and only if the column \(c + 1\) is upper.

(ii) Column 0 is lower.

(iii) Column 1 is upper.

**Proof:** For \(x \in \mathbb{Z}/(d_1 + d_2)\mathbb{Z}\), we have \(0 \leq xd_1 \mod (d_1 + d_2) < d_2\) if and only if \(d_1 \leq (x + 1)d_1 \mod (d_1 + d_2) < d_1 + d_2\), which proves (i). (ii) follows, because \(0 \leq 0 = 0d_1 \mod (d_1 + d_2) < d_2\). Finally, (i) and (ii) imply (iii).

The columns \(x, x + 1, \ldots, x + l - 1\) form a block of length \(l\), if column \(x\) is lower, column \(x + l\) is lower, and none of the columns \(x + 1, \ldots, x + l - 1\) is lower. The block that contains column 0 is denoted by \(B_0\). Let \(l\) be the length of block \(B_1\) and let column \(x\) be the unique lower column that belongs to block

![Fig. 1: Neighborhood of (a) a lower, (b) a middle, and (c) an upper vertex.](image)
Fig. 2: The distance graph $G_{85}^{[8,3]}$. Note that the vertices of column 0 are drawn twice. In order to simplify the drawing, we adopt the convention that such a vertex is adjacent to the union of the neighbors of the two copies, i.e. vertex 22 is adjacent to the vertices 19, 30, 14, and 25.

Figure 3 shows the blocks of $G_{85}^{[12,5]}$.

Fig. 3: Blocks of $G_{85}^{[12,5]}$. Note that 4 equals $-1$ in $\mathbb{Z}/5\mathbb{Z}$, that is, $B_4 = B_{-1}$.

Lemma 3

(i) The length of a block is either $\left\lfloor \frac{d_1}{d_2} \right\rfloor + 1$ or $\left\lceil \frac{d_1}{d_2} \right\rceil + 1$.

(ii) The length of $B_0$ is $\left\lfloor \frac{d_1}{d_2} \right\rfloor + 1$.

(iii) The length of $B_{-1}$ is $\left\lceil \frac{d_1}{d_2} \right\rceil + 1$.
(iv) The number of blocks is $d_2$.

**Proof:** Let $x, x + 1, \ldots, x + l - 1$ be the columns of a block $B$ of length $l$. By definition and Lemma 2 (i), $x$ is the unique lower column of block $B$, $x + 1$ is the unique upper column of block $B$, and $x + l$ is a lower column. Hence, for all $y \in \mathbb{Z}$ and $x + 1 \leq k \leq x + l - 1$, we have $(k, y) - (k + 1, y) = d_2$ and therefore $(x + 1, y) - (x + l, y) = d_2(l - 1)$. Since column $x + 1$ is upper and column $x + l$ is lower, we have $d_1 - d_2 + 1 \leq (x + 1, y) - (x + l, y) \leq d_1 - d_2 - 1$, which implies (i).

If $B = B_0$, then $x = 0$ and $(x + 1, y) \equiv d_1 \pmod{d_1 + d_2}$ for all $y \in \mathbb{Z}$. Hence $(x + 1, y) - (x + l, y) \leq d_1$. Together with $(x + 1, y) - (x + l, y) = d_2(l - 1)$, this implies (ii).

If $B = B_1$, then $x + l = 0$ and $(x + l, y) \equiv d_2 \pmod{d_1 + d_2}$ for all $y \in \mathbb{Z}$. Since column $x + 1$ is upper, we have $(x + 1, y) - (x + l, y) \geq d_1$. Together with $(x + 1, y) - (x + l, y) = d_2(l - 1)$, this implies (iii).

Since the function $f : \{0, \ldots, d_1 + d_2 - 1\} \rightarrow \{0, \ldots, d_1 + d_2 - 1\}$ with $f(x) = xd_1 \pmod{d_1 + d_2}$ is bijective for $\gcd(d_1, d_2) = 1$, there are exactly $d_2$ lower columns and therefore $d_2$ blocks, which proves (iv). \qed

### 3 Hamiltonian paths of $G_n^D$

The main result of this section is the following.

**Theorem 4** For every $D = \{d_1, d_2\} \subseteq \mathbb{N}$ with $d_1 > d_2$ and $\gcd(d_1, d_2) = 1$, there is some $n_0 \in \mathbb{N}$ such that for all integers $n$ with $n \geq n_0$, the distance graph $G_n^D$ has a Hamiltonian path with endvertices 0 and $n - 1$.

As before let $D = \{d_1, d_2\}$ for two positive integers $d_1$ and $d_2$ such that $\gcd(d_1, d_2) = 1$ and $d_1 > d_2$. For two lower vertices $(x, y)$ and $(x', y')$ with $x \neq x'$ and $y < y'$ in the distance graph $G_n^D$, a path in $G_n^D$ with endvertices $(x, y)$ and $(x', y')$ whose vertex set consists of all vertices in the rows $y, y + 1, \ldots, y' - 1$ and the vertex $(x', y')$ is called an $(x, y)$-$(x', y')$-climbing path of $G_n^D$. See Figure 5 for an illustration.

Before we proceed to the proof of Theorem 4 we establish a series of lemmas concerning the existence of climbing paths.

**Lemma 5** If $B_i$ is a block of even length in $G_n^D$, then $G_n^D$ has an $(x_i, y)$-$(x_{i+1}, y + 2)$-climbing path for all $y$.

**Proof:** Let

$$P : (x_i+1 - 1, y), (x_{i+1} - 1, y + 1), (x_{i+1} - 1, y + 1), (x_{i+1} - 2, y),$$

$$(x_{i+1} - 3, y), (x_{i+1} - 2, y + 1), (x_{i+1} - 3, y + 1), (x_{i+1} - 4, y),$$

$$\ldots, (x_i + 3, y), (x_i + 4, y + 1), (x_i + 3, y + 1), (x_i + 2, y).$$

The sequence

$$(x_i, y), (x_i - 1, y), \ldots, (x_{i+1}, y),$$

$$(x_i + 1, y), (x_i + 2, y + 1), (x_i + 1, y + 1), \ldots, (x_{i+1} + 1, y + 1),$$

$$((x_{i+1}, y + 2)$$

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Lemma 6 If $B_{i-1}$ is a block of even length in $G^D$, then $G^D$ has an $(x_i, y)-(x_{i-1}, y+2)$-climbing path for all $y$.

Proof: Let

$$P : (x_{i-1} + 3, y + 1), (x_{i-1} + 2, y), (x_{i-1} + 3, y), (x_{i-1} + 4, y + 1),$$

$$(x_{i-1} + 5, y + 1), (x_{i-1} + 4, y), (x_{i-1} + 5, y), (x_{i-1} + 6, y + 1),$$

$$\ldots, (x_i - 1, y + 1), (x_i - 2, y), (x_i - 1, y), (x_i, y + 1).$$

Fig. 4: $P$ for a block $B_i$ of length 8.

Fig. 5: An $(x_i, y)-(x_{i+1}, y+2)$-climbing path for block $B_i$ of length 8.

Fig. 6: $P$ for a block $B_{i-1}$ of length 8.
The sequence
\[(x_i, y), (x_i + 1, y), \ldots, (x_{i-1} + 1, y), (x_{i-1}, y + 1), (x_{i-1} + 1, y + 1), (x_{i-1} + 2, y + 1), P, (x_i + 1, y + 1), (x_i + 2, y + 1), \ldots, (x_{i-1} - 1, y + 1), (x_{i-1}, y + 2)\]
defines an \((x_i, y)-(x_{i-1}, y + 2)\)-climbing path of \(G^D\). See Figures 6 and 7 for an illustration. \(\square\)

**Lemma 7** If \(G^D\) has at least \(j + 2\) blocks for some \(j \geq 1\) and for some \(i \in \mathbb{Z}/d_2\mathbb{Z}\), the blocks \(B_1, B_{i+1}, \ldots, B_{i+j}\) of \(G^D\) are such that \(B_i\) and \(B_{i+j}\) are of odd length and \(B_{i+1}, \ldots, B_{i+j-1}\) are of even length at least 4, then \(G^D\) has an \((x_i, y)-(x_{i+j+1}, y + 3)\)-climbing path for all \(y\).

**Proof:** By Lemma 3 the blocks \(B_i\) and \(B_{i+j}\) are of length at least 3.

Let
\[P_{i+j} : (x_{i+j+1} - 1, y), (x_{i+j+1} + 1, y + 1), (x_{i+j+1} + 2, y + 1), (x_{i+j+1} - 1, y + 1), (x_{i+j+1} - 2, y), (x_{i+j+1} - 3, y), (x_{i+j+1} - 2, y + 1), (x_{i+j+1} - 3, y + 1), (x_{i+j+1} - 4, y), \ldots, (x_{i+j} + 2, y), (x_{i+j} + 3, y + 1), (x_{i+j} + 2, y + 1), (x_{i+j} + 1, y)\]

For \(1 \leq q \leq j - 1\), let
\[P_{i+q} : (x_{i+q} + 3, y + 2), (x_{i+q} + 2, y + 1), (x_{i+q} + 3, y + 1), (x_{i+q} + 4, y + 2), (x_{i+q} + 5, y + 2), (x_{i+q} + 4, y + 1), (x_{i+q} + 5, y + 1), (x_{i+q} + 6, y + 2), \ldots, (x_{i+q+1} - 3, y + 2), (x_{i+q+1} - 4, y + 1), (x_{i+q+1} - 3, y + 1), (x_{i+q+1} - 2, y + 2)\]
and let
\[P'_{i+q} : P_{i+q}, (x_{i+q+1} - 1, y + 2), (x_{i+q+1} - 2, y + 1), (x_{i+q+1} - 1, y + 1), (x_{i+q+1} + 1, y + 1), (x_{i+q+1} + 1, y + 1), (x_{i+q+1} + 1, y + 2), (x_{i+q+1} + 1, y + 2), (x_{i+q+1} + 2, y + 2)\]

Note that \(P_{i+q}\) is empty if \(B_{i+q}\) is of length 4. Furthermore, let
\[P_i : (x_{i+1} - 2, y + 2), (x_{i+1} - 3, y + 1), (x_{i+1} - 4, y + 1), (x_{i+1} - 3, y + 2), (x_{i+1} - 4, y + 2), (x_{i+1} - 5, y + 1), (x_{i+1} - 6, y + 1), (x_{i+1} - 5, y + 2), \ldots, (x_i + 3, y + 2), (x_i + 2, y + 1), (x_i + 3, y + 1), (x_i + 2, y + 2)\]
Note that $P_i$ is empty if $B_i$ is of length 3.

Now, the sequence

$$(x_i, y), (x_i - 1, y), \ldots, (x_{i+j+1}, y), P_{i+j},$$

$$(x_{i+j}, y), (x_{i+j} - 1, y), \ldots, (x_{i+1} - 1, y),$$

$$(x_{i+1}, y + 1), (x_{i+1} + 1, y + 1), (x_{i+1}, y + 2), (x_{i+1} + 1, y + 2), (x_{i+1} + 2, y + 2),$$

$$P'_{i+1}, P'_{i+2}, \ldots, P'_{i+j-1},$$

$$(x_{i+j} + 3, y + 2), (x_{i+j} + 4, y + 2), \ldots, (x_{i+j+1}, y + 2),$$

$$(x_{i+j+1} + 1, y + 1), (x_{i+j+1} + 2, y + 1), \ldots, (x_i + 1, y + 1),$$

$$(x_i + 1, y), (x_i + 2, y), \ldots, (x_{i+1} - 2, y),$$

$$(x_{i+1} - 1, y + 1), (x_{i+1} - 2, y + 1), (x_{i+1} - 1, y + 2), P_i,$$

$$(x_i + 1, y + 2), (x_i, y + 2), \ldots, (x_{i+j+1} + 1, y + 2),$$

$$(x_{i+j+1}, y + 3)$$

defines an $(x_i, y)$-$(x_{i+j+1}, y + 3)$-climbing path of $G^D$. See Figures 8 and 9 for an illustration.

**Lemma 8** If $G^D$ has at least $j + 2$ blocks for some $j \geq 1$ and for some $i \in \mathbb{Z}/d_2\mathbb{Z}$, the blocks $B_i, B_{i+1}, \ldots, B_{i+j}$ of $G^D$ are such that $B_i$ and $B_{i+j}$ are of length 3 and $B_{i+1}, \ldots, B_{i+j-1}$ are of length 2, then $G^D$ has an $(x_i, y)$-$(x_{i+j+1}, y + j + 2)$-climbing path for all $y$.

**Proof:** Note that $x_{i+j+1} = x_i + 2j + 4$. For $1 \leq q \leq j - 1$, let

$$P_q : (x_i + 2j + 2, y + q), (x_i + 2j + 3, y + q + 1), (x_i + 2j + 4, y + q + 2),$$

$$(x_i + 2j + 5, y + q + 2), \ldots, (x_{i+j+1}, y + j + 2).$$
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\[(x_i + 2j + 5, y + q + 1), (x_i + 2j + 6, y + q + 1), \ldots, (x_i, y + q + 1),
(x_i + 1, y + q), (x_i + 2, y + q), \ldots, (x_i + 2j - 2q + 2, y + q),
(x_i + 2j - 2q + 1, y + q + 1), (x_i + 2j - 2q + 2, y + q + 1), \ldots, (x_i + 2j + 1, y + q + 1).\]

\[\text{Fig. 10: } P_q \text{ for } j = 4 \text{ and } q = 1.\]

Now, the sequence

\[(x_i, y), (x_i - 1, y), \ldots, (x_i + 2j + 4, y),
(x_i + 2j + 3, y), (x_i + 2j + 4, y + 1), (x_i + 2j + 3, y + 1), (x_i + 2j + 4, y + 2),
(x_i + 2j + 5, y + 1), (x_i + 2j + 6, y + 1), \ldots, (x_i, y + 1),
(x_i + 1, y), (x_i + 2, y), \ldots, (x_i + 2j + 2, y),
(x_i + 2j + 1, y + 1),
P_1, P_2, \ldots, P_{j-1},
(x_i + 2j + 2, y + j),
(x_i + 2j + 3, y + j + 1)(x_i + 2j + 2, y + j + 1), \ldots, (x_i + 3, y + j + 1),
(x_i + 2, y + j), (x_i + 1, y + j),
(x_i + 2, y + j + 1)(x_i + 1, y + j + 1), \ldots, (x_i + 2j + 5, y + j + 1),
(x_i + 2j + 4, y + j + 2)\]

defines an \((x_i, y)-(x_{i+j+1}, y + j + 2)\)-climbing path of \(D^G\). See Figures 10 and 11 for an illustration. \(\square\)

We are now in a position to prove the main result of this section. A path \(P\) in \(D^G\) with \(V(P) = [\min(V(P)), \max(V(P))]\) is called \textit{special}, if the endvertices of \(P\) are \(\min(V(P))\) and \(\max(V(P))\).

\textbf{Proof of Theorem 4.} If \(d_2 = 1\), then the statement of the theorem is trivial. Hence we assume that \(d_2 > 1\). The idea of the proof is to show the existence of two distinct positive integers \(p_1\) and \(p_2\) with
Let $d_1, d_2$ be such that $d_1 + d_2 + 1$ has a Hamiltonian path with endvertices $0$ and $d_1 + d_2$. Hence, for $p_1$, we choose $d_1 + d_2$.

For $p_2$, we show that there is a positive integer $p_2$ with $p_2 \equiv -1 \pmod{d_1 + d_2}$, such that $G_{d_1+d_2+1}$ has a special path of length $p_2$, thus $\gcd(p_1, p_2) = 1$.

Let $x'$ be such that $x'd_1 \equiv -1 \pmod{d_1 + d_2}$. By definition and Lemma 3(i), column $x'$ is upper and column $x' - 1$ is lower. In order to get a special path with endvertices $(0, 0)$ and $(x', y')$ for some $y'$, we concatenate climbing paths to form a $(0, 0)$-$(x' - 1, y')$-climbing path and append the path $(x' - 2, y'), (x' - 3, y'), \ldots, (x, y')$.

Let $k$ be such that the block $B_k$ contains column $x'$, that is, $x_k = x' - 1$. Since column $x'$ is upper, column $x' - 2$ belongs to block $B_{k-1}$.

Since $\gcd(d_1, d_2) = 1$, at least one of $d_1$ and $d_2$ is odd.

**Case 1** One of $d_1$ and $d_2$ is even and $G^D$ has at most 2 blocks of odd length.

Since $d_1 + d_2$ is odd, the number of blocks of odd length is odd, that is, it equals 1.

We first assume that all blocks $B_0, B_1, \ldots, B_{k-1}$ are of even length. By Lemma 5, there exists an $(x_i, 2i)$-$(x_{i+1}, 2i + 2)$-climbing path $P_i$ for $0 \leq i \leq k - 1$. Since $x' - 1 = x_k$, the concatenation of the paths $P_0, P_1, \ldots, P_{k-1}$ forms a $(0, 0)$-$(x' - 1, y')$-climbing path for $y' = 2k$.
Next, we assume that all blocks $B_{k}, B_{k+1}, \ldots, B_{-1}$ are of even length. Then, by Lemma 5 there exists an $(x_{i+1}, 2(d_{1} + d_{2} - i) - 2)\cdot (x_{i}, 2(d_{1} + d_{2} - i))$-climbing path $P_{i}$ for $k \leq i \leq d_{1} + d_{2} - 1$. Since $x' - 1 = x_{k}$, the concatenation of the paths $P_{d_{1} + d_{2} - 1}, P_{d_{1} + d_{2} - 2}, \ldots, P_{k}$ forms a $(0, 0)\cdot (x' - 1, y')$-climbing path for $y' = 2(d_{1} + d_{2} - k)$. This concludes the first case.

**Case 2** One of $d_{1}$ and $d_{2}$ is even and $G^{D}$ has at least 3 blocks of odd length.

Since $d_{1} + d_{2}$ is odd, the number of blocks of odd length is odd. This implies that one of the two sequences $B_{0}, B_{1}, \ldots, B_{-1}$ and $B_{0}, B_{1}, \ldots, B_{-1}$, $B_{0}, B_{1}, \ldots, B_{k-1}$ has an even number of blocks with odd length. We call this sequence $S$. We can partition $S$ into subsequences $S_{1}, S_{2}, \ldots, S_{t}$, where each subsequence is either a block of even length or a sequence $B_{i}, B_{i+1}, \ldots, B_{i+j}$ of blocks $i \in \mathbb{Z}/d_{2}\mathbb{Z}$ and $j \geq 1$, such that block $B_{i}$ has odd length, block $B_{i+j}$ has odd length, and blocks $B_{i+1}, \ldots, B_{i+j-1}$ have even length. For a subsequence $S_{q}$, $1 \leq q \leq t$, that consists of one block $B_{i}$ with $i \in \mathbb{Z}/d_{2}\mathbb{Z}$, Lemma 5 implies that there exists an $(x_{i}, y)-(x_{i+1}, y + 2)$-climbing path $P_{q,y}$ for every $y$. If $\frac{d_{1}}{d_{2}} < 2$, then Lemma 3 implies that the lengths of the blocks are 2 and 3. For a subsequence $S_{q}$, $1 \leq q \leq t$, that consists of at least two blocks $B_{i}, B_{i+1}, \ldots, B_{i+j}$ with $i \in \mathbb{Z}/d_{2}\mathbb{Z}$ and $j \geq 1$, Lemma 8 implies that there exists an $(x_{i}, y)-(x_{i+j+1}, y + j + 2)$-climbing path $P_{q,y}$ for every $y$. If $\frac{d_{1}}{d_{2}} \geq 2$, then Lemma 3 implies that the lengths of the blocks are at least 3. For a subsequence $S_{q}$, $1 \leq q \leq t$, that consists of at least two blocks $B_{i}, B_{i+1}, \ldots, B_{i+j}$ with $i \in \mathbb{Z}/d_{2}\mathbb{Z}$ and $j \geq 1$, Lemma 7 implies that there exists an $(x_{i}, y)-(x_{i+j+1}, y + 3)$-climbing path $P_{q,y}$ for every $y$. The concatenation of the paths $P_{1,y_{1}}, P_{2,y_{2}}, \ldots, P_{1,y_{t}}$ forms a $(0, 0)\cdot (x' - 1, y')$-climbing path for $y_{1} = 0$, suitable $y_{t}$'s, where $2 \leq q \leq t$, and $y' = y_{t}$. This concludes the second case.

If both $d_{1}$ and $d_{2}$ are odd, then $d_{1} + d_{2}$ is even, which implies that the number of blocks of odd length is even and exactly those vertices are even integers that are in a column with an even index. This implies that $x'$ is odd and $x_{k} = x' - 1$ is even. Since column 0 and column $x' - 1$ are lower, the sequence $B_{0}, B_{1}, \ldots, B_{k-1}$ has an even number of blocks with odd length.

**Case 3** Both $d_{1}$ and $d_{2}$ are odd and $G^{D}$ has at most 2 blocks of odd length.

Since $d_{2} \geq 2$, $G$ has exactly 2 blocks of odd length. This implies that one of the two sequences $B_{0}, B_{1}, \ldots, B_{-1}$ and $B_{k}, B_{k+1}, \ldots, B_{-1}$ has only blocks of even length. Now we are in the same situation as in Case 1. Arguing as in Case 1 completes this case.

**Case 4** Both $d_{1}$ and $d_{2}$ are odd and $G^{D}$ has at least 4 blocks of odd length.

Since the sequence $B_{0}, B_{1}, \ldots, B_{k-1}$ has an even number of blocks of odd length, we are in the same situation as in Case 2. Arguing as in Case 2 completes this case, which concludes the proof of the theorem.

\[ \square \]

4 Hamiltonian cycles of $G^{D}_{n}$

The main results of this section are the following.

**Theorem 9** For every $D = \{d_{1}, d_{2}\} \subseteq \mathbb{N}$ with $d_{1} > d_{2}$, $d_{1}d_{2}$ odd, and $\gcd(d_{1}, d_{2}) = 1$, there is some $n_{0} \in \mathbb{N}$ such that for all even integers $n$ with $n \geq n_{0}$, the distance graph $G^{D}_{n}$ has a Hamiltonian cycle.
Theorem 10 For every $D = \{d_1, d_2\} \subseteq \mathbb{N}$ with $d_1 > d_2$, $d_1d_2$ even, and $\gcd(d_1, d_2) = 1$, there is some $n_0 \in \mathbb{N}$ such that for all integers $n$ with $n \geq n_0$, the distance graph $G_D^n$ has a Hamiltonian cycle.

Note that the distance graphs considered in Theorem 9 are necessarily bipartite. Therefore, they can only have a Hamiltonian cycle if their order is even.

As in Section 3 we establish several lemmas before proceeding to the proofs of Theorems 9 and 10.

For two lower vertices $(x, y)$ and $(x', y')$ with $x \neq x'$, $0 \not\in \{x' + 1, x' + 2, \ldots, x\}$, and $y < y'$ in the distance graph $G_D$, a set of vertex disjoint paths $R_y, R_{y+1}, \ldots, R_{y'-1}$ in $G_D$ is called an $(x, y)$-$(x', y')$-path-collection of $G_D$, if it satisfies the following conditions:

- for $y \leq i < y'$, $P_i$ has the endvertices $(0, i)$ and $(-1, i + 1)$,
- for $y \leq i < y'$, the path $(0, i), (1, i), \ldots, (x', i)$ is a subpath of $P_i$,
- for $y \leq i < y'$, the path $(x + 1, i + 1), \ldots, (-1, i + 1)$ is a subpath of $P_i$,
- the union of the vertex sets of the paths consists of all vertices in the rows $y + 1, y + 2, \ldots, y' - 1$, the vertices $(0, y), (1, y), \ldots, (x - 1, y)$, and the vertices $(x', y'), (x' + 1, y'), \ldots, (1, y')$, and
- no edge of the form $\{(-1, z), (0, z')\}$ for some $z, z' \in \mathbb{Z}$ is in the union of the edge sets of the paths.

See Figures 13, 15, and 16 for an illustration. Note, that $(x, y)$ does not belong to any path of an $(x, y)$-$(x', y')$-path-collection.

Lemma 11 If for some $i \neq -1$, $B_i$ is a block of even length in $G_D$, then $G_D$ has an $(x_{i+1}, y)$-$(x_i, y + 1)$ path collection for all $y$.

Proof: Let

$$P : (x_i + 3, y + 1), (x_i + 2, y), (x_i + 3, y), (x_i + 4, y + 1),$$
$$\quad (x_i + 5, y + 1), (x_i + 4, y), (x_i + 5, y), (x_i + 6, y + 1),$$
$$\quad \ldots, (x_{i+1} - 1, y + 1), (x_{i+1} - 2, y), (x_{i+1} - 1, y), (x_{i+1}, y + 1).$$

Fig. 12: $P$ for a block $B_i$ of length 8.
The sequence

\[(0, y), (1, y), \ldots, (x_i + 1, y),
(x_i, y + 1), (x_i + 1, y + 1), (x_i + 2, y + 1), P_i,
(x_i + 1 + 1, y + 1), (x_i + 2 + 1, y + 1), \ldots, (-1, y + 1)\]

defines an \((x_{i+1}, y)-(x_i, y + 1)\)-path-collection in \(G^D\). See Figures 12 and 13 for an illustration. □

**Lemma 12** If for some \(i \in Z/dZ\) and for some \(j \geq 1\), the blocks \(B_i, B_{i+1}, \ldots, B_{i+j}\) of \(G^D\) are such that \(-1 \notin \{i, i+1, \ldots, i+j\}\), \(B_i\) and \(B_{i+j}\) are of odd length and \(B_{i+1}, \ldots, B_{i+j-1}\) are of even length at least 4, then \(G^D\) has an \((x_{i+j+1}, y)-(x_i, y + 2)\)-path-collection for all \(y\).

**Proof:** By Lemma 3 the blocks \(B_i\) and \(B_{i+j}\) are of length at least 3.

Let

\[
P_i : (x_i + 1, y), (x_i + 2, y + 1), (x_i + 3, y + 1), (x_i + 2, y),
(x_i + 3, y), (x_i + 4, y + 1), (x_i + 5, y + 1), (x_i + 4, y),
\ldots, (x_{i+1} - 2, y), (x_{i+1} - 1, y + 1), (x_{i+1}, y + 1), (x_{i+1} - 1, y).
\]

For \(1 \leq q \leq j - 1\), let

\[
P_{i+q} : (x_{i+q}, y), (x_{i+q} + 1, y),
(x_{i+q} + 2, y), (x_{i+q} + 3, y + 1), (x_{i+q} + 4, y + 1), (x_{i+q} + 3, y),
(x_{i+q} + 4, y), (x_{i+q} + 5, y + 1), (x_{i+q} + 6, y + 1), (x_{i+q} + 5, y),
\ldots, (x_{i+q+1} - 2, y), (x_{i+q+1} - 1, y + 1), (x_{i+q+1}, y + 1), (x_{i+q+1} - 1, y).
\]

Furthermore, let

\[
P_{i+j} : (x_{i+j} + 2, y), (x_{i+j} + 3, y + 1), (x_{i+j} + 4, y + 1), (x_{i+j} + 3, y),
(x_{i+j} + 4, y), (x_{i+j} + 5, y + 1), (x_{i+j} + 6, y + 1), (x_{i+j} + 5, y),
\ldots, (x_{i+j+1} - 3, y), (x_{i+j+1} - 2, y + 1), (x_{i+j+1} - 1, y + 1), (x_{i+j+1} - 2, y).
\]

For \(1 \leq q \leq j\), let

\[
Q_{i+q} : (x_{i+q-1} + 4, y + 2), (x_{i+q-1} + 5, y + 2), \ldots, (x_{i+q} + 2, y + 2),
(x_{i+q} + 1, y + 1), (x_{i+q} + 2, y + 1), (x_{i+q} + 3, y + 2).
\]

Fig. 13: An \((x_{i+1}, y)-(x_i, y + 1)\)-path-collection for a block \(B_i\) of length 8.
Now, $R_y$ and $R_{y+1}$, where

$$R_y : (0, y), (1, y), \ldots, (x_i, y),$$
$$P_i,$$
$$P_{i+1}, P_{i+2}, \ldots, P_{i+j-1},$$
$$(x_{i+j}, y), (x_{i+j} + 1, y), P_{i+j}, (x_{i+j+1} - 1, y),$$
$$(x_{i+j+1}, y + 1), (x_{i+j+1} + 1, y + 1), \ldots, (-1, y + 1)$$

and

$$R_{y+1} : (0, y + 1), (1, y + 1), \ldots, (x_i + 1, y + 1),$$
$$(x_i, y + 2), (x_i + 1, y + 2), (x_i + 2, y + 2), (x_i + 3, y + 2),$$
$$Q_{i+1}, Q_{i+2}, \ldots, Q_{i+j},$$
$$(x_{i+j+1} - 1, y + 2), (x_{i+j+1}, y + 2), \ldots, (-1, y + 2)$$

define an $(x_{i+j+1}, y)$-$(x_i, y + 2)$-path-collection of $G^D$. See Figures 14 and 15 for an illustration. □

**Lemma 13** If for some $i \in \mathbb{Z}/d_2\mathbb{Z}$ and for some $j \geq 1$, the blocks $B_i, B_{i+1}, \ldots, B_{i+j}$ of $G^D$ are such that $-1 \notin \{i, i + 1, \ldots, i + j\}$, $B_i$ and $B_{i+j}$ are of length 3 and $B_{i+1}, \ldots, B_{i+j-1}$ are of length 2, then $G^D$ has an $(x_{i+j+1}, y)$-$(x_i, y + j + 1)$-path-collection for all $y$.

**Proof:** Note that $x_{i+j+1} = x_i + 2j + 4$. For $0 \leq q \leq j - 1$, let

$$R_{i+q} : (0, y + q), (1, y + q), \ldots, (x_i + 1, y + q),$$

...
Proof: for all \(y\) blocks of \(G\) if for some \(l \geq 3\) and \(2\) \((\text{a block of even length or a sequence that consists of one even block})\), Lemma 13 implies that there exists an \((i, y + j)\)-path-collection for every \(y\) and for all \(i\).

Now, \(R_i, R_{i+1}, \ldots, R_{i+q}\) define an \((x_{i+j+1}, y)\)-(\(x_i, y + j + 1\))-path-collection of \(G_D\). See Figure 16 for an illustration.

\[
\begin{align*}
R_{i+j} & : (0, y + j), (1, y + j), \ldots, (x_i + 1, y + j), \\
R_{i+3} & : (x_i + 2, y + j), (x_i + 3, y + j), \ldots, (x_i + 2j + 3, y + j), \\
R_{i+2} & : (x_i + 2j + 4, y + j), (x_i + 2j + 5, y + j + 1), \ldots, (1, y + j + 1).
\end{align*}
\]

Fig. 16: An \((x_{i+j+1}, y)-(x_i, y + j + 1)\)-path-collection for \(j = 4\).

**Lemma 14** If for some \(i \in \mathbb{Z}/d_2\mathbb{Z}\) and for some \(j \geq 0\), the sequence \(S = B_i, B_{i+1}, \ldots, B_{i+j}\) of blocks of \(G_D\) is such that \(-1 \not\in \{i, i+1, \ldots, i+j\}\) and the number of blocks of odd length among \(B_i, B_{i+1}, \ldots, B_{i+j}\) is even, then \(G_D\) has an \((x_{i+j+1}, y)-(x_i, y + \Delta y)\)-path-collection for some \(\Delta y\) and for all \(y\).

**Proof:** By definition, the union of suitable path-collections is a path-collection: If for some \(x, x', x'', y, y', y''\), \(G_D\) has an \((x, y)-(x', y')\)-path-collection and an \((x', y')-(x'', y'')\)-path-collection, then \(G_D\) has an \((x, y)-(x'', y'')\)-path-collection. We can partition \(S\) into subsequences, where each subsequence is either a block of even length or a sequence \(B_k, B_{k+1}, \ldots, B_{k+l}\) of blocks with \(k \in \mathbb{Z}/d_2\mathbb{Z}\) and \(l \geq 1\), such that blocks \(B_k\) and \(B_{k+1}\) have odd length and blocks \(B_{k+1}, \ldots, B_{k+l-1}\) have even length. For a subsequence that consists of one even block \(B_k\) with \(k \in \mathbb{Z}/d_2\mathbb{Z}\), Lemma 11 implies that there exists a \((x_k, y + 1)\)-(\(x_k, y + 1\)) path collection for every \(y\). If \(\frac{a_1}{a_2} < 2\), then Lemma 3 implies that the lengths of the blocks are \(2\) and \(3\). For a subsequence that consists of at least two blocks \(B_{k}, B_{k+1}, \ldots, B_{k+i}\) with \(k \in \mathbb{Z}/d_2\mathbb{Z}\) and \(l \geq 1\), Lemma 13 implies that there exists an \((x_k+y+1, y)-(x_k, y + l + 1)\)-path-collection for every \(y\). If \(\frac{a_1}{a_2} > 2\), then Lemma 3 implies that the lengths of the blocks are at least \(3\). For a subsequence that consists
of at least two blocks \(B_k, B_{k+1}, \ldots, B_{k+l}\) with \(k \in \mathbb{Z}/d_2\mathbb{Z}\) and \(l \geq 1\), Lemma 12 implies that there exists an \((x_{k+i+1}, y)\)\(-(x_k, y + 2)\)-path-collection for every \(y\). Hence, a suitable union of path-collections forms an \((x_{i+j+1}, y)\)\-(\(x_i, y + \Delta y\))\-path-collection for a suitable \(\Delta y\) and all \(y\).

\[\square\]

**Lemma 15** If for some \(-i \in \mathbb{Z}/d_2\mathbb{Z}\), the blocks \(B_{-i}, B_{-i+1}, \ldots, B_{-1}\) of \(G^D\) are such that \(B_{-i}\) is of odd length and \(B_{-i+1}, \ldots, B_{-1}\) are of even length at least 4, then for all \(y\), \(G^D\) has a path with endvertices \((-1, y + 1)\) and \((-1, y + 2)\) that consists of all vertices of rows \(y\) and \(y + 1\) and the vertices \((x_{-i}, y + 2), (x_{-i} + 1, y + 2), \ldots, (-1, y + 2)\).

**Proof:** For \(1 \leq q \leq i - 1\), let

\[Q_{-q} : (x_{-q+1} - 3, y), (x_{-q+1} - 4, y), \ldots, (x_{-q}, y), (x_{-q} - 1, y), (x_{-q} - y + 1), (x_{-q} - 2, y)\]

and let

\[Q_{-i} : (x_{-i+1} - 3, y), (x_{-i+1} - 2, y + 1), (x_{-i+1} - 3, y + 1), (x_{-i+1} - 4, y), (x_{-i+1} - 5, y), (x_{-i+1} - 4, y + 1), (x_{-i+1} - 5, y + 1), (x_{-i+1} - 6, y), \ldots, (x_{-i} + 2, y), (x_{-i} + 3, y + 1), (x_{-i} + 2, y + 1), (x_{-i} + 1, y)\]

Furthermore, let for \(1 \leq q \leq i - 1\)

\[P_{-q} : (x_{-q}, y + 2), (x_{-q} + 1, y + 2), (x_{-q} + 2, y + 2), (x_{-q} + 1, y + 1), (x_{-q} + 2, y + 1), (x_{-q} + 3, y + 2), (x_{-q} + 4, y + 2), (x_{-q} + 3, y + 1), (x_{-q} + 4, y + 1), (x_{-q} + 5, y + 2), \ldots, (x_{-q+1} - 2, y + 2), (x_{-q+1} - 3, y + 1), (x_{-q+1} - 2, y + 1), (x_{-q+1} - 1, y + 2)\]

Now, the sequence

\[(-1, y + 1), (-2, y), Q_{-1},Q_{-2}, \ldots, Q_{-i}, \ldots, (x_{-i}, y), (x_{-i} - 1, y), \ldots, (-1, y), (0, y + 1), (1, y + 1), \ldots, (x_{-i} + 1, y + 1), (x_{-i}, y + 2), (x_{-i} + 1, y + 2), \ldots, (x_{-i+1} - 1, y + 2), P_{-i+1}, P_{-i+2}, \ldots, P_{-1}\]
defines a path that satisfies the conditions of the lemma. See Figures 17 and 18 for an illustration.

**Lemma 16** For every $D = \{d_1, d_2\} \subseteq \mathbb{N}$ with $d_1 > d_2$, $d_1 \text{ and } d_2$ even, and $\gcd(d_1, d_2) = 1$, there is some $n \in \mathbb{N}$ with $n \equiv 0 \pmod{d_1 + d_2}$ such that $G^D$ has a special cycle $C$ of order $n+1$ with $V(C) = [0, n]$.

**Proof:** Clearly, vertex $n$ is in column 0. Since $d_1 d_2$ is even and $\gcd(d_1, d_2) = 1$, we obtain that $d_1 + d_2$ is odd and hence the number of blocks of odd length is odd, i.e. at least 1. Let $i \in \mathbb{Z}/d_2\mathbb{Z}$, such that block $B_i$ is of odd length and the blocks $B_{i+1}, \ldots, B_{i-1}$ are of even length. Clearly, by Lemma 3, the length of the blocks $B_0, \ldots, B_{i-1}$ of odd length is even, by Lemma 14, $G^D$ has an $(x_i, 2)$-$(0, y')$-path-collection $R$ for some $y'$. Note, that if $G^D$ has only one block of odd length, then $R = \emptyset$. In this case we define $y' = 2$. Let

$$P = \left( Q \cup \bigcup_{R \in \mathcal{R}} R \right) + \bigcup_{y=1}^{y'-2} \{(−1, y), (0, y+1)\}.$$  

**Fig. 19:** The path $P$. 

A cycle $C$ in $G^D$ is called special, if $V(C) = [\min(V(C)), \max(V(C))]$.
By construction, $P$ is a path with endvertices $(-1, y'-1)$ and $(-1, y')$ that consists of all vertices of rows $0, 1, \ldots, y'$. The vertex $(0, y')$ has the neighbors $(1, y'-1)$ and $(1, y')$ in $P$. Since the vertex $(1, y')$ is an upper vertex, $(1, y')$ has the neighbors $(0, y')$ and $(2, y')$ in $P$ and $\{(1, y'-1), (2, y')\} \in E(G^D)$. Now,

$$C = P + \{\{(1, y'-1), (2, y')\}, \{(-1, y'-1), (0, y')\}, \{(-1, y'), (0, y'+1)\}, \{(0, y'+1), (1, y')\}\}$$

is a special cycle of $G^D$ of order $n+1$ with $n = (y'+1)(d_1 + d_2)$ and $V(C) = [0, n]$. See Figures 19 and 20 for an illustration.

**Lemma 17** For every $D = \{d_1, d_2\} \subseteq \mathbb{N}$ with $d_1 > d_2$, $d_1d_2$ odd, and $\gcd(d_1, d_2) = 1$, there is some $n \in \mathbb{N}$ with $n \equiv 0 \pmod{d_1 + d_2}$ such that $G^D$ has a special cycle $C$ of order $n+2$ with $V(C) = [0, n+1]$.

**Proof:** Clearly, vertex $n$ is in column 0. First we assume that $d_2 = 1$. In that case, $G^D$ has only one block and the vertex $n+1$ is in column $-1$. Let $P = \emptyset$ for $d_1 = 3$, otherwise let

$$P : \quad (1, 0), (2, 1), (3, 1), (2, 0), \quad (3, 0), (4, 1), (5, 1), (4, 0), \quad \ldots, (-5, 0), (-4, 1), (-3, 1), (-4, 0).$$

The sequence

$$C : \quad (0, 0), P, (-3, 0), (-2, 0), (-1, 1), (-2, 1), (-1, 2), (0, 2), (1, 1), (0, 1), (-1, 0), (0, 0)$$

defines a special cycle of $G^D$ of order $2(d_1 + d_2) + 2$ with $V(C) = [0, 2(d_1 + d_2) + 1]$. See Figure 21 for an illustration.
Now we assume that $d_2 > 1$. Hence, by Lemma 3 $G^D$ has more than one block. This implies that vertex $n+1$ is lower. Let $k \in \mathbb{Z}/d_2\mathbb{Z}$, such that vertex $n+1$ belongs to block $B_k$. Since $d_1 + d_2$ is even, exactly those vertices are even integers that are in a column with an even index. Since vertex $n+1$ is lower and an odd integer, the number of blocks among $B_k, B_{k+1}, \ldots, B_{-1}$ of odd length is odd, i.e. at least one. Let $i \in \mathbb{Z}/d_2\mathbb{Z}$ be such that block $B_i$ is of odd length and the blocks $B_{i+1}, B_{i+2}, \ldots, B_{-1}$ are of even length. Clearly, by Lemma 3 the length of the blocks $B_{i+1}, B_{i+2}, \ldots, B_{-1}$ are at least 4. By Lemma 15 $G^D$ has a path $Q_1$ with endvertices $(-1,1)$ and $(-1,2)$ that consists of all vertices of rows 0 and 1 and the vertices $(x_i,2), (x_i + 1,2), \ldots, (-1,2)$. Since the number of blocks of $B_k, B_{k+1}, \ldots, B_{-1}$ of odd length is even, by Lemma 14 $G^D$ has an $(x_i,2)-(x_k,y')$-path-collection $\mathcal{R}_1$ for some $y'$. Note, that if $i = k$, then $\mathcal{R}_1 = \emptyset$. In this case we define $y' = 2$. By the same arguments, $G^D$ has a path $Q_2$ with endvertices $(-1,y' + 2)$ and $(-1,y' + 3)$ that consists of all vertices of rows $y' + 1$ and $y' + 2$ and the vertices $(x_i,y' + 3), (x_{i+1},y' + 3), \ldots, (-1,y' + 3)$ and $G^D$ has an $(x_i,y' + 3)-(x_k,2y' + 1)$-path-collection $\mathcal{R}_2$.

By definition, for every $y' + 1 \leq y \leq 2y'$, the edges $\{(0,y'),(1,y')\}$ and $\{(x_k - 1,y),(x_k,y)\}$ belong to $Q_2$ or a path in $\mathcal{R}_2$. Furthermore, the path

$$P_0 : (x_k + 1,2y'), (x_k,2y' + 1), (x_k + 1,2y' + 1), (x_k + 2,2y' + 1)$$

is a subpath of a path in $\{Q_2\} \cup \mathcal{R}_2$. Let

$$P_1 : (0,y'), (1,y'), \ldots, (x_k - 1,y')$$

and let

$$P_2 : (x_k,2y' + 1), (x_k + 1,2y' + 1), (x_k,2y' + 2), (x_k - 1,2y' + 1), (x_k - 2,2y' + 1), \ldots, (1,2y' + 1).$$

Now,

$$C = (Q_1 \cup Q_2 \cup \mathcal{R}_1 \cup \mathcal{R}_2)
- \left( E(P_0) \cup \bigcup_{y=y'+1}^{2y'} \{(0,y),(1,y)\} \cup \bigcup_{y=y'+1}^{2y'} \{(x_k - 1,y),(x_k,y)\} \right)
+ E(P_1) \cup E(P_2)
+ \bigcup_{y=1}^{y'} \{(-1,y),(0,y+1)\}$$
defines a special cycle of $G^D$ of order $n + 2$ with $n = (2y' + 2)(d_1 + d_2) + 2$ and $V(C) = [0, n + 1]$. See Figures 22 and 23 for an illustration.

Let $C$ be a special cycle of $G^D$ and let $n' = \max(V(C))$. If for all $a, b \in V(C)$ with $n' - d_1 + 1 \leq a < b \leq n'$, $\{a, b\} \neq \{n' - 2d_2, n' - d_2\}$, and $\{a, b\} \in D$, we have $\{a, b\} \in E(C)$, then we call $C$ good.

We are now in a position to prove the main results of this section.

**Proof of Theorem 9** If $D = \{1, 3\}$, then the result follows by induction on $n$. $C : 0, 1, 2, 3, 0$ is a Hamiltonian cycle of $G^D_4$. Let $C_n$ be a Hamiltonian cycle of $G^D_n$. Since the vertex $n - 1$ has degree 2 in
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Fig. 23: The cycle \( C \) in the proof of Lemma 17.

\[ \begin{align*}
G_n^D, \{n - 2, n - 1\} &\in E(C_n). \text{ Hence,} \\
C_{n+2} &= C_n + \{\{n - 2, n + 1\}, \{n + 1, n\}, \{n, n - 1\}\} - \{\{n - 2, n - 1\}\}
\end{align*} \]

is a Hamiltonian cycle of \( G_{n+2}^D \).

Hence we can assume that \( D \neq \{1, 3\} \). Note that we can shift special cycles: If \( C : v_0, \ldots, v_l, v_0 \) is a special cycle in \( G^D \), then also \( C + h : v_0 + h, \ldots, v_l + h, v_0 + h \) is a special cycle in \( G^D \). Furthermore, we can merge special cycles: If \( C_1 \) and \( C_2 \) are special cycles with \( \min(V(C_2)) = \max(V(C_1)) + 1 \), \( \{a, b\} \in E(C_1), \{c, d\} \in E(C_2) \), and \( \{a, c\}, \{b, d\} \in E(G^D) \), then

\[(C_1 \cup C_2) + \{\{a, c\}, \{b, d\}\} - \{\{a, b\}, \{c, d\}\}\]

is a special cycle with vertex set \( [\min(V(C_1)), \max(V(C_2))] \). If for \( i \leq a < b \leq j \), \( \{a, b\} \) is an edge of \( G^D \) and at least one of \( a, b \) has degree 2 in \( G^D[i, j] \), then the edge \( \{a, b\} \) belongs to every special cycles \( C \) of \( G^D \) with \( V(C) = [i, j] \).

**Claim 1** If \( C_1 \) and \( C_2 \) are good special cycles of \( G^D \) with \( \min(V(C_2)) = \max(V(C_1)) + 1 \) and \( D \neq \{1, 3\} \), then there is a good special cycle \( C \) with \( V(C) = [\min(V(C_1)), \max(V(C_2))] \).

**Proof of Claim:** Let \( n' = \max(V(C_1)) \).

Case 1 \( d_1 \neq 2d_2 + 1 \).
Since \( d_1 \neq 2d_2 + 1 \) and \( C_1 \) is good, \( e_1 = \{ n' - d_1 + 1, n' - d_1 + d_2 + 1 \} \in E(C_1) \). Clearly, \( e_2 = \{ n' + 1, n' + d_2 + 1 \} \in E(C_2) \). Hence,
\[
C = (C_1 \cup C_2) + \{ n' - d_1 + 1, n' + 1 \}, \{ n' - d_1 + d_2 + 1, n' + d_2 + 1 \} - \{ e_1, e_2 \}
\]
is a good special cycle with \( V(C) = [\min(V(C_1)), \max(V(C_2))] \). This concludes the first case.

**Case 2** \( d_1 = 2d_2 + 1 \).

Since \( D \neq \{1, 3\} \), we have \( d_2 > 1 \). Since \( d_1 = 2d_2 + 1 \), and \( C_1 \) is good, \( e_1 = \{ n' - d_1 + 2, n' - d_1 + d_2 + 2 \} \in E(C_1) \). Since \( d_2 > 1 \), \( e_2 = \{ n' + 2, n' + d_2 + 2 \} \in E(C_2) \). Hence,
\[
C = (C_1 \cup C_2) + \{ n' - d_1 + 2, n' + 2 \}, \{ n' - d_1 + d_2 + 2, n' + d_2 + 2 \} - \{ e_1, e_2 \}
\]
is a good special cycle with \( V(C) = [\min(V(C_1)), \max(V(C_2))] \). This concludes the second case and the proof of Claim 1.

**Claim 2** \( G^D \) has a good special cycle of order \( 2 \pmod{d_1 + d_2} \).

**Proof of Claim 2.** By Lemma 17, \( G^D \) has a special cycle of order \( 2 \pmod{d_1 + d_2} \). Let \( C_1 \) be a special cycle of \( G^D \) of order \( 2 \pmod{d_1 + d_2} \) and let \( n' = \max(V(C_1)) \). It follows from \( 25 \) that \( G^D \) has a special cycle of order \( d_1 + d_2 \). Note that every vertex in \( \{ j, j + 1, \ldots, j + d_1 + d_2 - 1 \} \) has degree 2 in \( G^D[i, j + d_1 + d_2 - 1] \), for \( j \in \mathbb{Z} \) and hence a special cycle of order \( d_1 + d_2 \) is good. Let \( C_2 \) be a special cycle of \( G^D \) of order \( d_1 + d_2 \) with \( \min(V(C_2)) = n' + 1 \). Since vertex \( n' \) has degree 2 in \( G^D[V(C_1)], \{ n' - d_2, n' \} \in E(C_1) \) and since vertex \( n' + 1 \) has degree 2 in \( G^D[V(C_2)], \{ n' + d_1 - d_2, n' + d_1 \} \in E(C_2) \). Hence,
\[
(C_1 \cup C_2) + \{ n' - d_2, n' + d_1 - d_2 \}, \{ n', n' + d_1 \} - \{ n' - d_2, n' \}, \{ n' + d_1 - d_2, n' + d_1 \}
\]
is a good special cycle of \( G^D \). This concludes the proof of Claim 2.

Let \( p_1 \) with \( p_1 \equiv 2 \pmod{d_1 + d_2} \), such that \( G^D \) has a good special cycle of order \( p_1 \). By Claim 2, such a \( p_1 \) exists. As said before, \( G^D \) has a good special cycle of order \( p_2 = d_1 + d_2 \). Since \( \gcd(p_1, p_2) = 2 \), it follows from the extended Euclidean algorithm that every sufficiently large even integer is a positive integral linear combination of \( p_1 \) and \( p_2 \). Therefore and by Claim 1 the desired result follows by shifting and merging copies of good special cycles of order \( p_1 \) and \( p_2 \).

**Proof of Theorem 10:** The proof is analogous to the proof of Theorem 9. Instead of using Lemma 17 we use Lemma 16. Proceeding as in the proof of Theorem 9 we obtain \( p_1 \) with \( p_1 \equiv 1 \pmod{d_1 + d_2} \) and hence \( \gcd(p_1, p_2) = 1 \). This clearly allows to establish the theorem for all sufficiently large \( n \) and not just for sufficiently large even \( n \).

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References


