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THE BIPARTITE RAMSEY NUMBERS $BR(C_8, C_{2n})$

Mostafa Gholami[1*] Yaser Rowshan[1]

[1] *Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan 66731-45137, Iran*

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For the given bipartite graphs $G_1, G_2, \ldots, G_t$, the multicolor bipartite Ramsey number $BR(G_1, G_2, \ldots, G_t)$ is the smallest positive integer b, such that any t-edge-coloring of $K_{b,b}$ contains a monochromatic subgraph isomorphic to G_i, colored with the ith color for some $1 \leq i \leq t$. We compute the exact values of the bipartite Ramsey numbers $BR(C_8, C_{2n})$ for $n \geq 2$.

Keywords: Ramsey numbers, Bipartite Ramsey numbers, Cycle

1 Introduction

Since 1956, when Erdös and Rado published the fundamental paper Erdös and Rado (1956), Ramsey theory has grown into one of the most active areas of research in combinatorics while interacting with graph theory, number theory, geometry, and logic Rosta (2004). Ramsey theory has many applications in several branches of mathematics. We refer to Graham et al.; Parsons to see these diverse applications. In particular, one can see the bipartite Ramsey numbers have many applications, and this motivated us to conduct a study on bipartite Ramsey numbers. The classical Ramsey number for the given numbers $n_1, \ldots, n_k$, is the smallest integer n in a way that there is some $1 \leq i \leq k$ for each k-coloring of the edges of complete graph K_n, such that there is a complete subgraph of size n_i whose edges are all the ith color. However, there is no loss to work in any class of graphs and their subgraphs instead of complete graphs. If $G_1, G_2, \ldots, G_t$ be bipartite graphs, the multicolor bipartite Ramsey number $BR(G_1, G_2, \ldots, G_t)$ is defined as the smallest positive integer b, such that any t-edge-coloring of the complete bipartite graph $K_{b,b}$ contains a monochromatic subgraph isomorphic to G_i, colored with the ith color for some i. The existence of such a positive integer is guaranteed by a result of Erdős and Rado Erdös and Rado (1956). Recently, new versions of Ramsey numbers, such as multipartite Ramsey numbers have been defined. One can refer to Day et al. (2001), Rowshan et al. (2021), Rowshan and Gholami (2023), Rowshan and their references for further studies.

*Corresponding Author.

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The exact values of the bipartite Ramsey numbers $BR(P_n, P_m)$ of two paths follow from the results of Faudree and Schelp Faudree and Schelp (1975) and Gyárfás and Lehel Gyárfás and Lehel (1973). The bipartite Ramsey numbers $BR(K_{1,n}, P_m)$ are given by Hatting and Henning in Hattingh and Henning (1998). The multicolor bipartite Ramsey numbers $BR(G_1, G_2, \ldots, G_t)$ when $G_1, G_2, \ldots, G_t$ are either stars and stripes, or stars and paths, has been studied in Raeisi (2015). In Bucic et al. (2019), authors have determined asymptotically the 4-colour bipartite Ramsey number of paths and cycles. The same authors have determined asymptotically the 3-colour bipartite Ramsey number of paths and cycles in Bucić et al. (2019). The three-colour bipartite Ramsey number $BR(G_1, G_2, P_3)$ is considered in Lakshmi and Sindhu (2020). New values of the bipartite Ramsey number $BR(C_4, K_{1,n})$ using induced subgraphs of the incidence graph of a projective plane are given in Hatala et al. (2021). Bipartite Ramsey numbers of $K_{t,s}$ in many colors and bipartite Ramsey numbers of cycles for random graphs have been discussed in Wang et al. (2021.) and Liu and Li (2021) respectively. Xuemei Zhang et.al have done a research on multicolor bipartite Ramsey numbers for quadrilaterals and stars Zhang et al. (2023). The exact value of $BR(nP_2, nP_2, ..., nP_2)$ has been obtained in Qiao et al. (2021). The exact value of $BR(P_3, P_3, P_n)$, and $BR(P_3, P_3, ..., P_n)$ for $n \geq 4r + 2$, and $BR(P_4, P_4, P_n)$ for $n \geq 4$ have been computed in Wang et al. (2021). We intend to compute the exact values of the multicolor bipartite Ramsey numbers $BR(C_8, C_{2n})$. Actually, we prove the following theorem:

Theorem 1.1. *For any $n \geq 2$, we have:*

$$BR(C_8, C_{2n}) = \begin{cases} 8 & n = 4, \\ n + 3 & \text{otherwise.} \end{cases}$$

In this paper, we are only concerned with undirected, simple, and finite graphs. We follow Bondy et al. (1976) for terminology and notations not defined here. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $v \in V(G)$ is denoted by $\deg_G(v)$, or simply by $\deg(v)$. The neighborhood $N_G(v)$ of a vertex v is the set of all vertices of G adjacent to v and satisfies $|N_G(v)| = \deg_G(v)$. The minimum and maximum degrees of vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Let C be a set of colors $\{c_1, c_2, ..., c_m\}$ and $E(G)$ be the edges of a graph G, an edge coloring $f : E \rightarrow C$ assigns each edge in $E(G)$ to a color in C. If an edge coloring uses k colors on a graph, it is known as a k-colored graph. As usual, C_n stands for a cycle on n vertices. The complete bipartite graph with bipartition (X, Y), where $|X| = m$ and $|Y| = n$ is denoted by $K_{m,n}$. We use $[X, Y]$ to indicate the set of edges between a bipartition (X, Y) of G. Let $W \subseteq V(G)$ be any subset of vertices of G, the induced subgraph $G\langle W\rangle$ is the graph whose vertex set is W and whose edge set consists of all of the edges in $E(G)$ that have both endpoints in W. The complement of a graph G, denoted by $\overline{G}$, is the graph with the same vertices as G and contains those edges which are not in G. G is n-colorable to $(G_1, G_2, \ldots, G_n)$ if there exists an n-edge decomposition $(H_1, H_2, \ldots, H_n)$ of G, where $G_i \nsubseteq H_i$ for each $i = 1, 2, \ldots, n$.

2 Some basic results

To prove our main results, namely Theorem 1.1, we need to establish some preliminary results. We start with the following simple but helpful lemma:

Lemma 2.1. *Let G be a subgraph of $K_{t,t}$ with a cycle $C : x_1y_1x_2 \ldots x_ky_kx_1$ of length $2k$, where $t \geq k + 1$. If x and y be two vertices of G not in C, where $x_i, x_{i+1} \in N_G(y)$ and $y_i, y_{i+1} \in N_G(x)$, or*

$xy \in E(G)$, in which $x_i \in N_G(y)$ and $y_i \in N_G(x)$ for some $i, 1 \leq i \leq k$, then G has a cycle of length $2k+2$.

Proof: Consider $C' = x_1y_1x_2 \dots y_{i-1}x_iyx_{i+1}y_ixy_{i+1}x_{i+2} \dots x_ky_kx_1$, where $x_i, x_{i+1} \in N_G(y)$ and $y_i, y_{i+1} \in N_G(x)$. Also, consider $C'' = x_1y_1x_2 \dots x_iyxy_ix_{i+1} \dots x_ky_kx_1$, where $xy \in E(G)$, $x_i \in N_G(y)$, and $y_i \in N_G(x)$. □

Lemma 2.2. *Let G be a spanning subgraph of $K_{t,t}$ with a cycle C of length $2k$, where $t \geq k+1$ and $k \geq 4$. Let x and y be the vertices of G not in C. Assume that x, y are adjacent to at least $k-1$ vertices of C where $xy \notin E(G)$, or x and y are adjacent to at least $\lceil \frac{k}{2} \rceil + 1$ vertices of C where $xy \in E(G)$. Then G has a cycle of length $2k+2$.*

Proof: If $xy \notin E(G)$, the lemma is proven by Zhang et al. in Zhang et al. (2013), hence we may assume that x, y are adjacent to at least $\lceil \frac{k}{2} \rceil + 1$ vertices of C where $xy \in E(G)$. Without loss of generality (W.l.g.), assume that $C_{2k} = x_1y_1x_2y_2 \dots x_ky_kx_1$. In this case, it is easy to check that there is at least one i, $1 \leq i \leq k$, such that xy_i and x_iy belong to $E(G)$. Therefore by Lemma 2.1, we have $C_{2k+2} \subseteq G$ and the proof is complete. □

In the following two theorems, the authors in Rui and Yongqi (2011) and Zhang et al. (2013) have determined the exact value of the bipartite Ramsey number of $BR(C_{2m}, C_{2n})$ for $m = 2, 3$, respectively.

Theorem 2.3. *For any $n \geq 2$, we have:*

$$BR(C_4, C_{2n}) = \begin{cases} 5 & n = 2, 3, \\ n+1 & \text{otherwise.} \end{cases}$$

Theorem 2.4. *For any $n \geq 3$, we have:*

$$BR(C_6, C_{2n}) = \begin{cases} 6 & n = 3, \\ n+2 & \text{otherwise.} \end{cases}$$

Proposition 2.5. *Let G be a subgraph of $K_{8,8}$ where $K_{3,4} \subseteq G$, then either $C_8 \subseteq G$ or $C_8 \subseteq \overline{G}$.*

Proof: Let $(X = \{x_1, \dots, x_8\}, Y = \{y_1, \dots, y_8\})$ be a bipartition of $K_{8,8}$ and $K_{3,4} \subseteq G[X_1, Y_1]$, where $X_1 = \{x_1, x_2, x_3\}$, $Y_1 = \{y_1, \dots, y_4\}$. Assume that $C_8 \nsubseteq \overline{G}$. Thus we have $|N_G(x) \cap Y_1| \leq 1$ for each $x \in X \setminus X_1$. Otherwise, $C_8 \subseteq G$. Consider $X_2 = X \setminus X_1$ and $Y_2 = Y_1 \cup \{y_5\}$. Since $BR(C_8, C_4) = 5$ and $C_8 \nsubseteq \overline{G}$, we have $C_4 \subseteq G[X_2, Y_2]$. Hence $y_5 \in V(C_4)$; otherwise, $C_8 \subseteq G$. W.l.g., we may assume that $C_4 = x_4y_4x_5y_5x_4$. Therefore we have $[\{x_4, x_5\}, \{y_1, y_2, y_3\}], [\{x_1, x_2, x_3\}, \{y_5\}] \subseteq \overline{G}$, otherwise, $C_8 \subseteq G$. If $|N_{\overline{G}}(y_5) \cap \{x_6, x_7, x_8\}| \geq 2$, then $C_8 \subseteq \overline{G}[X_2, \{y_1, y_2, y_3, y_5\}]$, a contradiction. So $|N_G(y_5) \cap \{x_6, x_7, x_8\}| \geq 2$. Let $x_6y_5, x_7y_5 \in E(G)$, hence $K_{4,3} \cong [\{x_4, x_5, x_6, x_7\}, \{y_1, y_2, y_3\}] \subseteq \overline{G}$. Therefore, $|N_{\overline{G}}(y) \cap \{x_4, x_5, x_6, x_7\}| \leq 1$ for each $y \in \{y_6, y_7, y_8\}$, that is $|N_G(y) \cap \{x_4, x_5, x_6, x_7\}| \geq 3$ for each $y \in \{y_6, y_7, y_8\}$. So, we have $K_{3,4} \cong [\{x_1, x_2, x_3\}, \{y_5, y_6, y_7, y_8\}] \subseteq \overline{G}$. Therefore for each $x \in \{x_4, x_5, x_6, x_7\}$, we have $|N_{\overline{G}}(x) \cap \{y_6, y_7, y_8\}| \leq 1$; otherwise, $C_8 \subseteq \overline{G}[\{x_1, x_2, x_3, x\}, \{y_5, y_6, y_7, y_8\}]$ which is a contradiction. Afterwards one can check that $C_8 \subseteq G[\{x_4, x_5, x_6, x_7\}, \{y_5, y_6, y_7, y_8\}]$ and the proof is complete. □

3 Proof of the main results

In this section, we compute the exact value of the bipartite Ramsey numbers $BR(C_8, C_{2n})$ for $n \geq 2$. Also, we guess that for $m, n \geq 3$, we have $BR(C_{2n}, C_{2m}) = \begin{cases} m+n & n = m, \\ m+n-1 & n \neq m. \end{cases}$ It should be noted that, after the authors posted the current article on arXiv, the correctness of the conjecture was proved by Yan and Peng Yan and Peng (2021).

In order to simplify the comprehension, let us split the proof of Theorem 1.1 into small parts. We begin with a simple but very useful general lower bound in the following theorem:

Theorem 3.1. *Rui and Yongqi (2011) We have* $BR(C_{2m}, C_{2n}) \geq m+n-1$ *for all* $m, n \geq 2$.

Proof: Let G_1 and G_2 denote vertex-disjoint induced subgraphs of $K_{m+n-2,m+n-2}$ ($m, n \geq 2$), which are isomorphic to complete bipartite graphs $K_{m-1,m+n-2}$ and $K_{n-1,m+n-2}$, respectively. Clearly, $E(K_{m+n-2,m+n-2}) = E(G_1) \cup E(G_2)$. As $C_{2m} \nsubseteq G_1$ and $C_{2n} \nsubseteq G_2$, it follows that $BR(C_{2m}, C_{2n}) \geq m+n-1$, as required. □

Lemma 3.2. *Rui and Yongqi (2011)* $BR(C_8, C_4) = 5$.

Lemma 3.3. *Zhang et al. (2013)* $BR(C_8, C_6) = 6$.

In the following two theorems, we determine the exact values of the bipartite Ramsey number of $BR(C_8, C_{2n})$ for $n = 4, 5$.

Theorem 3.4. $BR(C_8, C_8) = 8$.

Proof: To prove the lower bound, consider a bipartition (X, Y) of $K_{7,7}$, where $X = \{x_1, x_2, \ldots, x_7\}$ and $Y = \{y_1, y_2, \ldots, y_7\}$. Let $(G, \overline{G})$ be a 2-edge-coloring of $K_{7,7}$, where G is given in Figure 1. Hence it is easy to see that $C_8 \nsubseteq G$ and $\overline{G} \cong G \setminus x_4y_4$, that is we have $C_8 \nsubseteq \overline{G}$. Therefore, $BR(C_8, C_8) \geq 8$.

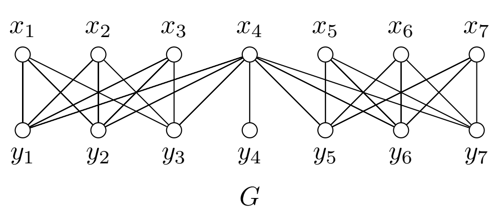

Fig. 1: Edge disjoint subgraphs G and $\overline{G}$ of $K_{7,7}$

To complete the proof, suppose on the contrary that $BR(C_8, C_8) > 8$, that is $K_{8,8}$ is 2-colorable to (C_8, C_8), say $C_8 \nsubseteq G$ and $C_8 \nsubseteq \overline{G}$ for some $G \subseteq K_{8,8}$. Since $BR(C_6, C_8) = 6$ and $C_8 \nsubseteq \overline{G}$, G has a subgraph $C \cong C_6$. Let (X, Y) be a bipartition of $K_{8,8}$, where $X = \{x_1, x_2, \ldots, x_8\}$ and $Y = \{y_1, y_2, \ldots, y_8\}$. Set $X_1 = V(C) \cap X$, $Y_1 = V(C) \cap Y$. W.l.g., we may assume that $X_1 = \{x_1, x_2, x_3\}$, $Y_1 = \{y_1, y_2, y_3\}$, and $C_6 = x_1y_1x_2y_2x_3y_3x_1$. Consider $X_2 = X \setminus \{x_1, x_2\}$ and $Y_2 = Y \setminus \{y_1, y_2\}$. Since $BR(C_6, C_8) = 6$, $|X_2| = |Y_2| = 6$, and $C_8 \nsubseteq \overline{G}$, $G[X_2, Y_2]$ has a subgraph $C' \cong C_6$. Let

$X_1' = V(C') \cap X_2$ and $Y_1' = V(C') \cap Y_2$. Now we consider the $|\{x_3, y_3\} \cap V(C')|$, there are three cases as follows:

Case 1. $|\{x_3, y_3\} \cap V(C')| = 2$.
W.l.g., we may assume that $X_1' = \{x_3, x_4, x_5\}$ and $Y_1' = \{y_3, y_4, y_5\}$. If $x_3y_3 \notin E(C')$, then we have $\{x_3y_4, x_3y_5, y_3x_4, y_3x_5\} \subseteq E(C')$ and there is at least one edge $x'y'$ between $\{x_4, x_5\}$ and $\{y_4, y_5\}$ in G. Since $x_3y_3 \in E(C)$ and x_3y', y_3x' are belong to $E(C')$, by Lemma 2.1 we have $C_8 \subseteq G[X_1 \cup \{x'\}, Y_1 \cup \{y'\}]$, which is a contradiction. Hence, assume that $x_3y_3 \in E(C')$, and w.l.g. assume that $C' = x_3y_3x_4y_4x_5y_5x_3$. Now since $C_8 \nsubseteq G$, and by Figure 2, it is easy to check that

$$\{x_1y_2, x_1y_4, x_2y_4, x_2y_5, x_4y_1, x_4y_5, x_5y_1, x_5y_2\} \subseteq E(\overline{G})$$

That is, $C_8 = x_1y_2x_5y_1x_4y_5x_2y_4x_1 \subseteq \overline{G}$, a contradiction again.

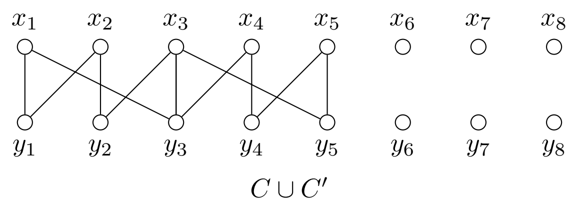

Fig. 2: $x_3y_3 \in E(C')$

Case 2. $|\{x_3, y_3\} \cap V(C')| = 1$.
W.l.g., we may assume that $x_3 \in V(C')$, $X_1' = \{x_3, x_4, x_5\}$, $Y_1' = \{y_4, y_5, y_6\}$, and $C' = x_3y_4x_4y_5x_5y_6x_3$. Since $C_8 \nsubseteq G$, by Figure 3, it is easy to check that $K_{2,3} \cong [\{x_4, x_5\}, \{y_1, y_2, y_3\}] \subseteq \overline{G}$, $K_{2,3} \cong [\{x_1, x_2\}, \{y_4, y_5, y_6\}] \subseteq \overline{G}$. Now we have the following claim:

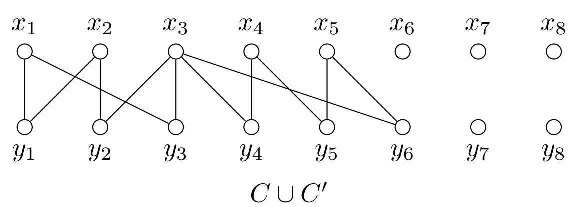

Fig. 3: $|\{x_3, y_3\} \cap V(C')| = 1$ $(x_3 \in V(C'))$

Claim 1. If there exists a vertex x of $X_2' = \{x_6, x_7, x_8\}$, such that $|N_G(x) \cap \{y_2, y_3\}| \neq 0$, then $|N_G(x) \cap Y_1'| = 0$.
Proof: Let $xy_2 \in E(G)$. For other cases, the proof is the same. Hence if xy_4 or xy_6 belong to $E(G)$, then we have $C_8 \subseteq G$, and if $xy_5 \in E(G)$, then we set $C' = x_3y_2xy_5x_4y_4x_3$ and the proof is the same

as case 1 when $Y_2 = Y \setminus \{y_1, y_3\}$. □

Similar to the proof of Claim 1, we have the following claim, which is easily verifiable.

Claim 2. If there exist a vertex x of $X_2' = \{x_6, x_7, x_8\}$, such that $|N_G(x) \cap \{y_4, y_6\}| \neq 0$, then $|N_G(x) \cap Y_1| = 0$.
Now by Claim 1, there are at least two vertices x_6 and x_7 of $X_2' = \{x_6, x_7, x_8\}$, such that $[\{x_6, x_7\}, \{y_2, y_3\}] \subseteq \overline{G}$; otherwise, we have $K_{4,3} \subseteq \overline{G}[\{x_1, x_2, x_6, x_7, x_8\}, Y_1']$ and the proof is complete by Proposition 2.5. Therefore by Claim 2, we have $|N_G(x) \cap \{y_4, y_6\}| = 0$ for at least one vertex of $\{x_6, x_7\}$. Otherwise, we have $K_{4,3} \cong [\{x_4, x_5, x_6, x_7\}, Y_1] \subseteq \overline{G}$ and the proof is complete by Proposition 2.5. Hence we may assume that $|N_G(x_7) \cap \{y_4, y_6\}| = 0$, that is $x_7y_4, x_7y_6 \in E(\overline{G})$. So it is easy to check that $x_1y_2, x_2y_3, x_4y_6, x_5y_4, x_6y_4, x_6y_5, x_6y_6 \in E(G)$; otherwise, $C_8 \subseteq \overline{G}$. Hence $x_6y_1 \in E(\overline{G})$; otherwise, $C_8 \subseteq G$. Now $x_7y_1 \in E(G)$, if not we have $K_{4,3} \cong [\{x_4, x_5, x_6, x_7\}, Y_1] \subseteq \overline{G}$ and the proof is complete by Proposition 2.5. We should also have $x_7y_5 \in E(\overline{G})$, if not $C_8 \subseteq G$, that is we have $K_{3,3} \cong [\{x_4, x_5, x_6\}, Y_1] \subseteq \overline{G}$ and $K_{3,3} \cong [\{x_1, x_2, x_7\}, Y_1'] \subseteq \overline{G}$. Now consider $N(x_8)$, by Claims 1 and 2, we have $|N_G(x_8) \cap \{y_2, y_3\}| = |N_G(x_8) \cap \{y_4, y_6\}| = 0$, otherwise we have $K_{4,3} \subseteq \overline{G}$ and the proof is complete by Proposition 2.5. Therefore $\{y_2, y_3, y_4, y_6\} \subseteq N_{\overline{G}}(x_8)$ and one can check that $C_8 = x_8y_6x_1y_5x_2y_4x_7y_2x_8 \subseteq \overline{G}$ and the proof is complete.

Case 3. $|\{x_3, y_3\} \cap V(C')| = 0$.
We may assume that $X_1' = \{x_4, x_5, x_6\}$, $Y_1' = \{y_4, y_5, y_6\}$, and $C' = x_4y_4x_5y_5x_6y_6x_4$. In this case, for each $x \in X_1$, $x' \in X_1'$, $y \in Y_1$, and $y' \in Y_1'$, we have $|N_G(x) \cap Y_1'| \leq 1$, $|N_G(x') \cap Y_1| \leq 1$, $|N_G(y') \cap X_1| \leq 1$, and $|N_G(y) \cap X_1'| \leq 1$. Otherwise, the proof is the same as case 2. Therefore it is easy to show that the following claim is true.

Claim 3. $K_{3,3} \setminus e \subseteq \overline{G}[X_1, Y_1']$ and $K_{3,3} \setminus e \subseteq \overline{G}[X_1', Y_1]$, in other words, there is at most one edge between $[X_1, Y_1']$ and $[X_1', Y_1]$ in G.
Now by Claim 3, one can check that $[X_1, Y_1] \subseteq G$ or $[X_1', Y_1'] \subseteq G$; otherwise, $C_8 \subseteq \overline{G}$. W.l.g., we may assume that $[X_1, Y_1] \subseteq G$. Hence we have $[X_1, Y_1'] \subseteq \overline{G}$ or $[X_1', Y_1] \subseteq \overline{G}$, if not, we have $C_8 \subseteq G$, which is a contradiction. W.l.g., assume that $[X_1, Y_1'] \subseteq \overline{G}$. Now consider x_7y_7 and w.l.g. assume that $x_7y_7 \in E(G)$. For other cases, the proof is the same. If $|N_G(x_7) \cap Y_1| \neq 0$, by Lemma 2.1, we have $|N_G(y_7) \cap X_1| = 0$, hence $K_{3,4} \cong [X_1, Y_1' \cup \{y_7\}] \subseteq \overline{G}$ and the proof is complete by Proposition 2.5. So let $|N_G(x_7) \cap Y_1| = 0$. Therefore by Claim 3, there exists one edge between $[X_1', Y_1]$ in G; otherwise, $K_{3,4} \subseteq \overline{G}$ and the proof is the same. Assume that $x_4y_1 \in E(G)$, if $|N_G(x_7) \cap Y_1'| = 0$ or $|N_G(y_7) \cap X_1| = 0$, then $K_{4,3} \cong [X_1 \cup \{x_7\}, Y_1'] \subseteq \overline{G}$ or $K_{3,4} \cong [X_1, Y_1' \cup \{y_7\}] \subseteq \overline{G}$, respectively and the proof is complete by Proposition 2.5. Hence w.l.g. assume that $x_7y_4, x_1y_7 \in E(G)$. Therefore one can check that $C_8 = x_1y_7x_7y_4x_4y_1x_2y_2x_1 \subseteq G$, a contradiction again.
Now by cases 1, 2, and 3, the proof is complete and the theorem holds. □

Theorem 3.5. $BR(C_{10}, C_8) = 8$.

Proof: The lower bound holds by Theorem 3.1. To complete the proof, by contrary suppose that $BR(C_{10}, C_8) > 8$, that is $K_{8,8}$ is 2-colorable to (C_{10}, C_8), say $C_{10} \nsubseteq G$ and $C_8 \nsubseteq \overline{G}$ for some $G \subseteq K_{8,8}$. Since $BR(C_8, C_8) = 8$ and $C_8 \nsubseteq \overline{G}$, G has a subgraph $C \cong C_8$. Let (X, Y) be a bipartition of $K_{8,8}$, where $X = \{x_1, x_2, \ldots, x_8\}$ and $Y = \{y_1, y_2, \ldots, y_8\}$. Set $X_1 = V(C) \cap X$ and $Y_1 = V(C) \cap Y$. W.l.g., we may assume that $X_1 = \{x_1, x_2, x_3, x_4\}$, $Y_1 = \{y_1, y_2, y_3, y_4\}$, and $C = x_1y_1x_2y_2x_3y_3x_4y_4x_1$. Since $C_8 \nsubseteq \overline{G}$, there is at least one edge other than $E(C)$ between X_1 and Y_1 in G, w.l.g. assume that $x_1y_2 \in E(G)$ (for other cases, the proof is the same). Consider $X_2 = X \setminus \{x_1, x_2\}$ and $Y_2 = Y \setminus \{y_1, y_2\}$. Since $BR(C_6, C_8) = 6$, $|X_2| = |Y_2| = 6$, and $C_8 \nsubseteq \overline{G}$, $G[X_2, Y_2]$ has a subgraph $C' \cong C_6$. Let $X'_1 = V(C') \cap X_1$, $Y'_1 = V(C') \cap Y_1$. Now consider the $(|X'_1|, |Y'_1|)$. By symmetry we note that $(|X'_1|, |Y'_1|) = (|Y'_1|, |X'_1|)$, so $(|X'_1|, |Y'_1|) \in \{(0,0), (1,0), (2,0), (1,1), (2,1), (2,2)\}$, now we have the following cases:

Case 1. $(|X'_1|, |Y'_1|) = (0,0)$.
Assume that $V(C') = \{x_5, x_6, x_7, y_5, y_6, y_7\}$ and $C' = x_5y_5x_6y_6x_7y_7x_5$. Set $X'_2 = \{x_5, x_6, x_7\}$ and $Y'_2 = \{y_5, y_6, y_7\}$. Since $C_{10} \nsubseteq G$, one can check that either $K_{3,4} \cong [X'_2, Y_1] \subseteq \overline{G}$ or $K_{4,3} \cong [X_1, Y'_2] \subseteq \overline{G}$; otherwise, assume to the contrary that there exists at least one edge between X'_2 and Y_1 in G, along with there exists at least one edge between X_1 and Y'_2 in G. Utilizing symmetry and without loss of generality, let's consider that $x_5y_1 \in E(G)$. Subsequently, let's posit that $|N_G(y_5) \cap X_1| \neq 0$. For other cases, the proof remains consistent. Since both x_5y_5 and x_5y_1 are edges in G, if x_1y_5 or x_2y_5 are also edges in G, then by lemma 2.1 it can be deduced tha $C_{10} \subseteq G$. Therefore, let's assume that either $x_3y_5 \in E(G)$ or $x_4y_5 \in E(G)$. In the instance of $x_3y_5 \in E(G)$, we have $C = x_5y_1x_2y_2x_1y_4x_4y_3x_3y_5x_5$ is a copy of C_{10} in G. The proof remains analogous in the scenario where $x_4y_5 \in E(G)$.
Hence, w.l.g., we may assume that $K_{3,4} \cong [X'_2, Y_1] \subseteq \overline{G}$. If there exists a vertex x of $X \setminus X'_2$, such that $|N_{\overline{G}}(x) \cap Y_1| \geq 2$, then $C_8 \subseteq \overline{G}$, a contradiction. Hence $|N_G(x) \cap Y_1| \geq 3$ for each $x \in X \setminus X'_2$, that is $|N_G(x_8) \cap Y_1| \geq 3$. Therefore by Lemma 2.1, we have $|N_G(y) \cap X_1| \leq 1$ for each $y \in Y \setminus Y_1$; otherwise, $C_{10} \subseteq G$. If there exist $y, y' \in \{y_5, y_6, y_7\}$, such that $|N_G(y) \cap X_1| = |N_G(y') \cap X_1| = 1$ and $N_G(y) \cap X_1 \neq N_G(y') \cap X_1$, one can check that $C_{10} \subseteq G$, a contradiction again (for example w.l.g., assume that $N_G(y_5) \cap X_1 = \{x_1\}$ and $N_G(y_6) \cap X_1 = \{x_2\}$. Hence $C_{10} := y_5x_1y_4x_4y_3x_3y_2x_2y_6x_6y_5 \subseteq G$). Therefore, w.l.g. we may assume that $\{x_1, x_2, x_3\} \subseteq N_{\overline{G}}(y)$ for each $y \in \{y_5, y_6, y_7\}$. If there are at least two vertices y_5, y_6 of Y'_2, such that $|N_{\overline{G}}(y_i) \cap X_1| = 4$ for $i = 5, 6$, then $C_8 \subseteq \overline{G}[X_1, Y \setminus Y_1]$. In other words, there are at least two vertices y_5, y_6 of Y'_2, such that $N_G(y_i) \cap X_1 = \{x_4\}$. Hence $x_8y_i \in E(\overline{G})$ for $i = 5, 6$, if not, $C_{10} \subseteq G$, a contradiction. Now one can check that $C_8 \subseteq \overline{G}[\{x_1, x_2, x_3, x_8\}, Y \setminus Y_1]$ and the proof is complete.

Case 2. $(|X'_1|, |Y'_1|) = (1,0)$.
Assume that $X'_1 = \{x_4\}$. For the case that $X'_1 = \{x_3\}$, the proof is the same. Hence w.l.g. assume that $V(C') = \{x_4, x_5, x_6, y_5, y_6, y_7\}$ and $C' = x_4y_5x_5y_6x_6y_7x_4$. Set $X''_2 = \{x_1, x_2, x_3\}$, $X'_2 = \{x_5, x_6\}$, and $Y'_2 = \{y_5, y_6, y_7\}$. Since $C_{10} \nsubseteq G$, we have $K_{3,3} \cong [X''_2, Y'_2] \subseteq \overline{G}$ and $K_{2,4} \cong [X'_2, Y_1] \subseteq \overline{G}$. Now consider the vertices $\{x_7, x_8\}$, one can check that for at least one vertex x of $\{x_7, x_8\}$, we have $|N_G(x) \cap Y_1| \geq 2$; otherwise, by$K_{2,4} \cong [X'_2, Y_1] \subseteq \overline{G}$ one can say that $C_8 \subseteq \overline{G}[X \setminus X_1, Y_1]$. W.l.g., assume that $|N_G(x_7) \cap Y_1| \geq 2$ and thus $|N_G(x_7) \cap Y'_2| = 0$; otherwise, $C_{10} \subseteq G$. Therefore, we have $K_{4,3} \cong [X''_2 \cup \{x_7\}, Y'_2] \subseteq \overline{G}$, that is $|N_{\overline{G}}(y_8) \cap (X''_2 \cup \{x_7\})| \leq 1$, if not, $C_8 \subseteq \overline{G}[X''_2 \cup \{x_7\}, Y_2]$. Since $|N_G(x_7) \cap Y_1| \geq 2$ and $|N_G(y_8) \cap X''_2| \geq 2$, if $x_7y_8 \in E(G)$, using Lemma 2.1, it can be easily checked that $C_{10} \subseteq G$. So let $x_7y_8 \in E(\overline{G})$, that is $X''_2 \subseteq N_G(y_8)$ and thus $|N_G(x_7) \cap Y_1| = 2$ and $N_G(x_7) \cap Y_1 = \{y_3, y_4\}$. Otherwise by Lemma 2.1, we have $C_{10} \subseteq G$ and the proof is complete. Similarly $|N_G(x_8) \cap Y_1| \leq 2$; otherwise, by Lemma 2.2, $C_{10} \subseteq G[X_1 \cup \{x_8\}, Y_1 \cup \{y_8\}]$. If $|N_G(x_8) \cap Y_1| \leq 1$, then $C_8 \subseteq \overline{G}[X \setminus X_1, Y_1]$. Hence $|N_G(x_8) \cap Y_1| = 2$ and $x_8y_8 \notin E(G)$, if not, one can by Lemma 2.1, check that $C_{10} \subseteq G[X_1 \cup \{x_8\}, Y_1 \cup \{y_8\}]$. Thus we have $Y \setminus Y_1 \subseteq N_{\overline{G}}(x_i)$ for $i = 7, 8$.

That is, $C_8 \subseteq \overline{G}[\{x_1,x_2,x_7,x_8\}, Y \setminus Y_1]$ and the proof is complete.

Case 3. $(|X_1'|,|Y_1'|)=(1,1)$.
W.l.g., we may assume that $V(C')=\{x,x_5,x_6,y,y_5,y_6\}$, where $x\in\{x_3,x_4\}$ and $y\in\{y_3,y_4\}$. If $xy\notin E(C)$, then $x=x_3$ and $y=y_4$. Let $x_3y_4\in E(C')$, and w.l.g. we may assume that $C'=x_3y_4x_5y_5x_6y_6x_3$. In this case, we have $C_{10}=x_1y_1x_2y_2x_3y_6x_6y_5x_5y_4x_1\subseteq G$, a contradiction. So let $x_3y_4\notin E(C')$ and w.l.g. we may assume that $C'=x_3y_5x_5y_4x_6y_6x_3$. Consider the following figure:

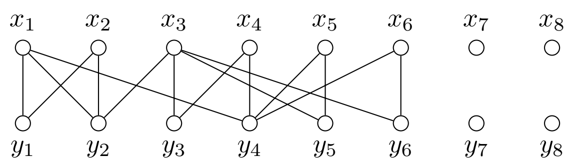

Fig. 4: $x=x_3, y=y_4$ and $x_3y_4\notin E(C')$

By Figure 4, it is easy to check that $x_iy_j\in E(\overline{G})$, where $i\in\{5,6\}$, $j\in\{1,2,3\}$. Similarly, $x_jy_i\in E(\overline{G})$, where $i\in\{5,6\}$, $j\in\{1,2,4\}$, and $x_2y_3, x_4y_2\in E(\overline{G})$. Hence we have $C_8\subseteq \overline{G}[\{x_2,x_4,x_5,x_6\},\{y_1,y_2,y_3,y_5\}]$, a contradiction too. So let $xy\in E(C)$, that is $xy\in\{x_3y_3,y_3x_4,x_4y_4\}$. In this case, the proof is the same as the case that $x=x_3$ and $y=y_4$, where $x_3y_4\in E(C')$ and we get a contradiction again.

Case 4. $(|X_1'|,|Y_1'|)=(2,0)$.
W.l.g., let $V(C')=\{x_3,x_4,x_5,y_5,y_6,y_7\}$ and $C'=x_3y_5x_4y_6x_5y_7x_3$. In this case, we have $C_{10}=x_1y_1x_2y_2x_3y_7x_5y_6x_4y_4x_1\subseteq G$, which is a contradiction.

Case 5. $(|X_1'|,|Y_1'|)=(2,1)$.
W.l.g., we may assume that $V(C')=\{x_3,x_4,x_5,y,y_5,y_6\}$, where $y\in\{y_3,y_4\}$. If $y=y_3$ or $y=y_4$ and $x_3y_4\in E(C')$, by considering the edges of C and C', in any case, it is easy to check that $C_{10}\subseteq G[V(C)\cup V(C')]$. For example, assume that $y=y_3$ and $C'=x_3y_3x_4y_5x_5y_6x_3$, hence we have $C_{10}:=y_6x_5y_5x_4y_4x_1y_1x_2y_2x_3y_6\subseteq G$. For other cases, the proof is the same.

Hence, assume that $y=y_4$, and $x_3y_4\notin E(C')$. W.l.g., assume that $C'=x_4y_4x_5y_5x_3y_6x_4$. Consider the following figure:

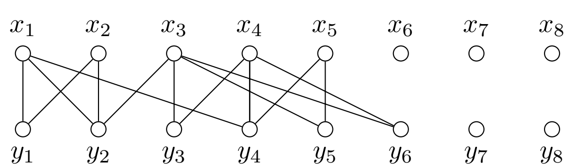

Fig. 5: $x_3,x_4\in X_1', y_4\in Y_1'$ and $x_3y_4\notin E(C')$

By Figure 5, it is easy to check that:

$$\{x_1y_3, x_1y_5, x_1y_6, x_2y_3, x_2y_5, x_2y_6, x_4y_5, x_5y_2, x_5y_3, x_5y_6\} \subseteq E(\overline{G})$$

Otherwise, $C_{10} \subseteq G$. For example by contrary assume that $x_1y_3 \in E(G)$, it can be seen that $C_{10} := x_1y_1x_2y_2x_3y_5x_5y_4x_4y_3x_1 \subseteq G$. For other cases, the proof is the same. Now, consider x_4y_2. If $x_4y_2 \in E(\overline{G})$, then we have $C_8 = x_1y_3x_5y_2x_4y_5x_2y_6x_1 \subseteq \overline{G}$, a contradiction. Hence assume that $x_4y_2 \in E(G)$, therefore $C_{10} = x_1y_1x_2y_2x_4y_6x_3y_5x_5y_4x_1 \subseteq G$, which is a contradiction again.

Case 6. $(|X_1'|, |Y_1'|) = (2, 2)$.
W.l.g., we may assume that $V(C') = \{x_3, x_4, x_5, y_3, y_4, y_5\}$. If $x_5y_5 \notin E(C')$, we have $x_5y_j, y_5x_j \in E(C')$ for $j = 3, 4$. Thus by Lemma 2.1, we have $C_{10} \subseteq G$, which is a contradiction. Now let $x_5y_5 \in E(C')$. If $x_4y_5 \in E(C')$, then by Lemma 2.1, the proof is the same. So let $x_4y_5 \notin E(C')$. Therefore, we have $x_3y_5 \in E(C')$. If $x_5y_3 \in E(C')$, the proof is the same. Hence $C' = x_3y_3x_4y_4x_5y_5x_3 \subseteq G$. Since $C_{10} \nsubseteq G$, and by Figure 6, it is easy to check that:

$$\{x_1y_3, x_1y_5, x_2y_3, x_2y_5, x_4y_1, x_4y_2, x_4y_5, x_5y_1, x_5y_2, x_5y_3\} \subseteq E(\overline{G}).$$

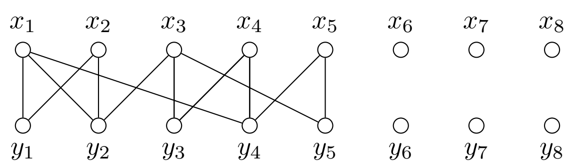

Fig. 6: $(|X_1'|, |Y_1'|) = (2, 2)$ and $x_4y_5, x_5y_3 \in E(\overline{G})$

Now we have the following claim:

Claim 4. $|N_G(x_5) \cap \{y_6, y_7, y_8\}| = 0$.
Proof: By contradiction we may assume that $|N_G(x_5) \cap \{y_6, y_7, y_8\}| \neq 0$, say $x_5y_6 \in E(G)$. In this case, we have $\{x_1, x_2, x_4\} \subseteq N_{\overline{G}}(y_6)$; otherwise, $C_{10} \subseteq G$ which is a contradiction. Now, since $\{x_1, x_2, x_4\} \subseteq N_{\overline{G}}(y_6)$, one can check that $C_8 = y_6x_4y_1x_5y_3x_1y_5x_2y_6 \subseteq \overline{G}$, a contradiction again. Hence the assumption does not hold and the claim follows. □

Therefore by Claim 4, $\{y_6, y_7, y_8\} \subseteq N_{\overline{G}}(x_5)$, and we have the following claim:

Claim 5. $\{x_1, x_2\} \subseteq N_G(y_i)$ for $i = 6, 7, 8$.
Proof: By contradiction, let there is at least one edge between $\{x_1, x_2\}$ and $\{y_6, y_7, y_8\}$ in $\overline{G}$, say $x_1y_6 \in E(\overline{G})$. Hence $C_8 = x_1y_3x_2y_5x_4y_1x_5y_6x_1 \subseteq \overline{G}$, a contradiction.
Now by Lemma 2.1 and Claim 4, for each $i = 6, 7, 8$, $x_4y_i \in E(\overline{G})$; otherwise, $C_{10} \subseteq G$, a contradiction. If there exists a vertex x of $\{x_6, x_7, x_8\}$, such that $|N_{\overline{G}}(x) \cap \{y_6, y_7, y_8\}| \geq 2$, then we have $C_8 \subseteq \overline{G}$,

a contradiction as well. Hence, for each $i = 6, 7, 8$, we have $|N_G(x) \cap \{y_6, y_7, y_8\}| \geq 2$. Assume that $x_6y_6, x_6y_7 \in E(G)$, therefore it is easy to check that $C_{10} \subseteq G[X_1 \cup \{x_6\}, Y_1 \setminus \{y_1\} \cup \{y_6, y_7\}]$, a contradiction again. □

Now by cases $1, 2, \ldots, 6$, the proof is complete and the theorem holds. □

In the following theorem, we determine the exact value of the bipartite Ramsey number $BR(C_8, C_{2n})$ for $n \geq 5$.

Theorem 3.6. $BR(C_8, C_{2n}) = n + 3$ *for each* $n \geq 5$.

Proof: The lower bound holds by Theorem 3.1. We use induction to prove the upper bound. For the base step of the induction, the theorem holds by Theorem 3.5. Suppose that $n \geq 6$ and $BR(C_8, C_{2n'}) = n' + 3$ for each $n' < n$. We will show that $BR(C_8, C_{2n}) \leq n + 3$. To complete the proof, by contrary suppose that $BR(C_8, C_{2n}) > n + 3$, that is there exists a subgraph G of $K_{t,t}$, such that neither $C_{2n} \subseteq G$ nor $C_8 \subseteq \overline{G}$, where $t = n + 3$. Since $BR(C_8, C_{2(n-1)}) = n + 2$ and $C_8 \nsubseteq \overline{G}$, G has a subgraph $C \cong C_{2(n-1)}$. Let (X, Y) be a bipartition of $K_{t,t}$, where $X = \{x_1, x_2, \ldots, x_t\}$ and $Y = \{y_1, y_2, \ldots, y_t\}$. Set $X_1 = V(C) \cap X$ and $Y_1 = V(C) \cap Y$. W.l.g., we may assume that $X_1 = \{x_1, x_2, \ldots x_{n-2}, x_{n-1}\}$, $Y_1 = \{y_1, y_2, \ldots, y_{n-2}, y_{n-1}\}$, and $C = x_1y_1x_2y_2 \ldots x_{n-2}y_{n-2}x_{n-1}y_{n-1}x_1$. Since $C_8 \nsubseteq \overline{G}$, there is at least one edge other than $E(C)$ between X_1 and Y_1 in G. Now we have the following cases.

Case 1. There exists $x_iy_k \in E(G)$ for some $i \in \{1, 2, \ldots, n-1\}$, where $k - i = 1(\mod n-1)$ or $i - k = 2(\mod n-1)$.
W.l.g., assume that $x_1y_2 \in E(G)$. For other cases, the proof is the same. Set $X_2 = \{x_{n-2}, x_{n-1}, \ldots, x_{n+3}\}$, $Y_2 = \{y_{n-2}, y_{n-1}, \ldots, y_{n+3}\}$, $X'' = \{x_1, x_2, x_{n-2}, x_{n-1}\}$, and $Y'' = \{y_1, y_2, y_{n-2}, y_{n-1}\}$. Since $BR(C_6, C_8) = 6$, $|X_2| = |Y_2| = 6$ and $C_8 \nsubseteq \overline{G}$, $G[X_2, Y_2]$ has a subgraph $C' \cong C_6$. Let $X'_1 = V(C') \cap X_1$ and $Y'_1 = V(C') \cap Y_1$. Now consider the $(|X'_1|, |Y'_1|)$, we note that $(|X'_1|, |Y'_1|) = (|Y'_1|, |X'_1|)$, hence we have $(|X'_1|, |Y'_1|) \in \{(0,0), (1,0), (2,0), (1,1), (2,1), (2,2)\}$. Now we have the following subcases:

Subcase 1-1. $(|X'_1|, |Y'_1|) = (0, 0)$.
W.l.g. assume that $V(C') = \{x_n, x_{n+1}, x_{n+2}, y_n, y_{n+1}, y_{n+2}\}$ and $C' = x_ny_nx_{n+1}y_{n+1}x_{n+2}y_{n+2}x_n$. Set $X'_2 = \{x_n, x_{n+1}, x_{n+2}\}$, $Y'_2 = \{y_n, y_{n+1}, y_{n+2}\}$. Since $C_{2n} \nsubseteq G$, one can check that either $K_{3,4} \cong [X'_2, Y''] \subseteq \overline{G}$ or $K_{4,3} \cong [X'', Y'_2] \subseteq \overline{G}$. Otherwise, let's assume the contrary: there is at least one edge between the sets X'_2 and Y'' within the graph G, and there exists at least one edge between X'' and Y'_2 in G. Without loss of generality and due to symmetry, let's assume that $x_ny_1 \in E(G)$. Now, let's assume that $|N_G(y_n) \cap X_1| \neq 0$. The proof remains the same for other cases. As $x_ny_n \in E(G)$ and $x_ny_1 \in E(G)$, if either x_1y_n or $x_2y_n \in E(G)$, then, according to Lemma 2.1 it can be concluded that $C_{2n} \subseteq G$. Therefore, let's suppose that either $x_{n-2}y_5n \in E(G)$ or $x_{-1}ny_n \in E(G)$. Assuming $x_{n-2}y_n \in E(G)$, it follows that $C = x_ny_1x_2y_2x_3y_3 \ldots x_{n-3}y_{n-3}x_{n-2}y_nx_{n+1}y_{n+1}x_{n+2}y_{n+2}x_n$ forms a copy of C_{2n} in G. The proof remains the same in the scenario where $x_{n-1}y_n \in E(G)$.
Now w.l.g. let $K_{3,4} \cong [X'_2, Y''\}] \subseteq \overline{G}$. So for each $x \in X \setminus X'_2$, we have $|N_{\overline{G}}(x) \cap Y''| \leq 1$; otherwise, $C_8 \subseteq \overline{G}$, which is a contradiction. That is, $|N_G(x) \cap Y''| \geq 3$ for each $x \in X''$ and $|N_G(x_{n+3}) \cap Y''| \geq 3$. Thus, one can assume that $|N_G(y) \cap X''| \leq 1$ for each $y \in Y \setminus Y_1$. If this were not the case, assume by contradiction that $|N_G(y) \cap X''| \geq 2$ for at least one $y \in Y \setminus Y_1$. Considering the situation, where $|N_G(x) \cap Y''| \geq 3$ holds true for every $x \in X''$ and $|N_G(x_{n+3}) \cap Y''| \geq 3$, it becomes evident that

there is an $i \in \{1, 2, n-2, n-1\}$, such that $x_i y_i x_{i+1} y_{i+1}$ forms a part of a copy of C_{2n-2} within $G[X_1, Y_1]$. In this configuration, $x_i, x_{i+1} \in N_G(y)$ and $y_i, y_{i+1} \in N_G(x_{n+3})$. By utilizing Lemma 2.1, it is concluded that $C_{2n} \subseteq G$, leading to a contradiction. If there exist $y, y' \in \{y_n, y_{n+1}, y_{n+2}\}$, such that $|N_G(y) \cap X''| = |N_G(y') \cap X''| = 1$ and $N_G(y) \cap X'' \neq N_G(y') \cap X''$, then one can check that $C_{2n} \subseteq G$, a contradiction too. W.l.g., we may assume that $\{x_1, x_2, x_{n-2}\} \subseteq N_{\overline{G}}(y)$ for each $y \in \{y_n, y_{n+1}, y_{n+2}\}$. If there are at least two vertices of $\{y_n, y_{n+1}, y_{n+2}\}$, say y_n, y_{n+1}, such that $|N_{\overline{G}}(y_i) \cap X''| = 4$ for $i = n, n+1$, then $C_8 \subseteq \overline{G}[X'', Y \setminus Y_1]$. In other words, there are at least two vertices of $\{y_n, y_{n+1}, y_{n+2}\}$, say y_n, y_{n+1}, such that $N_G(y_i) \cap X'' = \{x_{n-1}\}$. Therefore $x_{n+3} y_i \in E(\overline{G})$ for $i = n, n+1$; otherwise, $C_{2n} \subseteq G$, a contradiction again. Now one can check that $C_8 \subseteq \overline{G}[\{x_1, x_2, x_{n-2}, x_{n+3}\}, Y \setminus Y_1]$ and the proof is complete.

Subcase 1-2. $(|X_1'|, |Y_1'|) = (1, 0)$.
W.l.g., assume that $V(C') = \{x_{n-1}, x_n, x_{n+1}, y_n, y_{n+1}, y_{n+2}\}$ and $C' = x_{n-1} y_n x_n y_{n+1} x_{n+1} y_{n+2} x_{n-1}$. Set $X_2'' = \{x_1, x_2, x_{n-2}\}$, $X_2' = \{x_n, x_{n+1}\}$, and $Y_2' = \{y_n, y_{n+1}, y_{n+2}\}$.
Hence, one can assume that $K_{3,3} \cong [X_2'', Y_2'] \subseteq \overline{G}$. Otherewise, let $|N_G(y) \cap X_2''| \neq 0$ for at least one $y \in Y_2'$. W.l.g, assume that $x_1 y_n \in E(G)$, the proof for other cases follows similarly. In this case, we observe that the cycle $C = y_n x_1 y_2 x_3 y_3 \ldots x_{n-2} y_{n-2} x_{n-1} y_{n+2} x_{n+1} y_{n+1} x_n y_n$ forms a copy of C_{2n} within G, which is a contradiction. Applying symmetry, we can similarly deduce that $K_{2,4} \cong [X_2', Y''] \subseteq \overline{G}$.
Now consider the vertices $\{x_{n+2}, x_{n+3}\}$. One can check that $|N_G(x) \cap Y''| \geq 2$ for at least one $x \in \{x_{n+2}, x_{n+3}\}$; otherwise, we have $C_8 \subseteq \overline{G}[\{x_n, x_{n+1}, x_{n+2}, x_{n+3}\}, Y'']$. W.l.g., we may assume that $|N_G(x_{n+2}) \cap Y''| \geq 2$, hence we have $|N_G(x_{n+2}) \cap Y_2'| = 0$, if not, $C_{2n} \subseteq G$. Therefore $K_{4,3} \cong [X_2'' \cup \{x_{n+2}\}, Y_2'] \subseteq \overline{G}$, and so $|N_{\overline{G}}(y_{n+3}) \cap (X_2'' \cup \{x_{n+2}\})| \leq 1$; otherwise, $C_8 \subseteq \overline{G}[X_2'' \cup \{x_{n+2}\}, Y_2' \cup \{y_{n+3}\}]$. Since $|N_G(x_{n+2}) \cap Y''| \geq 2$ and $|N_G(y_{n+3}) \cap X_2''| \geq 2$, if $x_{n+2} y_{n+3} \in E(G)$ then by Lemma 2.1, we have $C_{2n} \subseteq G$. Hence $x_{n+2} y_{n+3} \in E(\overline{G})$, that is $X_2'' \subseteq N_G(y_{n+3})$. Thus $|N_G(x_{n+2}) \cap Y''| = 2$ and $N_G(x_{n+2}) \cap Y'' = \{y_{n-2}, y_{n-1}\}$; otherwise, by Lemma 2.1, we have $C_{2n} \subseteq G$ and the proof is complete. Similarly $|N_G(x_{n+3}) \cap Y''| \leq 2$, if not, by Lemma 2.1, $C_{2n} \subseteq G[X_1 \cup \{x_{n+3}\}, Y_1 \cup \{y_{n+3}\}]$. If $|N_G(x_{n+3}) \cap Y''| \leq 1$, we have $C_8 \subseteq \overline{G}[X \setminus X_1, Y'']$. So $|N_G(x_{n+3}) \cap Y''| = 2$ and $x_{n+3} y_{n+3} \notin E(G)$; otherwise, by Lemma 2.1, it is easy to check that $C_{2n} \subseteq G[X_1 \cup \{x_{n+3}\}, Y_1 \cup \{y_{n+3}\}]$. Thus we have $Y \setminus Y_1 \subseteq N_{\overline{G}}(x_i)$ for $i = n+2, n+3$. That is, we have $C_8 \subseteq \overline{G}[\{x_1, x_2, x_{n+2}, x_{n+3}\}, Y \setminus Y_1]$, and the proof is complete.

Subcase 1-3. $(|X_1'|, |Y_1'|) = (1, 1)$.
W.l.g., assume that $V(C') = \{x, x_n, x_{n+1}, y, y_n, y_{n+1}\}$, where $x \in \{x_{n-2}, x_{n-1}\}$ and $y \in \{y_{n-2}, y_{n-1}\}$. If $xy \notin E(C)$, then $x = x_{n-2}$ and $y = y_{n-1}$. Let $x_{n-2} y_{n-1} \in E(C')$ and assume that $C' = x_{n-2} y_{n-1} x_n y_n x_{n+1} y_{n+1} x_{n-2}$. In this case, we have $C_{2n} = x_1 y_1 x_2 y_2 \ldots x_{n-2} y_{n+1} x_{n+1} y_n x_n y_{n-1} x_1 \subseteq G$, a contradiction. Assume that $x_{n-2} y_{n-1} \notin E(C')$ and w.l.g. let $C' = x_{n-2} y_n x_n y_{n-1} x_{n+1} y_{n+1} x_{n-2}$. By considering Figure 7, it is easy to check that:

$$\{x_2 y_{n-2}, x_2 y_n, x_{n-1} y_n, x_{n-1} y_{n+1}, x_n y_{n-3}, x_n y_{n+1}, x_{n+1} y_{n-2}, x_{n+1} y_{n-1}\} \subseteq E(\overline{G})$$

Therefore one can check that $C_8 = x_2 y_n x_{n-1} y_{n+1} x_n y_{n-3} x_{n+1} y_{n-2} x_2 \subseteq \overline{G}$, a contradiction again. So, assume that $xy \in \{x_{n-2} y_{n-2}, x_{n-1} y_{n-2}, x_{n-1} y_{n-1}\}$, that is $xy \in E(C)$. Hence the proof is the same as the case that $x = x_{n-1}, y = y_{n-1}$, where $xy \in E(C')$, now we can check that $C_{2n} \subseteq G$, which is a contradiction.

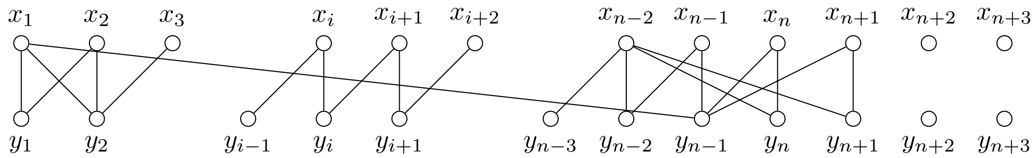

Fig. 7: $(|X'_1|, |Y'_1|) = (1, 1)$ and $x_{n-2}y_{n-1} \notin E(C')$

Subcase 1-4. $(|X'_1|, |Y'_1|) = (2, 0)$.
Assume that $V(C') = \{x_{n-2}, x_{n-1}, x_n, y_n, y_{n+1}, y_{n+2}\}$ and $C' = x_{n-2}y_nx_{n-1}y_{n+1}x_ny_{n+2}x_{n-2}$. In this case, we have:

$$C_{2n} = x_1y_1 \dots y_{n-3}x_{n-2}y_{n+2}x_ny_{n+1}x_{n-1}y_{n-1}x_1 \subseteq G$$

Which is a contradiction.

Subcase 1-5. $(|X'_1|, |Y'_1|) = (2, 1)$.
W.l.g., we may assume that $V(C') = \{x_{n-2}, x_{n-1}, x_n, y, y_n, y_{n+1}\}$, where $y \in \{y_{n-2}, y_{n-1}\}$. In this case, by considering the edges of C' and using Lemma 2.1, it is easy to check that in any case $C_{2n} \subseteq G$, unless for the case that $y = y_{n-1}$ and $x_{n-2}y_{n-1} \in E(\overline{G})$. W.l.g., we may assume that $C' = x_{n-1}y_{n-1}x_ny_nx_{n-2}y_{n+1}x_{n-1}$. Consider the Figure 8.

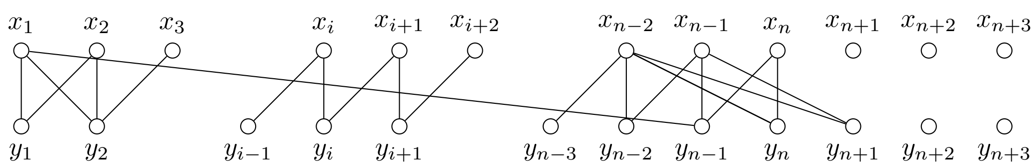

Fig. 8: $(|X'_1|, |Y'_1|) = (2, 1)$, $y = y_{n-1}$ and $x_{n-2}y_{n-1} \in E(\overline{G})$

By Figure 8, it is easy to check that:

$$\{x_1y_{n-2}, x_1y_n, x_1y_{n+1}, x_2y_{n-2}, x_2y_n, x_2y_{n+1}, x_{n-1}y_n, x_ny_{n-3}, x_ny_{n-2}, x_ny_{n+1}\} \subseteq E(\overline{G})$$

Otherwise, we have $C_{2n} \subseteq G$. Now consider $x_{n-1}y_{n-3}$. If $x_{n-1}y_{n-3} \in E(\overline{G})$, we have $C_8 = x_1y_{n-2}x_ny_{n-3}x_{n-1}y_nx_2y_{n+1}x_1 \subseteq \overline{G}$, a contradiction. Hence assume that $x_{n-1}y_{n-3} \in E(G)$, therefore we have $C_{2n} = x_1y_{n-1}x_ny_nx_{n-2}y_{n-2}x_{n-1}y_{n-3}x_{n-3}y_{n-4} \dots x_3y_2x_2y_1x_1 \subseteq G$, which is a contradiction again.

Subcase 1-6. $(|X'_1|, |Y'_1|) = (2, 2)$.
W.l.g., we may assume that $V(C') = \{x_{n-2}, x_{n-1}, x_n, y_{n-2}, y_{n-1}, y_n\}$. If $x_ny_n \notin E(C')$, then $x_ny_j, y_nx_j \in E(C')$ for $j \in \{n-2, n-1\}$ and by Lemma 2.1, we have $C_{2n} \subseteq G$. Hence assume that $x_ny_n \in E(C')$. Hence, it can be concluded that $|N_G(x_n) \cap \{y_{n-2}, y_{n-1}\}| \geq 1$ and $|N_G(y_n) \cap$

$\{x_{n-2}, x_{n-1}\}| \geq 1$. Considering that $x_{n-2}y_{n-2}, y_{n-2}x_{n-1}, x_{n-1}y_{n-1} \in E(C)$, if we have either $x_{n-1}y_n \in E(C')$ or $x_ny_{n-2} \in E(C')$, then the proof follows the same logic, as indicated by Lemma 2.1. Therefore, it can be say that $C' = x_{n-2}y_{n-2}x_{n-1}y_{n-1}x_ny_nx_{n-2} \subseteq G$. Since $C_{2n} \nsubseteq G$, by Figure 9, one can check that:

$$\{x_1y_{n-2}, x_1y_n, x_2y_{n-2}, x_2y_n, x_{n-1}y_{n-3}, x_{n-1}y_n, x_ny_{n-3}, x_ny_{n-2}\} \subseteq E(\overline{G})$$

Now we have the following claim:

Claim 6. $|N_G(x_n) \cap \{y_{n+1}, y_{n+2}, y_{n+3}\}| = 0$.
Proof: By contradiction assume that $|N_G(x_n) \cap \{y_{n+1}, y_{n+2}, y_{n+3}\}| \neq 0$, say $x_ny_{n+1} \in E(G)$. Hence $\{x_1, x_{n-1}\} \subseteq N_{\overline{G}}(y_{n+1})$; otherwise, $C_{2n} \subseteq G[X_1 \cup \{x_n\}, Y_1 \cup \{y_{n+1}\}]$ which is a contradiction. Since $\{x_1, x_{n-1}\} \subseteq N_{\overline{G}}(y_{n+1})$, we have $C_8 = x_1y_nx_2y_{n-2}x_ny_{n-3}x_{n-1}y_{n+1}x_1 \subseteq \overline{G}$, a contradiction again. So the assumption does not hold and the claim holds. □

Therefore by Claim 6, $\{y_{n+1}, y_{n+2}, y_{n+3}\} \subseteq N_{\overline{G}}(x_n)$. Now we have the following claim:

Claim 7. $\{x_1, x_2\} \subseteq N_G(y_i)$ for $i \in \{n+1, n+2, n+3\}$.
Proof: By contradiction we may assume that $x_1y_{n+1} \in E(\overline{G})$ (for other cases, the proof is identical). Therefore $C_8 = y_{n+1}x_1y_{n-2}x_2y_nx_{n-1}y_{n-3}x_ny_{n+1} \subseteq \overline{G}$, a contradiction. □

Now by Lemma 2.1 and Claim 7, for $i \in \{n+1, n+2, n+3\}$ we have $\{x_{n-1}y_i, x_{n-1}y_1, x_ny_1\} \subseteq E(\overline{G})$; otherwise, $C_{2n} \subseteq G$, a contradiction. If there exists a vertex x of $\{x_{n+1}, x_{n+2}, x_{n+3}\}$, such that $|N_{\overline{G}}(x) \cap \{y_{n+1}, y_{n+2}, y_{n+3}\}| \geq 2$, then $C_8 \subseteq \overline{G}$, a contradiction too. Hence, for each $i \in \{n+1, n+2, n+3\}$, we have $|N_G(x) \cap \{y_{n+1}, y_{n+2}, y_{n+3}\}| \geq 2$. W.l.g., we may assume that $x_{n+1}y_n, x_{n+1}y_{n+1} \in E(G)$ and thus one can check that $C_{2n} \subseteq G[X_1 \cup \{x_{n+1}\}, (Y_1 \setminus \{y_1\}) \cup \{y_n, y_{n+1}\}]$, which is a contradiction.

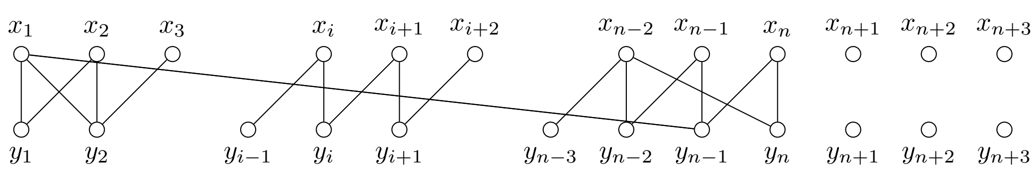

Fig. 9: $(|X'_1|, |Y'_1|) = (2, 2)$, $x_ny_n \in E(C')$

Case 2. For any $i \in \{1, 2, \ldots, n-1\}$, each $x_iy_k \in E(\overline{G})$ if $k - i = 1(\mod n-1)$ or $i - k = 2(\mod n-1)$.
Set $X_2 = \{x_{n-2}, x_{n-1}, \ldots, x_{n+3}\}$ and $Y_2 = \{y_{n-2}, y_{n-1}, \ldots, y_{n+3}\}$. Since $BR(C_6, C_8) = 6$, $|X_2| = |Y_2| = 6$, and $C_8 \nsubseteq \overline{G}$, $G[X_2, Y_2]$ has a subgraph $C' \cong C_6$. Let $X'_1 = V(C') \cap X_1$ and $Y'_1 = V(C') \cap Y_1$. As the same as case 1, we have the following subcases:

Subcase 2-1. $(|X'_1|, |Y'_1|) = (0, 0)$.
W.l.g., assume that $V(C') = \{x_n, x_{n+1}, x_{n+2}, y_n, y_{n+1}, y_{n+2}\}$ and $C' = x_ny_nx_{n+1}y_{n+1}x_{n+2}y_{n+2}x_n$.

Set $X_2'' = \{x_n, x_{n+1}, x_{n+2}\}$, $Y_2'' = \{y_n, y_{n+1}, y_{n+2}\}$. Since $C_8 \nsubseteq \overline{G}$, there is at least one edge between X_1 and Y_2'', or at least one edge between X_2'' and Y_1. W.l.g., assume that $x_n y_{n-1} \in E(G)$. For other cases, the proof is the same. Hence we have $K_{3,2} \cong [\{x_{n-3}, x_{n-2}, x_{n-1}\}, \{y_n, y_{n+1}\}] \subseteq \overline{G}$. Similarly we have $y_1x_{n+1}, y_1x_{n+2}, y_{n-2}x_{n+1}, y_{n-2}x_{n+2} \in E(\overline{G})$. Therefore $C_8 \subseteq \overline{G}$, a contradiction.

Subcase 2-2. $(|X_1'|, |Y_1'|) = (1, 0)$.
Assume that $V(C') = \{x_{n-1}, x_n, x_{n+1}, y_n, y_{n+1}, y_{n+2}\}$ and $C' = x_{n-1}y_nx_ny_{n+1}x_{n+1}y_{n+2}x_{n-1} \subseteq G$. Now one can check that $K_{2,4} \cong [\{x_n, x_{n+1}\}, \{y_1, y_{n-3}, y_{n-2}, y_{n-1}\}] \subseteq \overline{G}$ and $x_{n-3}y_n, x_{n-3}y_{n+2} \in E(\overline{G})$. Otherwise, we have $C_{2n} \subseteq G$, a contradiction. For example, by contrary assume that $x_ny_1 \in E(G)$, so it can be seen that $C_{2n} := x_ny_1x_2y_2x_2 \dots x_{n-2}y_{n-2}x_{n-1}y_{n+2}x_{n+1}y_{n+1}x_n \subseteq G$. For other cases, the proof is the same.
Since $x_{n-4}y_{n-3}, x_{n-3}y_{n-2}, x_{n-2}y_{n-1} \in E(\overline{G})$, if $x_{n-2}y_n$ or $x_{n-2}y_{n+2}$ belong to $E(\overline{G})$, then we have $C_8 \subseteq \overline{G}$. Hence assume that $x_{n-2}y_n, x_{n-2}y_{n+2} \in E(G)$ and thus $x_{n-4}y_n \in E(\overline{G})$, if not, $C_{2n} \subseteq G$. So $C_8 = y_1x_{n+1}y_{n-2}x_{n-3}y_nx_{n-4}y_{n-3}x_ny_1 \subseteq \overline{G}$, a contradiction too.

Subcase 2-3. $(|X_1'|, |Y_1'|) = (2, 0)$.
Let us make the assumption, without loss of generality, that $V(C') = \{x_{n-2}, x_{n-1}, x_n, y_n, y_{n+1}, y_{n+2}\}$. Furthermore, leveraging of symmetry and without loss of generality, we can assume that C' has a form like $C' = x_{n-2}y_nx_{n-1}y_{n+1}x_ny_{n+2}x_{n-2}$. In this scenario, the following contradiction arises:

$$C_{2n} = x_1y_1x_2y_2 \dots x_{n-3}y_{n-3}x_{n-2}y_{n+2}x_ny_{n+1}x_{n-1}y_{n-1}x_1 \subseteq G$$

Subcase 2-4. $(|X_1'|, |Y_1'|) = (1, 1)$.
W.l.g., assume that $V(C') = \{x, x_n, x_{n+1}, y, y_n, y_{n+1}\}$, where $x \in \{x_{n-2}, x_{n-1}\}$ and $y \in \{y_{n-2}, y_{n-1}\}$. If $xy \notin E(C)$, then $x = x_{n-2}$ and $y = y_{n-1}$. Let $x_{n-2}y_{n-1} \in E(C')$ and assume that $C' = x_{n-2}y_{n-1}x_ny_nx_{n+1}y_{n+1}x_{n-2}$. In this case, we have $C_{2n} = x_1y_1x_2y_2 \dots x_{n-2}y_{n+1}x_{n+1}y_nx_ny_{n-1}x_1 \subseteq G$, a contradiction. Assume that $x_{n-2}y_{n-1} \notin E(C')$ and w.l.g. let $C' = x_{n-2}y_nx_ny_{n-1}x_{n+1}y_{n+1}x_{n-2}$. Since $C_{2n} \nsubseteq G$, it is easy to check that:

$$\{x_1y_n, x_1y_{n+1}, x_{n-1}y_n, x_{n-1}y_{n+1}, x_ny_{n-2}, x_ny_{n-3}, x_{n+1}y_{n-2}, x_{n+1}y_n, x_ny_{n+1}\} \subseteq E(\overline{G})$$

For example, by contrary assume that $x_1y_n \in E(G)$, hence it can be checked that G has at least one copy of C_{2n}, namely $C_{2n} := x_1y_1x_2y_2 \dots x_{n-3}y_{n-3}x_{n-2}y_{n+1}x_{n+1}y_{n-1}x_ny_nx_1$. For other cases, the proof is the same.
Since $x_{n-1}y_{n-3} \in E(\overline{G})$, we have $C_8 = x_1y_{n+1}x_{n-1}y_{n-3}x_ny_{n-2}x_{n+1}y_nx_1 \subseteq \overline{G}$, a contradiction again. Therefore, assume that $xy \in \{x_{n-2}y_{n-2}, x_{n-1}y_{n-2}, x_{n-1}y_{n-1}\}$, that is $xy \in E(C)$. Hence the proof is the same as the case that $x = x_{n-1}, y = y_{n-1}$ where $xy \in E(C')$,that is we can check that $C_{2n} \subseteq G$, a contradiction.

Subcase 2-5. $(|X_1'|, |Y_1'|) = (2, 1)$.
W.l.g., we may assume that $V(C') = \{x_{n-2}, x_{n-1}, x_n, y, y_n, y_{n+1}\}$ where $y \in \{y_{n-2}, y_{n-1}\}$. In this case by considering the edges of C' and by Lemma 2.1, it is easy to check that $C_{2n} \subseteq G$ for any case, unless for the case that $y = y_{n-1}$ and $x_{n-2}y_{n-1} \in E(\overline{G})$. W.l.g., we may assume that $C' = x_{n-1}y_{n-1}x_ny_nx_{n-2}y_{n+1}x_{n-1}$. Now it is easy to check that:

$$\{x_1y_{n-2}, x_1y_n, x_1y_{n+1}, x_2y_n, x_2y_{n+1}, x_{n-1}y_n, x_ny_{n-3}, x_ny_{n-2}, x_ny_{n+1}\} \subseteq E(\overline{G})$$

Otherwise, $C_{2n} \subseteq G$. For example, by contrary assume that $x_1y_{n-2} \in E(G)$, hence it can be checked that G has at least one copy of C_{2n}, namely $C_{2n} := x_1y_1x_2y_2 \dots x_{n-3}y_{n-3}x_{n-2}y_nx_ny_{n-1}x_{n-1}y_{n-2}x_1$. For other cases, the proof is the same.
Since $x_{n-1}y_{n-3} \in E(\overline{G})$, we have $C_8 = x_1y_{n-2}x_ny_{n-3}x_{n-1}y_nx_2y_{n+1}x_1 \subseteq \overline{G}$, a contradiction again.

Subcase 2-6. $(|X_1'|, |Y_1'|) = (2, 2)$.
W.l.g., we may assume that $V(C') = \{x_{n-2}, x_{n-1}, x_n, y_{n-2}, y_{n-1}, y_n\}$. If $x_ny_n \notin E(C')$, then $x_ny_j, y_nx_j \in E(C')$ for $j \in \{n-2, n-1\}$, and by Lemma 2.1 we have $C_{2n} \subseteq G$. Hence assume that $x_ny_n \in E(C')$. Now, one can say that $|N_G(x_n) \cap \{y_{n-2}, y_{n-1}\}| \geq 1$ and $|N_G(y_n) \cap \{x_{n-2}, x_{n-1}\}| \geq 1$. Now, since $x_{n-2}y_{n-2}, y_{n-2}x_{n-1}, x_{n-1}y_{n-1} \in E(C)$, if $x_{n-1}y_n \in E(C')$ or $x_ny_{n-2} \in E(C')$, then by Lemma 2.1, the proof is complete. Therefore, one can assume that $C' = x_{n-2}y_{n-2}x_{n-1}y_{n-1}x_ny_nx_{n-2} \subseteq G$. Since $C_{2n} \nsubseteq G$, one can check that:

$$\{x_1y_{n-2}, x_1y_n, x_{n-1}y_{n-3}, x_{n-1}y_n, x_ny_{n-3}, x_ny_{n-2}\} \subseteq E(\overline{G})$$

Set $X_3 = \{x_{n+1}, x_{n+2}, x_{n+3}\}$, $Y_3 = \{y_{n+1}, y_{n+2}, y_{n+3}\}$. There exists at least one vertex of X_3 or Y_3, say y_{n+1}, such that $|N_{\overline{G}}(y_{n+1}) \cap X_3| \geq 2$; otherwise, we have $C_6 \subseteq G[X_3, Y_3]$ and the proof is the same as subcase 2-1. Assume that $x_{n+1}y_{n+1}, x_{n+2}y_{n+1} \in E(\overline{G})$. Therefore, for each $x \in \{x_{n+1}, x_{n+2}\}$, we have $|N_{\overline{G}}(x) \cap \{y_{n-3}, y_{n-2}, y_n\}| \geq 2$; otherwise, $C_{2n} \subseteq G$, a contradiction. If $|N_{\overline{G}}(x) \cap \{y_{n-3}, y_{n-2}, y_n\}| \geq 2$ for some $x \in \{x_{n+1}, x_{n+2}\}$ or $N_{\overline{G}}(x_{n+1}) \cap \{y_{n-3}, y_{n-2}, y_n\} \neq N_{\overline{G}}(x_{n+2}) \cap \{y_{n-3}, y_{n-2}, y_n\}$, then we have $C_8 \subseteq \overline{G}$, a contradiction. That is, we have $|N_{\overline{G}}(x) \cap \{y_{n-3}, y_{n-2}, y_n\}| = 1$ for each $x \in \{x_{n+1}, x_{n+2}\}$ and one can check that $N_{\overline{G}}(x) \cap \{y_{n-3}, y_{n-2}, y_n\} = \{y_n\}$. Therefore we have $x_1y_{n+1}, x_{n-1}y_{n+1} \in E(G)$, if not, $C_{2n} \subseteq G$, which is a contradiction. Now, we have $C_n = x_1y_1x_2y_2 \dots x_{n-}y_{n-3}x_{n-2}y_nx_ny_{n-1}x_{n-1}y_{n+1}x_1 \subseteq G$, a contradiction again.
Hence by Cases $1, 2$, the proof is complete and the theorem holds. □

Therefore by Lemmas 3.2 and 3.3 and by Theorems 3.1, 3.4, 3.5 and 3.6, Theorem 1.1 holds.

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