On a Classification of Smooth Fano Polytopes

Benjamin Assarf Michael Joswig Andreas Paffenholz[‡]

TU Darmstadt, Germany

Abstract. The d-dimensional simplicial, terminal, and reflexive polytopes with at least $3d-2$ vertices are classified. In particular, it turns out that all of them are smooth Fano polytopes. This improves previous results of Casagrande (2006) and Øbro (2008). Smooth Fano polytopes play a role in algebraic geometry and mathematical physics. This text is an extended abstract of Assarf et al. (2012).

Résumé. Nous classifions les polytopes simpliciaux, terminaux et réflexifs de dimension d avec au moins $3d-2$ sommets. En particulier, tous ces polytopes se trouvent être des polytopes de Fano lisses. Nous améliorons des résultats antérieurs de Casagrande (2006) et d'Øbro (2008). Les polytopes de Fano lisses apparaissent en géométrie algébrique et en physique mathématique. Ce texte est un résumé étendu de Assarf et al. (2012).

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1 Introduction

A *lattice polytope* P is a convex polytope whose vertices lie in a lattice N contained in the vector space $\mathbb{R}^d$. Fixing a basis of N describes an isomorphism to $\mathbb{Z}^d$. Throughout this paper, we restrict our attention to the standard lattice $N = \mathbb{Z}^d$. A d-dimensional lattice polytope $P \subset \mathbb{R}^d$ is called *reflexive* if it contains the origin **0** as an interior point and its polar polytope is a lattice polytope in the dual lattice $M := \mathrm{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^d$. A lattice polytope P is *terminal* if **0** and the vertices are the only lattice points in $P \cap \mathbb{Z}^d$. It is *simplicial* if each face is a simplex. We say that P is a *smooth Fano polytope* if $P \subseteq \mathbb{R}^d$ is simplicial with **0** in the interior and the vertices of each facet form a lattice basis of $\mathbb{Z}^d$.

In algebraic geometry, reflexive polytopes correspond to *Gorenstein toric Fano varieties*. The toric variety X_P of a polytope P is determined by the face fan of P, that is, the fan spanned by all faces of P; see (Ewald, 1996) or (Cox et al., 2011) for details. The toric variety X_P is $\mathbb{Q}$-*factorial* (some multiple of a Weil divisor is Cartier) if and only if the polytope P is simplicial. In this case the *Picard number* of X equals $n-d$, where n is the number of vertices of P. The polytope P is smooth if and only if the variety X_P is a manifold (that is, it has no singularities). Note that the notions detailed above are not entirely standardized in the literature. For example, our definitions agree with (Nill, 2005), but disagree with (Kreuzer and Nill, 2009).

Our main result is a classification of those simplicial, terminal, and reflexive lattice polytopes with at least $3d-2$ vertices. We show that such a polytope is lattice equivalent to a direct sum of del Pezzo polytopes,

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[‡]Email: `{assarf,joswig,paffenholz}@mathematik.tu-darmstadt.de`

pseudo del Pezzo polytopes, or a (possibly skew) bipyramid over (pseudo) del Pezzo polytopes. In particular, a simplicial, terminal, and reflexive polytope with at least $3d-2$ vertices is necessarily smooth Fano. The precise statement can be found in Theorem 2 below.

This extends results of Casagrande who proved that the number of a d-dimensional simplicial, terminal, and reflexive lattice polytopes does not exceed $3d$; she also showed that, up to lattice equivalence, only one type exists which attains this bound (and the dimension d is even) (Casagrande, 2006). Moreover, our result also extends Øbro's classification of all polytopes of the named kind with $3d-1$ vertices (Øbro, 2008). Our proof employs techniques similar to those used by (Øbro, 2008) and (Nill and Øbro, 2010), but requires more organization since a greater variety of possibilities occurs. Translated into the language of toric varieties our main result establishes that any d-dimensional terminal $\mathbb{Q}$-factorial Gorenstein toric Fano variety with Picard number at least $2d-2$ decomposes as a (possibly trivial) toric fiber bundle with known fiber and base space; the precise statement is Corollary 4. As a key benefit of our systematic approach a certain general pattern emerges, and we state this as Conjecture 3 below. Like our main result this conjecture also admits a direct translation to toric varieties.

The interest in structural results of this type originates in applications of algebraic geometry to mathematical physics. For instance, (Batyrev and Borisov, 1996) use reflexive polytopes to construct pairs of mirror symmetric Calabi-Yau manifolds. Up to unimodular equivalence, there exists only a finite number of such polytopes in each dimension, and they have been classified up to dimension 4, see (Batyrev, 1991), (Kreuzer and Skarke, 1997, 2002). Smooth reflexive polytopes have been classified up to dimension 8 by (Øbro, 2007); see (Brown and Kasprzyk, 2009–2012) for data. By enhancing Øbro's implementation within the `polymake` framework (Gawrilow and Joswig, 2000) this classification was extended to dimension 9 (Lorenz and Paffenholz, 2008); from that site the data is available in `polymake` format.

In this extended abstract we will only summarize the essential ideas for the proofs. In addition, we will detail the 6-dimensional case. For full proofs we refer to the paper (Assarf et al., 2012).

2 Lattice Polytopes

A polytope $P \subset \mathbb{R}^d$ is a *lattice polytope* if its *vertex set* $\mathrm{Vert}(P)$ is contained in $\mathbb{Z}^d$ (more generally, contained in some lattice $N \subseteq \mathbb{R}^d$). See (Ewald, 1996) for background on lattice polytopes. P is called *reflexive*, if P contains the origin in its interior and its dual P^* is a lattice polytope in the dual lattice. P is *terminal* if $P \cap N = \mathrm{Vert}(P) \cup \{\mathbf{0}\}$. More generally, P is *canonical* if the origin is the only interior lattice point in P. Two lattice polytopes are *lattice equivalent* if one can be mapped to the other by a transformation in $\mathrm{GL}_d\mathbb{Z}$ followed by a lattice translation.

We start out with listing all possible types of 2-dimensional terminal and reflexive lattice polytopes in Figure 1. Up to lattice equivalence five cases occur which we denote as P_6, P_5, P_{4a}, P_{4b}, and P_3, respectively; one hexagon, one pentagon, two quadrangles, and a triangle; see (Ewald, 1996, Thm. 8.2). All of them are *smooth Fano* polytopes, that is, the origin lies in the interior and the vertex set of each facet forms a lattice basis. The only 1-dimensional reflexive polytope is the interval $[-1, 1]$.

Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be polytopes with the origin in their respective relative interiors. The polytope

$$P \oplus Q \;=\; \mathrm{conv}(P \cup Q) \quad \subset \mathbb{R}^{d+e}$$

is the *direct sum* of P and Q. This construction also goes by the name "linear join" of P and Q. Clearly, forming direct sums is commutative and associative. Notice that the polar polytope $(P \oplus Q)^* = P^* \times Q^*$ is

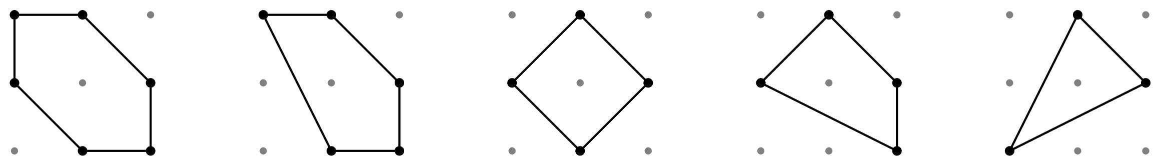

(a) P_6 (b) P_5 (c) P_{4a} (d) P_{4b} (e) P_3

Figure 1: The 2-dimensional reflexive and terminal lattice polytopes

the direct product. An important special case is the *proper bipyramid* $[-1,1] \oplus Q$ over Q. More generally, we consider the possibly *skew* bipyramids

$$\mathrm{BP}(Q, v, w) \;:=\; \mathrm{conv}((\{0\} \times Q) \cup \{w, v-w\}) \,,$$

where $v \in Q \cap \mathbb{Z}^e$ is a lattice point in Q and w is orthogonal to the affine hull of Q with $|w| = 1$. In particular, choosing $v = 0$ recovers the proper bipyramid. The relevance of these constructions for simplicial, terminal, and reflexive polytopes stems from the following lemma; see also (Ewald, 1996, §V.7.7) and Figure 2 below. The reader can find the simple proof in (Assarf et al., 2012, Lemmas 2,3,4).

Lemma 1 *Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ both be lattice polytopes. Then the direct sum $P \oplus Q \subset \mathbb{R}^{d+e}$ is simplicial, terminal, or reflexive if and only if P and Q are.*

In particular, this applies to the case that $P = [-1,1] \oplus Q$ is a proper bipyramid over a $(d-1)$-dimensional lattice polytope Q. More generally, P is a simplicial, terminal, or reflexive skew bipyramid if and only if Q has the corresponding property.

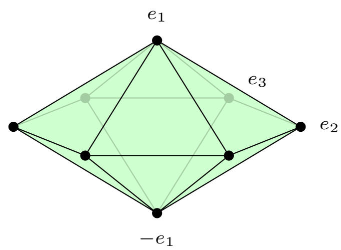
The latter case of the lemma occurs frequently in the classification. Let $e_1, e_2, \dots, e_d$ be the standard basis of $\mathbb{Z}^d$ in $\mathbb{R}^d$. Here and throughout we abbreviate $\mathbf{1} = (1, 1, \dots, 1)$. For even d the d-polytopes

$$\mathrm{DP}(d) \;=\; \mathrm{conv}\{\pm e_1, \pm e_2, \dots, \pm e_d, \pm \mathbf{1}\} \quad \subset \mathbb{R}^d$$

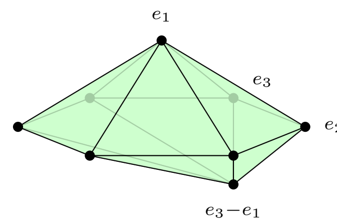
with $2d+2$ vertices form a 1-parameter family of smooth Fano polytopes; see (Ewald, 1996, §V.8.3). They are usually called *del Pezzo* polytopes. If $-\mathbf{1}$ is not a vertex the resulting polytopes are sometimes called *pseudo del Pezzo*. Notice that the 2-dimensional del Pezzo polytope $\mathrm{DP}(2)$ is lattice equivalent to the hexagon P_6 shown in Figure 1, and the 2-dimensional pseudo del Pezzo polytope is lattice equivalent to the pentagon P_5. While the definition of $\mathrm{DP}(d)$ also makes sense in odd dimensions, the polytopes obtained are not simplicial.

For *centrally symmetric* smooth Fano polytopes (Voskresenskiĭ and Klyachko, 1984) provide a classification result. They showed that every centrally symmetric smooth Fano polytope can be written as a sum of line segments and del Pezzo polytopes. This was later generalized to simplicial and reflexive pseudo-symmetric polytopes by (Ewald, 1988, 1996) in the smooth case, and by (Nill, 2006, Thm. 0.1) in the general case. A polytope is *pseudo-symmetric* if there exists a facet F, such that $-F = \{-v \mid v \in F\}$ is also a facet. They proved that any pseudo-symmetric simplicial and reflexive polytope is lattice equivalent to a direct sum of a (possibly trivial) cross polytope, del Pezzo polytopes, and pseudo del Pezzo polytopes.

A direct sum of d intervals $[-1,1] \oplus [-1,1] \oplus \cdots \oplus [-1,1]$ is the same as the regular cross polytope $\mathrm{conv}\{\pm e_1, \pm e_2, \dots, \pm e_d\}$. The direct sum of several intervals with a polytope Q is the same as an iterated proper bipyramid over Q. Casagrande showed that any simplicial and reflexive d-polytope P has at most $3d$ vertices, and if it does have exactly $3d$ vertices then d is even, and P is a centrally symmetric smooth Fano



(a) Proper bipyramid over P_6



(b) Skew bipyramid over P_6

Figure 2: The smooth Fano 3-polytopes with $3d-1=8$ vertices. Combinatorially, both are bipyramids over P_6.

polytope (Casagrande, 2006, Thm. 3). Thus, in this case P is lattice equivalent to a direct sum of $\frac{d}{2}$ copies of $P_6 \cong \mathrm{DP}(2)$.

Øbro classified the simplicial, terminal, and reflexive d-polytopes with $3d-1$ vertices (Øbro, 2008). Up to lattice equivalence, there is the interval $[-1,1]$ in dimension 1 and the pentagon P_5 in dimension 2. Forming suitable direct sums and (skew) bipyramids gives more smooth Fano d-polytopes with $3d-1$ vertices via

$$P_5 \oplus P_6^{\oplus(\frac{d}{2}-1)} \quad \text{for even } d \text{, and} \quad \mathrm{BP}(P_6^{\oplus(\frac{d-1}{2})}, v, e_d) \quad \text{for odd } d \text{ and } v \in \mathbb{Z}^{d-1} \cap P_6^{\oplus(\frac{d-1}{2})}\,.$$

Note that, up to lattice isomorphism, there are only two choices for v, either 0, which gives a proper bipyramid, or some vertex, which results in a skew bipyramid. The 3-dimensional cases are shown in Figure 2. Up to lattice equivalence, these are the only d-dimensional simplicial, terminal, and reflexive polytopes with $3d-1$ vertices (Øbro, 2008, Thm. 1). It turns out that all these polytopes are smooth Fano. Our main result is the following classification, which is a summary of (Assarf et al., 2012, Thm. 7).

Theorem 2 *For even $d \geq 6$ there are three combinatorial types of d-dimensional simplicial, terminal, and reflexive polytopes with $3d-2$ vertices. These three cases split into eleven types up to lattice equivalence. For odd $d \geq 5$ there is only one combinatorial type that splits into five types up to lattice equivalence.*

For $d=1$ there is one combinatorial type, for $d=2$ there is one combinatorial type with two different lattice realizations, for $d=3$ there is one combinatorial type with 4 different lattice realizations, and, finally, for $d=4$ there are three combinatorial types with ten different lattice realizations; see (Batyrev, 1999).

We list the types explicitly. To this end we label the vertices of P_5 by $v_1, v_2, \ldots, v_5$ and those of P_6 with $w_1, w_2, \ldots, w_6$ in clockwise order. For P_5, let v_1 be the unique vertex such that $-v_1 \notin P_5$. For even $d \geq 4$ the three combinatorial types are

$$P_5^{\oplus 2} \oplus P_6^{\oplus(\frac{d}{2}-2)}\,, \qquad \mathrm{DP}(4) \oplus P_6^{\oplus(\frac{d}{2}-2)}\,, \qquad \text{and} \qquad \mathrm{BP}(\mathrm{BP}(P_6^{\oplus\frac{d-2}{2}}, x, a), y, b)\,,$$

for a lattice point x of $P_6^{\oplus\frac{d-2}{2}}$, a lattice point y of $\mathrm{BP}(P_6^{\oplus\frac{d-2}{2}}, x, a)$ and transversal vectors a, b. The last case splits, up to lattice equivalence, into eight types if $d=4$ and nine if $d \geq 6$. The relevant choices of x, y are

$$(0,0)\,, \quad (0,c)\,, \quad (0,w_1)\,, \quad (w_1,w_1)\,, \quad (w_1,w_2)\,, \quad (w_1,w_3)\,, \quad (w_1,w_4)\,, \quad \text{and} \quad (w_1,c)$$

for $d=4$, where all w_i are vertices of some copy of P_6; here c denotes one of the two apices of the bipyramid $\mathrm{BP}(P_6^{\oplus\frac{d-2}{2}}, x, a)$. For $d \geq 6$ we can additionally choose two vertices in different copies of P_6. It is a key step in our proof to recognize these (proper or skew) bipyramids. The fact that the group of lattice automorphisms of P_6, which is isomorphic to the dihedral group of order 12, acts sharply transitively on adjacent pairs of vertices then entails the classification up to lattice equivalence. For odd $d \geq 5$ the one combinatorial type is $\mathrm{BP}(P_5 \oplus P_6^{\oplus \frac{d-3}{2}}, x, a)$ for some lattice point $x \in P_5 \oplus P_6^{\oplus \frac{d-3}{2}}$. The five different lattice isomorphism types are realized by choosing x in $\{0, v_1, v_2, v_3, w_1\}$.

We do believe that the list of the classifications obtained so far follows a pattern.

Conjecture 3 *Let P be a d-dimensional smooth Fano polytope with n vertices such that $n \geq 3d - k$ for $k \leq \frac{d}{3}$. If d is even then P is lattice equivalent to $Q \oplus P_6^{\oplus(\frac{d-3k}{2})}$ where Q is a $3k$-dimensional smooth Fano polytope with $n - 3d + 9k \geq 8k$ vertices. If d is odd then P is lattice equivalent to $Q \oplus P_6^{\oplus(\frac{d-3k-1}{2})}$ where Q is a $(3k{+}1)$-dimensional smooth Fano polytope with $n - 3d + 9k - 3 \geq 8k - 3$ vertices.*

This conjecture is best possible in the following sense: The k-fold direct sum of skew bipyramids over P_6 yields a smooth Fano polytope of dimension $d = 3k$ with $8k = 3d - k$ vertices, but it has no copy of P_6 as a direct summand. However, it does contain $P_6^{\oplus k}$ as a subpolytope of dimension $2k = \frac{2}{3}d$.

If the conjecture above holds the full classification of the smooth Fano polytopes of dimension at most nine Lorenz and Paffenholz (2008) would automatically yield a complete description of all d-dimensional smooth Fano polytopes with at least $3d - 3$ vertices.

3 Toric Varieties

Reading a lattice point $a \in \mathbb{Z}^d$ as the exponent vector of the monomial $z^a = z_1^{a_1} z_2^{a_2} \ldots z_d^{a_d}$ in the Laurent polynomial ring $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_d^{\pm d}]$ provides an isomorphism from the additive group of $\mathbb{Z}^d$ to the multiplicative group of Laurent monomials. This way the maximal spectrum X_σ of a lattice cone σ becomes an *affine toric variety*. If Σ is a fan of lattice cones, gluing the duals of the cones along common faces yields a *(projective) toric variety* X_Σ. This complex algebraic variety admits a natural action of the embedded dense torus corresponding to the (dual of) the trivial cone $\{\mathbf{0}\}$ which is contained in each cone of Σ. If $P \in \mathbb{R}^d$ is a lattice polytope containing the origin, then the *face fan*

$$\Sigma(P) \;=\; \{\mathrm{pos}(F) \;\mid F \text{ face of } P\}$$

is such a fan of lattice cones. We denote the associated toric variety by $X_P = X_{\Sigma(P)}$. The face fan of a polytope is isomorphic to the normal fan of its polar. Two lattice polytopes P and Q are lattice equivalent if and only if X_P and X_Q are isomorphic as toric varieties.

Let P be a full-dimensional lattice polytope containing the origin as an interior point. Then the toric variety X_P is smooth if and only if P is smooth in the sense of the definition given above, that is, the vertices of each facet of P are required form a lattice basis. A smooth compact projective toric variety X_P is a *toric Fano variety* if its anticanonical divisor is very ample. This holds if and only if P is a smooth Fano polytope; see (Ewald, 1996, §VII.8.5).

We now describe the toric varieties arising from the polytopes listed in our Theorem 2. For the list of two-dimensional toric Fano varieties we use the same notation as in Figure 1; see (Ewald, 1996, §VII.8.7). The toric variety X_{P_3} is the complex projective plane $\mathbb{P}_2$. The toric variety $X_{P_{4a}}$ is isomorphic to a direct product $\mathbb{P}_1 \times \mathbb{P}_1$ of lines, and $X_{P_{4b}}$ is the smooth *Hirzebruch surface* $\mathcal{H}_1$. The toric variety X_{P_5} is a blow-up of $\mathbb{P}_2$ at two points or, equivalently, a blow-up of $\mathbb{P}_1 \times \mathbb{P}_1$ at one torus invariant point. The toric varieties associated with the del Pezzo polytopes $\mathrm{DP}(d)$ are called *del Pezzo varieties*; notice that this notion also occurs with a different meaning in the literature. The toric variety X_{P_6} is a del Pezzo surface or, equivalently, a blow-up of $\mathbb{P}_2$ at three non-collinear torus invariant points.

Two polytope constructions play a role in our classification, direct sums and (skew) bipyramids. We want to translate them into the language of toric varieties. Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ both be full-dimensional lattice polytopes containing the origin. Then the toric variety $X_{P\oplus Q}$ is isomorphic to the direct product $X_P \times X_Q$. In particular, for $P = [-1, 1]$ we have that the toric variety

$$X_{[-1,1]\oplus Q} \;=\; \mathbb{P}_1 \times X_Q$$

over the regular bipyramid over Q is a direct product with the projective line $\mathbb{P}_1 \cong X_{[-1,1]}$. More generally, the toric variety of a skew bipyramid over Q is a toric fiber bundle with base space $\mathbb{P}_1$ and generic fiber X_Q; see (Ewald, 1996, §VI.6.7). An example is the smooth Hirzebruch surface $\mathcal{H}_1 \cong X_{P_{4b}}$, which is a (projective) line bundle over $\mathbb{P}_1$.

In order to translate Theorem 2 to toric varieties we need a few more definitions. For the sake of brevity we explain these in polytopal terms and refer to (Ewald, 1996) for the details. A toric variety X_P associated with a canonical lattice d-polytope P is $\mathbb{Q}$-*factorial* (or *quasi-smooth*) if P is simplicial; see (Ewald, 1996, §VI.3.9). In this case the *Picard number* equals $n - d$ where n is the number of vertices of P; see (Ewald, 1996, §VII.2.17). We call this toric variety a 2-*stage fiber bundle over* Z if X is a fiber bundle with base space Y such that Y itself is a fiber bundle with base space Z. The following is now a corollary of Theorem 2.

Corollary 4 *Let X be d-dimensional terminal $\mathbb{Q}$-factorial Gorenstein toric Fano variety with Picard number $2d-2$. We assume $d \geq 4$.*

If d is even, then X is isomorphic to

i. *a 2-stage toric fiber bundle such that the base spaces of both stages are projective lines and the generic fiber of the second stage is the direct product of $\frac{d-2}{2}$ copies of the del Pezzo surface X_{P_6}, or*

ii. *the direct product of two copies of X_{P_5} and $\frac{d}{2} - 2$ copies of X_{P_6} or*

iii. *the direct product of the del Pezzo fourfold $X_{\mathrm{DP}(4)}$ and $\frac{d}{2} - 2$ copies of X_{P_6}.*

If d is odd then X is isomorphic to

iv. *a toric fiber bundle over a projective line with generic fiber isomorphic to the direct product of X_{P_5} and $\frac{d-3}{2}$ copies of X_{P_6}.*

All fiber bundles in the preceding result may or may not be trivial. Classifying the polytopes in Theorem 2 up to lattice equivalence is tantamount to classifying the associated toric varieties up to toric isomorphism. As detailed above there is one type for $d = 1$, two types for $d = 2, 3$, ten for $d = 4$, five for any odd dimension $d \geq 5$ and eleven types for even dimensions $d \geq 6$. For $d = 6$ this is explained in detail in Section 5 below. In dimensions up to and including 4 this is known from work of Batyrev (1991, 1999).

4 Special Facets and η-Vectors

In this section we will describe our major technical tools. This follows the approach of Øbro (2008). Let $P \subset \mathbb{R}^d$ be a reflexive lattice d-polytope with vertex set $\mathrm{Vert}(P)$. In particular, the origin $\mathbf{0}$ is an interior point. We let $v_P \;:=\; \sum_{v\in\mathrm{Vert}(P)} v$ be the *vertex sum* of P. As P is a lattice polytope v_P is a lattice point.

$\mathrm{ecc}(P)$	2	1	1	0	0	0	0
η_1	d	d	d	d	d	d	d
η_0	d	d	$d-1$	d	d	$d-1$	$d-2$
η_{-1}	$d-2$	$d-3$	$d-1$	$d-3$	$d-4$	$d-2$	d
η_{-2}	0	1	0	0	2	1	0
η_{-3}	0	0	0	1	0	0	0

Table 1: List of possible η-vectors of simplicial, terminal, and reflexive d-polytopes with $3d-2$ vertices, where $\mathrm{ecc}(P)$ denotes the eccentricity of P. Marked with a gray background are the η-vectors, which do not occur.

Now, a facet F of P is called *special* if the *face cone* $\mathrm{pos}\, F$ spanned by F contains v_P. Since the fan $\Sigma(P)$ generated by the face cones is complete, a special facet always exists. However, it is not necessarily unique. For instance, if P is centrally symmetric, we have $v_P = \mathbf{0}$, and each facet is special.

Since P is reflexive, for each facet F of P the primitive outer facet normal vector u_F satisfies $\langle u_F, x\rangle \leq 1$ for all points $x \in P$ and the set $\{x \in \mathbb{R}^d \mid \langle u_F, x\rangle = 1\}$ is the affine hull of F. We define

$$H(F,k) := \left\{x \in \mathbb{R}^d \;\middle|\; \langle u_F, x\rangle = k\right\}, \quad V(F,k) := H(F,k) \cap \mathrm{Vert}(P), \quad \text{and} \quad \eta_k^F := |V(F,k)|$$

for any integer $k \leq 1$. The sequence of numbers $\eta^F = (\eta_1^F, \eta_0^F, \eta_{-1}^F, \dots)$ is the *η-vector* of P with respect to F (we usually omit F in the notation). We omit any trailing zeros so that η has finite length. We have

$$\mathrm{Vert}(P) = \bigcup_{k\leq 1} V(F,k) \subseteq \bigcup_{k\leq 1} H(F,k).$$

Thus $\eta_1^F + \eta_0^F + \eta_{-1}^F + \cdots = |\,\mathrm{Vert}(P)|$ is the number of vertices of P. If a vertex v is contained in $V(F,k)$ we call the number k the *level* of v with respect to F. As P is simplicial we have $\eta_1 = d$ for any facet F. Furthermore, one can show that for any facet F any vertex on level 0 is contained in a facet adjacent to F. Looking at a special facet and evaluating

$$0 \leq \langle u_F, v_P\rangle = \langle u_F, \sum_{k\leq 1}\sum_{v\in V(F,k)} v\rangle = \sum_{k\leq 1}\sum_{v\in V(F,k)} \langle u_F, v\rangle = d + \sum_{k\leq -1}\sum_{v\in V(F,k)} \langle u_F, v\rangle \qquad (1)$$

shows that there can only be at most d many vertices below level 0. Thus, P has at most $3d$ vertices, implying the upper bound of (Casagrande, 2006). This allows to deduce a list of potential η-vectors from (1). Now we assume that P has exactly $3d-2$ vertices. *A priori*, the potential cases are listed in Table 1. The maximum level of v_P is 2. Our classification shows that not all of the η-vectors listed actually occur. Some can be ruled out by a direct argument, some only *a posteriori*. Those that do not occur are marked in gray in the table.

Our overall proof strategy is as follows. It turns out that the level of v_P is the same for each special facet. Hence, this is an invariant of the polytope, which we call the *eccentricity* $\mathrm{ecc}(P)$. We look at the three possible cases separately. We choose a special facet F of P. As a refinement, we consider separate cases according to the η-vector of F. A key is the observation, that we can, up to lattice equivalence, restrict the possible choices for vertices in levels $1, 0$, and -1 of F. This is summarized in the proposition below; see (Assarf et al., 2012, Prop. 32). Given this initial distribution of the vertices we want to determine the remaining vertices. Sometimes this turns out to be quite difficult. In this cases we switch to a special neighboring facet with a different η-vector which is easier to analyze or already have been analyzed. With $\mathrm{opp}(F)$ we denote the set of all vertices which lie in a facet adjacent to F but which are not vertices of F itself.

Proposition 5 *Let P be a d-dimensional simplicial, terminal, and reflexive polytope such that F is a special facet. Up to lattice equivalence, we can assume that $F = \mathrm{conv}\{e_1, e_2, \ldots, e_d\}$ and there is a map $\phi : \mathrm{Vert}(F) \to \mathrm{Vert}(F) \cup \{0\}$ such that:*

i. if $\eta_0^F = d$, then

$$\begin{aligned} V(F,0) &= \{\phi(e_1) - e_1,\, \phi(e_2) - e_2,\, \ldots,\, \phi(e_d) - e_d\} \\ V(F,-1) &\subseteq \{-e_1,\, -e_2,\, \ldots,\, -e_d\}\,. \end{aligned}$$

ii. If $\eta_0^F = d-1$ and $\mathrm{opp}(F) = V(F,0)$, then, for $a, b \in [d] \setminus \{1,2\}$ not necessarily distinct,

$$\begin{aligned} V(F,0) &= \{-e_1 - e_2 + e_a + e_b,\, \phi(e_3) - e_3,\, \ldots,\, \phi(e_d) - e_d\} \\ V(F,-1) &\subseteq \{-e_1,\, -e_2,\, \ldots,\, -e_d\} \,\cup\, \{-e_1 - e_2 + e_s \mid s \in [d]\} \end{aligned}$$

iii. If $\eta_0^F = d-1$ and $\mathrm{opp}(F) \neq V(F,0)$, then

$$\begin{aligned} V(F,0) &= \{\phi(e_2) - e_2,\, \phi(e_3) - e_3,\, \ldots,\, \phi(e_d) - e_d\} \\ V(F,-1) &\subseteq \{-e_1,\, -e_2,\, \ldots,\, -e_d\} \,\cup\, \{-2e_1 - e_r + e_s + e_t \mid r,s,t \in [d] \text{ pairwise distinct },\, r \neq 1\}\,. \end{aligned}$$

The first case above occurs in (Øbro, 2008). This result allows us to control most of the vertices of a simplicial, terminal, and reflexive polytope if η_0 is given. In this way an approach to the classification is by examining choices for the vertices on the levels k for $k \leq -2$.

5 The Classification Explained in Dimension Six

In this section we will explicitly list the 6-dimensional simplicial, terminal, and reflexive polytopes with exactly $3 \cdot 6 - 2 = 16$ vertices. This is the smallest even dimension in which all eleven types up to lattice equivalence arise. This list in dimension 6 is already subsumed in the classifications (Brown and Kasprzyk, 2009–2012) and (Lorenz and Paffenholz, 2008); and we will refer to the latter. Here we will organize the polytopes in a way such that it fits the line of arguments in (Assarf et al., 2012). Additional comments are meant to give the reader an idea about the organization of our proof.

Throughout let P be a d-dimensional simplicial, terminal, and reflexive polytope with $3d-2$ vertices such that F is a special facet. The vertex sum v_P lies on level 0, 1 or 2 with respect to F. Throughout we assume that d is even and $d \geq 4$. It turns out that each such polytope P contains a copy of the hexagon P_6 as a subpolytope, albeit not necessarily as a direct summand. So we normalized the examples in a way that P_6 always lies in in the coordinate subspace $\mathrm{lin}\{e_1, e_2\}$. This way the differences among the examples are particularly easy to spot.

5.1 Polytopes of Eccentricity 2

The classification becomes more involved the more symmetric P is. The most eccentric case occurs if the vertex sum lies on level 2, and this is the easiest. Table 1 tells us that there is only one kind of η-vector, namely $\eta^F = (d, d, d-2)$. What makes this case simpler than others is that we immediately have $\eta_0 = d$, which forces that the vertices on F form a lattice basis, and the vertices on level 0 can be determined (Øbro, 2008). In this case the partial description of the vertices in Proposition 5 is already good enough to get the full picture with little extra effort. It turns out that P is lattice equivalent to $P_5^{\oplus 2} \oplus P_6^{\oplus \frac{d}{2}-2}$ or to a skew bipyramid over a $(d-1)$-dimensional smooth Fano polytope with $3(d-1)-1 = 3d-4$ vertices.

Example 6 *For $d = 6$ the first case is $P \cong P_6 \oplus P_5 \oplus P_5$ such that $v_P = e_3 + e_5$. Here and in the examples below, we list the vertices sorted by level.*

$$e_1, e_2, e_3, e_4, e_5, e_6$$
$$\pm(e_1 - e_2), \pm(e_3 - e_4), \pm(e_5 - e_6)$$
$$-e_1, -e_2, -e_4, -e_6$$

In the database (Lorenz and Paffenholz, 2008) this occurs as `F.6D.6552`. The polytope has 24 special facets.

If the polytope P is not of the type above then, for $d = 6$, the polytope P is a double skew bipyramid over $P_6 \oplus P_6$. Four more cases arise depending on the relative positions of the apices of the two bipyramids. To form a skew bipyramid we need to pick a vertex of the base. Since the group of lattice automorphisms of P_6 acts transitively on the vertices, we may assume that the first skew bipyramid is $\mathrm{BP}(P_6^{\oplus 2}, e_1, e_5)$. The three distinct relative positions of two vertices of P_6 lead to the next three cases.

Example 7 *For $d = 6$ the second type is given by $\mathrm{BP}(\mathrm{BP}(P_6^{\oplus 2}, e_1, e_5), e_1, e_6)$ such that $v_P = 2e_1$.*

$$e_1, e_2, e_3, e_4, e_5, e_6$$
$$\pm(e_1 - e_2), \pm(e_3 - e_4), e_1 - e_5, e_1 - e_6$$
$$-e_1, -e_2, -e_3, -e_4$$

In the database this occurs as `F.6D.5346`. The polytope has 48 special facets.

Example 8 *For $d = 6$ the third type is given by $\mathrm{BP}(\mathrm{BP}(P_6^{\oplus 2}, e_1, e_5), e_2, e_6)$ such that $v_P = e_1 + e_2$.*

$$e_1, e_2, e_3, e_4, e_5, e_6$$
$$\pm(e_1 - e_2), \pm(e_3 - e_4), e_1 - e_5, e_2 - e_6$$
$$-e_1, -e_2, -e_3, -e_4$$

In the database this occurs as `F.6D.5680`. The polytope has 24 special facets.

Example 9 *For $d = 6$ the fourth type is given by $\mathrm{BP}(\mathrm{BP}(P_6^{\oplus 2}, e_1, e_5), e_3, e_6)$ such that $v_P = e_1 + e_3$.*

$$e_1, e_2, e_3, e_4, e_5, e_6$$
$$\pm(e_1 - e_2), \pm(e_3 - e_4), e_1 - e_5, e_3 - e_6$$
$$-e_1, -e_2, -e_3, -e_4$$

In the database this occurs as `F.6D.5553`. The polytope has 16 special facets.

The final case in this section differs from the above in that the base vertex of the second skew bipyramid is an apex of the first stage.

Example 10 *For $d = 6$ the fifth type is given by $\mathrm{BP}(\mathrm{BP}(P_6^{\oplus 2}, e_1, e_5), e_5, e_6)$ such that $v_P = e_1 + e_5$.*

$$e_1, e_2, e_3, e_4, e_5, e_6$$
$$\pm(e_1 - e_2), \pm(e_3 - e_4), e_1 - e_5, e_5 - e_6$$
$$-e_1, -e_2, -e_3, -e_4$$

In the database this occurs as `F.6D.5685`. The polytope has 24 special facets.

5.2 Polytopes of Eccentricity 1

If the vertex sum lies on level one, then the situation is still somewhat benign. Our proof strategy is to first consider polytopes P with a special facet that have η-vector $(d, d, d-3, 1)$. In (Assarf et al., 2012, Prop. 36) we show that in this case P, again, must be a skew bipyramid. Notice, however, that our classification shows *a posteriori* that this case does not occur. Table 1 then says that the only choice left is $\eta = (d, d-1, d-1)$. In this situation (Assarf et al., 2012, Prop. 39) shows that, once more, P is a double bipyramid.

In the first two cases the first stage is a proper bipyramid. For the second stage then the base vertex can either be in the base of the first stage or an apex.

Example 11 *For* $d = 6$ *the sixth type is given by* $\mathrm{BP}(\mathrm{BP}(P_6^{\oplus 2}, 0, e_5), e_1, e_6)$ *such that* $v_P = e_1$.

$$\begin{gathered} e_1, e_2, e_3, e_4, e_5, e_6 \\ \pm(e_1 - e_2), \pm(e_3 - e_4), e_1 - e_6 \\ -e_1, -e_2, -e_3, -e_4, -e_5 \end{gathered}$$

In the database this occurs as `F.6D.5711`*. The polytope has 48 special facets.*

Example 12 *For* $d = 6$ *the seventh type is given by* $\mathrm{BP}(\mathrm{BP}(P_6^{\oplus 2}, 0, e_5), e_5, e_6)$ *such that* $v_P = e_5$.

$$\begin{gathered} e_1, e_2, e_3, e_4, e_5, e_6 \\ \pm(e_1 - e_2), \pm(e_3 - e_4), e_5 - e_6 \\ -e_1, -e_2, -e_3, -e_4, -e_5 \end{gathered}$$

In the database this occurs as `F.6D.6558`*. The polytope has 72 special facets.*

For $v_P \in H(F, 1)$ there is only one choice of a double bipyramid where both stages are skew.

Example 13 *For* $d = 6$ *the eighth type is given by* $\mathrm{BP}(\mathrm{BP}(P_6^{\oplus 2}, e_2, e_5), e_1 - e_2, e_6)$ *such that* $v_P = e_1$.

$$\begin{gathered} e_1, e_2, e_3, e_4, e_5, e_6 \\ \pm(e_1 - e_2), \pm(e_3 - e_4), e_2 - e_5, e_1 - e_2 - e_6 \\ -e_1, -e_2, -e_3, -e_4 \end{gathered}$$

In the database this occurs as `F.6D.5702`*. The polytope has 48 special facets.*

5.3 Polytopes of Eccentricity 0

If the vertex sum of P is zero all facets are special. The easy subcase occurs when all η-vectors of P are of type $(d, d-2, d)$. We show that in this case P is centrally symmetric (Assarf et al., 2012, Prop. 40). Extending arguments of (Nill, 2006, Thm. 0.1) we show that such a polytope is lattice equivalent to a double proper bipyramid over $P_6^{\oplus \frac{d-2}{2}}$ or $\mathrm{DP}(4) \oplus P_6^{\oplus \frac{d}{2}-2}$.

Example 14 *If* $d = 6$ *the ninth type occurs for* $P \cong P_6 \oplus \mathrm{DP}(4)$.

$$\begin{gathered} e_1, e_2, e_3, e_4, e_5, e_6 \\ \pm(e_1 - e_2), \pm(e_3 + e_4 - e_5 - e_6) \\ -e_1, -e_2, -e_3, -e_4, -e_5, -e_6 \end{gathered}$$

In the database this occurs as `F.6D.3154`*. All 180 facets are special, and all of them have the same η-vector* $(6, 4, 6)$.

Example 15 *If $d = 6$ the tenth case is the direct sum of two hexagons P_6 and two line segments. In our notation, this means that $P \cong \mathrm{BP}(\mathrm{BP}(P_6^{\oplus 2}, 0, e_5), 0, e_6)$.*

$$\begin{gathered} e_1, e_2, e_3, e_4, e_5, e_6 \\ \pm(e_1 - e_2), \pm(e_3 - e_4) \\ -e_1, -e_2, -e_3, -e_4, -e_5, -e_6 \end{gathered}$$

In the database this occurs as `F.6D.6765`*. All 144 facets are special, and all of them have the same η-vector* $(6, 4, 6)$.

It remains to discuss the situation where $v_P = \mathbf{0}$ but P is not centrally symmetric. This is by far the most complicated case in our proof. It contributes to this complexity that we need to discuss four candidates of η-vectors. First, $\eta = (6, 6, 3, 0, 1)$ is excluded (Assarf et al., 2012, Prop. 4). Second, $\eta = (6, 6, 2, 2)$ is essentially reduced to a bipyramid (Assarf et al., 2012, Lem. 43) (but this case does not exist *a posteriori*). So this leaves two more η-vectors. Surprisingly, they lead to the same polytopes.

Example 16 *If $d = 6$ the final eleventh type occurs for $P \cong \mathrm{BP}(\mathrm{BP}(P_6^{\oplus 2}, e_1, e_5), -e_1, e_6)$. Up to lattice equivalence this is the only case in which $v_P = \mathbf{0}$ but P is not centrally symmetric.*

$$\begin{gathered} e_1, e_2, e_3, e_4, e_5, e_6 \\ \pm(e_1 - e_2), \pm(e_3 - e_4), e_1 - e_5 \\ -e_1, -e_2, -e_3, -e_4 \\ -e_1 - e_6 \end{gathered}$$

In the database this occurs as `F.6D.5713`*. All 144 facets are special, where 96 of them have the η-vector* $(6, 5, 4, 1)$ *and the other 48 the η-vector reads* $(6, 4, 6)$*. For instance, the facet which is induced by $\langle \mathbf{1}, x \rangle = 1$ has the η-vector* $(6, 5, 4, 1)$*, and the facet induced by $\langle \mathbf{1} - 2e_1 - 2e_6, x \rangle = 1$ has the η-vector* $(6, 4, 6)$.

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