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Contact graphs of boxes with unidirectional contacts[*]

Daniel Gonçalves[1] Vincent Limouzy[2] Pascal Ochem[1]

[1] *LIRMM, Université de Montpellier, CNRS, Montpellier, France*
[2] *Université Clermont Auvergne, Clermont Auvergne INP, CNRS, Mines Saint-Etienne,* LIMOS*, F-63000 Clermont-Ferrand, France*

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This paper is devoted to the study of particular geometrically defined intersection classes of graphs. Those were previously studied by Magnant and Martin, who proved that these graphs have arbitrary large chromatic number, while being triangle-free. We give several structural properties of these graphs, and we raise several questions.

Keywords: intersection graphs

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1 Introduction

A lot of graph classes studied in the literature are defined by a geometric model, where vertices are represented by geometric objects (e.g. intervals on a line, disks in the plane, chords inscribed in a circle...) and the adjacency of two vertices is determined according to the relation between the corresponding objects. A large amount of graph classes consider the intersection relation (e.g. interval graphs, disk graphs or circle graphs). However some other relations might be considered such as the containment, the overlap or also the contact between objects. Recently several groups of authors started to study graph classes defined by contact models, as for example Contact of Paths in a grid (CPG) [10], Contact of L shapes in $\mathbb{R}^2$ or even contact of triangles in the plane [9]. In this note we consider a new class defined by a contact model. More precisely we consider the class of graphs defined by contact of axis parallel boxes in $\mathbb{R}^d$ where the contact occurs on $(d-1)$-dimensional object in only one direction (CBU) .

When considering graph defined by axis-parallel boxes in $\mathbb{R}^d$ and the adjacency relation is given by the intersection it corresponds to the important notion of boxicity introduced by Roberts [31], when the adjacency relation is given by the containment relation it correspond to comparability graphs and it is connected to the poset dimension introduced by Dushnik & Miller [12]

The motivation for this class of graphs originate from an article of Magnant and Martin [26] where a wireless channel assignment is considered. The problem considers rectangular rooms in a building and asks to find a channel assignment for each room. In order to avoid interferences rooms sharing the same

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wall, floor or ceiling need to use different channels. The question was to determine whether a constant number of channels would suffice to answer this problem. The first negative answer was provided by Reed and Allwright [30] that a constant number of channels is not sufficient. Magnant and Martin strengthened their result that for any integer k there exists a *building* that requires exactly k channels. In addition, their construction only requires floor-ceiling contacts.

We provide the first structural properties of this class. We first establish some links with the well-known notion of boxicity in Section 4. Then in Section 7 we consider the recognition problem and we prove that it is NP-complete to determine if a graph is d-CBU for any integer $d \geq 3$. Then we provide a characterization in terms of an acyclic orientation of the class of general CBU. Thanks to this characterization, it is immediate to realize that the class of CBU constitutes a proper sub-class of Hasse diagram graph (A Hasse diagram graph is the undirected graph obtained from a Hasse diagram associated to a poset). Finally we prove in Section 9 that several well studied optimization problems remains NP-hard on either 2- or 3-CBU graphs.

2 Preliminaries

We consider $\mathbb{R}^d$ and d orthogonal vectors $e_1, \ldots, e_d$ and we introduce a new class of geometric intersection graphs. Here, the vertices correspond to interior disjoint d-dimensional axis-parallel boxes in $\mathbb{R}^d$, and two such boxes are only allowed to intersect on a $(d-1)$-dimensional box orthogonal to e_1. This class of graphs is denoted by *d-CBU*, for *Contact* graphs of d-dimensional *Boxes* with *Unidirectional* contacts. We denote *CBU* the union of d-CBU for all d.

Note that 1-CBU correspond to the forests of paths.

Claim 1 *For every $d \geq 1$, d-CBU graphs are triangle-free.*

Indeed, note that orienting the edges according to vector e_1 and labeling each arc with the coordinate of the corresponding $(d-1)$-hyperplane, one obtains an acyclic orientation such that for every vertex, all the outgoing arcs have the same label, all the ingoing arcs have the same label, and the label of ingoing arcs is smaller than the label of outgoing arcs. We call such a labeling of the arcs an *homogeneous arc labeling*. Note that an oriented cycle cannot admit such a labeling. A triangle abc oriented acyclically is, up to automorphism, such that $d^+(a) = 2$, $d^+(b) = 1$, and $d^+(c) = 0$. Now ab and ac should have the same label, such as ac and bc, but ab and bc should be distinct, a contradiction. Thus a triangle cannot admit a homogeneous arc labeling. This completes the proof of the claim.

With similar arguments one obtains the following for short cycles. See Figure 1.

Claim 2 *For any homogeneous arc labeling of a graph G, its restriction to a short cycle is as follows.*

- *For a 4-cycle, the orientation is either such that there are two sources and two sinks, or it is such that there is one source and one sink linked by two oriented paths of length 2.*
- *For a 5-cycle, the orientation is such that there is one source and one sink linked by two oriented paths, one of length 2 and one of length 3.*

3 Relation with Cover Graphs

An undirected graph is a *cover graph* if it is the underlying graph of the Hasse diagram of some partial order. It was shown by Brightwell [5] and also by Nešetřil and Rödl [27, 28, 29] that deciding whether a

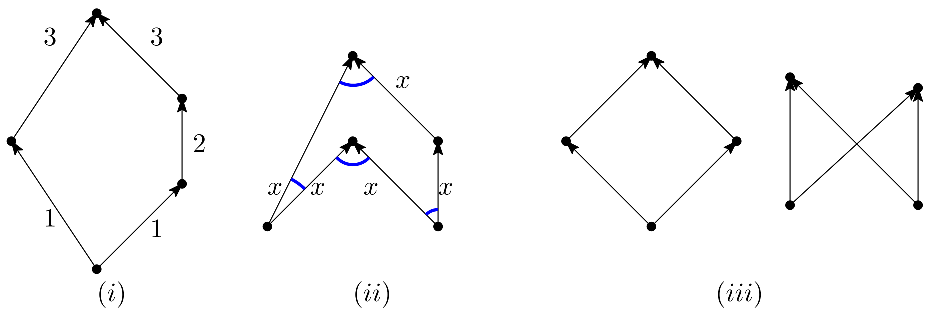

Figure 1: (i) Example of a good orientation of a C_5 with some valid labels, (ii) example of bad orientation. Once a label x is fixed for one arc, this label is propagated to all the arcs leading to the conclusion that $x < x$. (iii) the two valid orientations of a C_4

graph is a cover graph is NP-complete. However, they came up with a simple characterization in terms of acyclic orientations.

Their characterization states that a graph is a cover graph if and only if there exists an acyclic orientation without a quasi-cycle. A *quasi-cycle*, being an orientation of a cycle $(v_1, v_2, \ldots, v_n)$ with the arcs (v_i, v_{i+1}) for all $1 \leq n-1$ plus the arc (v_1, v_n). From a homogeneous arc labelling, the orientation provided by the labelling clearly fulfills the above defined condition.

Claim 3 *For any homogeneous arc labeling of a graph G, the orientation of G does not contain any quasi-cycle.*

Corollary 4 *The class of CBU graphs is contained in the class of cover graphs.*

From the previous result, it is natural to ask whether both classes are equivalent. The following remark provides the answer.

Remark 5 *The class of CBU graphs is strictly contained in the class of cover graphs. In Lemma 16 we will exhibit a graph that is not CBU but is a cover graph.*

We will see in the following that an orientation of a triangle-free graph G, fulfilling the conditions of Claim 2 and of Claim 3 may not admit a homogeneous arc labeling.

4 Boxicity

The *boxicity* $box(G)$ of a graph G, is the minimum dimension d such that G admits an intersection representation with axis-aligned boxes. Of course, the graphs in d-CBU have boxicity at most d. The converse cannot hold for the graphs containing a triangle, as those are not in CBU. However, some relations hold for triangle-free graphs. Let us begin with bipartite graphs.

Theorem 6 *Every bipartite graphs of boxicity b belongs to $(b+1)$-CBU.*

Proof: Consider a bipartite graph G with vertex sets A and B. Consider a boxicity b representation of G and slightly expand each box in such a way that the intersection graph remains unchanged (we stop the expansion of the boxes before creating new intersections). Now, any two intersecting boxes intersect on a

b-dimensional box. Assume that this representation is drawn in the space spanned by $e_2, \ldots, e_{b+1}$, and let us set for the first dimension (spanned by e_1) that the vertices of A and B, correspond to the intervals $[0, 1]$ and $[1, 2]$, respectively. As A and B are independent sets, it is clear that the boxes in the representation are interior disjoint and that any two intersecting boxes intersect on a b-dimensional box orthogonal to e_1. The obtained representation is thus a $(b + 1)$-CBU representation of G. $\square$

Theorem 6 does not extend to triangle-free graphs. We will see in the following section that there exists triangle-free graphs with bounded boxicity that are not d-CBU, for any value d. Actually, Lemma 16 tells that there exists such graphs with girth 5. In other words, for a $3 \leq g \leq 5$, there is no function f_g such that every graph G, of girth at least g and of boxicity b belongs to $(f_g(b))$-CBU.

Problem 7 *For $g \geq 6$, is there a function f_g such that every graph G, of girth at least g and of boxicity b belongs to $(f_g(b))$-CBU?*

By Theorem 12, we know that if f_6 exists, then $f_6(2)$ is at least 3. Nevertheless, the following theorem shows that subdividing the edges enables to consider every triangle-free graph. An intersection representation is said *proper* if two objects intersect if and only if some point of the representation belongs to these 2 objects, only.

Theorem 8 *For every graph G having a proper intersection representation with axis-parallel boxes in $\mathbb{R}^b$, the 1-subdivision of G belongs to $(b + 1)$-CBU.*

Proof: Consider such a representation of G and slightly expand each box in such a way that the intersection graph remains unchanged, and any two intersecting boxes intersect on a b-dimensional box. Assume that this representation is drawn in the space spanned by $e_2, \ldots, e_{b+1}$, and for the first dimension (spanned by e_1) let us consider any vertex ordering, $v_1, \ldots, v_n$. For the first dimension, a vertex v_i, corresponds to the interval $[2i, 2i + 1]$. Clearly, none of these boxes intersect. Let us now add the boxes for the vertices added by subdividing the edges of G. For any edge $v_i v_j$, in the space spanned by $e_2, \ldots, e_{b+1}$, the expansion ensured that the intersection of v_i and v_j contains a box B_{ij}, that does not intersect any other box of the representation. If $i < j$, the subdivision vertex of $v_i v_j$, is represented by $[2i + 1, 2j] \times B_{i,j}$. The obtained representation is clearly a $(b + 1)$-CBU representation of the subdivision of G. $\square$

Corollary 9 *For every triangle-free graph G of boxicity b, the 1-subdivision of G belongs to $(b+1)$-CBU.*

5 Planar graphs

While planar graphs have boxicity at most 3 [35, 18, 4], many subclasses of planar graphs are known to have boxicity at most 2. This is the case for 4-connected planar graphs [34], and their subgraphs. The subgraphs of 4-connected graphs include every triangle-free planar graph (see Lemma 4.1 in [20]). As observed earlier, for those the representation is necessarily proper. For general planar graphs, the representation in $\mathbb{R}^3$ provided in [18] is clearly proper. So Theorem 8 implies the following.

Corollary 10 *For every planar graph G, the 1-subdivision of G belongs to 4-CBU. Furthermore, if G is triangle-free then it even belongs to 3-CBU.*

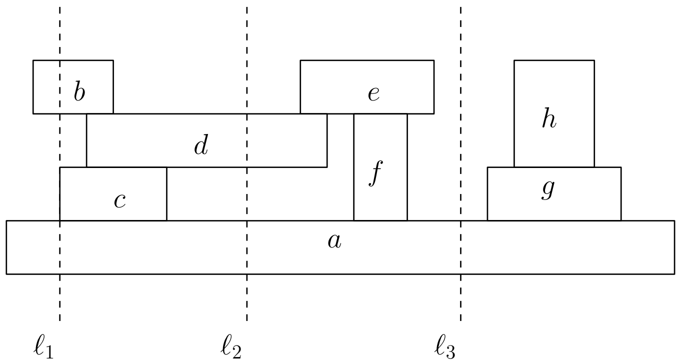

Figure 2: The top boxes with respect to ℓ_1, ℓ_2, and ℓ_3 are b, d, and a, respectively. The top sequence of this 2-CBU representation is $a, b, d, e, a, g, h, g, a$.

5.1 2-CBU graphs

Given a 2-CBU representation of a graph G, and a vertical line ℓ, the *top box* of this representation with respect to ℓ is the highest box intersecting ℓ. Now, the *top sequence* of a 2-CBU representation is the sequence of top boxes obtained when parsing the representation with ℓ from left to right (see Figure 2).

One can easily see that 2-CBU graphs are planar graphs, and that every forest is a 2-CBU graph. Actually, this class contains every triangle-free outerplanar graph.

Theorem 11 *Every triangle-free outerplanar graph is 2-CBU.*

Proof: Let us prove that for any connected outerplanar graph G, and any facial walk $v_1, v_2, \ldots, v_k, v_{k+1} = v_1$ on the outerboundary of G (with separating vertices appearing several times in this walk), there exists a 2-CBU representation of G with top sequence $v_1, v_2, \ldots, v_k, v_{k+1}$.

We proceed by induction on the number of vertices in G. The statement clearly holds if G has only one vertex. A connected triangle-free outerplanar graph with more vertices contains either a vertex v_i of degree one, or a cycle $v_i, \ldots, v_j$ of length at least four (i.e. $j - i \geq 3$), whose vertices $v_{i+1}, \ldots, v_{j-1}$ have degree two in G.

In the first case we can add the box of v_i in the representation of $G \setminus v_i$ obtained by induction (see Figure 3, left). In the second case we consider the representation of $G \setminus \{v_{i+1}, \ldots, v_{j-1}\}$ obtained by induction. Note that v_i and v_j now appear consecutively in its outerboundary, such as in the top sequence. It is thus easy to add the boxes of $v_{i+1}, \ldots, v_{j-1}$ in the representation (see Figure 3, right). □

Theorem 12 *There are series-parallel graphs of girth 6 that are not 2-CBU.*

Proof: Consider a graph G with two vertices, a and b, linked by 9 disjoint ab-paths of length three ax_iy_ib, for $i \in \{1, \ldots, 9\}$. Then for each edge x_iy_i add a length 5 path from x_i to y_i. The obtained graph has girth 6 and is series-parallel (see Figure 5).

Note that in a 2-CBU representation of G, the box of a (resp. b) has at most four neighbors such that their intersection contains a corner of a (resp. b). Thus, there exists an $i \in \{1, \ldots, 9\}$ such that one side of x_i is contained in one side of a, and one side of y_i is contained in one side of b. Now, whatever the

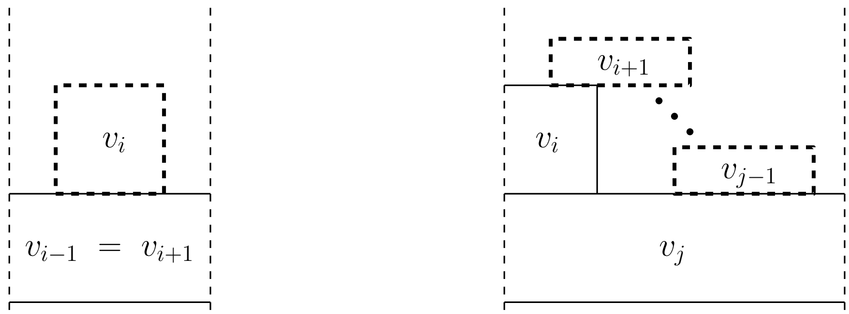

Figure 3: Left part : Adding a degree one vertex v_i in the representation. This is done in the vertical stripe where the neighbor of v_i, $v_{i-1} = v_{i+1}$, is the top box. Right part : Adding $v_{i+1}, \ldots, v_{j-1}$. This is done in the vertical stripe where v_i and v_j are the top box, successively.

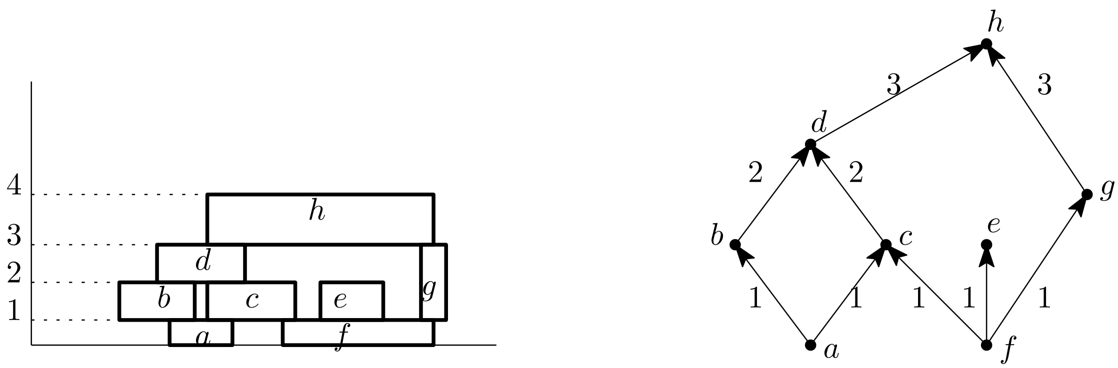

Figure 4: An example of 2-CBU graph and its associated acyclic orientation

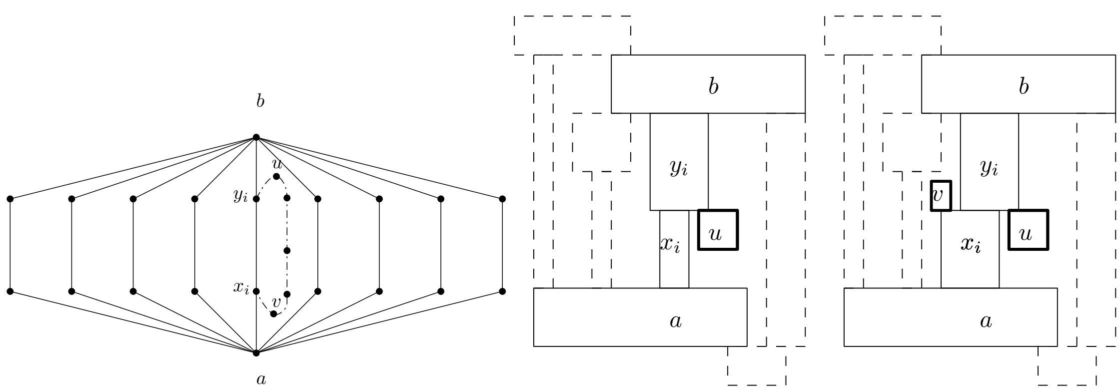

Figure 5: The series-parallel graph G of Theorem 12.

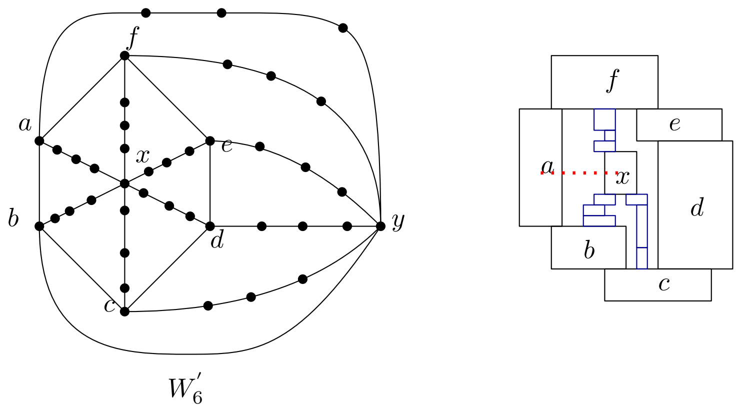

Figure 6: The graph W'_6.

way x_i and y_i intersect (a side of x_i may be contained in a side of y_i, or the other way around, or also their intersection may contain a corner of each box), it is not possible to have the length 5 x_iy_i-path (see Figure 5, right). If a side of x_i is contained in a side of y_i there is no place left around x_i to draw a third neighbor. If the intersection of x_i and y_i contains a corner of x_i and a corner of y_i, there is space to draw a third neighbor for these vertices, say u and v respectively, but in that case the uv-path should go around a or b, but it would intersect the paths ax_jy_jb with $j \neq i$. Thus G does not admit a 2-CBU representation. □

Problem 13 *Is there a girth g such that every series parallel graph of girth at least g belongs to 2-CBU ?*

For planar graphs, the following theorem shows that such a bound on the girth does not exist. Let us denote by W_g^2 the double wheel graph, obtained from a cycle C_g by adding two non-adjacent vertices, each of them being adjacent to every vertex of C_g. An edge incident to one of these two vertices (i.e., an edge not contained in C_g) is called a *ray*. Now, let W'_g be the graph obtained from W_g^2 by subdividing $\lfloor g/2 \rfloor$ times every ray (see Figure 6). This graph is planar and has girth g.

Theorem 14 *The graph W'_g does not belong to 2-CBU.*

Proof: For any 2-CBU representation of the cycle C of length g there is a rectangle R, for example the one with the leftmost right side, such that none of the top or bottom side of R is incident to the inner region. It is thus impossible to connect R with a ray in the inner region. On the other hand, there is no planar embedding of W'_g where C bounds an inner face. □

5.2 3-CBU graphs

Bipartite planar graphs are known to be contact graphs of axis-aligned segments in $\mathbb{R}^2$ [3, 8], and their boxicity is thus at most two. By Theorem 6, we thus have the following.

Corollary 15 *Every bipartite planar graph belongs to 3-CBU.*

The following lemma tells us that this property does not generalize, in a strong sense, to triangle-free planar graphs.

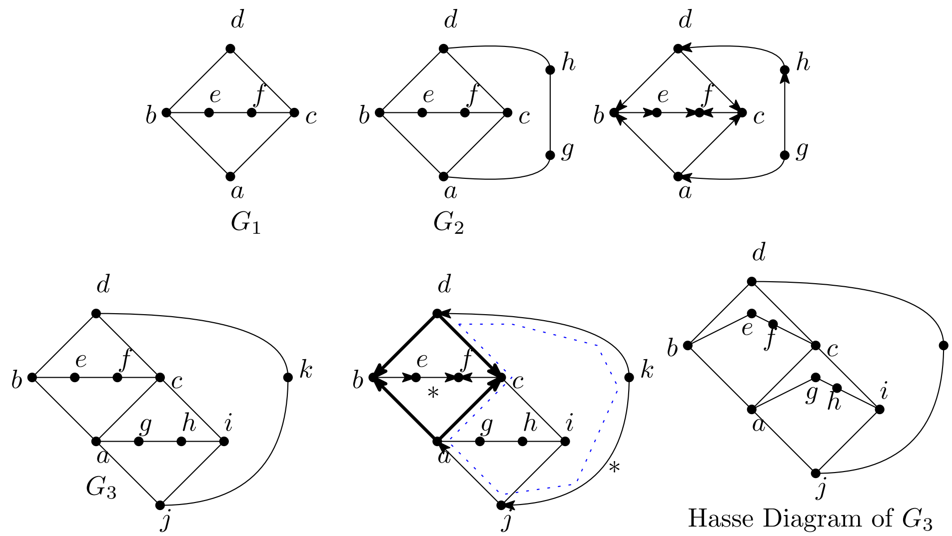

Figure 7: The graph G_3 is planar and non CBU. It is however a cover graph.

Lemma 16 *There exists a girth 4 planar graph that is not CBU.*

Proof: Let us consider the graph G_1 represented in Figure 7. One can show that this graph does not admit any valid CBU orientation where in the C_4 induced by a, b, c and d, a is a source and d is a sink (nor the converse). Let us assume that there exists a CBU orientation such that a is a source and d is a sink. By fixing the orientation of edges ab and ac from a to b and from a to c respectively, and by Claim 2, the edges be and cf have to be oriented from b to e and from c to f. The edge ef is oriented in any direction, *w.l.o.g.* let us say from e to f. But in that case, in the C_5 induced by b, e, f, c and d contains two sources and two sinks, which is not a valid CBU orientation (By Claim 2).

By adding a path of length 3 between a and d, we obtain the graph G_2. From the previous observation, we can conclude the same property for vertices b and c. Hence, in the valid orientation of G_2, a and d are sources and b and c are sinks (or the converse). Let us now consider the graph G_3 obtained by gluing two copies of G_2 in a special manner (see Figure 7). Let us remark that the graph obtained is planar. Let us consider, *w.l.o.g.*, that a and d are sources and b and c are sinks in the C_4 induced by a, b, c, and d. Recall that a valid orientation of a C_5 contains exactly one source and one sink. Thus, in the C_5 induced by the vertices a, c, d, k, and j, the vertex c has to be a sink from the already fixed orientation. Hence, it forces the edge aj to be oriented from j to a and the edge dk from k to d (the edge k, j can be oriented in any direction), since the length of a path from a source to a sink in an orientation is exactly 2 for one path and 3 for the other.

Then in the partial orientation obtained, we can conclude that in the C_4 induced by $a, c, i,$ and j, the vertex c will be a sink and vertex j will be a source. However, as mentioned in the beginning of this proof, this orientation will not lead to valid orientation, since j and c play the same role as a and d in G_1. Hence, G_3 does not admit a valid CBU orientation. $\square$

Remark 17 *The graph G_3 used in the proof of Lemma 16 is actually a cover graph. In Figure 7, the bottom picture depicts its Hasse diagram.*

Since every planar graph with girth at least 10 has circular chromatic number at most $5/2$ [14], the forthcoming Theorem 35 implies that such a graph necessarily belongs to CBU.

Problem 18 *What is the lowest $g \in [5, \ldots, 10]$ such that every planar graph G with girth at least g belongs to CBU.*

6 Structural properties of d-CBU and CBU

Theorem 19 *For every $d \geq 1$, the class of d-CBU graphs is strictly contained in the class of $(d+1)$-CBU graphs.*

The case $d = 1$ follows from the earlier observation that 1-CBU graphs correspond to forests of paths, and from the many examples of 2-CBU graphs provided above. For $d \geq 2$, the following structural lemma allows us to translate the strict containment of boxicity b bipartite graphs, to the strict containment of d-CBU. Indeed, it is known that the graph obtained from the complete bipartite graph $K_{2b,2b}$ by removing a perfect matching has boxicity exactly b [33].

Lemma 20 *Given a connected bipartite graph B, with parts X and Y, let B' be the graph obtained from B, by adding a path xzy and by connecting x and y to every vertex in X and Y, respectively. Then, B has boxicity at most d if and only if B' belongs to $(d+1)$-CBU.*

Proof: Let us begin with the simpler "only if" part. We proceed as in the proof of Theorem 6 in order to obtain a $(d+1)$-CBU representation of B such that every vertex of X (resp. Y) corresponds to $[0, 1]$ (resp. $[1, 2]$) in the space spanned by e_1. Then it suffices to add the boxes for x, y and z. For a sufficiently large Ω, x is represented by $[-1, 0] \times [-\Omega, +\Omega] \times \ldots \times [-\Omega, +\Omega]$, y is represented by $[2, 3] \times [-\Omega, +\Omega] \times \ldots \times [-\Omega, +\Omega]$, and z is represented by $[0, 2] \times [\Omega - 1, \Omega] \times \ldots \times [\Omega - 1, \Omega]$.

For the "if" part, consider a $(d+1)$-CBU representation of B', and the homogeneous arc labeling of B' induced by this representation. We first prove that all the arcs between X and Y are oriented in the same direction. Towards a contradiction, consider a path $x_1y_2x_3$ with $x_1, x_3 \in X$ and $y_2 \in Y$, and such that the edges are oriented from x_1 to y_2, and from y_2 to x_3. This forces the remaining edges of the 4-cycle $yx_1y_2x_3$ to be oriented from x_1 to y, and from y to x_3 (see Claim 2). Now we cannot orient the edge y_2x, xz, zy in such a way to fulfill Claim 2 for the 5-cycles $xzyx_1y_2$ and $xzyx_3y_2$. Indeed for the first one, yz should be oriented from y to z, while for the second one it should be oriented from z to y, a contradiction.

This orientation ensures that the labels of all the arcs is the same. This implies that there is an hyperplane $\mathcal{H}$ orthogonal to e_1 such that for any pair of intersecting boxes $x' \in X$ and $y' \in Y$, their intersection belongs to $\mathcal{H}$. This implies that projecting the $(d+1)$-CBU representation (restricted to B) along e_1 leads to a boxicity d representation of B. □

It is clear that CBU is hereditary (i.e. closed under induced subgraphs) but actually it is also closed under subgraphs.

Theorem 21 *For any subgraph H of G, $G \in$ CBU implies that $H \in$ CBU. More precisely, if there is a complete bipartite graph $K_{a,b}$ such that $V(K_{a,b}) \subseteq V(G)$, and such that $E(H) = E(G) \setminus E(K_{a,b})$, then if G belongs to d-CBU then H belongs to $(d+1)$-CBU.*

Proof: Let A, B be the parts of $K_{a,b}$. Given a CBU representation of G in $\mathbb{R}^d$ we are going to build a CBU representation of H in $\mathbb{R}^{d+1}$. For this, the first d intervals defining each d-box remain unchanged while the last interval is $[0,1]$ for the vertices in A, $[2,3]$ for the vertices in B, and $[0,3]$ for the remaining vertices. It is now easy to check that two boxes intersect if and only if they intersect in G and if they are not adjacent in $K_{a,b}$. It is also clear that the intersections occur on planes orthogonal to e_1. □

The graph class CBU is also closed by the addition of false twins.

Theorem 22 *For any graph G and any vertex v of G, consider the graph G^v obtained from G by adding a new vertex v' such that $N(v') = N(v)$. Then $G \in$ CBU if and only if $G^v \in$ CBU. Furthermore, if $G \in d$-CBU then $G^v \in (d+1)$-CBU.*

Proof: The "if" part is obvious as G is an induced subgraph of G^v. For the "only if" part, given a CBU representation of G in $\mathbb{R}^d$ we are going to build a CBU representation of G^v in $\mathbb{R}^{d+1}$. For this, the first d intervals defining each d-box remain unchanged, and those of v' are the same as those of v. The last interval is $[0,1]$ for v, $[2,3]$ for v', and $[0,3]$ for all the remaining vertices. It is now easy to check that two boxes intersect if and only if they intersected and if one of them is distinct from v or v'. It is also clear that the intersections occur on planes orthogonal to e_1. □

Shift graphs were introduced by P. Erdős and A. Hajnal in [17] (see Theorem 6 therein). Those are the graphs H_m whose vertices are the ordered pairs (i,j) satisfying $1 \le i < j \le m$, and where two pairs (i,j) and (k,l) form an edge if and only if $j = k$ or $l = i$. Note that such graphs admit a homogeneous arc labeling ℓ defined by $\ell(\{(i,j),(j,k)\}) = j$, and by orienting any edge $\{(i,j),(j,k)\}$ from (i,j) to (j,k).

Theorem 23 *The graph H_m belongs to $(m-1)$-CBU. Furthermore, H_m has a CBU representation such that in the first dimension the vertex (i,j) corresponds to interval $[i,j]$.*

Proof: This clearly holds for the one vertex graph H_2. By induction on m consider a representation of H_{m-1}, add a false twin for every vertex $(i, m-1)$ and modify the first interval of these new twins, so that the interval $[i, m-1]$ becomes $[i,m]$. These boxes correspond to the vertices (i,m) with $i < m-1$. For the vertex $(m-1,m)$, one should add a box $[m-1,m] \times [-\Omega,+\Omega] \times \ldots \times [-\Omega,+\Omega]$, for a sufficiently large Ω. To deal with the intersections between this box and the boxes of the other vertices (i,m), we add a new dimension such that vertex $(m-1,m)$ has interval $[1,2]$, the vertices $(i,m-1)$ have interval $[1,2]$, the vertices (i,m) with $i < m-1$ have interval $[3,4]$, and all the other vertices have interval $[1,4]$. □

Theorem 24 *For every n-vertex graph G the following properties are equivalent.*

a) G belongs to CBU.

b) G admits a homogeneous arc labeling.

c) G is the subgraph of a graph H_m^t, obtained from the shift graph H_m by iteratively adding t false twins, for some values m, t such that $m + t \le n + 1$.

d) G belongs to $(2n-1)$-CBU.

Proof: We have already seen that $a) \Rightarrow b)$. Let us show $b) \Rightarrow c)$. Consider a homogeneous arc labeling of G, with labels in $[2, m-1]$, for the minimum m. By minimality of m, note that all the labels are used, and thus $m-2 \leq n-1$. Let H_m^t be the graph obtained from the shift graph H_m by adding $t_{i,j}$ false twins of vertex (i,j) if there are $t_{i,j}+1$ vertices of G whose incoming arcs are labeled i, and whose outgoing arcs are labeled j. For the vertices without incoming (resp. outgoing) arcs assume that those are labeled 1 (resp. m). Consider now an injective mapping $\gamma : V(G) \longrightarrow V(H_m^t)$, such that any vertex with incoming and outgoing arcs labeled i, j is mapped to (i,j) or one of its twins. This mapping ensures us that G is a subgraph of H_m^t. Indeed, for any two adjacent vertices u, v of G linked by an edge labelled j oriented from u to v, their incoming and outgoing arcs are labeled i, j and j, k respectively, for some $i < j < k$, and thus the vertices $\gamma(u)$ and $\gamma(v)$ of H_m^t are adjacent, as they correspond to or are twins of (i,j) and (j,k).

We now show $c) \Rightarrow d)$. Consider a graph H_m^t containing G as a subgraph, for some m, t such that $m+t \leq n+1$. By Theorem 23 and Theorem 22 we have that H_m^t belongs to $(m-1+t)$-CBU, and so to n-CBU. Starting from H_m^t one can obtain G by successively deleting $n-1$ stars $K_{1,b}$, so by Theorem 21, we have that G belongs to $(2n-1)$-CBU. Finally, $d) \Rightarrow a)$ is obvious. □

It is easy to see that every complete bipartite graph belongs to 3-CBU. By Theorem 21, removing stars $K_{1,b}$ centered on the smallest part, one obtains that every n-vertex bipartite graph belongs to $(\lfloor n/2 \rfloor + 3)$-CBU. One can reach a slightly better bound from Theorem 6, and the fact that for every graph G, $box(G) \leq \lfloor n/2 \rfloor$ [31].

Corollary 25 *Every bipartite graph G belongs to CBU. Furthemore, if $|V(G)| = n$ then G belongs to $(\lfloor n/2 \rfloor + 1)$-CBU.*

As already mentioned, some bipartite graphs have arbitrary large boxicity, and thus there is no fixed d such that every bipartite graph belongs to d-CBU. For large girth graphs it is a different.

Theorem 26 *For any $g \geq 3$, there exist graphs of girth g not contained in CBU.*

Proof: Indeed, for any $g \geq 3$ there exist graphs of girth g with fractional chromatic number at least 4 [16]. (Actually, their fractional chromatic number is arbitrarily large). By Theorem 39, such graphs cannot belong to CBU. □ Nevertheless, the following remains open.

Problem 27 *Are there integers d, g such that every girth g graph G of CBU belongs to d-CBU?*

The remarks above imply that testing if a bipartite graph belongs to CBU is obvious (computable in constant time), while for girth g graphs the question is more involved, as CBU has such graphs included and some other excluded. The following section treats the computational problem of recognizing CBU graphs.

7 Recognition

Computing the boxicity of a bipartite graph is a difficult problem. It is known that deciding whether a bipartite graph has boxicity two is NP-complete [24]. Furthermore, it is proven in [1] that it is not possible to approximate the boxicity of a bipartite graph within a $O(n^{0.5-\varepsilon})$-factor in polynomial time, unless $NP = ZPP$. By Lemma 20, for every bipartite graph B there is a graph B' (obtained in polynomial time) such that the minimum value d for which B' belongs to d-CBU is exactly $d = box(B) + 1$.

Corollary 28 *It is NP-complete to decide whether a graph belongs to 3-CBU. Furthermore, unless $NP = ZPP$, one cannot approximate in polynomial time and within a $O(n^{0.5-\varepsilon})$-factor, the minimum value d for which an input graph G belongs to d-CBU.*

This implies that for most values d the problem of deciding whether an input graph belongs to d-CBU, cannot be computed in polynomial time, unless $NP = ZPP$. The hypothesis $NP = P$ being stronger than $NP = ZPP$, it would be stronger to know that it is NP-complete to decide if an input graph belongs to d-CBU.

Problem 29 *For which values d, is it NP-complete to decide whether a graph belongs to d-CBU? Are there values d, in particular for $d = 2$, for which the problem is polynomial?*

By Lemma 20, this problem would be solved, for $d \geq 3$, if the following problem admits a positive answer.

Problem 30 *For any $d \geq 3$, is it NP-complete to decide whether a bipartite graph B has boxicity at most d?*

Another computational problem is testing the membership in CBU.

Problem 31 *Is it polynomial to decide whether a graph belongs to CBU?*

We have seen that some triangle-free planar graphs, or some graphs with arbitrary large girth, are not in CBU. We can thus restrict the problem.

Problem 32 *Is it polynomial to decide whether a planar graph G belongs to CBU? For some $g \geq 3$, is it polynomial to decide whether a graph G of girth at least g belongs to CBU?*

7.1 Recognition through forbidden induced subgraphs

As CBU and d-CBU are closed under induced subgraphs, they are characterized by a set of minimal excluded induced subgraphs, $\mathcal{F}_{CBU}$ and $\mathcal{F}_{d-CBU}$. If one of these sets is finite, then recognizing the corresponding class becomes polynomial-time tractable. Thus by Corollary 28, the set $\mathcal{F}_{3-CBU}$ (resp. $\mathcal{F}_{d-CBU}$ for $d \geq 4$) is not finite, unless $P = NP$ (resp. unless $NP = ZPP$). For the set $\mathcal{F}_{2-CBU}$ (resp. $\mathcal{F}_{CBU}$), we are sure that it is infinite. Indeed, Theorem 14 (resp. Theorem 26) provides an infinite sequence of graphs $(G_i)_{i\geq 0}$ not in 2-CBU (resp. not in CBU) such that the girth of G_i is at least i. If there was an n such that every graph in $\mathcal{F}_{2-CBU}$ (resp. $\mathcal{F}_{CBU}$) has at most n vertices, then to exclude G_{n+1} one would need to have a tree in $\mathcal{F}_{2-CBU}$ (resp. $\mathcal{F}_{CBU}$). This is not the case as for every tree T, we have that $T \in 2\text{-}CBU \subseteq CBU$.

7.2 Recognition through homogeneous arc labelings

By Theorem 24, a graph G belongs to CBU if and only if it admits a homogeneous arc labeling. If we are given an orientation of a graph G it is simple to check whether this orientation admits such labeling. For example, one can use linear programming. For each arc uv, set a variable ℓ_{uv} corresponding to a label, and for any two incident arcs, add a constraint. For two arcs uv and uw (resp. uv and wv), the constraint is $\ell_{uv} = \ell_{uw}$ (resp. $\ell_{uv} = \ell_{wv}$). For two arcs uv and vw, the constraint is $\ell_{uv} + 1 \leq \ell_{vw}$. Problem 31 thus reduces to deciding whether a graph G admits an orientation that is homogeneously labelable. In the following we characterize such orientations.

A cycle $(v_0, v_1, \ldots, v_{n-1})$ is said *badly oriented* if there is a vertex v_i whose incident arcs are $v_{i-1}v_i$ and v_iv_{i+1}, and if there is no vertex v_j whose incident arcs are $v_{j+1}v_j$ and v_jv_{j-1} (indices being considered $\mod n$).

Theorem 33 *An orientation of a graph G admits a homogeneous labeling if and only if there is no badly oriented cycle.*

Proof: For the "only if" part, consider a badly oriented cycle $(v_0, v_1, \ldots, v_{n-1})$ with arcs $v_{n-1}v_0$ and v_0v_1, but with no vertex $v_j \neq v_0$ whose incident arcs are $v_{j+1}v_j$ and v_jv_{j-1}. This latter condition implies that in any homogeneous labeling the sequence of labels for the edges (without considering their orientation) $v_0v_1, v_1v_2, \ldots, v_{n-2}v_{n-1}, v_{n-1}v_0$ is non-decreasing, while the former condition implies that the label of v_0v_1 is greater than the one of $v_{n-1}v_0$, a contradiction. Thus this orientation of G does not allow any homogeneous labeling.

For the "if" part, consider a graph G oriented without badly oriented cycle, and consider a source u, and let us denote $v_1, \ldots, v_n$ its out-neighbors. If for every vertex v_i, u is its unique in-neighbor, then by recurrence on the number of vertices we assume that $G \setminus \{u\}$ has a homogeneous labeling, and we label the arcs incident to u with a sufficiently small value, say $-\Omega$. In that case it is easy to check that this labeling is homogeneous.

Otherwise, let v_i and u' be vertices such that G has arcs from both u and u' toward vertex v_i. In that case, consider the oriented graph G' obtained from $G \setminus \{u\}$ by adding the arcs $u'v_1, \ldots, u'v_n$, if missing.

Claim 34 *G' has no badly oriented cycle.*

Proof: If G' had a badly oriented cycle C, this one should go through a newly added arc $u'v_j$. If $v_i \notin C$, by replacing the arc $u'v_j$ by the path (u', v_i, u, v_j) one would obtain a badly oriented cycle in G, a contradiction. We thus assume that $v_i \notin C$, and now by replacing the arc $u'v_j$ by the path (u', v_i, u, v_j) we obtain a badly oriented closed walk W (that is a walk where there are consecutive "forward" arcs, but no consecutive "backward" arcs). Let us denote P and P' the sub-paths of $C \setminus \{u'v_j\} \subsetneq G$ linking v_i and v_j, and linking u' and v_i, respectively.

Let us show that if the edge incident to v_i in P' is oriented from v_i to the other end, denoted v, then this arc is backward with respect to C. Indeed, the cycle $C_{P'}$ of G formed by P' and the arc $u'v_i$, has consecutive arcs oriented in the same direction, $u'v_i$ and v_iv, and (as G contains no badly oriented cycles) has consecutive arcs oriented in the other direction. The latter pair of arcs belonging both to $P' \subset C$, they are forward with respect to C, thus v_iv is backward.

Similarly, let us show that if the edge incident to v_i in P is oriented from v_i to the other end, denoted w, then this arc is backward with respect to C. Indeed, they cycle C_P of G formed by P and the arcs $uv_i$and uv_j, has consecutive arcs oriented in the same direction, uv_i and v_iw, and (as G contains no badly oriented cycles) has consecutive arcs oriented in the other direction. The latter pair of arcs belong both to $P \subset C$, or they are the arcs incident to v_j. In the former case, these arcs are forward with respect to C, thus v_iv is backward. In the latter case, replacing uv_j with $u'v_j$, one has that the incident arcs of v_j in C are oriented in the same direction. this direction is thus the forward direction, and in that case also v_iv is backward.

We thus have that the arcs incident to v_i cannot be oriented in the same direction (they would form consecutive backward arcs in C), and they are not both oriented from v_i to the other end (they would be both backwards although they have distinct directions). Now we distinguish cases according to the position of the consecutive forward arcs in C. We have that:

a) there are two consecutive forward arcs in $P \cup \{u'v_j\}$, or

b) there are two consecutive forward arcs in $P' \cup \{u'v_j\}$.

In case a), the cycle C_P of G has consecutive forward arcs (by replacing if necessary the arc $u'v_j$ with uv_j). Since this cycle is not badly oriented it also contains consecutive backward arcs. According to the orientation of the arcs, those backwards arcs cannot be the arcs incident to u, or those incident to v_i. Thus they belong both to $P \cup \{uv_j\}$, but this would imply that C also contains consecutive backward arcs, a contradiction.

In case b), the cycle $C_{P'}$ of G has consecutive forward arcs (by replacing if necessary the arc $u'v_j$ with $u'v_i$). Since this cycle is not badly oriented it also contains consecutive backward arcs. According to the orientation of the arcs, those backwards arcs cannot be the arcs incident to v_i. This would imply that C also contains consecutive backward arcs, a contradiction.

This concludes the proof of the claim □

So now, by recurrence on the number of vertices we can assume that G' has a homogeneous labeling, and let ℓ be the label of the arcs outgoing from u'. In that case one can derive a labeling of G by keeping the same labels, and by setting the label ℓ for the arcs outgoing from u. It is easy to check that this labeling is homogeneous. □

Note that Theorem 33 provides another proof that CBU contains every bipartite graph. Indeed, orienting all the edges from one part toward the other, the direction of the arcs alternate along any cycle, and so there is no badly oriented cycle. Actually, we can go a little further.

Theorem 35 *Every graph G with circular chromatic number $\chi_c(G) \leq 5/2$ belongs to CBU.*

Proof: A graph G with circular chromatic number $\chi_c(G) \leq 5/2$ has a homomorphism into the circular complete graph $K_{5/2}$ that is the 5-cycle. As this graph belongs to CBU the theorem follows from Theorem 37. □

Note that we cannot replace $5/2$ by $8/3$ in Theorem 35, as one can easily check that every orientation of $K_{8/3}$ contains a badly oriented cycle.

Problem 36 *What is the largest c such that every graph G with $\chi_c(G) \leq c$ (or with $\chi_c(G) < c$) belongs to CBU.*

Theorem 37 *Given two graphs G, H such that there is an homomorphism $\gamma : V(G) \longrightarrow V(H)$, then if $H \in$ CBU we have that $G \in$ CBU.*

Proof: By Theorem 24, the graph H admits a homogeneous arc labeling, ℓ_H. Orient the edges of G in such a way that $uv \in E(G)$ is oriented as the edge $\gamma(u)\gamma(v) \in E(H)$, that is from u to v if and only if $\gamma(u)\gamma(v)$ is oriented from $\gamma(u)$ to $\gamma(v)$ in H. Similarly we copy the labeling of H's arcs by setting $\ell_G(uv) = \ell_H(\gamma(u)\gamma(v))$. One can easily check that this is a homogeneous arc labeling of G, and thus that G belongs to CBU. □

8 Chromatic Number and Independent Sets

While 2-CBU graphs have chromatic number at most 3 (by Grötzsch's theorem), 3-CBU graphs have unbounded chromatic number.

Theorem 38 (Magnant and Martin [26]) *For any $\chi \geq 1$, there exists a graph in 3-CBU with chromatic number χ.*

However, these graphs have bounded fractional chromatic number, and thus have linear size independent sets. Indeed, G. Simonyi and G. Tardos [32] showed that shift graphs have fractional chromatic number less than 4. As such a bound extends by adding a false twin and by taking a subgraph, we have the following.

Theorem 39 *For any graph $G \in$ CBU, $\chi_f(G) < 4$, and $\alpha(G) > |V(G)|/4$.*

For planar graphs in CBU, this bound on χ_f can be improved by one, but not more.

Theorem 40 *For every planar graph G in CBU we have $\chi_f(G) \leq \chi(G) \leq 3$. On the other hand, for every $n \equiv 2 \pmod 3$ there is a n-vertex planar graph G in CBU such that $\alpha(G) = (n+1)/3$, and thus $\chi_f(G) \geq n/\alpha(G) = 3 - \frac{3}{n+1}$.*

Proof: The first statement follows from Grötzsch's theorem. The second statement follows from graphs constructed by Jones [22], which were proved to have independence number $\alpha(G) = (n+1)/3$. Those graphs form a sequence $J_1, J_2, \ldots$ such that J_1 is the 5-cycle (a_1, b_1, c_1, d, e), and such that J_{i+1} is obtained from J_i by adding three vertices $a_{i+1}, b_{i+1}, c_{i+1}$ such that $N(a_{i+1}) = \{b_i, b_{i+1}\}$, $N(b_{i+1}) = \{a_{i+1}, c_{i+1}\}$, and $N(c_{i+1}) = \{a_i, c_i, b_{i+1}\}$ (see Figure 8). It is already known that those graphs are planar, and it does only remain to show that they belong to CBU. Let us do so by exhibiting a homogeneous arc labeling ℓ. This labeling is such that for any $i \geq 1$ we orient the edges a_ib_i and b_ic_i toward b_i, we orient the edges x_iy_{i+1}, for $x, y \in \{a, b, c\}$, from x_i towards y_{i+1}, and we set $\ell(a_ib_i) = \ell(a_ic_{i+1)} = 2i$, $\ell(c_ib_i) = \ell(c_ic_{i+1)} = 2i$, and $\ell(b_ia_{i+1)} = 2i + 1$. By examining Figure 8 it is clear that this is a homogeneous arc labeling. $\square$

Although 2-CBU lies in the intersection of CBU and planar graphs, it might be the case that the fractional chromatic number of graphs in 2-CBU is bounded by some $c < 3$. Indeed, Jones graphs J_i, for a sufficiently large i, seem to not be in 2-CBU.

Problem 41 *Is there a $c < 3$ such that every graph G in 2-CBU has fractional chromatic number $\chi_f(G) \leq c$?*

A positive answer to this question, would give support to two conjectures. Let $\mathcal{P}_{g\geq 5}$ be the set of planar graph with girth at least five, and let $\mathcal{P}^f_{g\geq 4}$ be the set of planar graph with girth at least four, where every 4-cycle bounds a face. Clearly $\mathcal{P}_{g\geq 5} \subsetneq \mathcal{P}^f_{g\geq 4}$, since these classes avoid Jones graphs it is conjectured that graphs in $\mathcal{P}_{g\geq 5}$, or more generally graphs in $\mathcal{P}^f_{g\geq 4}$, have fractional chromatic number at most c, for some $c < 3$ [13, 15]. However, our problem is not a sub-case of these conjectures (as $K_{2,t}$ belongs to 2-CBU $\setminus \mathcal{P}^f_{g\geq 4}$), nor a super-case (as $\mathcal{P}_{g\geq 5} \setminus$ 2-CBU is not empty, by Theorem 14).

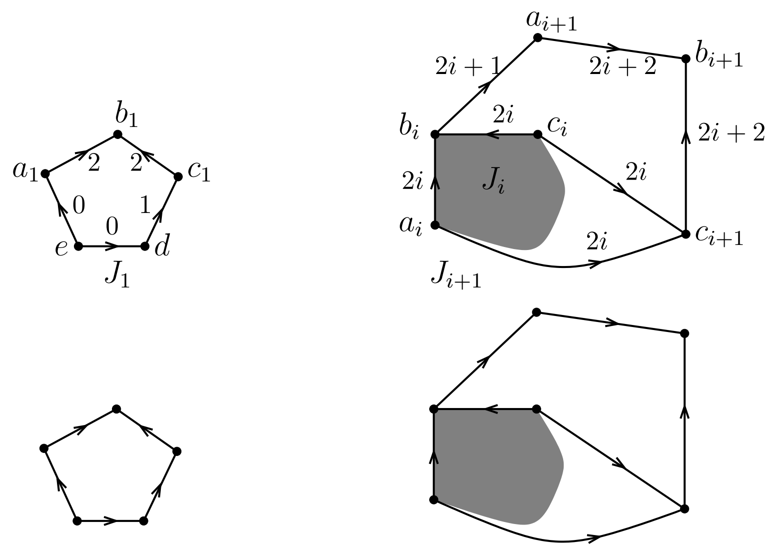

Figure 8: The Jones graphs J_1 and J_{i+1}, with a homogeneous arc labeling. For every $i \geq 1$, this embedding is such that the path $a_i b_i c_i$ is on the outer-boundary. Thus, adding vertices $a_{i+1}, b_{i+1}, c_{i+1}$ does not break planarity.

9 Computational hardness for many problems

We have seen (c.f. Theorem 8, Corollary 9, and Corollary 10) that many 1-subdivided graphs belong to CBU, or even to 3- or 4-CBU. For (≥ 2)-subdivided graphs, the picture is even simpler.

Theorem 42 *For every graph G, if we subdivide every edge at least twice, the obtained graph belongs to 3-CBU.*

Proof: Let us denote $v_1, \ldots, v_n$ the vertices of G, and let $m = |E(G)|$. To construct a CBU representation for any (≥ 2)-subdivision, we start by assigning each vertex v_i to the box $[3i, 3i+1] \times [n-i, n-i+1] \times [0, 2m]$. Then consider each edge e of G in any given order. For the k^{th} edge e assume it links v_i and v_j, for some $i < j$, and assume e is replaced by the path $(v_i, u_1, \ldots, u_r, v_j)$ for some $r \geq 2$. Here, u_1 is assigned to $[3i+1, 3i+2] \times [n-j, n-i+1] \times [2k-1, 2k]$, while the vertices u_ℓ with $2 \leq \ell \leq r$ are assigned to $[3i+2+(\ell-2)(3j-3i-2)/(r-1), 3i+2+(\ell-1)(3j-3i-2)/(r-1)] \times [n-j, n-j+1] \times [2k-1, 2k]$ (see Figure 9). One can easily check that the obtained representation is a 3-CBU representation of the subdivided graph. □

Corollary 43 *The problems of* Minimum Feedback Vertex Set *and* Cutwidth *are NP-hard, even when restricted to 3-CBU graphs. The problems* Maximum Cut, Minimum Vertex Cover, Minimum Dominating Set, *and* Minimum Independent Dominating Set *are APX-hard, even when restricted to 3-CBU graphs.*

Proof: For Minimum Feedback Vertex Set and Cutwidth, this follows from the fact that these problems are NP-hard, and that for any instance, subdividing an edge does not change the solution. For

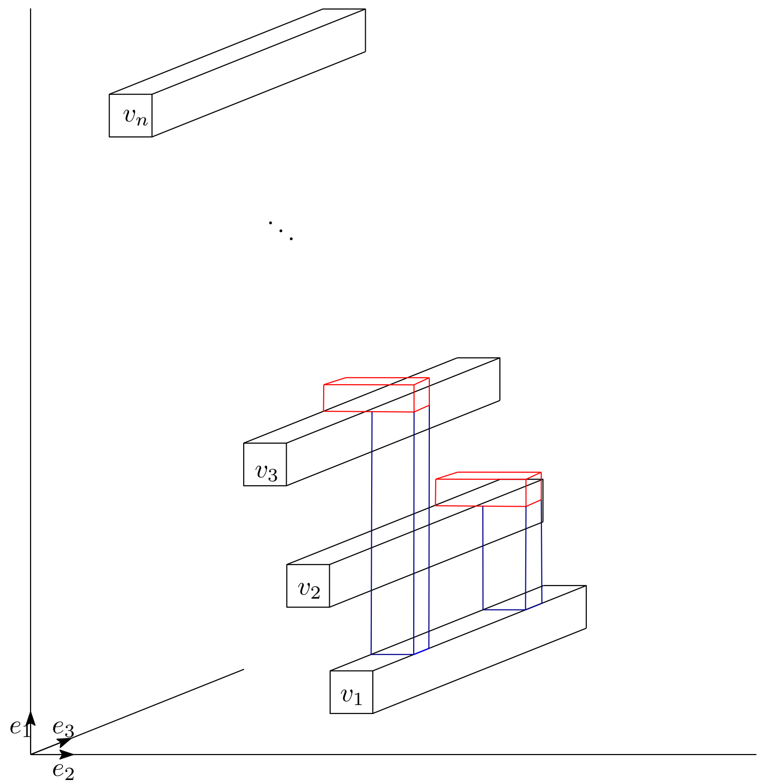

Figure 9: Construction of a 3-CBU representation of a 2-subdivision of a graph.

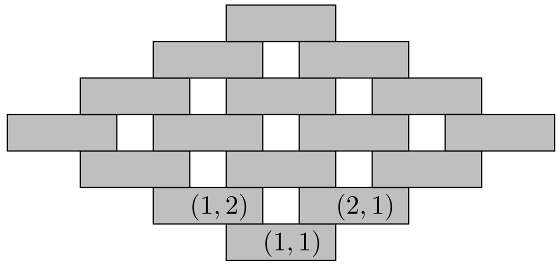

Figure 10: 2-CBU representation of the 4×4 grid.

MAXIMUM CUT, it follows from its APX-hardness and the fact that the maximum cut of a graph G and its 2-subdivision $G_{\text{2-sub}}$ verify $mc(G) = mc(G_{\text{2-sub}}) - 2|E(G)|$ and $3|E(G)|/2 = |E(G_{\text{2-sub}})|/2 \leq mc(G_{\text{2-sub}}) \leq |E(G_{\text{2-sub}})| = 3|E(G)|$. The other problems are shown APX-hard even when restricted to 6-subdivided graphs [6]. □

When restricted to 2-CBU some of these problems become simpler to handle, as every graph in 2-CBU is planar. Indeed, the MAXIMUM CUT problem turns out to be polynomial time solvable [11], while MINIMUM VERTEX COVER, MINIMUM DOMINATING SET, and MINIMUM INDEPENDENT DOMINATING SET admit PTAS [2, 25] (with standard techniques), such as MINIMUM FEEDBACK VERTEX SET [23]. However, many problems remain NP-hard when restricted to 2-CBU.

Theorem 44 *The problems* MAXIMUM INDEPENDENT SET, MINIMUM VERTEX COVER, MINIMUM DOMINATING SET, HAMILTONIAN PATH, *and* HAMILTONIAN CYCLE *are NP-complete, even when restricted to 2-CBU graphs.*

Proof: As these problems belong to NP, it remains to show that they are NP-hard for 2-CBU graphs. Let us first show that the induced subgraphs of grids (so called *grid graphs*) belong to 2-CBU. Consider the $n \times n$ grid G such that $V(G) = \{1, \ldots, n\} \times \{1, \ldots, n\}$, and such that the neighbors of any vertex (i, j) are $\{(i, j-1)(i-1, j), (i, j+1), (i+1, j)\} \cap \{1, \ldots, n\} \times \{1, \ldots, n\}$. Since it suffices to delete some boxes to obtain an induced subgraph, the claim follows by constructing a 2-CBU representation for any such grid G. This construction is obtained by mapping any vertex (i, j) to the box $[i + j - 1, i + j] \times [2i - 2j, 2i - 2j + 3]$ (see Figure 10). As DOMINATION [7], HAMILTONIAN PATH, and HAMILTONIAN CYCLE [21] are NP-hard for grid graphs, those problems are NP-hard for 2-CBU graphs.

For the problems MAXIMUM INDEPENDENT SET and MINIMUM VERTEX COVER, we have to consider a variant of grid graphs, the graph $R'(n_1, n_2)$ depicted in Figure 11, and it is easy to see how to modify the construction above in order to obtain a 2-CBU representation of this type of graphs. Again, this implies that every induced subgraph of such a graph belongs to 2-CBU. As the problems MAXIMUM INDEPENDENT SET and MINIMUM VERTEX COVER are NP-hard for this class (see the proof of Theorem 10 in [19]), those problems are NP-hard for 2-CBU graphs □

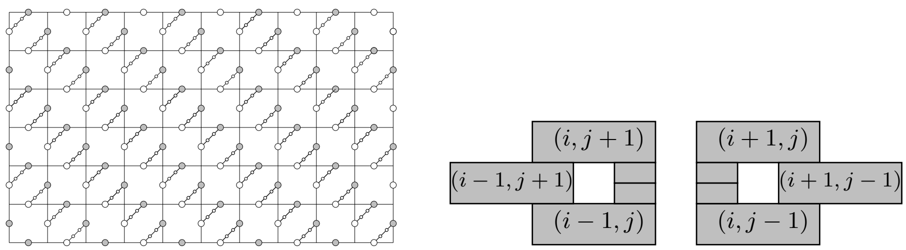

Figure 11: The graph $R'(n_1, n_2)$ and the local modification to obtain its 2-CBU representation. From the 2-CBU representation of the grid given above, one has to delete the box of every vertex (i, j), where i and j are even, and if $i + j \equiv 2 \bmod 4$ one has to replace the box by 4 smaller boxes.

References

[1] Abhijin Adiga, Diptendu Bhowmick, and L. Sunil Chandran. The hardness of approximating the boxicity, cubicity and threshold dimension of a graph. *Discret. Appl. Math.*, 158(16):1719–1726, 2010.

[2] Brenda S. Baker. Approximation algorithms for NP-complete problems on planar graphs. *Journal of the ACM*, 41:153–180, 1994.

[3] Irith Ben-Arroyo Hartman, Ilan Newman, and Ran Ziv. On grid intersection graphs. *Discret. Math.*, 87:41–52, 1991.

[4] David Bremner, William Evans, Fabrizio Frati, Laurie Heyer, Stephen G. Kobourov, William J. Lenhart, Giuseppe Liotta, David Rappaport, and Sue H. Whitesides. On representing graphs by touching cuboids. In Walter Didimo and Maurizio Patrignani, editors, *Graph Drawing*, pages 187–198, Berlin, Heidelberg, 2013. Springer Berlin Heidelberg.

[5] Graham R. Brightwell. On the complexity of diagram testing. *Order*, 10:297–303, 1993.

[6] Miroslav Chlebík and Janka Chlebíková. The complexity of combinatorial optimization problems on d-dimensional boxes. *SIAM Journal on Discrete Mathematics*, 21, 2007.

[7] Brent N. Clark, Charles J. Colbourn, and David S. Johnson. Unit disk graphs. *Discrete Mathematics*, 86(1):165–177, 1990.

[8] Hubert de Fraysseix, Patrice Ossona de Mendez, and Janos Pach. Representation of planar graphs by segments. *Intuit. Geom. (Szeged, 1991), Colloq. Math. Soc. János Bolyai*, 63:109–117, 1994.

[9] Hubert de Fraysseix, Patrice Ossona de Mendez, and Pierre Rosenstiehl. On triangle contact graphs. *Probability and Computing*, 3, 1994.

[10] Zakir Deniz, Esther Galby, Andrea Munaro, and Bernard Ries. On contact graphs of paths on a grid. In Therese C. Biedl and Andreas Kerren, editors, *Graph Drawing and Network Visualization - 26th International Symposium, GD 2018, Barcelona, Spain*, volume 11282 of *LNCS*, pages 317–330. Springer, 2018.

[11] Josep Diaz and Marcin Karminski. Max-cut and max-bisection are NP-hard on unit disk graphs. *Theo. Comp. Sci.*, 377:271–276, 2007.

[12] Ben Dushnik and Edwin Wilkinson Miller. Partially ordered sets. *Amer. J. Math.*, 63, 1941.

[13] Zdeněk Dvořák and Xiaolan Hu. Fractional coloring of planar graphs of girth five. *SIAM Journal on Discrete Mathematics*, 34(1):538–555, 2020.

[14] Zdeněk Dvořák and Luke Postle. Density of 5/2-critical graphs. *Combinatorica*, 37(5):863–886, 2017.

[15] Zdeněk Dvořák, Jean-Sébastien Sereni, and Jan Volec. Fractional coloring of triangle-free planar graphs. *The Electronic Journal of Combinatorics*, pages P4–11, 2015.

[16] Paul Erdős. Graph theory and probability. *Canadian Journal of Mathematics*, 11:34–38, 1959.

[17] Paul Erdős and András Hajnal. Some remarks on set theory. ix. combinatorial problems in measure theory and set theory. *Michigan Mathematical Journal*, 11(2):107, 1964.

[18] Stefan Felsner and Mathew C. Francis. Contact representations of planar graphs with cubes. In *Proceedings of the Twenty-Seventh Annual Symposium on Computational Geometry*, SoCG '11, page 315–320, New York, NY, USA, 2011. Association for Computing Machinery.

[19] Matthew C. Francis, Daniel Gonçalves, and Pascal Ochem. The maximum clique problem in multiple interval graphs. *Algorithmica*, 71:812–836, 2015.

[20] Daniel Gonçalves, Lucas Isenmann, and Claire Pennarun. Planar graphs as L-intersection or L-contact graphs. In *Proceedings of the 2018 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 172–184, 2018.

[21] Alon Itail, Christos H. Papadimitriou, and Jayme Luiz Szwarcfiter. Hamilton paths in grid graphs. *SIAM Journal On Computing*, 11, 1982.

[22] Kathryn Fraughnaugh Jones. Independence in graphs with maximum degree four. *Journal of Combinatorial Theory, Series B*, 37(3):254–269, 1984.

[23] Jon Kleinberg and Amit Kumar. Wavelength conversion in optical networks. *Journal of Algorithms*, 38(1):25–50, 2001.

[24] Jan Kratochvíl. A special planar satisfiability problem and a consequence of its np-completeness. *Discrete Applied Mathematics*, 52(3):233–252, 1994.

[25] Richard J. Lipton and Robert E. Tarjan. Applications of a planar separator theorem. *SIAM J. Comput.*, 9:615–627, 1980.

[26] Colton Magnant and Daniel L. Martin. Coloring rectangular blocks in 3-space. *Discussiones Mathematicae Graph Theory*, 31, 2011.

[27] Jaroslav Nešetřil and Vojtech Rödl. Complexity of diagrams. *Order*, 3:321–330, 1987.

[28] Jaroslav Nešetřil and Vojtech Rödl. Erratum: Complexity of diagrams. *Order*, 10:393 – 393, 1993.

[29] Jaroslav Nešetřil and Vojtech Rödl. More on the complexity of cover graphs. *Commentationes Mathematicae Universitatis Carolinae*, 36, 1995.

[30] Bruce A. Reed and David Allwright. Painting the office. *Mathematics-in-Industry Case Studies Journal*, 1, 2008.

[31] Fred S. Roberts. On the boxicity and cubicity of a graph. In William T. Tutte, editor, *Recent Progress in Combinatorics*, pages 301–310, 1969.

[32] Gábor Simonyi and Gábor Tardos. On directed local chromatic number, shift graphs, and borsuk-like graphs. *Journal of Graph Theory*, 66(1):65–82, 2011.

[33] L. Sunil Chandran, Anita Das, and Chintan D. Shah. Cubicity, boxicity, and vertex cover. *Discrete Mathematics*, 309(8):2488–2496, 2009.

[34] Carsten Thomassen. Plane representations of graphs. *Progress in graph theory (Bondy and Murty, eds.)*, pages 336–342, 1984.

[35] Carsten Thomassen. Interval representations of planar graphs. *J. Combin. Theory Ser. B*, 40:9–20, 1986.