Bijective proof of a conjecture on unit interval posets

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In a recent preprint, Matherne, Morales and Selover conjectured that two different representations of unit interval posets are related by the famous zeta map in q, t-Catalan combinatorics. This conjecture was proved recently by Gélinas, Segovia and Thomas using induction. In this short note, we provide a bijective proof of the same conjecture with a reformulation of the zeta map using left-aligned colored trees, first proposed in the study of parabolic Tamari lattices.

Keywords: Dyck path, zeta map, unit interval poset, plane tree

1 Introduction

The study of unit interval posets started in statistics (Wine and Freund, 1957) and psychology (Scott, 1964). However, it has since been connected to other objects in algebraic combinatorics, such as $(3 + 1)$-free posets Skandera and Reed (2003) and positroids Chavez and Gotti (2018). The connection to $(3 + 1)$-free posets is particularly important, as these posets are at the center of the Stanley–Stembridge conjecture in Stanley and Stembridge (1993), which states that the chromatic symmetric function of the incomparability graph of a $(3 + 1)$-free poset has only positive coefficients when expanded in the basis of elementary symmetric functions. Unit interval posets thus receive some attention as it can be used to study the structure of $(3+1)$-free posets (Skandera and Reed, 2003; Guay-Paquet, Morales, and Rowland, 2014; Lewis and Zhang, 2013).

It is known in Wine and Freund (1957) that unit interval posets are counted by Catalan numbers, and researchers have given two different bijections to represent a unit interval poset by Dyck paths (Skandera and Reed, 2003; Guay-Paquet, Morales, and Rowland, 2014). In a recent preprint (Matherne, Morales, and Selover), it was conjectured that the two bijections are related by the famous zeta map in q, t-Catalan combinatorics (Haglund, 2008), which was first given in (Andrews, Krattenthaler, Orsina, and Papi, 2002). In this short note, we settle this conjecture using bijections (see Theorem 4.6), in contrast to a recent inductive proof of the same conjecture by Gélinas, Segovia, and Thomas. To this end, we first introduce a bijection Ξ_{poset} between unit interval posets and plane trees. We then show that the two different known bijections from unit interval posets are conjugates of special cases of some bijections Ξ_{steep} and Ξ_{bounce} from Ceballos, Fang, and Mühle (2020) constructed for the study of parabolic Tamari lattices. Using the link between $\Xi_{\text{steep}}, \Xi_{\text{bounce}}$ and the zeta map established in the same article, we conclude our proof.

The rest of the article is organized as follows. In Section 2 we give the basic definitions. Then we give our bijection between unit interval posets and plane trees in Section 3, along with the related bijections from Ceballos, Fang, and Mühle (2020). Finally in Section 4 we recall the two known bijections between unit interval posets and Dyck paths, and then establish their link with the bijections in Section 3, leading to a bijective proof of our main result (Theorem 4.6).

2   Preliminary

For convenience, we write $[n]$ for the set $\{1, 2, \ldots, n\}$. We consider finite partially order sets (or *poset* for short) of the form $P = (P_{\mathrm{elem}}, \preceq)$, where P_{elem} is a finite set and $\preceq$ a partial order on P_{elem}. Given a set S of real numbers $x_1 < x_2 < \cdots < x_n$, we define a partial order $\preceq_S$ on $[n]$ by taking $i \preceq_S j$ if and only if $x_i + 1 < x_j$. The order $\preceq_S$ can also be seen as defined on intervals of unit length starting at the x_i's, where $i \preceq_S j$ if the interval starting with x_i is on the left of that with x_j without overlap. A poset P with n elements is a *unit interval order* if there is some $S \subset \mathbb{R}$ with n elements such $P \cong ([n], \preceq_S)$. In this case, we call S a *starting set* of P. We sometimes represent unit interval orders as $([n], \preceq_S)$ for some S hereinafter.

We have the following characterization of unit interval orders.

Theorem 2.1 (Scott (1964, Theorem 2.1))**.** *A poset P is a unit interval poset if and only if it is $(3+1)$-free and $(2+2)$-free, that is, the order induced on any four elements cannot be a chain of 3 elements plus an incomparable element, or two disjoint chains each containing 2 elements.*

Unit interval orders are counted by Catalan numbers.

Proposition 2.2 (Wine and Freund (1957))**.** *The number of unit interval posets with n elements is the n-th Catalan number* $\mathrm{Cat}_n = \frac{1}{2n+1}\binom{2n+1}{n}$.

By definition, we may represent a unit interval poset P by a set of real numbers S, though the choice of S is clearly not unique. In the following, for convenience, we use this perspective of (starting points of) intervals, which is arguably easier to manipulate. We denote by $\mathcal{P}_n$ the set of unit interval posets with n elements.

There are many other families of combinatorial objects counted by Catalan numbers, such as Dyck paths and plane trees. A *Dyck path* is a lattice path formed by north steps $\uparrow = (0, 1)$ and east steps $\rightarrow = (1, 0)$, starting at $(0, 0)$ and ending on the diagonal $y = x$ while staying weakly above it. The *size* of a Dyck path is the number of its north steps. We denote by $\mathcal{D}_n$ the set of Dyck paths of size n. We define plane trees recursively: a *plane tree* T is either a single node (a *leaf*) or an *internal node* u linked by edges to a sequence of plane trees called *sub-trees*. In the latter case, the node u is also called the *root* of T, and the roots of sub-trees are called *children* of u. We denote by $\mathcal{T}_n$ the set of plane trees with n non-root nodes. We recall the well-known fact that $|\mathcal{T}_n| = |\mathcal{D}_n| = \mathrm{Cat}_n$. For a node u in a plane tree T, its *depth*, denoted by $d(u)$, is the distance (number of edges) between u and the root of T.
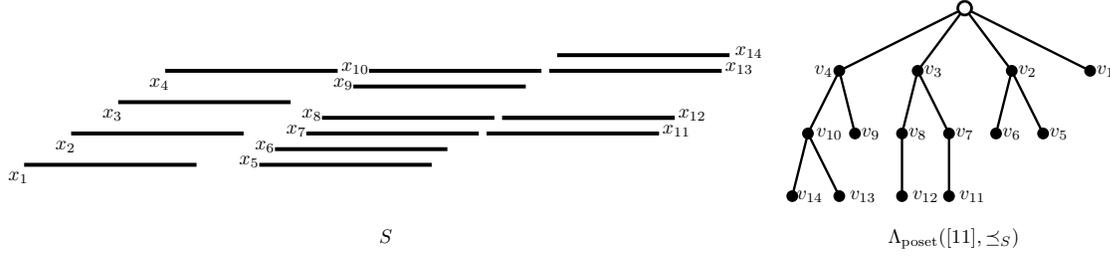
Fig. 1: Example of Λ_{poset} from a unit interval poset defined by a set of unit intervals to a plane tree

3 Unit interval posets and plane trees

We start by a new bijection between unit interval posets and plane trees.

Construction 3.1. Let $S = \{x_1 < \cdots < x_n\}$ be a starting set of a unit interval poset $P = ([n], \preceq_S)$. We define a plane tree T as follows (see Figure 1 for an example). We set $x_0 = x_1 - 2$. We denote by v_i the node of T corresponding to x_i. For $i \in [n]$, the parent of v_i is v_j if and only if j is the largest index such that $j \preceq_S i$. By the definition of x_0, the parent of each v_i is well-defined, and all nodes are descendants of x_0. We then order children of the same node from left to right with *decreasing* index. We thus conclude that T is a well-defined plane tree. We note that T depends only on $\preceq_S$. We define $\Lambda_{\mathrm{poset}}(P) = T$.

Construction 3.2. Given a plane tree T with n non-root nodes, we define an order relation $\preceq_S$ on $[n]$ induced by some $S \subset \mathbb{R}$. Let r_T be the root of T, and m the maximal arity of T, *i.e.*, the maximal number of children of nodes in T. Given a node u in T, if it is the i-th child of its parent *from right to left*, then we define $c(u) = i$. We observe that $1 \leq c(u) \leq m$. For a non-root node u in T, let $u_0 = r_T, u_1, \ldots, u_{d(u)} = u$ be the nodes on the unique path from the root $u_0 = r_T$ to $u_{d(u)} = u$, where $d(u)$ is the depth of u. We then define a real number x_u associated to u using base $(m + 2)$:

$$x_u = d(u) + (0.c(u_1)c(u_2)\cdots c(u_{d(u)}))_{(m+2)} = \ell + \sum_{i=1}^{d(u)} c(u_i)(m + 2)^{-i}. \tag{1}$$

It is clear from (1) that x_u is strictly decreasing from left to right for nodes of the same depth in T. Let S be the set of x_u for non-root nodes u in T. We define $\Xi_{\mathrm{poset}}(T) = ([n], \preceq_S)$.

As an example of Construction 3.2, let T be the tree on the right part of Figure 1, with nodes labeled as in the figure, and S the set of numbers in the construction. The maximal arity of T is $m = 4$, achieved by the root. The number $x_{v_{12}} \in S$ for the node v_{12} is thus $x_{v_{12}} = 3 + (0.321)_6 = 769/216$, as v_{12} is of depth 3, and it is the first child *from right to left* of its parent v_8, itself the second child *from right to left* of its parent v_3, which is the third child *from right to left* of the root.

The following lemma gives a direct connection between a plane tree and its corresponding unit interval poset obtained from Ξ_{poset}.

Lemma 3.3. *For $T \in \mathcal{T}_n$, we take S to be the set constructed in Construction 3.2, and x_u the number constructed from a non-root node u of T. Then for non-root nodes u, v, the parent of u in T is v if and only if x_v is the largest element in S smaller than $1 + x_u$.*

Proof: Suppose that T has maximal arity m. Let v' be the parent of u. By (1), we have $1 < x_u - x_{v'} < 1 + (m+2)^{-\ell+1}$, with ℓ the distance from v' to the root. Then, x_v satisfying $x_{v'} \leq x_v < 1 + x_u$ means $0 \leq x_v - x_{v'} < (m+2)^{-\ell+1}$, which means $v = v'$ by (1), as it requires v and v' to have the same distance to the root, but in this case $|x_v - x_{v'}|$ cannot be smaller than $(m+2)^{-\ell+1}$ if $v \neq v'$. We thus have the equivalence. □

We now show that Λ_{poset} and Ξ_{poset} are bijections and are inverse to each other.

Proposition 3.4. *For all $n \geq 1$, the map Λ_{poset} is a bijection from $\mathcal{P}_n$ to $\mathcal{T}_n$, with Ξ_{poset} its inverse.*

Proof: By Proposition 2.2, we have $|\mathcal{P}_n| = \mathrm{Cat}_n = |\mathcal{T}_n|$. We thus only need to show that $\Lambda_{\mathrm{poset}} \circ \Xi_{\mathrm{poset}} = \mathrm{id}$. Given a plane tree $T \in \mathcal{T}_n$, let $P = ([n], \preceq_S) = \Xi_{\mathrm{poset}}(T)$ and $T' = \Lambda_{\mathrm{poset}}(P)$. Let $S = \{x_1 < \cdots < x_n\}$ be the set of real numbers constructed in Construction 3.2 for $\preceq_S$. Given $i \in [n]$, let u_i be the node in T that gives rise to x_i in S, and u'_i the node in T' that represents x_i. By Lemma 3.3 and Construction 3.1, for any $i, j \in [n]$, the node u_i is the parent of u_j in T if and only if u'_i is the parent of u'_j in T'. Then, T' has the same order of siblings as T, as in both we order siblings with decreasing order in their corresponding real numbers in S. We thus conclude that $T = T'$. □

We now restate two previously known bijections between plane trees and Dyck paths that will be used to prove our main result.

Construction 3.5. Let $n \geq 1$ and $T \in \mathcal{T}_n$, we construct a Dyck path D by taking the *clockwise* contour walk starting from the top of the root of T (see the Dyck path on the right of Figure 2). During the walk, when we pass an edge for the first (resp. second) time, we append $\uparrow$ (resp. $\rightarrow$) to D. We define $\Xi_{\mathrm{steep}}(T) = D$. The map Ξ_{steep} is a bijection from $\mathcal{T}_n$ to $\mathcal{D}_n$ for all $n \geq 1$, and we denote its inverse by Λ_{steep}.

Construction 3.6. Let $n \geq 1$ and $T \in \mathcal{T}_n$, we construct a Dyck path D by appointing the number of north steps at each x-coordinate (see the Dyck path on the left of Figure 2). Let α_ℓ be the number of nodes of depth ℓ (thus distance ℓ to the root), and $\ell_{\max}$ the maximal depth of nodes in T. We take $s_\ell = \alpha_1 + \cdots + \alpha_\ell$. Given $1 \leq k \leq n - 1$, there is a unique way to write $k = s_\ell - r$ with $0 \leq r < \alpha_\ell$ and $1 \leq \ell \leq \ell_{\max}$. In this case, the number of north steps in D on $x = k$ is the number of children of the $(r+1)$-st node of depth ℓ. The number of north steps in D on $x = 0$ is the number of children of the root. We define $\Xi_{\mathrm{bounce}}(T) = D$. The map Ξ_{bounce} is a bijection from $\mathcal{T}_n$ to $\mathcal{D}_n$ for all $n \geq 1$, see Ceballos, Fang, and Mühle (2020, Lemma 3.18), and we denote its inverse by $\Lambda_{\mathrm{bounce}} = \Xi_{\mathrm{bounce}}^{-1}$.

Proposition 3.7 (Ceballos, Fang, and Mühle (2020, Theorem IV)). *Let ζ be the zeta map from $\mathcal{D}_n$ to $\mathcal{D}_n$ with $n \geq 1$. We have $\zeta = \Xi_{\mathrm{bounce}} \circ \Lambda_{\mathrm{steep}}$.*

See Figure 2 for an illustration of Proposition 3.7. Here, we do not give the original definition of the zeta map ζ in Andrews, Krattenthaler, Orsina, and Papi (2002), and will instead be using Proposition 3.7 in the following, as it is better suited to our approach.

Remark 3.8. Albeit the same notation, the maps Ξ_{steep} and Ξ_{bounce} are only special cases of the ones from Ceballos, Fang, and Mühle (2020, Construction 3.10 and Equation 14). More precisely, the central objects of Ceballos, Fang, and Mühle (2020) are called *LAC trees*, which are plane trees with non-root nodes divided into regions algorithmically. We are only using the special case where the region d consists of nodes of depth d, which is also the one used in Ceballos, Fang, and Mühle (2020, Section 3.3) to prove Proposition 3.7. In full generality, the map Ξ_{steep} maps a LAC tree to a *steep pair*, which is a pair of
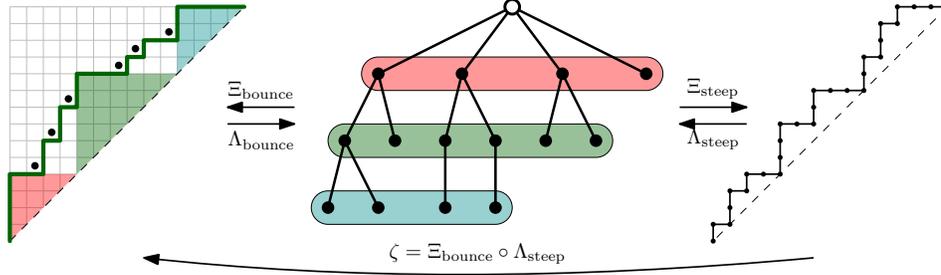
Fig. 2: The zeta map as composition of bijections mediated by trees. The nodes of the same depth of the tree are grouped together. For the Dyck path on the left, the number of north steps on $x = k$ is the number of children of the k-th node in the tree, ordered by increasing depth, then from right to left. The one on the right comes from a clockwise contour walk.

nested Dyck paths with a *steep* upper path, *i.e.*, without two consecutive east steps except on the maximal height. The map Ξ_{bounce} sends a LAC tree to a *bounce pair*, which is a pair of nested Dyck path with a *bounce* lower path, *i.e.*, a concatenation of sub-paths of the form $\uparrow^k \to^k$ for all k. This explains the words "steep" and "bounce" in their notation.

Remark 3.9. The bijection Ξ_{steep} is classical, except that we do the contour walk from right to left instead of from left to right. The bijection Ξ_{bounce} is quite close to the classical bijection between plane trees and Łukasiewicz words, as given in Flajolet and Sedgewick (2009, Section I.5.3)), which can be seen as sequences $(z_0, z_1, \ldots, z_{n-1})$ with $z_i \geq -1$ such that $\sum_{i=0}^{k} z_i \geq 0$ for $0 \leq k < n-1$, and the sum of all z_i's is -1. However, in the classical bijection of Łukasiewicz, we deal with nodes in a depth-first order, but in Ξ_{bounce} they are processed in a breadth-first order.

4 Unit interval posets and the zeta map

In the following, we give the two representations of unit interval posets as Dyck paths, and we detail how they are related to the bijections detailed in Section 3, leading to a proof of our main result (Theorem 4.6).

We start by the first representation, which was first defined implicitly using anti-adjacency matrices of unit interval posets in Skandera and Reed (2003), only involving the poset structure. The following form was first given in Chavez and Gotti (2018).

Construction 4.1. Given $P = ([n], \preceq_S)$ a unit interval poset with $S \subset \mathbb{R}$, we define a Dyck path D as follows. Let $S^+ = \{x + 1 \mid x \in S\}$. Without loss of generality, we can choose S such that $S \cap S^+ = \varnothing$. Then suppose that $S \cup S^+ = \{y_1 < \cdots < y_{2n}\}$. For $i \in [2n]$, the i-th step of D is $\uparrow$ if $y_i \in S$, and is $\to$ otherwise. We define $\varphi(P) = D$. See Figure 3 for an example.

This form of the map φ in Construction 4.1 was given in Chavez and Gotti (2018, Section 5) and proved in Lemma 5.7 therein to be equivalent to the original definition, which is the one used in Matherne, Morales, and Selover, Section 2.3. Although defined here using the set S, the map φ does not depend on the choice of S.

Proposition 4.2. *We have $\varphi \circ \Xi_{\mathrm{poset}} = \Xi_{\mathrm{bounce}}$, and φ is a bijection from $\mathcal{P}_n$ to $\mathcal{D}_n$ for all $n \geq 1$.*
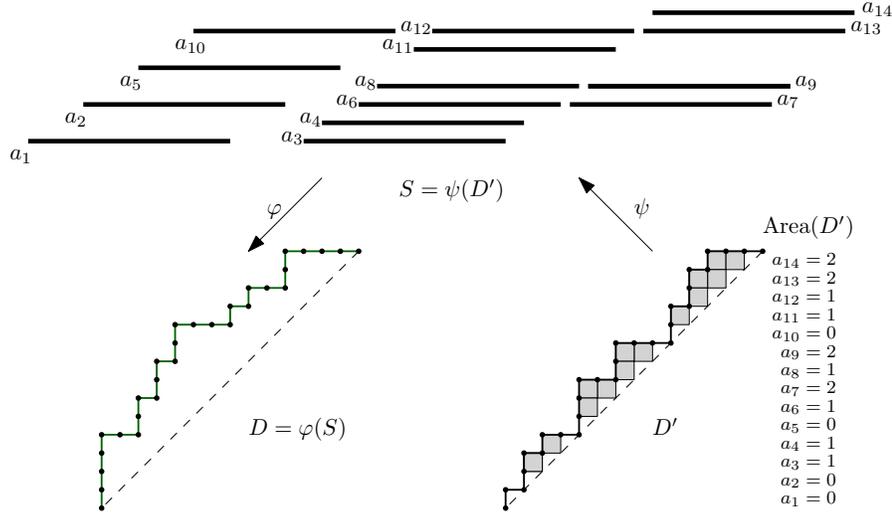
Fig. 3: Example of the bijections φ and ψ

Proof: Let $T \in \mathcal{T}_n$ and $D = \Xi_{\text{bounce}}(T)$. We take $P = \Xi_{\text{poset}}(T) = ([n], \preceq_S) \in \mathcal{P}_n$ with $S = \{x_1 < \cdots < x_n\}$ given in Construction 3.2. Let $D' = \varphi(P)$. To show that $D' = D$, we first notice that a Dyck path is determined by the number of north steps on each line $x = k$ for $0 \leq k \leq n - 1$. Thus, we only need to show that for each k, D' has the same number of north steps on $x = k$ as D has according to Construction 3.6.

For the line $x = 0$, we observe that the number of elements in S strictly smaller than $x_1 + 1$ must be those with integer part equal to 1, thus those from the children of the root of T. For $1 \leq k \leq n - 1$, from Construction 4.1, we know that the number of north steps on the line $x = k$ is the number of elements x_i such that $x_{k-1}+1 < x_i < x_k+1$, as x_k+1 corresponds to the k-th east step in D. Let u_k be the node in T corresponding to x_k. By Lemma 3.3, the nodes in T corresponding to such x_i are children of u_k, meaning that the total number of north steps on $x = k$ is the number of children of u_k. Furthermore, suppose that u_k is of depth ℓ, then we can write $k = s_\ell - r$ with $0 \leq r < \alpha_\ell$ as in Construction 3.6. We know that u_k is the $(r + 1)$-st node of depth ℓ in T, as in Construction 3.1 the values corresponding to nodes of the same depth are decreasing from left to right. We thus conclude that $D = D'$, and $\varphi \circ \Xi_{\text{poset}} = \Xi_{\text{bounce}}$. As both Ξ_{bounce} (Ceballos, Fang, and Mühle, 2020, Lemma 3.18) and Ξ_{poset} (Proposition 3.4) are bijections, we conclude that φ is also a bijection. □

The second presentation was first defined in Guay-Paquet, Morales, and Rowland (2014) for $(3 + 1)$-free posets, and it takes a simpler form for unit interval posets. A more explicit presentation is given in Guay-Paquet (2013, Section 2).

Construction 4.3. For $n \geq 1$, let $D \in \mathcal{D}_n$ be a Dyck path of size n. Its *area vector*, denoted by $\text{Area}(D)$, is a vector $(a_1, \ldots, a_n)$ with a_i the number of full unit squares with the upper edge on $y = i$ between D and the diagonal $y = x$. Such area vectors are characterized by the conditions $a_1 = 0$ and $a_i \leq a_{i-1} + 1$ for $2 \leq i \leq n$. It is clear that the area vector determines the Dyck path. We then define a poset $P = ([n], \preceq)$ by taking $i \prec j$ for $i, j \in [n]$ such that

- Either $a_i + 2 \leq a_j$;

- Or $a_i + 1 = a_j$ and $i < j$.

We define $\psi(D) = P$, and its inverse $\psi^{-1}(P) = D$. See Figure 3 for an example.

The following result ensures that ψ is well-defined. It is a special case of Matherne, Morales, and Selover, Theorem 5.2, which is obtained by combining Guay-Paquet, Morales, and Rowland (2014, Remark 3.2, Proposition 3.11). A self-contained direct proof of this special case can be found in Gélinas, Segovia, and Thomas, Section 6.

Proposition 4.4 (Special case of Matherne, Morales, and Selover, Theorem 5.2). *The map ψ is a bijection from Dyck paths to unit interval posets preserving sizes.*

Proposition 4.5. *We have $\psi = \Xi_{\mathrm{poset}} \circ \Lambda_{\mathrm{steep}}$.*

Proof: Let $P = \psi(D)$, $T = \Lambda_{\mathrm{steep}}(D)$ and $P' = \Xi_{\mathrm{poset}}(D)$. We write $P = ([n], \preceq)$ and $P' = ([n], \preceq_S)$, with S given in Construction 3.2. We show that P is isomorphic to P' after an implicit relabeling.

For $i \in [n]$, let u_i be the i-th non-root node in T that is visited in the clockwise contour walk of T. We note that this order on nodes of T is different from the one used before, for instance in Construction 3.1. Suppose that $\mathrm{Area}(D) = (a_1, \ldots, a_n)$. By Construction 3.5, the depth of u_i is $a_i + 1$. We now define a partial order $\preceq_T$ on non-root nodes in T by taking $u_i \preceq_T u_j$ if and only if $i \preceq j$ in P. We recall that the depth of v in T is denoted by $d(v)$. By Construction 4.3, two non-root nodes v, w in T satisfy $v \prec_T w$ if and only if

- Either $d(v) + 2 \leq d(w)$;

- Or $d(v) + 1 = d(w)$, and the parent of w is either v or on the left of v.

Now, for v a non-root node in T, we take x_v defined in (1). Suppose that there are two non-root nodes v, w of T such that $x_v + 1 < x_w$. Then we have $d(v) + 1 \leq d(w)$. There are two possibilities: either $d(v) + 1 = d(w)$, or $d(v) + 2 \leq d(w)$. In the first case, let w' be the parent of w, we have $x_{w'} + 1 < x_w$. By Lemma 3.3, we know that $x_v \leq x_{w'}$. From (1), we know that either $v = w'$ or v is on the right of w', as x_v is decreasing from left to right for nodes in T of the same depth. Combining with the second case, we conclude that $x_v + 1 < x_w$ if and only if $v \prec_T w$. By Construction 3.2, we have $P \cong P'$, which concludes the proof. □

Theorem 4.6 (Matherne, Morales, and Selover, Conjecture 7.1). *We have $\varphi \circ \psi = \zeta$.*

Proof: Combining Proposition 4.2 and 4.5, we have $\varphi \circ \psi = \Xi_{\mathrm{bounce}} \circ \Lambda_{\mathrm{steep}}$, and we conclude by Proposition 3.7. □

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