Bijective proof of a conjecture on unit interval posets

Wenjie Fang

LIGM, Univ Gustave Eiffel, CNRS, ESIEE Paris, Marne-la-Vallée, France

revisions 20th Jan. 2023, 15th Jan. 2024; accepted 30th Jan. 2024.

In a recent preprint, Matherne, Morales and Selover conjectured that two different representations of unit interval posets are related by the famous zeta map in $q,t$-Catalan combinatorics. This conjecture was proved recently by Gelinas, Segovia and Thomas using induction. In this short note, we provide a bijective proof of the same conjecture with a reformulation of the zeta map using left-aligned colored trees, first proposed in the study of parabolic Tamari lattices.

Keywords: Dyck path, zeta map, unit interval poset, plane tree

1 Introduction

The study of unit interval posets started in statistics [Wine and Freund 1957] and psychology [Scott 1964]. However, it has since been connected to other objects in algebraic combinatorics, such as $(3 + 1)$-free posets [Skandera and Reed 2003] and positroids [Chavez and Gotti 2018]. The connection to $(3 + 1)$-free posets is particularly important, as these posets are at the center of the Stanley–Stembridge conjecture in [Stanley and Stembridge 1993], which states that the chromatic symmetric function of the incomparability graph of a $(3 + 1)$-free poset has only positive coefficients when expanded in the basis of elementary symmetric functions. Unit interval posets thus receive some attention as it can be used to study the structure of $(3+1)$-free posets [Skandera and Reed 2003; Guay-Paquet, Morales, and Rowland 2014; Lewis and Zhang 2013].

It is known in [Wine and Freund 1957] that unit interval posets are counted by Catalan numbers, and researchers have given two different bijections to represent a unit interval poset by Dyck paths [Skandera and Reed 2003; Guay-Paquet, Morales, and Rowland 2014]. In a recent preprint [Matherne, Morales, and Selover], it was conjectured that the two bijections are related by the famous zeta map in $q,t$-Catalan combinatorics [Haglund 2008], which was first given in [Andrews, Krattenthaler, Orsina, and Papi 2002]. In this short note, we settle this conjecture using bijections (see Theorem 4.6), in contrast to a recent inductive proof of the same conjecture by [Gelinas, Segovia, and Thomas]. To this end, we first introduce a bijection $\Xi_{\text{poset}}$ between unit interval posets and plane trees. We then show that the two different known bijections from unit interval posets are conjugates of special cases of some bijections $\Xi_{\text{steep}}$ and $\Xi_{\text{bounce}}$ from Ceballos, Fang, and Mühle [2020] constructed for the study of parabolic Tamari lattices. Using the link between $\Xi_{\text{steep}}, \Xi_{\text{bounce}}$ and the zeta map established in the same article, we conclude our proof.
The rest of the article is organized as follows. In Section 2 we give the basic definitions. Then we give our bijection between unit interval posets and plane trees in Section 3, along with the related bijections from Ceballos, Fang, and Mühle (2020). Finally in Section 4 we recall the two known bijections between unit interval posets and Dyck paths, and then establish their link with the bijections in Section 3, leading to a bijective proof of our main result (Theorem 4.6).

Acknowledgment We thank Adrien Segovia for bringing this conjecture to our attention. We also thank the anonymous reviewers for their detailed and helpful comments. This work is not supported by any funding with precise predefined goal, but it is supported by the publicly funded laboratory LIGM of Université Gustave Eiffel.

2 Preliminary

For convenience, we write $[n]$ for the set $\{1, 2, \ldots, n\}$. We consider finite partially order sets (or poset for short) of the form $P = (P_{\text{elem}}, \preceq)$, where $P_{\text{elem}}$ is a finite set and $\preceq$ a partial order on $P_{\text{elem}}$. Given a set $S$ of real numbers $x_1 < x_2 < \cdots < x_n$, we define a partial order $\preceq_S$ on $[n]$ by taking $i \preceq_S j$ if and only if $x_i + 1 < x_j$. The order $\preceq_S$ can also be seen as defined on intervals of unit length starting at the $x_i$'s, where $i \preceq_S j$ if the interval starting with $x_i$ is on the left of that with $x_j$ without overlap. A poset $P$ with $n$ elements is a unit interval order if there is some $S \subset \mathbb{R}$ with $n$ elements such $P \cong ([n], \preceq_S)$. In this case, we call $S$ a starting set of $P$. We sometimes represent unit interval orders as $([n], \preceq_S)$ for some $S$ hereinafter.

We have the following characterization of unit interval orders.

**Theorem 2.1 (Scott (1964, Theorem 2.1)).** A poset $P$ is a unit interval poset if and only if it is (3+1)-free and (2+2)-free, that is, the order induced on any four elements cannot be a chain of 3 elements plus an incomparable element, or two disjoint chains each containing 2 elements.

Unit interval orders are counted by Catalan numbers.

**Proposition 2.2 (Wine and Freund (1957)).** The number of unit interval posets with $n$ elements is the $n$-th Catalan number $\text{Cat}_n = \frac{1}{2n+1} \binom{2n}{n}$.

By definition, we may represent a unit interval poset $P$ by a set $S$ of real numbers, though the choice of $S$ is clearly not unique. In the following, for convenience, we use this perspective of (starting points of) intervals, which is arguably easier to manipulate. We denote by $\mathcal{P}_n$ the set of unit interval posets with $n$ elements.

There are many other families of combinatorial objects counted by Catalan numbers, such as Dyck paths and plane trees. A Dyck path is a lattice path formed by north steps $\uparrow = (0, 1)$ and east steps $\to = (1, 0)$, starting at $(0, 0)$ and ending on the diagonal $y = x$ while staying weakly above it. The size of a Dyck path is the number of its north steps. We denote by $\mathcal{D}_n$ the set of Dyck paths of size $n$. We define plane trees recursively: a plane tree $T$ is either a single node (a leaf) or an internal node $u$ linked by edges to a sequence of plane trees called sub-trees. In the latter case, the node $u$ is also called the root of $T$, and the roots of sub-trees are called children of $u$. We denote by $\mathcal{T}_n$ the set of plane trees with $n$ non-root nodes. We recall the well-known fact that $|\mathcal{T}_n| = |\mathcal{D}_n| = \text{Cat}_n$. For a node $u$ in a plane tree $T$, its depth, denoted by $d(u)$, is the distance (number of edges) between $u$ and the root of $T$. 
Bijective proof of a conjecture on unit interval posets

3 Unit interval posets and plane trees

We start by a new bijection between unit interval posets and plane trees.

Construction 3.1. Let $S = \{x_1 < \cdots < x_n\}$ be a starting set of a unit interval poset $P = ([n], \preceq_S)$. We define a plane tree $T$ as follows (see Figure 1 for an example). We set $x_0 = x_1 - 2$. We denote by $v_i$ the node of $T$ corresponding to $x_i$. For $i \in [n]$, the parent of $v_i$ is $v_j$ if and only if $j$ is the largest index such that $j \preceq_S i$. By the definition of $x_0$, the parent of each $v_i$ is well-defined, and all nodes are descendants of $x_0$. We then order children of the same node from left to right with decreasing index. We thus conclude that $T$ is a well-defined plane tree. We note that $T$ depends only on $\preceq_S$. We define $\Lambda_{\text{poset}}(P) = T$.

Construction 3.2. Given a plane tree $T$ with $n$ non-root nodes, we define an order relation $\preceq_S$ on $[n]$ induced by some $S \subset \mathbb{R}$. Let $r_T$ be the root of $T$, and $m$ the maximal arity of $T$, i.e., the maximal number of children of nodes in $T$. Given a node $u$ in $T$, if it is the $i$-th child of its parent from right to left, then we define $c(u) = i$. We observe that $1 \leq c(u) \leq m$. For a non-root node $u$ in $T$, let $u_0 = r_T, u_1, \ldots, u_{d(u)} = u$ be the nodes on the unique path from the root $u_0 = r_T$ to $u_{d(u)} = u$, where $d(u)$ is the depth of $u$. We then define a real number $x_u$ associated to $u$ using base $(m + 2)$:

$$x_u = d(u) + (0.c(u_1)c(u_2)\cdots c(u_{d(u)}))(m+2)^{-1} = \ell + \sum_{i=1}^{d(u)} c(u_i)(m+2)^{-i}. \quad (1)$$

It is clear from (1) that $x_u$ is strictly decreasing from left to right for nodes of the same depth in $T$. Let $S$ be the set of $x_u$ for non-root nodes $u$ in $T$. We define $\Xi_{\text{poset}}(T) = ([n], \preceq_S)$.

As an example of Construction 3.2, let $T$ be the tree on the right part of Figure 1, with nodes labeled as in the figure, and $S$ the set of numbers in the construction. The maximal arity of $T$ is $m = 4$, achieved by the root. The number $x_\nu_{12} \in S$ for the node $\nu_{12}$ is thus $x_\nu_{12} = 3 + (0.321)_{6} = 769/216$, as $\nu_{12}$ is of depth 3, and it is the first child from right to left of its parent $\nu_8$, itself the second child from right to left of its parent $\nu_3$, which is the third child from right to left of the root.

The following lemma gives a direct connection between a plane tree and its corresponding unit interval poset obtained from $\Xi_{\text{poset}}$.

Lemma 3.3. For $T \in \mathcal{T}_n$, we take $S$ to be the set constructed in Construction 3.2 and $x_u$ the number constructed from a non-root node $u$ of $T$. Then for non-root nodes $u, v$, the parent of $u$ in $T$ is $v$ if and only if $x_v$ is the largest element in $S$ smaller than $1 + x_u$.
We now show that $\Lambda_{\text{poset}}$ and $\Xi_{\text{poset}}$ are bijections and are inverse to each other.

**Proposition 3.4.** For all $n \geq 1$, the map $\Lambda_{\text{poset}}$ is a bijection from $\mathcal{P}_n$ to $\mathcal{T}_n$, with $\Xi_{\text{poset}}$ its inverse.

**Proof:** By Proposition 2.2, we have $|\mathcal{P}_n| = \text{Cat}_n = |\mathcal{T}_n|$. We thus only need to show that $\Lambda_{\text{poset}} \circ \Xi_{\text{poset}} = \text{id}$. Given a plane tree $T \in \mathcal{T}_n$, let $P = (|n|, \preceq_S) = \Xi_{\text{poset}}(T)$ and $T' = \Lambda_{\text{poset}}(P)$. Let $S = \{x_1 < \cdots < x_n\}$ be the set of real numbers constructed in Construction 3.2 for $\preceq_S$. Given $i \in [n]$, let $u_i$ be the node in $T$ that gives rise to $x_i$ in $S$, and $u'_i$ the node in $T'$ that represents $x_i$. By Lemma 3.3 and Construction 3.1, for any $i, j \in [n]$, the node $u_i$ is the parent of $u_j$ in $T$ if and only if $u'_j$ is the parent of $u'_i$ in $T'$. Then, $T'$ has the same order of siblings as $T$, as in both we order siblings with decreasing order in their corresponding real numbers in $S$. We thus conclude that $T = T'$.

We now restate two previously known bijections between plane trees and Dyck paths that will be used to prove our main result.

**Construction 3.5.** Let $n \geq 1$ and $T \in \mathcal{T}_n$, we construct a Dyck path $D$ by taking the clockwise contour walk starting from the top of the root of $T$ (see the Dyck path on the right of Figure 2). During the walk, when we pass an edge for the first (resp. second) time, we append $↑$ (resp. $→$) to $D$. We define $\Xi_{\text{steep}}(T) = D$. The map $\Xi_{\text{steep}}$ is a bijection from $\mathcal{T}_n$ to $\mathcal{D}_n$ for all $n \geq 1$, and we denote its inverse by $\Lambda_{\text{steep}}$.

**Construction 3.6.** Let $n \geq 1$ and $T \in \mathcal{T}_n$, we construct a Dyck path $D$ by appointing the number of north steps at each $x$-coordinate (see the Dyck path on the left of Figure 2). Let $\alpha_\ell$ be the number of nodes of depth $\ell$ (thus distance $\ell$ to the root), and $\ell_{\text{max}}$ the maximal depth of nodes in $T$. We take $s_\ell = \alpha_1 + \cdots + \alpha_\ell$. Given $1 \leq k \leq n - 1$, there is a unique way to write $k = s_\ell - r$ with $0 \leq r < \alpha_\ell$ and $1 \leq \ell \leq \ell_{\text{max}}$. In this case, the number of north steps in $D$ on $x = k$ is the number of children of the $(r + 1)$-st node of depth $\ell$. The number of north steps in $D$ on $x = 0$ is the number of children of the root. We define $\Xi_{\text{bounce}}(T) = D$. The map $\Xi_{\text{bounce}}$ is a bijection from $\mathcal{T}_n$ to $\mathcal{D}_n$ for all $n \geq 1$, see Ceballos, Fang, and Muhle (2020, Lemma 3.18), and we denote its inverse by $\Lambda_{\text{bounce}} = \Xi_{\text{bounce}}^{-1}$.

**Proposition 3.7.** (Ceballos, Fang, and Muhle (2020, Theorem IV)). Let $\zeta$ be the zeta map from $\mathcal{D}_n$ to $\mathcal{D}_n$ with $n \geq 1$. We have $\zeta = \Xi_{\text{bounce}} \circ \Lambda_{\text{steep}}$.

See Figure 2 for an illustration of Proposition 3.7. Here, we do not give the original definition of the zeta map $\zeta$ in Andrews, Krattenthaler, Orsina, and Papi (2002), and will instead be using Proposition 3.7 in the following, as it is better suited to our approach.

**Remark 3.8.** Albeit the same notation, the maps $\Xi_{\text{steep}}$ and $\Xi_{\text{bounce}}$ are only special cases of the ones from Ceballos, Fang, and Muhle (2020, Construction 3.10 and Equation 14). More precisely, the central objects of Ceballos, Fang, and Muhle (2020) are called LAC trees, which are plane trees with non-root nodes divided into regions algorithmically. We are only using the special case where the region $d$ consists of nodes of depth $d$, which is also the one used in Ceballos, Fang, and Muhle (2020, Section 3.3) to prove Proposition 3.7. In full generality, the map $\Xi_{\text{steep}}$ maps a LAC tree to a steep pair, which is a pair of
Bijective proof of a conjecture on unit interval posets

ζ = Ξbounce ◦ Λsteep

Fig. 2: The zeta map as composition of bijections mediated by trees. The nodes of the same depth of the tree are grouped together. For the Dyck path on the left, the number of north steps on x = k is the number of children of the k-th node in the tree, ordered by increasing depth, then from right to left. The one on the right comes from a clockwise contour walk.

nested Dyck paths with a steep upper path, i.e., without two consecutive east steps except on the maximal height. The map Ξbounce sends a LAC tree to a bounce pair, which is a pair of nested Dyck path with a bounce lower path, i.e., a concatenation of sub-paths of the form ↑k→k for all k. This explains the words “steep” and “bounce” in their notation.

Remark 3.9. The bijection Ξsteep is classical, except that we do the contour walk from right to left instead of from left to right. The bijection Ξbounce is quite close to the classical bijection between plane trees and Łukasiewicz words, as given in Flajolet and Sedgewick (2009, Section I.5.3)), which can be seen as sequences (z0, z1, ..., zn−1) with zi ≥ −1 such that ∑zi−1 k=0ū0k ≥ 0 for 0 ≤ k < n − 1, and the sum of all zi’s is −1. However, in the classical bijection of Łukasiewicz, we deal with nodes in a depth-first order, but in Ξbounce they are processed in a breadth-first order.

4 Unit interval posets and the zeta map

In the following, we give the two representations of unit interval posets as Dyck paths, and we detail how they are related to the bijections detailed in Section 3, leading to a proof of our main result (Theorem 4.6).

We start by the first representation, which was first defined implicitly using anti-adjacency matrices of unit interval posets in Skandera and Reed (2003), only involving the poset structure. The following form was first given in Chavez and Gotti (2018).

Construction 4.1. Given P = ([n], ≤S) a unit interval poset with S ⊂ R, we define a Dyck path D as follows. Let S+ = {x + 1 | x ∈ S}. Without loss of generality, we can choose S such that S ∩ S+ = ∅. Then suppose that S ∪ S+ = {y1 < ⋯ < y2n}. For i ∈ [2n], the i-th step of D is ↑ if yi ∈ S, and is → otherwise. We define ϕ(P) = D. See Figure 3 for an example.

This form of the map ϕ in Construction 4.1 was given in Chavez and Gotti (2018, Section 5) and proved in Lemma 5.7 therein to be equivalent to the original definition, which is the one used in Matherne, Morales, and Selover Section 2.3. Although defined here using the set S, the map ϕ does not depend on the choice of S.

Proposition 4.2. We have ϕ ◦ Ξposet = Ξbounce, and ϕ is a bijection from Pn to Dn for all n ≥ 1.
D = φ(S)

S = ψ(D)

Area(D')

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ

ψ
Bijective proof of a conjecture on unit interval posets

- Either \( a_i + 2 \leq a_j \);
- Or \( a_i + 1 = a_j \) and \( i < j \).

We define \( \psi(D) = P \), and its inverse \( \psi^{-1}(P) = D \). See Figure 3 for an example.

The following result ensures that \( \psi \) is well-defined. It is a special case of Matherne, Morales, and Selover Theorem 5.2, which is obtained by combining Guay-Paquet, Morales, and Rowland (2014, Remark 3.2, Proposition 3.11). A self-contained direct proof of this special case can be found in Gélinas, Segovia, and Thomas Section 6.

**Proposition 4.4** (Special case of Matherne, Morales, and Selover Theorem 5.2). The map \( \psi \) is a bijection from Dyck paths to unit interval posets preserving sizes.

**Proposition 4.5.** We have \( \psi = \Xi_{\text{poset}} \circ \Lambda_{\text{steep}} \).

**Proof:** Let \( P = \psi(D), T = \Lambda_{\text{steep}}(D) \) and \( P' = \Xi_{\text{poset}}(D) \). We write \( P = ([n], \preceq) \) and \( P' = ([n], \preceq_S) \), with \( S \) given in Construction 3.2. We show that \( P \) is isomorphic to \( P' \) after an implicit relabeling.

For \( i \in [n] \), let \( u_i \) be the \( i \)-th non-root node in \( T \) that is visited in the clockwise contour walk of \( T \). We note that this order on nodes of \( T \) is different from the one used before, for instance in Construction 3.1. Suppose that \( \text{Area}(D) = (a_1, \ldots, a_n) \). By Construction 3.5, the depth of \( u_i \) is \( a_i + 1 \). We now define a partial order \( \preceq_T \) on non-root nodes in \( T \) by taking \( u_i \preceq_T u_j \) if and only if \( i \leq j \) in \( P \). We recall that the depth of \( v \) in \( T \) is denoted by \( d(v) \). By Construction 4.3, two non-root nodes \( v, w \) in \( T \) satisfy \( v \prec_T w \) if and only if

- Either \( d(v) + 2 \leq d(w) \);
- Or \( d(v) + 1 = d(w) \), and the parent of \( w \) is either \( v \) or on the left of \( v \).

Now, for \( v \) a non-root node in \( T \), we take \( x_v \) defined in (1). Suppose that there are two non-root nodes \( v, w \) of \( T \) such that \( x_v + 1 < x_w \). Then we have \( d(v) + 1 \leq d(w) \). There are two possibilities: either \( d(v) + 1 = d(w) \), or \( d(v) + 2 \leq d(w) \). In the first case, let \( w' \) be the parent of \( w \), we have \( x_{w'} + 1 < x_w \).

By Lemma 3.3, we know that \( x_v \leq x_{w'} \). From (1), we know that either \( v = w' \) or \( v \) is on the right of \( w' \), as \( x_v \) is decreasing from left to right for nodes in \( T \) of the same depth. Combining with the second case, we conclude that \( x_v + 1 < x_w \) if and only if \( v \prec_T w \). By Construction 3.2, we have \( P \cong P' \), which concludes the proof.

**Theorem 4.6** (Matherne, Morales, and Selover Conjecture 7.1). We have \( \varphi \circ \psi = \zeta \).

**Proof:** Combining Proposition 4.2 and 4.5, we have \( \varphi \circ \psi = \Xi_{\text{bounce}} \circ \Lambda_{\text{steep}} \), and we conclude by Proposition 3.7.

---

**References**


