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Extending partial edge colorings of iterated cartesian products of cycles and paths

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We consider the problem of extending partial edge colorings of iterated cartesian products of even cycles and paths, focusing on the case when the precolored edges satisfy either an Evans-type condition or is a matching. In particular, we prove that if $G = C^d_{2k}$ is the dth power of the cartesian product of the even cycle C_{2k} with itself, and at most $2d-1$ edges of G are precolored, then there is a proper $2d$-edge coloring of G that agrees with the partial coloring. We show that the same conclusion holds, without restrictions on the number of precolored edges, if any two precolored edges are at distance at least 4 from each other. For odd cycles of length at least 5, we prove that if $G = C^d_{2k+1}$ is the dth power of the cartesian product of the odd cycle C_{2k+1} with itself ($k \geq 2$), and at most $2d$ edges of G are precolored, then there is a proper $(2d+1)$-edge coloring of G that agrees with the partial coloring. Our results generalize previous ones on precoloring extension of hypercubes [Journal of Graph Theory 95 (2020) 410–444].

Keywords: Precoloring extension, Edge coloring, Cartesian product, List coloring

1 Introduction

An *(edge) precoloring* (or *partial edge coloring*) of a graph G is a proper edge coloring of some subset $E' \subseteq E(G)$; *a t-edge precoloring* is such a coloring with t colors. A t-precoloring φ of G is *extendable* if there is a proper t-edge coloring f of G such that $f(e) = \varphi(e)$ for any edge e that is colored under φ; f is called an *extension* of φ. In general, the problem of extending a given edge precoloring is an $\mathcal{NP}$-complete problem, already for 3-regular bipartite graphs [8, 11].

Edge precoloring extension problems seem to have been first considered in connection with the problem of completing partial Latin squares and the well-known Evans' conjecture that every $n \times n$ partial Latin square with at most $n-1$ non-empty cells is completable to a Latin square [10]. By a well-known correspondence, the problem of completing a partial Latin square is equivalent to asking if a partial edge coloring with $\Delta(G)$ colors of a balanced complete bipartite graph G is extendable to a $\Delta(G)$-edge coloring, where $\Delta(G)$ as usual denotes the maximum degree. Evans' conjecture was proved for large n by Häggkvist [13], and in full generality by Andersen and Hilton [1], and, independently, by Smetaniuk [16].

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Another early reference on edge precoloring extension is [14], where the authors study the problem from the viewpoint of polyhedral combinatorics. More recently, the problem of extending a precoloring of a matching has been considered in [9]. In particular, it is conjectured that for every graph G, if φ is an edge precoloring of a matching M in G using $\Delta(G)+1$ colors, and any two edges in M are at distance at least 2 from each other, then φ can be extended to a proper $(\Delta(G)+1)$-edge coloring of G; here, by the *distance* between two edges e and e' we mean the number of edges in a shortest path between an endpoint of e and an endpoint of e'; a *distance-t matching* is a matching where any two edges are at distance at least t from each other. In [9], it is proved that this conjecture holds for e.g. bipartite multigraphs and subcubic multigraphs, and in [12] it is proved that a version of the conjecture with the distance increased to 9 holds for general graphs.

Quite recently, with motivation from results on completing partial Latin squares, questions on extending partial edge colorings of d-dimensional hypercubes Q_d were studied in [7]. Among other things, a characterization of partial edge colorings with at most d precolored edges that are extendable to d-edge colorings of Q_d is obtained, thereby establishing an analogue for hypercubes of the characterization by Andersen and Hilton [1] of $n \times n$ partial Latin squares with at most n non-empty cells that are completable to Latin squares. In particular, every partial d-edge coloring with at most $d-1$ colored edges is extendable to a d-edge coloring of Q_d. This line of investigation was continued in [5, 6] where similar questions are investigated for trees.

In [4], similar questions are investigated for *cartesian products* of graphs. The *cartesian product* $G\square H$ of two graphs G and H is the graph with vertex set $V(G\square H) = \{(u,v) : u \in V(G), v \in V(H)\}$, and where (u,v) is adjacent to (u',v') if and only if $u = u'$ and $vv' \in E(H)$, or $uu' \in E(G)$ and $v = v'$.

In [4], Evans-type edge precoloring extension results are obtained for the cartesian products of complete and complete bipartite graphs with K_2, respectively, as well as for the product of K_2 with graphs of small maximum degree and trees. Moreover, similar results for the cartesian product of K_2 with a general regular (triangle-free) graph, where the precolored edges are required to be independent, were obtained.

In this paper, we continue the study of questions on precoloring extension of cartesian products of graphs with a focus on iterated cartesian products of graphs. Denote by G^d the dth power of the cartesian product of G with itself. We pose the following question.

Problem 1.1 *Let G be a graph where every precoloring of at most $\chi'(G)-k$ edges, where $k \geq 1$, can be extended to a proper $\chi'(G)$-edge coloring. Is it true that every precoloring of at most $\chi'(G^d)-k$ edges of G^d can be extended to a $\chi'(G^d)$-edge coloring of G^d?*

The result of [7] for hypercubes deals with the case when $G = K_2$ (as well as $G = C_4$), so a positive answer to Problem 1.1 would be a far-reaching generalization of this result.

In this paper, we study Problem 1.1 for graphs with maximum degree two. We verify that it has a positive answer for even as well as for odd cycles of length at least 5, and therefore also for paths. The case of odd cycles of length 3 appears to be more difficult, and it remains an open problem whether Problem 1.1 has a positive answer in this case.

Even though any partial edge coloring of an odd cycle is extendable, we shall restrict ourselves to the case when at most $\chi'(G)-1$ edges in a graph G are precolored, since for all connected graphs except odd cycles and stars, there are examples of partial edge colorings with $\chi'(G)$ precolored edges that are not extendable. In fact, in [4] it was proved that every partial $\chi'(G)$-edge coloring of G is extendable if and only G is isomorphic to a star $K_{1,n}$ or an odd cycle.

For even cycles, we additionally prove that any precoloring of a distance-4 matching in C_{2k}^d is extendable to a proper $2k$-edge coloring. Here the argument relies heavily on the fact that C_{2k}^d is Class 1, and we do not know whether a similar result hold for odd cycles.

2 Preliminaries

Before we prove our results, let us introduce some terminology and auxiliary results.

If φ is an edge precoloring of G and an edge e is colored under φ, then we say that e is *φ-colored*. A color c *appears* at a vertex v under φ if there is an edge incident with v that is colored c; otherwise, c is *missing* at v.

If the edge coloring φ uses t colors and $1 \leq a, b \leq t$, then a path or cycle in G is called *(a,b)-colored under φ* if its edges are colored by colors a and b alternately. We also say that such a path or cycle is *bicolored under φ*. By switching colors a and b on a maximal (a,b)-colored path or an (a,b)-colored cycle, we obtain another proper t-edge coloring of G; this operation is called an *interchange* or a *swap*.

In the above definitions, we often leave out the reference to an explicit coloring φ, if the coloring is clear from the context.

If G_1 and G_2 are subgraphs of G, and f_i is a proper edge coloring of G_i, then we say that f_1 *has no conflicts* with f_2 if no vertex is incident with two edges e_1 and e_2 such that $f_1(e_1) = f_2(e_2)$.

By construction, $G = C_r^d$ decomposes into d subgraphs in terms of its edges, each consisting of r^{d-1} disjoint copies of C_r; these subgraphs are called *dimensions*. Each subgraph of a dimension which is isomorphic to C_r is called a *layer*, and each component of $G - E(D)$, where D is a dimension, is called a *plane* of G. If $d = 2$, then layers and planes are identical objects.

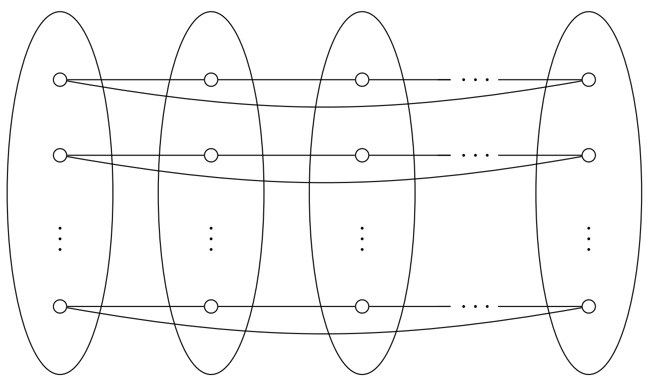

Fig. 1: An illustration of dimensions, layers and planes. Each cycle forms a *layer*, all the cycles together form a *dimension*, and the components obtained by removing all the edges from the cycles are the *planes*.

In Figure 1, the edge-induced subgraph consisting of all vertices and drawn edges form a dimension, each cycle is a layer, and each connected component in the subgraph obtained by removing all drawn edges is a plane.

Two planes are *adjacent* if there is an edge with endpoints in both planes. Similarly an edge e not contained in a plane is *incident* to the plane if one endpoint of e is contained in the plane, and we say that a layer edge is *between* two planes if it is incident with both planes.

Two vertices of two distinct planes are *corresponding* if they are joined by an edge; similarly for edges. Given edge colorings of two distinct planes, we say that the planes are colored *correspondingly* if corresponding edges have the same color.

We shall also need some standard definitions on list edge coloring. Given a graph G, assign to each edge e of G a set $\mathcal{L}(e)$ of colors. Such an assignment $\mathcal{L}$ is called a *list assignment* for G and the sets $\mathcal{L}(e)$ are referred to as *lists* or *color lists*. If all lists have equal size k, then $\mathcal{L}$ is called a k*-list assignment*. Usually, we seek a proper edge coloring φ of G, such that $\varphi(e) \in \mathcal{L}(e)$ for all $e \in E(G)$. If such a coloring φ exists then G is $\mathcal{L}$*-colorable* and φ is called an $\mathcal{L}$*-coloring*. Denote by $\chi'_L(G)$ the minimum integer t such that G is $\mathcal{L}$-colorable whenever $\mathcal{L}$ is a t-list assignment. If $\chi'_L(G) \leq t$, then G is t*-edge-choosable*. The following lemmas are well-known and easy to prove.

Lemma 2.1 *Every even cycle is* 2*-edge-choosable.*

Lemma 2.2 *If L is a 2-list assignment for the edges of an odd cycle C, then C is L-colorable, unless all lists are identical.*

We shall also use the well-known proposition that paths are edge-list colorable from a list assignment where every edge except the first one gets a list of size at least two.

3 Extension of $2d - 1$ precolored edges of C_{2k}^d

In this section, we prove the following theorem.

Theorem 3.1 *If $G = C_{2k}^d$ is the dth power of the cartesian product of the even cycle C_{2k} with itself, and φ is a proper partial edge coloring of G with at most $2d - 1$ precolored edges, then φ can be extended to a proper $2d$-edge coloring of G.*

As mentioned in the introduction, every connected graph except odd cycles and stars have a partial edge coloring with $\chi'(G)$ precolored edges that is not extendable. Thus, since $\chi'(G) = 2d$, the bound on the number of precolored edges here is best possible.

Proof Proof of Theorem 3.1: The proof proceeds by induction on d, the case $d = 1$ being trivial. We shall prove a series of lemmas that together will imply the theorem. In the proofs of these lemmas we shall consider a specified dimension D_1, and the subgraph $G - E(D_1)$ consisting of $2k$ planes $Q_1, \dots, Q_{2k}$, where Q_i is adjacent to Q_{i+1} (here, and in the following, indices are taken modulo $2k$).

We shall assume that every precoloring of a plane of $G - E(D_1)$ with at most $2d - 3$ precolored edges is extendable to a proper edge coloring using $2d - 2$ colors, and prove that a given precoloring φ of G with at most $2d - 1$ precolored edges is extendable to a proper $2d$-edge coloring of G.

We shall distinguish between the following cases, each of which is dealt with in a lemma below.

- There is a dimension of G that contains no precolored edges.
- Each dimension of G contains precolored edges, and there is a dimension with at most two precolored edges, the colors of which do not appear on edges in any other dimension of G.
- Every dimension of G contains edges with colors that also appear on edges in another dimension, or at least three precolored edges.

□

Lemma 3.2 *If there is a dimension of G that contains no precolored edges, then φ is extendable.*

Proof: Suppose that D_1 is a dimension in G that contains no precolored edges, and consider the subgraph $G - E(D_1)$.

Suppose first that all precolored edges are contained in one plane, say Q_1. Let c_1 and c_2 be two colors used by φ (if just one color appears under φ, then c_2 is any color from $\{1, \dots, 2d\} \setminus \{c_1\}$). From the restriction of φ to Q_1, we define an edge precoloring φ' of Q_1 by removing the colors c_1 and c_2 from any edge of Q_1 φ-colored by these colors. Then, by the induction hypothesis, φ' is extendable to a $(2d-2)$-coloring of Q_1 using colors $\{1, \dots, 2d\} \setminus \{c_1, c_2\}$. Next we recolor the edges φ-precolored c_1 and c_2 by these colors, and thereafter color all other planes correspondingly. Thus, we can define a list assignment L for the edges of D_1, by for each edge $e \in E(D_1)$, letting $L(e)$ be the set of all colors from $\{1, \dots, 2d\}$ that do not appear on edges that are adjacent to e. By Lemma 2.1, we can properly color the edges of D_1 from these lists to obtain a proper coloring that has no conflicts with the coloring of $G - E(D_1)$, and thus φ is extendable.

Next, we consider the case when exactly two planes, say Q_1 and Q_i contain all precolored edges. Since at most $2d-1$ colors appear under φ, there is a color $c_1 \in \{1, \dots, 2d\}$ that is not used by φ. Furthermore, let c_2 be a color appearing on some edge in the plane with the largest number of precolored edges, say Q_1. Let φ' be the coloring obtained from φ by removing color c_2 from any edge colored c_2 under φ. Then the restrictions of φ' to Q_1 and Q_i, respectively, are extendable to proper $(2d-2)$-edge colorings using colors $\{1, \dots, 2d\} \setminus \{c_1, c_2\}$. By recoloring any edge φ-colored c_2 by the color c_2, we obtain proper edge colorings f_1 and f_i of Q_1 and Q_i, respectively.

Now, either $i \neq 2$ or $i \neq 2k$; suppose the former holds. Then we color Q_2 correspondingly to how Q_1 is colored under f_1, and we color all other uncolored Q_j's correspondingly to how Q_i is colored under f_i. Now, since Q_{2j-1} and Q_{2j} are colored correspondingly, for every edge e with one endpoint in Q_{2j-1} and one endpoint in Q_{2j}, there is a color $\{1, \dots, 2d\} \setminus \{c_1\}$ that does not appear at an endpoint of e. Thus, by coloring all such edges by such a color and then coloring all other edges of D_1 by the color c_1, we obtain an extension of φ.

Lastly, let us consider the case when at least three planes contain all precolored edges. As before, let c_1 be a color that is not used by φ. Since at least three planes contain precolored edges, each plane contains at most $2d-3$ precolored edges, and two adjacent planes contain precolored edges from at most $2d-2$ colors. This implies that for each $j = 1, \dots, k$ there is a color c'_j in $\{1, \dots, 2d\} \setminus \{c_1\}$ that is not used in the restriction of φ to Q_{2j-1} and Q_{2j}. Thus for $j = 1, \dots, k$, we can extend the restriction of φ to Q_{2j-1} and Q_{2j} using the $2d-2$ colors in $\{1, \dots, 2d\} \setminus \{c_1, c'_j\}$. For $j = 1, \dots k$, we then color all edges of D_1 between Q_{2j-1} and Q_{2j} by c'_j, and all other edges of D_1 by the color c_1. This yields an extension of φ. □

Lemma 3.3 *If each dimension of G contains precolored edges and there is a dimension with at most two precolored edges, the colors of which do not appear on edges in any other dimension of G, then φ is extendable.*

Proof: Assume first that there is a dimension D_1 containing only one precolored edge and that there is a plane Q_1 in $G - E(D_1)$ containing all other precolored edges. Suppose that the precolored edge of

D_1 is colored c_1. As in the proof of the preceding lemma, there is an extension of the restriction of φ to Q_1 using colors $\{1,\dots,2d\} \setminus \{c_1\}$. Next, we color all other planes of $G - E(D_1)$ correspondingly, which implies that every edge of D_1 is adjacent to edges of $2d-2$ different colors, so by Lemma 2.1, φ is extendable.

Let us now consider the case when there is no such dimension containing only one precolored edge and a plane containing all other precolored edges. Let D_1 be a dimension containing at most two edges that are precolored by colors not appearing on any other edges under φ. Our assumption implies that every plane in $G - E(D_1)$ contains at most $2d-3$ precolored edges, and that there are two colors c_1, c_2 that do not appear on any edge in $G - E(D_1)$, and at most two edges of D_1 are colored by colors from $\{c_1, c_2\}$.

Suppose first that there is an extension of the restriction of φ to D_1 using colors c_1 and c_2. Since every plane in $G - E(D_1)$ contains at most $2d-3$ precolored edges, every plane has a proper edge coloring using colors $\{1,\dots,2d\} \setminus \{c_1, c_2\}$ that agrees with φ. Hence, φ is extendable.

Suppose now that the restriction of φ to D_1 is not extendable using colors c_1 and c_2. Then there are at least two precolored edges in D_1, and so $G - E(D_1)$ contains at most $2d-3$ precolored edges and there is a color $c_3 \in \{1,\dots,2d\} \setminus \{c_1, c_2\}$ that does not appear on any edge under φ.

Since the restriction of φ to D_1 is not extendable, there are planes $Q_1, Q_2, \dots, Q_i$ in $G - E(D_1)$ such that Q_1 and Q_i are incident with precolored edges of D_1 and $Q_2, \dots, Q_{i-1}$ are not. Without loss of generality we assume that Q_1 is incident with an edge of D_1 that is precolored c_1. We take an extension of Q_1 using colors $\{1,\dots,2d\} \setminus \{c_1, c_3\}$, and for $j = 2, \dots, i-1$, we take an extension of the restriction of φ to Q_j using colors $\{1,\dots,2d\} \setminus \{c_2, c_3\}$. Moreover, we color every path in D_1 with vertices in $Q_1, \dots, Q_i$ using colors c_3 and c_2 alternately, and starting with c_3 at Q_1.

If Q_i is incident with an edge of D_1 colored c_2, then i is even, and we take extensions of the restriction of φ to Q_i and Q_{i+1} using colors $\{1,\dots,2d\} \setminus \{c_2, c_3\}$, and for $j = i+2, \dots, 2d$, we take extensions of the restrictions of φ to Q_j using colors $\{1,\dots,2d\} \setminus \{c_1, c_3\}$. Moreover, we color every path in D_1 with vertices in $Q_{i+1}, \dots, Q_{2k}$ using colors c_3 and c_1 alternately, and starting with c_3 at Q_{i+1}. Finally, we color all edges between Q_1 and Q_{2k} by the color c_1, and all edges between Q_i and Q_{i+1} by the color c_2. This yields an extension of φ.

If Q_i is incident with an edge colored c_1, then i is odd, and we proceed similarly, but take extensions of the restrictions of φ to Q_i and Q_{i+1} using colors $\{1,\dots,2d\} \setminus \{c_1, c_2\}$, for $j = i+2, \dots 2k-1$, we take extensions of the restrictions of φ to Q_j using colors $\{1,\dots,2d\} \setminus \{c_2, c_3\}$, and an extension of the restriction of φ to Q_{2k} using colors $\{1,\dots,2d\} \setminus \{c_1, c_3\}$. We then color the paths of D_1 so that the resulting coloring is proper and agrees with φ. □

Lemma 3.4 *If each dimension of G contains edges with colors that also appear on edges in another dimension, or at least three precolored edges, then φ is extendable.*

Proof: Since at most $2d-1$ edges are precolored, there is a dimension D_1 with just one precolored edge e. Suppose $\varphi(e) = c_1$. Since at least one color appears on at least two edges, there are two colors $c_2, c_3 \in \{1,\dots,2d\} \setminus \{c_1\}$ that do not appear on any edge under φ.

We consider some different cases.

Case 1. *All precolored edges except e lie in the same plane*:

Let Q_1 be a plane containing all precolored edges except e. By removing the color from every edge of Q_1 that is φ-colored c_1, we obtain a precoloring that by the induction hypothesis is extendable to a proper edge coloring of Q_1 using colors $\{1,\dots,2d\} \setminus \{c_1,c_2\}$. Denote this coloring by f. By recoloring the edges of Q_1 that are φ-colored c_1, we obtain, from f, an extension f' of the restriction of φ to Q_1. We color all other planes of $G - E(D_1)$ correspondingly.

Suppose first that e is incident with Q_1. Then, by Lemma 2.1, there is proper edge coloring g of D_1 that has no conflicts with f'. Moreover, since just one edge of D_1 is precolored, we can choose this coloring so that $g(e) = c_1$. Hence, φ is extendable.

Next, we consider the case when e is not incident with Q_1. Let C be the cycle in D_1 containing e. If no vertex of C is incident with an edge colored c_1 under f', then we may proceed as in the preceding paragraph. Otherwise, the endpoints of e are incident with two edges e_1 and e_2 of $G - E(D_1)$ colored c_1. Note that e, e_1, e_2 are contained in a 4-cycle in G, the fourth edge of which we denote by e_3. As before, we properly color the edges of D_1 so that the resulting coloring g' of G is proper. Moreover, we choose g' so that $g'(e) = g'(e_3) = c_2$. By swapping colors on the bicolored 4-cycle with edges e, e_1, e_2, e_3 we finally obtain an extension of φ.

Case 2. *All precolored edges except* e *lie in exactly two planes*:

Let Q_1 and Q_i be the two planes containing the precolored edges distinct from e.

Suppose first that e is not adjacent to Q_1 or Q_i. Since each of Q_1 and Q_i contains at most $2d-3$ precolored edges, there are extensions f_1 and f_i of the restrictions of φ to Q_1 and Q_i, respectively, using colors $\{1,\dots,2d\} \setminus \{c_2,c_3\}$. Next, we properly color each plane Q of $G - E(D_1)$ that is distinct from Q_1 and Q_i, by colors $\{1,\dots,2d\} \setminus \{c_2,c_3\}$, so that all these planes are colored correspondingly. Moreover, color all edges of D_1 alternately using colors c_2 and c_3 and starting with color c_2 for all edges of D_1 with endpoints in Q_1 and Q_2. Then e is contained in a bicolored 4-cycle, and by swapping colors on this cycle, we obtain an extension of φ.

Let us now consider the case when e is adjacent to exactly one of Q_1 and Q_i. Suppose e.g. that e is incident with Q_1 and Q_2, so $i \neq 2$. Since each of Q_1 and Q_i contains at most $2d-3$ precolored edges, there are extensions f_1 and f_i of the restrictions of φ to Q_1 and Q_i, respectively, using colors $\{1,\dots,2d\} \setminus \{c_2,c_3\}$. Next, we color all uncolored planes in $G - E(D_1)$ correspondingly to how Q_1 is colored, color all cycles of D_1 properly using colors c_2 and c_3, and starting with color c_2 for all edges with endpoints in Q_1 and Q_2. Since Q_1 and Q_2 are colored correspondingly, there is a bicolored 4-cycle with colors c_1, c_2 containing e. By swapping colors on this 4-cycle, we obtain an extension of φ.

Suppose now that e_1 is incident with both planes containing precolored edges, say Q_1 and Q_2. By removing the color from any edge that is colored c_1 under φ, we obtain a precoloring φ' of G such that the restrictions of φ' to Q_1 and Q_2, respectively, are extendable to proper edge colorings using colors $\{1,\dots,2d\} \setminus \{c_1,c_3\}$. By recoloring any edge of Q_1 and Q_2 that is φ-colored c_1, we obtain proper edge colorings f_1 and f_2 of Q_1 and Q_2, respectively, using colors $\{1,\dots,2d\} \setminus \{c_3\}$. We color the edges of D_1 between Q_1 and Q_2 by c_3, except for e which is colored c_1.

Next, we color $Q_3,\dots,Q_{2k-1}$ correspondingly to how Q_2 is colored under f_2 and Q_{2k} correspondingly to how Q_1 is colored. Now, since Q_2 and Q_3 are colored correspondingly, for each edge e' of D_1 between Q_2 and Q_3, there is a color $c' \in \{1,\dots,2d\}$ that does not appear on an adjacent edge in Q_2 or on the edge between Q_1 and Q_2. We color every such edge e' between Q_2 and Q_3 by such a color c', and thereafter color all edges of every path of D_1 from Q_1 to Q_{2k} alternately by the colors used on the edges between Q_1 and Q_2, and Q_2 and Q_3 respectively. This yields a proper partial edge coloring where the endpoints of every edge of D_1 between Q_1 and Q_{2k} are incident to edges with $2d-1$ colors from $\{1,\dots,2d\}$. Thus we can properly color every such edge so that we get an extension of φ using colors $1,\dots,2d$.

Case 3. *At least three planes in $G-E(D_1)$ contain precolored edges*:

The assumption implies that every plane in $G-E(D_1)$ contains at most $2d-4$ precolored edges, and any two adjacent planes contain altogether at most $2d-3$ precolored edges. Without loss of generality, we assume that e is incident with Q_1 and Q_2. Since every vertex of Q_1 and Q_2 has degree $2d-2$, and $Q_1 \cup Q_2$ contains at most $2d-3$ precolored edges, there are uncolored corresponding edges $e_1 \in E(Q_1)$ and $e_2 \in E(Q_2)$ that are adjacent to e but not to any edge in $Q_1 \cup Q_2$ precolored c_1. From the restriction of φ to $G-E(D_1)$, we define a new precoloring φ' of $G-E(D_1)$ by in addition coloring e_1 and e_2 by the color c_1.

Now, since each component of $G-E(D_1)$ contains at most $2d-3$ φ'-precolored edges, by the induction hypothesis, there is an extension of φ' to $G-E(D_1)$ using colors $\{1,\dots,2d\}\setminus\{c_2,c_3\}$. Next, we properly color the edges of D_1 using colors c_2 and c_3, and starting with color c_2 for the edges with endpoints in Q_1 and Q_2. Now, since e_1 and e_2 are both colored c_1, there is a bicolored 4-cycle with edges e_1, e_2 and e with colors c_1 and c_2. By swapping colors on this 4-cycle, we obtain an extension of φ. □

4 Extending a precoloring of $2d$ edges in C^d_{2k+1}

In this section, we prove the following theorem for the iterated cartesian product of odd cycles of length at least 5.

Theorem 4.1 *If $G = C^d_{2k+1}$ is the dth power of the cartesian product of the odd cycle C_{2k+1} with itself ($k \geq 2$), and φ is a proper partial edge coloring of G with at most $2d$ precolored edges, then φ can be extended to a proper $(2d+1)$-edge coloring of G.*

As for the case of even cycles, (for $d \geq 2$) it is easily seen that the number of precolored edges here is best possible, because $\chi'(G) = 2d+1$.

Proof Proof of Theorem 4.1: The proof of this theorem is similar to the proof of Theorem 3.1, so we shall omit or just sketch some parts which are similar to techniques in that proof. Particularly in the last parts of the proof, to avoid tedious repetition we omit parts which are very similar to techniques that have been described in more detail earlier in the proof.

We proceed by induction on d, the case $d=1$ being trivial. As in the proof of Theorem 3.1, we shall prove a series of lemmas that together will imply the theorem. Since odd cycles are not 2-edge-colorable, the proof is longer and more difficult than the proof of that theorem. In the proofs of these lemmas we shall consider a specified dimension D_1, and the subgraph $G-E(D_1)$ consisting of $2k+1$ planes $Q_1,\dots,Q_{2k+1}$, where Q_i is adjacent to Q_{i+1} (here, and in the following, indices are taken modulo $2k+1$).

We shall assume that every edge precoloring of a plane of $G-E(D_1)$ with at most $2d-2$ precolored edges is extendable to a proper edge coloring using $2d-1$ colors, and prove that a given precoloring φ of G with at most $2d$ precolored edges is extendable to a proper $(2d+1)$-edge coloring of G. To that end, we shall distinguish between the following different cases.

- There is a dimension of G that contains no precolored edges.

- Every dimension of G contains precolored edges, and there is a dimension with at most two precolored edges, the colors of which do not appear on edges in any other dimension of G.
- Every dimension of G contains edges with colors that also appear on edges in other dimensions, or at least three precolored edges, and one dimension contains only one precolored edge.
- Every dimension of G contains two precolored edges, at least one of which has a color appearing on edges in another dimension.

□

Lemma 4.2 *If there is a dimension of G that contains no precolored edges, then φ is extendable.*

Proof: Suppose that D_1 is a dimension in G that contains no precolored edges. We consider some different cases.

Case 1. *All precolored edges are contained in one plane:*
Suppose that all precolored edges are contained in one plane, say Q_1. Let c_1 and c_2 be two colors used by φ (if just one color appears under φ, then c_2 is any color from $\{1, \dots, 2d+1\} \setminus \{c_1\}$). By removing the colors c_1 and c_2 from any edge colored by these colors, we obtain an edge precoloring φ' of Q_1 that is extendable to a $(2d-1)$-coloring of Q_1 using colors $\{1, \dots, 2d+1\} \setminus \{c_1, c_2\}$. Next we recolor the edge precolored c_1 and c_2, respectively, using these colors, and thereafter color all other planes correspondingly. Since all planes are colored correspondingly, we can apply Lemma 2.2 to properly color the edges of each layer of D_1 to obtain an extension of φ.

Case 2. *All precolored edges are contained in two planes:*
Suppose that Q_1 and Q_i contain all precolored edges. We shall consider three different cases.

Suppose first that $2d-1$ precolored edges are contained in the same plane, say Q_1, and that one edge e_i in Q_i is colored c_1. Let c_2 be a color appearing on some edge in Q_1. From the restriction of φ to Q_1 we define a precoloring φ' of Q_1 by removing color c_2 from every edge φ-colored c_2. Then φ' is extendable to a proper $(2d-1)$-edge coloring using $2d-1$ colors from $\{1, \dots 2d+1\} \setminus \{c_2\}$. By recoloring every edge of Q_1 that is φ-colored c_2 by the color c_2, we obtain a proper edge coloring f of Q_1.

Let e_1 be the edge of Q_1 corresponding to e_i of Q_i. If $f(e_1) = c_1$, then we color all planes in $G - E(D_1)$ correpondingly to how Q_1 is colored. By Lemma 2.2, we may then color the edges of D_1 to obtain an extension of φ. If, on the other hand, $f(e_1) = c_3 \neq c_1$, then we define a proper edge coloring of Q_i by coloring it correspondingly to Q_1 but permuting the colors in $\{1, \dots, 2d+1\}$ so that c_1 is mapped to c_3 and vice versa, and all other colors are mapped to themselves. This yields a proper edge coloring of Q_i that agrees with the restriction of φ to Q_i. We color all other Q_j's correspondingly to how Q_i is colored, and applying Lemma 2.2, we obtain an extension of φ, as before.

Let us now assume that both Q_1 and Q_i contain at most $2d-2$ precolored edges, respectively, and at most $2d-1$ colors, say $1, \dots, 2d-1$, are used by φ. By the induction hypothesis, the restrictions of φ to Q_1 to Q_i are extendable to $(2d-1)$-edge colorings f_1 and f_i, respectively, using colors $1, \dots, 2d-1$. We color all other Q_j's correspondingly to how Q_i is colored, and using Lemma 2.2 we obtain an extension of φ.

On the other hand, if both Q_1 and Q_i contain at most $2d-2$ precolored edges, respectively, but in total $2d$ colors $1,\dots,2d$ are used by φ, then every color appears on exactly one edge under φ. Hence, we may assume that one edge of Q_1, but not Q_i, is colored, say, 1, and similarly, one edge of Q_i is colored $2d$. By the induction hypothesis, there is an extension f_1 of the restriction of φ to Q_1 using colors $1,\dots,2d-1$, and an extension f_i of the restriction of φ to Q_i using colors $2,\dots,2d$.

Now, either $i \neq 2$ or $i \neq 2k+1$; suppose that the former holds. We define a proper edge coloring f_2 of Q_2 using colors $2,\dots,2d$ by coloring Q_2 correspondingly to Q_1 but using color $2d$ instead of 1, and then coloring all other planes of $G-E(D_1)$ correspondingly to how Q_i is colored. By the construction of f_2, for each layer edge e of D_1 between Q_1 and Q_2, there is a color in $\{2,\dots,2d\}$ that does not appear at an endpoint of e. We color every such layer edge by this color, and then color the edges of every cycle in D_1 by colors 1 and $2d+1$ alternately, and starting with color 1 at Q_2. This yields an extension of φ.

Case 3. *All precolored edges are contained in at least three planes:*
Let $Q_{j_1}, Q_{j_2}, \dots, Q_{j_s}$ be the planes of $G-E(D_1)$ that contain precolored edges, where $j_1 \leq j_2 \leq \dots \leq j_s \leq 2k+1$. Note that any two planes contain precolored edges of altogether at most $2d-1$ colors, and that there are two planes Q_{j_i} and $Q_{j_{i+1}}$ that contain precolored edges of altogether at most $2d-2$ colors. We assume that Q_{j_1} and Q_{j_s} are two such planes.

Consider an arbitrary cycle C in D_1. We partition the edges of C into paths $P_{12},\dots,P_{(s-1)s},P_{s1}$ where $P_{r(r+1)}$ has its endpoints in Q_{j_r} and $Q_{j_{r+1}}$. Now, for each path $P_{r(r+1)}$, there are two colors $c_{r(r+1)}, c'_{r(r+1)} \in \{1,\dots,2d+1\}$ so that none of these colors appear in the restriction of φ to $Q_{j_r} \cup Q_{j_{r+1}}$. For $r=1,\dots s-1$, we color each path $P_{r(r+1)}$ alternately by colors $c_{r(r+1)}$ and $c'_{r(r+1)}$, so that the resulting edge coloring is proper. Now, by assumption we have that Q_{j_1} and Q_{j_s} contain edges of altogether at most $2d-2$ colors. Hence, there are two colors c and c' that do not appear on edges in Q_{j_1} or Q_{j_s}, nor on an edge of D_1 that is incident with Q_{j_1}. We color the edges in the path of C from Q_{j_s} to Q_{j_1} by colors c and c' so that the resulting coloring is proper.

Next, we color all uncolored edges of D_1 correspondingly to how C is colored. Now, each Q_j is incident with edges of D_1 of two colors that do not appear on edges of Q_j under φ, and, moreover, each Q_j contains at most $2d-2$ precolored edges. Hence, by the induction hypothesis, the restriction of φ to each Q_j can be extended to a proper edge coloring using colors that do not appear on edges of D_1 that are incident with Q_j. In conclusion, φ is extendable. □

Lemma 4.3 *If there is a dimension of G with at most two precolored edges, the colors of which do not appear on edges in any other dimension of G, then φ is extendable.*

Proof: Let D_1 be a dimension containing at most two precolored edges, the colors of which do not appear on any edges in $G-E(D_1)$.

We first consider the case when only one color c_1 appears on the precolored edges of D_1.

Case 1. *Only one color c_1 appears on the precolored edge(s) of D_1:*
In this case the argument breaks into several subcases.

Case 1.1. *All precolored edges of $G-E(D_1)$ are contained in one plane:*
Let Q_1 be a plane in $G-E(D_1)$ containing all precolored edges except the ones of D_1. As before, there is an extension of the restriction of φ to Q_1 using colors $\{1,\dots,2d+1\} \setminus \{c_1\}$ (by removing the

colors of edges colored by some color $c_2 \neq c_1$, taking an extension of the resulting precoloring of Q_1 using colors $\{1, \dots, 2d+1\} \setminus \{c_1, c_2\}$ and then recoloring the edges that are φ-colored c_2). Next, we color all other planes of $G - E(D_1)$ correspondingly. Now, since all planes of $G - E(D_1)$ are colored correspondingly, every edge of D_1 is adjacent to edges of $2d - 2$ different colors, so by Lemma 2.2, φ is extendable.

Case 1.2. *All precolored edges of $G - E(D_1)$ are contained in two planes:*
Since at most two edges of D_1 are precolored, and only two planes contain precolored edges, there are two planes Q_j and Q_{j+1}, at most one of which contains precolored edges, and such that there is no precolored edge between Q_j and Q_{j+1}. Suppose e.g. Q_{j+1} contains no precolored edges. Let $c_2 \in \{1, \dots, 2d+1\}$ be a color such that no edge of G is precolored c_2. We take an extension of the restriction of φ to Q_j using colors $\{1, \dots, 2d+1\} \setminus \{c_1, c_2\}$, color Q_{j+1} correspondingly, and then color all edges between Q_j and Q_{j+1} by the unique color in $\{1, \dots, 2d+1\} \setminus \{c_1, c_2\}$ missing at its endpoints. Now, unless there are two precolored edges of D_1 that are contained in the same layer P and at even distance in the path P' obtained from P by removing the edge between Q_j and Q_{j+1}, we can color all edges of D_1 alternately by colors c_1 and c_2, and then color all remaining planes of $Q - E(D_1)$ by colors in $\{1, \dots, 2d+1\} \setminus \{c_1, c_2\}$ so that the resulting edge coloring is proper and agrees with φ.

Alternatively, if the distance between the two precolored edges of D_1 is even (in P'), then we select two additional planes Q_r and Q_{r+1}, containing no precolored edges between them and such that at most one of Q_r and Q_{r+1} contains precolored edges. We may then repeat the above coloring procedure for Q_r and Q_{r+1}; we leave the details to the reader.

Case 1.3. *All precolored edges of $G - E(D_1)$ are contained in at least three planes:*
Suppose first that there is only one precolored e of $G - E(D_1)$, and let $Q_{j_1}, Q_{j_2}, \dots, Q_{j_s}$ be the planes of $G - E(D_1)$ that contain precolored edges, where $j_1 \leq j_2 \leq \dots \leq j_s$. Note that any two planes contain precolored edges of altogether at most $2d-2$ colors. Now as in Case 3 of the proof of the preceding lemma, we color the edges of the paths between pairs of planes with precolored edges by picking two colors that do not appear in the restrictions of φ to these planes. Naturally, we pick these colors so that in the path containing e, the resulting coloring agrees with φ. Thereafter, we take extensions of the restrictions of φ to the planes $Q_{j_1}, Q_{j_2}, \dots, Q_{j_s}$, so that the resulting coloring is proper. Hence, φ is extendable.

Suppose now that $G - E(D_1)$ contains two precolored edges e_1 and e_2. Then there are colors c_2, c_3 that do not appear on any edges of G under φ.

Now, if e_1 and e_2 are corresponding edges or are not incident with a common plane, then we may proceed as in the preceding paragraph, but possibly pick three colors when coloring the paths between planes with precolored edges to ensure that the obtained coloring of D_1 is proper and agrees with φ. This is possible since any two planes contain at most $2d - 3$ precolored edges.

It remains to consider the case when e_1 and e_2 are incident with exactly one common plane Q_1. Suppose that e_1 in addition is incident with Q_2. If there are at most $2d-4$ precolored edges in $Q_1 \cup Q_2$ and at most $2d - 4$ precolored edges in $Q_{2k+1} \cup Q_1$, then there are independent edges e'_1 and e'_2 in Q_1, adjacent to e_1 and e_2, respectively, and such that neither these edges, nor the corresponding edges of Q_2 and Q_{2k+1}, respectively, are precolored. From the restriction of φ to $Q_{2k+1} \cup Q_1 \cup Q_2$, we define a precoloring φ' by coloring all these four edges of $Q_{2k+1} \cup Q_1 \cup Q_2$ by the color c_1. We may now obtain an extension of φ' by proceeding as in Case 3 of the preceding lemma, and thereafter swap colors on two bicolored 4-cycles containing e_1 and e_2, respectively, to obtain an extension of φ.

Suppose now instead that $Q_1 \cup Q_{2k+1}$, say, contain exactly $2d-3$ precolored edges. If there exist independent edges e_1' and e_2' in Q_1, as described in the preceding paragraph, then we may proceed as in that case, so suppose that there are no two such edges.

Then, since both $Q_{2k+1} \cup Q_1$ and $Q_1 \cup Q_2$ contain at most $2d-3$ precolored edges and every Q_j is $(2d-2)$-regular, the endpoints of e_1 and e_2 in Q_1 must be adjacent. Now, it is easy to see that this implies that there is either an edge e_1' adjacent to e_1 but not to e_2, such that e_1' and the corresponding edge of Q_2 are not precolored, or an uncolored edge e_2' adjacent to e_2 but not to e_1, and such that e_2' and the corresponding edge of Q_{2k+1} are not precolored. Suppose, for instance, that such an edge e_1' exists.

Consider the precoloring φ' obtained from the restriction of φ to $Q_{2k+1} \cup Q_1 \cup Q_2$ by in addition coloring e_1' and also the corresponding edge of Q_2 by the color c_1. Now, since there is no uncolored edge e_2' as described above, it follows that all $d-2$ edges $a_1, \dots, a_{d-2}$ adjacent to e_2 in Q_1 satisfy that either a_i, or the corresponding edge of Q_{2k+1}, is φ'-precolored or adjacent to an edge colored c_1 under φ'. Moreover, since $Q_{2k+1} \cup Q_1$ is triangle-free and contains at most $2d-2$ φ'-precolored edges, every precolored edge of $Q_{2k+1} \cup Q_1$ satisfies this condition. Thus by properly coloring the uncolored edges adjacent to e_2, except the one adjacent to e_1, by colors from $\{1, \dots, 2d+1\} \setminus \{c_1, c_2, c_3\}$, we obtain a precoloring φ'' from φ'. Then every plane in $G - E(D_1)$ contains at most $2d-2$ precolored edges under φ''. Furthermore any extension of the restriction of φ'' to $Q_{2k+1} \cup Q_1$ using colors $\{1, \dots, 2d+1\} \setminus \{c_2, c_3\}$ does not use c_1 on an edge adjacent to e_2. Now, since there is exactly one precolored edge of $G - E(D_1)$ that is not contained in $Q_{2k+1} \cup Q_1$, we may once again proceed as in Case 3 of Lemma 4.2 and color the edges of D_1 appropriately to obtain an extension of φ'' where no edge adjacent to e_2 is colored c_1. Thereafter we may swap colors on a bicolored 4-cycle and recolor e_2 to obtain an extension of φ.

Case 2. *The precolored edges of D_1 are colored differently:*

Suppose now that D_1 contains two precolored edges, colored c_1 and c_2, respectively, and that c_3 is a color that does not appear on any edge under φ. If there is an extension of the restriction of φ to D_1 using colors $\{c_1, c_2, c_3\}$, such that all edges of D_1 are colored correspondingly, then there are extensions of the restrictions of φ to all the planes $G - E(D_1)$ using colors that do not appear on incident edges of D_1. Hence, φ is extendable.

On the other hand, if there is no such extension of the restriction of φ, then the two precolored edges e_1 and e_2 of D_1 are incident with the same pair of planes, say Q_1 and Q_2. Now, if $Q_1 \cup Q_2$ contains at most $2d-4$ precolored edges, then there are uncolored corresponding edges $e_1' \in E(Q_1)$ and $e_2' \in E(Q_2)$ that are adjacent to e_2, but not to e_1. We may now color these edges c_2 and remove the color from e_2 to obtain the precoloring φ' from φ, and then proceed as in Case 3 when only one edge of D_1 is precolored to obtain an extension of φ'. Thereafter we swap colors on a bicolored 4-cycle to obtain an extension of φ.

If, on the other hand, $Q_1 \cup Q_2$ contains at least $2d-3$ precolored edges, then there is at most one edge in $G - E(D_1) \cup E(Q_1) \cup E(Q_2)$ that is precolored. Without loss of generality, we assume that Q_3 contains no precolored edge. By the induction hypothesis, there is an extension of the restriction of φ to $Q_1 \cup Q_2$ using colors $\{1, \dots, 2d+1\} \setminus \{c_1, c_2\}$. We color Q_3 correspondingly to how Q_2 is colored, and every edge between Q_2 and Q_3 by the color in $\{1, \dots, 2d+1\} \setminus \{c_1, c_2\}$ missing at its endpoints. All other edges in D_1 are colored c_1, c_2 alternately so that the coloring agrees with φ. Finally, we color all hitherto uncolored planes using colors $\{1, \dots, 2d+1\} \setminus \{c_1, c_2\}$ so that the resulting coloring agrees with φ. In conclusion, φ is extendable. □

Lemma 4.4 *If each dimension of G contains precolored edges and there is a dimension with exactly one precolored edge, the color of which does appear on edges in other dimensions of G, then φ is extendable.*

Proof: Let D_1 be a dimension containing only one precolored edge e, colored, say c_1, and consider the subgraph $G - E(D_1)$ consisting of $2k+1$ planes $Q_1, \dots, Q_{2k+1}$. Since at least one color appears on at least two edges, there are two colors $c_2, c_3 \in \{1, \dots, 2d+1\}$ that do not appear on any edge under φ.

Case 1. *All precolored edges of $G - E(D_1)$ are contained in one plane:*
In this case, we may proceed as in Case 1 of Lemma 3.4, but use Lemma 2.2 instead of Lemma 2.1. We omit the details.

Case 2. *All precolored edges of $G - E(D_1)$ are contained in two planes:*
Let Q_1 and Q_i be the two planes containing the precolored edges distinct from e.

Let us first consider the case when e is not incident to Q_1 or Q_i. Since each of Q_1 and Q_i contains at most $2d-2$ precolored edges, there are extensions f_1 and f_i of the restrictions of φ to Q_1 and Q_i, respectively, using colors $\{1, \dots, 2d+1\} \setminus \{c_2, c_3\}$. Therafter we color the edges of D_1 properly and correspondingly using colors $\{c_1, c_2, c_3\}$ so that the coloring agrees with the restriction of φ to D_1 and has no conflicts with f_1 or f_i. Finally we color the remaining planes of $G - E(D_1)$, as to obtain an extension of φ.

Suppose now that e is incident with Q_1 and Q_2, and $i \neq 2$. Then either Q_3 or Q_{2k+1} contains no precolored edges; suppose Q_3. (The case when Q_{2k+1} has this property is similar.) As in the preceding paragraph, there are extensions f_1 and f_i of the restrictions of φ to Q_1 and Q_i, respectively, using colors $\{1, \dots, 2d+1\} \setminus \{c_2, c_3\}$. We color Q_2 and Q_3 correspondingly to how Q_1 is colored. Next, we color every edge of D_1 between Q_2 and Q_3 by a color in $\{1, \dots, 2d+1\} \setminus \{c_2, c_3\}$ missing at its endpoints, and then color all other edges of D_1 alternately by colors c_2 and c_3 so that all edges between Q_1 and Q_2 are colored c_2. The remaining uncolored planes of G are properly colored using colors $\{1, \dots, 2d+1\} \setminus \{c_2, c_3\}$. Now, if e is adjacent to edges colored c_1, then we swap colors on a bicolored 4-cycle containing e and two edges colored c_1 to obtain an extension of φ; otherwise we simply recolor e to obtain an extension of φ.

Suppose now that e is incident with Q_1 and Q_2, and $i = 2$. By removing the color from any edge that is colored c_1 under φ, we obtain a precoloring φ' of G. The restriction of φ' to Q_1 and Q_2 are extendable to proper edge colorings, respectively, using colors $\{1, \dots, 2d+1\} \setminus \{c_1, c_2\}$. By recoloring any edge of Q_1 and Q_2 that is φ-colored c_1 by the color c_1 we obtain edge colorings f_1 of Q_1 and f_2 of Q_2, respectively.

Next, we color every edge between Q_1 and Q_2 by the color c_2 except that e is colored c_1. Thereafter, we color $Q_3, \dots, Q_{2k}$ correspondingly to how Q_2 is colored, and Q_{2k+1} correspondingly to how Q_1 is colored. Now, for every vertex x of Q_2, there are colors $c_x, c'_x \in \{1, \dots, 2d+1\}$ that do not appear at x in Q_2 or on the incident edge between Q_1 and Q_2. We color every path in D_1 from Q_2 to Q_{2k} by colors c_x and c'_x alternately, and thereafter color every edge between Q_{2k} and Q_{2k+1} by the color of the edge in the same layer between Q_1 and Q_2. Finally, we color the edges between Q_1 and Q_{2k+1} by a color missing at its endpoints to obtain an extension of φ.

Case 3. *All precolored edges of $G - E(D_1)$ are contained in at least three planes:*

The assumption implies that every plane in $G - E(D_1)$ contains at most $2d - 3$ precolored edges. Assume that e is incident with Q_1 and Q_2.

Suppose first that $Q_1 \cup Q_2$ contains altogether at most $2d-3$ precolored edges. Then there are uncolored corresponding edges $e_1 \in E(Q_1)$ and $e_2 \in E(Q_2)$ that are adjacent to e but not to any edge in $Q_1 \cup Q_2$ φ-colored c_1. From the restriction of φ to $G - E(D_1)$ we define a new precoloring φ' by coloring e_1 and e_2 by the color c_1. Thereafter we may proceed as in Case 3 of the proof of Lemma 4.2 to obtain a proper $(2d+1)$-edge coloring of G which is an extension of φ' and where the edges of D_1 are colored correspondingly. Thus, by swapping colors on a bicolored 4-cycle we obtain an extension of φ.

Suppose now that $Q_1 \cup Q_2$ contains altogether $2d - 2$ precolored edges. If there are uncolored corresponding edges $e_1 \in E(Q_1)$ and $e_2 \in E(Q_2)$ that are adjacent to e but not to any edge in $Q_1 \cup Q_2$ φ-colored c_1, then we proceed as in the preceding paragraph. So assume that there are no such edges e_1 and e_2. Then there are $2d-2$ edges $e_1, \dots, e_{2d-2}$ in Q_1 that are adjacent to e and such that each of these edges satisfies that

(i) e_j or the corresponding edge of Q_2 is precolored by a color distinct from c_1, or

(ii) e_j or the corresponding edge of Q_2 is adjacent to an edge precolored c_1.

Moreover, since $Q_1 \cup Q_2$ is triangle-free and contains at most $2d-2$ precolored edges, every precolored edge in $Q_1 \cup Q_2$ satisfies one of these conditions. Now, for $j = 1, 2$, from the restriction of φ to Q_j, we define a new precoloring φ_j of Q_j by coloring every edge of Q_1 and Q_2 that is adjacent to e and does not satisfy (i) or (ii) by a color in $\{1, \dots, 2d+1\} \setminus \{c_1, c_2, c_3\}$ so that the resulting coloring is proper and agrees with φ. Now, by the induction hypothesis, φ_j is extendable to a proper edge coloring of Q_j using colors $\{1, \dots, 2d+1\} \setminus \{c_2, c_3\}$. Note that no edge of Q_1 or Q_2 adjacent to e is colored c_1 in these colorings. Thus we may color all edges between Q_1 and Q_2 by c_2 except e which is colored c_1.

Next, suppose that Q_r, $r \notin \{1, 2\}$, is the third plane containing a precolored edge. Then Q_{r+1} or Q_{r-1} contains no precolored or hitherto colored edges, suppose Q_{r+1}. We take an extension of the restriction of φ to Q_r using colors $\{1, \dots, 2d+1\} \setminus \{c_2, c_3\}$, and color all other uncolored planes correspondingly to how Q_r is colored. Thereafter we color the edges between Q_r and Q_{r+1} by the unique color in $\{1, \dots, 2d+1\} \setminus \{c_2, c_3\}$ missing at it endpoints. Finally, we color all remaining uncolored edges of D_1 by colors c_2, c_3 alternately so that the resulting coloring is proper. This yields an extension of φ. □

Lemma 4.5 *If each dimension of G contains exactly two precolored edges, at least one of which is colored by a color appearing on precolored edges in other dimensions, then φ is extendable.*

Proof: Let D_1 be a dimension containing two precolored edges e_1 and e_2. By assumption, at most $2d-1$ colors appear on edges under φ, so let let c_3, c_4 be two colors from $\{1, \dots, 2d+1\}$ that do not appear on any edges under φ.

Case 1. *All precolored edges of $G - E(D_1)$ are contained in one plane:*

Suppose that all precolored edges except e_1 and e_2 lie in one component Q_1 of $G - E(D_1)$. Without loss of generality, we assume that $\{\varphi(e_1), \varphi(e_2)\} \subseteq \{c_1, c_2\}$. We define a new precoloring φ' from the restriction of φ to Q_1 by removing the colors c_1 and c_2 from any edges of Q_1 colored by these colors. Now, by the induction hypothesis φ' is extendable to a proper $(2d-1)$-edge coloring of Q_1 using colors $\{1, \dots, 2d+1\} \setminus \{c_1, c_2\}$. By recoloring the edges of Q_1 that are φ-colored c_1 and c_2 by colors c_1 and

c_2, respectively, we obtain a proper edge coloring f' of Q_1. Next, we color all other planes of $G - E(D_1)$ correspondingly and define a list assignment for the edges of D_1 by assigning every edge the set of colors from $\{1, \dots, 2d+1\}$ not appearing on its adjacent edges. Then each edge of D_1 receives a list of 3 colors except for the two edges e_1 and e_2 that are precolored. Thus, there is an extension of φ.

Case 2. *All precolored edges of $G - E(D_1)$ are contained in two planes:*
We shall consider several different subcases.

Case 2.1. *The precolored edges of D_1 have the same color under φ:*
Suppose that $\varphi(e_1) = \varphi(e_2) = c_1$. If e_1 and e_2 are both incident with the same two planes, then we may apply arguments which are similar to the ones in Case 2 of the proof of Lemma 4.4. Consequently, assume that e_1 and e_2 are incident with at most one common plane.

Subcase 2.1.1. *e_1 and e_2 are both incident with exactly one common plane:*
Suppose that e_1 and e_2 are both incident with the common plane Q_1, and that e_1 is also incident with Q_2. If Q_1 and Q_2 contain all precolored edges of $G - E(D_1)$, then a similar argument as in the subcase of Case 2 of Lemma 4.4 when $Q_1 \cup Q_2$ contains all precolored edges of $G - E(D_1)$ again applies, so we omit the details here as well.

It remains to consider the following subcases:

(a) Q_1 contains precolored edges, but neither of Q_2 and Q_{2k+1}.

(b) Either Q_{2k+1} or Q_2, but not Q_1, contains precolored edges.

(c) Both Q_2 and Q_{2k+1} contain precolored edges.

(a) holds:
Suppose that Q_1 and Q_i contain all precolored edges of $G - E(D_1)$, where $i \notin \{1, 2, 3, 2k+1\}$. If e_1 and e_2 are adjacent via an uncolored edge e in Q_1, then since Q_1 contains at most $2d-3$ precolored edges, there is a color $c \in \{1, \dots, 2d+1\} \setminus \{c_1, c_3, c_4\}$ that does not appear on any edge adjacent to e. Thus the precoloring φ' obtained from φ by in addition coloring e by the color c is proper. On the other hand, if there is no such edge, then we set $\varphi' = \varphi$.

Now, since both Q_1 and Q_i contain at most $2d-2$ φ'-precolored edges, there are extensions f_1 and f_i, respectively, of the restrictions of φ' to Q_1 and Q_i, respectively, using colors $\{1, \dots, 2d+1\} \setminus \{c_3, c_4\}$. We color Q_2, Q_3, Q_{2k+1} correspondingly to how Q_1 is colored, and thereafter color every edge between Q_2 and Q_3 by the color in $\{1, \dots, 2d+1\} \setminus \{c_3, c_4\}$ missing at its endpoints.

Next, we color all hitherto uncolored edges of D_1 alternately using colors c_3, c_4 so that all edges between Q_1 and Q_2 have color c_3, and also color all hitherto uncolored planes properly using colors $\{1, \dots, 2d+1\} \setminus \{c_3, c_4\}$. The obtained coloring is proper and agrees with φ' except for e_1 and e_2. Now, if neither e_1 of e_2 are adjacent to an edge colored c_1, then we simply recolor them; otherwise, we swap on one or two bicolored cycles to obtain an extension of φ; note that if both e_1 and e_2 are adjacent to edges colored c_1, then these cycles are disjoint. Hence, φ is extendable.

(b) holds:
If instead either Q_{2k+1} or Q_2, but not Q_1, contains precolored edges, then a similar argument as in (a) applies, so we omit the details.

(c) holds:

Assume that Q_2 and Q_{2k+1} contain all precolored edges of $G - E(D_1)$, and let u_{2k+1} and u_2 be the vertices of Q_{2k+1} and Q_2 that are incident with e_1 and e_2, respectively.

If both Q_{2k+1} and Q_2 contain at most $2d-4$ precolored edges, then there are uncolored edges $e'_{2k+1} \in E(Q_{2k+1})$ and $e'_2 \in E(Q_2)$ that are incident with u_{2k+1} and u_2, respectively, not adjacent to any edges of $Q_{2k+1} \cup Q_2$ precolored c_1, and such that the corresponding edges of Q_1 are independent. We color e'_{2k+1}, e'_2, and also the corresponding edges of Q_1 by the color c_1. Together with φ, this defines a precoloring of $Q_{2k+1} \cup Q_1 \cup Q_2$, which by the induction hypothesis is extendable to a proper edge coloring using colors $\{1, \dots, 2d+1\} \setminus \{c_3, c_4\}$. We color all edges between Q_1 and Q_2 with color c_4, and all edges between Q_{2k+1} and Q_1 by the color c_3. Next, we color Q_{2k} and Q_3 correspondingly to how Q_{2k+1} and Q_2 are colored, respectively. Thereafter we color all edges between Q_{2k+1} and Q_{2k} by the color in $\{1, \dots, 2d+1\} \setminus \{c_3, c_4\}$ missing at its endpoints, and similarly for Q_2 and Q_3. Finally, we color all uncolored edges of D_1 alternately by colors c_3, c_4, color all hitherto uncolored planes using colors $\{1, \dots, 2d+1\} \setminus \{c_3, c_4\}$ and swap on two bicolored cycles containing e_1 and e_2, respectively, to obtain an extension of φ.

Suppose now instead that one of Q_{2k+1} and Q_2 contains $2d-3$ precolored edges, say Q_{2k+1}. Then Q_2 contains exactly one precolored edge e. By removing the color from any edge of Q_{2k+1} that is precolored c_1, we obtain a precoloring φ' from the restriction of φ to Q_{2k+1}. φ' is extendable to a proper coloring of Q_{2k+1} using colors $\{1, \dots, 2d+1\} \setminus \{c_1, c_3\}$ and by recoloring the edges of Q_{2k+1} φ-colored c_1 by the color c_1 we obtain an extension f_{2k+1} of the restriction of φ to Q_{2k+1}.

Next, we color Q_1 correspondingly to Q_{2k+1} except that we color any edge of Q_1 corresponding to an edge colored c_1 by the color c_3. We color the edges between Q_{2k+1} and Q_1 by an arbitrary color in $\{1, \dots, 2d+1\} \setminus \{c_1, c_3\}$ not appearing at its endpoints, except that e_2 is colored c_1.

Suppose first that e is precolored c_1. Then we color Q_2 correspondingly to how Q_1 is colored, but color e by color c_1. Next we color all edges between Q_1 and Q_2 by an arbitrary color in $\{1, \dots, 2d+1\}$ missing at its endpoints except that e_1 is colored c_1. Thereafter, we color Q_3 correspondingly to how Q_1 is colored, and all remaining uncolored planes correspondingly to how Q_{2k+1} is colored. We may then color the hitherto uncolored edges of D_1 appropriately to obtain an extension of φ.

Suppose now that the precolored edge of Q_2 is colored $c_2 \neq c_1$. Let e' be the edge of Q_1 corresponding to e, and assume that e' is colored c' in the hiherto constructed coloring. We color Q_2 correspondingly to how Q_1 is colored but permute the colors c_2 and c' in the coloring of Q_2. Thereafter, we color Q_3 correspondingly to Q_2 except that we permute colors c_2 and c', and finally we color the remaining uncolored edges of G by proceeding as in the preceding paragraph.

Subcase 2.1.2 *e_1 and e_2 are not incident with a common plane:*

Suppose that e_1 is incident with Q_1 and Q_2 and e_2 is incident with Q_j and Q_{j+1}, and all these four planes are distinct. If all precolored edges are contained in $Q_1 \cup Q_2$, then as before we may then select corresponding uncolored edges e'_j and e'_{j+1} of Q_j and Q_{j+1} that are adjacent to e_2. Next, we consider the precoloring φ' obtained from φ by coloring e'_j and e'_{j+1} by c_1 and removing the color c_1 from e_2. We may now apply similar arguments as in the subcase of Case 3 of Lemma 4.4 when $Q_1 \cup Q_2$ contains $2d-2$ precolored edges to obtain an extension of φ'. In particular, since there is at least one plane in G that is distinct from Q_1, Q_2, Q_j, Q_{j+1} that contains no φ'-colored edges, we can color the edges of D_1

so that all edges between Q_j and Q_{j+1} have the same color. We may then swap on a bicolored 4-cycle to obtain an extension of φ.

Suppose now that exactly one of the planes Q_1 and Q_2, and exactly one of the planes Q_j and Q_{j+1} contain precolored edges. Assume e.g. that Q_1 and Q_j contain no precolored edges (the other cases are analogous). We pick an edge e'_2 in Q_2 that is uncolored and adjacent to e_1, but not adjacent to any other edge precolored c_1, and a similar edge e'_{j+1} of Q_{j+1}; since each of these planes contains at most $2d-3$ precolored edges, such edges exist. From the restriction of φ to $Q_2 \cup Q_{j+1}$ we define a new precoloring φ' by in addition coloring e'_2 and e'_{j+1} c_1. Now, by the induction hypothesis, there are extensions f_2 and f_{j+1} of the restrictions of φ' to Q_2 and Q_{j+1}, respectively, using colors $\{1, \dots, 2d+1\} \setminus \{c_3, c_4\}$.

Let us now color the other planes of $G - E(D_1)$. Without loss of generality, we assume that $j + 1 < 2k + 1$. We color Q_1 and Q_{2k+1} correspondingly to how Q_2 is colored, Q_j correspondingly to how Q_{j+1} is colored, and all other planes arbitrarily using colors $\{1, \dots, 2d+1\} \setminus \{c_3, c_4\}$. Thereafter we color all edges between Q_1 and Q_{2k+1} by a color in $\{1, \dots, 2d+1\} \setminus \{c_3, c_4\}$ missing at its endpoints, and all other edges of D_1 alternately using colors c_3, c_4 and starting with color c_3 at Q_1. Finally, we swap colors on two bicolored 4-cycles containing e_1 and e_2, respectively, to obtain an extension of φ.

Finally, we consider the case when Q_1 may contain precolored edges, but none of Q_2, Q_j, Q_{j+1} contain precolored edges. We define a precoloring φ' from the restriction of φ to Q_1 by selecting an edge $e'_1 \in E(Q_1)$ adjacent to e_1 and coloring it c_1, as before. Thereafter, we take an extension of φ' using colors $\{1, \dots, 2d+1\} \setminus \{c_3, c_4\}$, and color Q_2 correspondingly. Next, we color all edges between Q_j and Q_{j+1} by the color c_1, and all other edges of D_1 alternately using colors c_3 and c_4, and starting with color c_3 at Q_{j+1}. We now obtain an extension of φ as before.

Case 2.2 *The precolored edges of D_1 are colored differently under φ:*
Suppose that $\varphi(e_1) = c_1$ and $\varphi(e_2) = c_2$. We shall consider some different cases.

Subcase 2.2.1 *e_1 and e_2 are both incident with two common planes:*
Suppose that e_1 and e_2 are both incident with the planes Q_1 and Q_2. Let u_1 and u_2 be the vertices in Q_1 that are incident with e_1 and e_2, respectively.

If none of Q_1 and Q_2 contain precolored edges, then we can select independent edges in Q_1 (and Q_2) that are incident with u_1 and u_2, respectively, and color them c_1 and c_2. Then we may proceed as in Case 3 of Lemma 4.2 to obtain an extension of the resulting precoloring φ' of $G - E(D_1)$, and thereafter we obtain an extension of φ, as before.

Let us now assume that all precolored edges are contained in $Q_1 \cup Q_2$. We first prove the following claim.

Claim 4.6 *Suppose $d \geq 3$. At least one of the following two statements hold.*
(i) There is an edge $e'_1 \in E(Q_1)$ incident with u_1 but not u_2, such that e'_1 and the corresponding edge e''_1 of Q_2 are uncolored and not adjacent to any edge colored c_1.
(ii) There is an edge $e'_2 \in E(Q_1)$ incident with u_2 but not u_1, such that e'_2 and the corresponding edge e''_2 of Q_2 are uncolored and not adjacent to any edge colored c_2.

Proof: Suppose that (i) is false. Since $G - E(D_1)$ is triangle-free and $(2d-2)$-regular, there are $2d-3$ edges $a_1, \dots, a_{2d-3} \in E(Q_1)$ incident with u_1, all of which are either precolored, adjacent to an edge colored c_1, or satisfies that the corresponding edge of Q_2 satisfies one of these conditions. Since $Q_1 \cup Q_2$

contains $2d-2$ precolored edges, it is easy to see that then (ii) must hold, so there is an edge e'_2 as desired. □ If $d = 2$, we note that the claim might fail if u_1 and u_2 are adjacent via an uncolored edge. However, in this case, it is trivial to verify that φ is extendable, since every precoloring of an odd cycle is extendable using 3 colors.

Suppose now that (ii) of Claim 4.6 holds, and let e'_2 and e''_2 be corresponding edges of Q_1 and Q_2 respectively, as described in the claim. From the restriction of φ to $Q_1 \cup Q_2$ we define a new precoloring φ' of $Q_1 \cup Q_2$ by in addition coloring e'_2 and e''_2 by the color c_2. We may now proceed as in the subcase of Case 2 of the proof of Lemma 4.4 when all the precolored edges are contained in $Q_1 \cup Q_2$, to obtain an extension of φ' (with c_3 in place of c_2). Thereafter, we swap colors on a bicolored 4-cycle containing e_2 to obtain an extension of φ.

It remains to consider the case when all precolored edges are contained in Q_1 and Q_i, where $i \neq 2$. Then either Q_3 or Q_{2k+1} contains no precolored edges, say Q_3.

Suppose first that Q_1 contains at most $2d-4$ precolored edges. Then since Q_1 is $(2r-2)$-regular, there are independent uncolored edges $e'_1 \in E(Q_1)$ and $e'_2 \in E(Q_1)$ incident with u_1 and u_2, respectively, and such that e'_1 is not adjacent to any edge of Q_1 φ-colored c_1, and e'_2 is not adjacent to any edge of Q_1 φ-colored c_2.

From the restriction of φ to Q_1 we define a new precoloring φ' of Q_1 by in addition coloring e'_1 by the color c_1, and e'_2 by the color c_2. Next, we take an extension of φ' using colors $\{1,\dots,2d+1\} \setminus \{c_3, c_4\}$, and color Q_2 and Q_3 correspondingly to how Q_1 is colored. Thereafter, we color the edges of D_1 as follows: color all edges between Q_2 and Q_3 by a color in $\{1,\dots,2d+1\} \setminus \{c_3, c_4\}$ missing at its endpoints, and color all other edges of D_1 alternately using colors c_3 and c_4 so that all edges between Q_1 and Q_2 are colored c_3. Thereafter, we color the planes $Q_4,\dots,Q_{2k+1}$ with colors $\{1,\dots,2d+1\} \setminus \{c_3, c_4\}$ so that the coloring agrees with φ. Now we may obtain an extension of φ by swapping colors on two bicolored 4-cycles containing e_1 and e_2, respectively.

Suppose now that Q_1 contains exactly $2d-3$ precolored edges. Then Q_i contains exactly one precolored edge. Moreover, as in the proof of Claim 4.6 it is straightforward that there is

- either an edge e'_1 satisfying (i) of Claim 4.6, or
- an edge e'_2 satisfying (ii) of Claim 4.6.

Suppose e.g. that (i) holds. Then from the restriction of φ to Q_1 we define a new precoloring φ' of Q_1 by in addition coloring e'_1 by the color c_1 and removing the color from any edge of Q_1 that is colored c_2.

Next, we take an extension of φ' using colors $\{1,\dots,2d+1\} \setminus \{c_2, c_3\}$, recolor the edges φ-colored c_2 by the color c_2, and thereafter color Q_2 and Q_3 correspondingly to how Q_1 is colored. Denote the obtained coloring by f. We color all edges between Q_1 and Q_2 by the color c_3 except e_2 which is colored c_2, and color the edges between Q_2 and Q_3 by a color in $\{1,\dots,2d+1\} \setminus \{c_2, c_3\}$ missing at its endpoints.

Now, let e''_i be the edge of Q_i that is precolored, and let e''_1 be the corresponding edge of Q_1. If $f(e''_1) = \varphi(e''_i)$, then we color all hitherto uncolored planes correspondingly to how Q_1 is colored, therafter color the remaining uncolored edges of D_1 and finally swap colors on a bicolored 4-cycle containing e_1 to obtain an extension of φ.

Otherwise, if $f(e''_1) \neq \varphi(e''_i)$, then we color all other planes correspondingly to how Q_1 is colored, except that we permute the colors $f(e''_1)$ and $\varphi(e''_i)$ in the colorings. We may now apply similar arguments as before to obtain an extension of φ; we leave the details to the reader.

Subcase 2.2.2 *e_1 and e_2 are incident with exactly one common plane:*

Suppose that e_1 is incident with Q_1 and Q_2, and e_2 with Q_1 and Q_{2k+1}. Let u_1 and u_2 be the vertices of Q_1 that are incident with e_1 and e_2, respectively. If neither of Q_1, Q_2, Q_{2k+1} contain precolored edges, then a similar argument as in the second paragraph of Subcase 2.2.1 applies. Thus it suffices to consider the following subcases:

(a) All precolored edges of $G - E(D_1)$ are contained in $Q_1 \cup Q_2$.

(b) Q_1 contains precolored edges, but neither of Q_2 and Q_{2k+1}.

(c) Q_2, but not Q_1 or Q_{2k+1}, contains precolored edges.

(d) All precolored edges of $G - E(D_1)$ are contained in $Q_2 \cup Q_{2k+1}$.

By symmetry, it suffices to consider these cases.

(a) holds:

We first consider the case when there is an edge $e_1' \in E(Q_1)$ adjacent to e_1, such that both e_1' and the corresponding edge $e_2' \in E(Q_2)$ are uncolored and not adjacent to any edge precolored c_1. If this holds, then from the restriction of φ to $Q_1 \cup Q_2$ we define a new precoloring φ' by coloring e_1' and e_2' by the color c_1, and also removing the color from any edge that is φ-colored c_2.

By the induction hypothesis, there is an extension of φ' using colors $\{1, \dots, 2d+1\} \setminus \{c_2, c_3\}$. From φ', we obtain an edge coloring f of $Q_1 \cup Q_2$ by recoloring the edges of Q_1 and Q_2 that are φ-colored c_2 by the color c_2. We color all edges between Q_1 and Q_2 by the color c_3, all the planes $Q_3, \dots, Q_{2k}$ correspondingly to how Q_2 is colored, and Q_{2k+1} correspondingly to how Q_1 is colored. Now the edges between Q_{2k+1} and Q_{2k} can be colored with the color c_3, and every other edge of D_1 by some appropriate color missing at its endpoints. Thus by swapping colors on a bicolored 4-cycle containing e_1 we obtain an extension of φ.

Suppose now that there is no edge $e_1' \in E(Q_1)$ adjacent to e_1, such that both e_1' and the corresponding edge $e_2' \in E(Q_2)$ are uncolored and not adjacent to any edge precolored c_1. Then, since $Q_1 \cup Q_2$ is $(2d-2)$-regular and contains $2d-2$ precolored edges, u_1 is incident with $2d-2$ edges $a_1, \dots, a_{2d-2}$ such that each a_i, or the corresponding edge of Q_2, is φ-colored by a color distinct from c_1, or uncolored and adjacent to an edge φ-colored c_1. In particular, if there is an edge of $Q_1 \cup Q_2$ colored c_2, then at most one edge in each of Q_1 and Q_2 is colored c_2. Moreover, since G is triangle-free and $Q_1 \cup Q_2$ contains exactly $2d-2$ precolored edges, an edge in $Q_1 \cup Q_2$ precolored c_2 is not adjacent to an edge precolored c_1 in $Q_1 \cup Q_2$.

If Q_1 contains an edge a precolored c_2, then from the restriction of φ to $Q_1 \cup Q_2$, we define a precoloring φ' by recoloring a and also the corresponding edge of Q_2 by the color c_1. Otherwise, if both Q_1 and Q_2 contain edges precolored c_2, then we define φ' by recoloring these edges by the color c_1. Now, by the induction hypothesis, there is an extension of the coloring φ' using colors $\{1, \dots, 2d+1\} \setminus \{c_2, c_3\}$. By recoloring the edges that were recolored c_1 by the color c_2 we obtain an extension of the restriction of φ to $Q_1 \cup Q_2$. Next, we color e_1 by the color c_1 and all other edges between Q_1 and Q_2 by the color c_3. We color Q_{2k+1} correspondingly to Q_1, and $Q_3, \dots, Q_{2k}$ correspondingly to how Q_2 is colored, and then color the hitherto uncolored edges of D_1 as before to obtain an extension of φ.

On the other hand, if no edge of Q_1 is colored c_2, then from the restriction of φ to $Q_1 \cup Q_2$, we define a new precoloring φ' by coloring all edges adjacent to e_1 that are not precolored or adjacent to an edge colored c_1 in $G - E(D_1)$ by an arbitrary color from $\{1, \dots, 2d+1\} \setminus \{c_1, c_2, c_3\}$ so that the resulting precoloring is proper. By the induction hypothesis, the obtained precoloring of Q_1 is extendable to a proper coloring using colors $\{1, \dots, 2d+1\} \setminus \{c_2, c_3\}$, and the precoloring of Q_2 is extendable using colors $\{1, \dots, 2d+1\} \setminus \{c_3, c_4\}$, where c_4 is some arbitrary color not appearing on an edge of Q_2. Note that no edge adjacent to e_1 is colored c_1 in this coloring. Hence, we can color Q_{2k+1} correspondingly to how Q_1 is colored, all edges between Q_1 and Q_{2k+1} by the color c_2, and all edges between Q_1 and Q_2 by the color c_3 except that e_1 is colored c_1. Since not other planes in $G - E(D_1)$ contain precolored edges, it is now straightforward to obtain an extension of φ from this partial coloring.

(b) holds:
Suppose that Q_1 and Q_i contain all precolored edges of $G - E(D_1)$, where $i \notin \{1, 2, 3, 2k+1\}$. We take an extension of the restriction of φ to $Q_1 \cup Q_i$ using colors $\{1, \dots, 2d+1\} \setminus \{c_3, c_4\}$, color Q_2, Q_3, Q_{2k+1} correspondingly to how Q_1 is colored, and all remaning planes in $G - E(D_1)$ by the colors $\{1, \dots, 2d+1\} \setminus \{c_3, c_4\}$ so that the coloring agrees with φ. Next, we color the edges of D_1: the edges between Q_2 and Q_3 we color with the color in $\{1, \dots, 2d+1\} \setminus \{c_3, c_4\}$ missing at its endpoints, and all other edges of D_1 are colored c_3 and c_4 alternately, and starting with color c_3 at Q_2. This yields a coloring that agrees with φ except for e_1 and e_2. We recolor these edges by c_1 and c_2, respectively, possibly by swapping on one or two bicolored 4-cycles if necessary, to obtain an extension of φ.

(c) holds:
The case when Q_2, but not Q_{2k+1} or Q_1, contains precolored edges can be dealt with as in the preceding pagragraph, so we omit the details here.

(d) holds:
If $d = 2$, then it is straightforward that φ is extendable, because any partial 3-edge coloring of an odd cycle is extendable. If $d > 2$, then since $Q_2 \cup Q_{2k+1}$ contains exactly $2d - 2$ precolored edges, it is straightforward that there are non-corresponding edges $e'_{2k+1} \in E(Q_{2k+1})$ and $e'_2 \in E(Q_2)$ that are uncolored, adjacent to e_1 and e_2, respectively, and not adjacent to any edges of $Q_{2k+1} \cup Q_2$ precolored c_2 and c_1, respectively.

From the restriction of φ to $Q_{2k+1} \cup Q_1 \cup Q_2$, we define a new precoloring φ' of $Q_{2k+1} \cup Q_1 \cup Q_2$ by in addition coloring e'_{2k+1} by c_2, e'_2 by the color c_1, and the corresponding edges of Q_1 by colors c_2 and c_1 respectively. By the induction hypothesis, there is an extension of φ' to $Q_{2k+1} \cup Q_1 \cup Q_2$ using colors $\{1, \dots, 2d+1\} \setminus \{c_3, c_4\}$. From this coloring we may now obtain an extension of φ by proceeding as before.

Subcase 2.2.3 *e_1 and e_2 are not incident with any common plane:*
Suppose that e_1 is incident with Q_1 and Q_2, and e_2 is incident with Q_j and Q_{j+1}. As in Subcase 2.1.2, we can distinguish between the following three cases:

- All precolored edges are contained in Q_1 and Q_2.
- One of Q_1 and Q_2, and one of Q_j and Q_{j+1}, contain precolored edges.
- At most one of the planes Q_1, Q_2, Q_j, Q_{j+1} contains precolored edges.

Moreover, in all these three subcases we may proceed precisely as in the corresponding subcases of Subcase 2.1.2. We omit the details.

Case 3. *All precolored edges of $G - E(D_1)$ are contained in at least three planes:*
In the case when no precolored edge of D_1 is incident with a plane containing precolored edges, then it is straightforward to obtain an extension by selecting uncolored edges in the planes that the precolored edges of D_1 are incident with, so throughout we assume that this is not the case. Note further that since G contains at least five precolored edges, $d \geq 3$.

Case 3.1. *The precolored edges of D_1 have the same color under φ:*
Suppose that $\varphi(e_1) = \varphi(e_2) = c_1$. We consider a number of different subcases.

Subcase 3.1.1 *e_1 and e_2 are incident with the same two planes:*
Assume that e_1 and e_2 are incident with the same two planes Q_1 and Q_2. If the endpoints of e_1 and e_2 are adjacent via uncolored edges in both Q_1 and Q_2, then we define the precoloring φ' from the restriction of φ to $Q_1 \cup Q_2$ by coloring these edges of Q_1 and Q_2 by the color c_1. We may now proceed as in Case 3 of Lemma 4.2 to obtain an extension of φ', and thereafter swap colors on a bicolored 4-cycle to obtain an extension of φ.

Otherwise, if the endpoints of e_1 and e_2 are not adjacent via uncolored edges in both Q_1 and Q_2. Then, since $d > 2$, $Q_1 \cup Q_2$ contains at most $2d-3$ precolored edges and any two vertices in G are contained in at most one 5-cycle, it is not hard to see that there are independent edges e_1' and e_2' in Q_1, adjacent to e_1 and e_2, respectively, and such that neither these edges, nor the corresponding edges of Q_2 are precolored or adjacent to edges precolored c_1 in $Q_1 \cup Q_2$. Hence, we may color these edges of Q_1 and Q_2 by the color c_1, and then proceed as in the preceding paragraph to obtain an extension of φ.

Subcase 3.1.2 *e_1 and e_2 are incident with one common plane:*
Suppose that e_1 and e_2 are incident with exactly one common plane Q_1, and that e_1 is also incident with Q_2. If there are at most $2d - 4$ precolored edges in $Q_1 \cup Q_2$ and at most $2d - 4$ precolored edges in $Q_{2k+1} \cup Q_1$, then there are independent edges e_1' and e_2' in Q_1, adjacent to e_1 and e_2, respectively, and such that neither these edges, nor the corresponding edges of Q_2 and Q_{2k+1}, respectively, are precolored or adjacent to edges precolored c_1 in $Q_1 \cup Q_2 \cup Q_{2k+1}$. Thus we may proceed as above to obtain an extension of φ.

Suppose now instead that $Q_1 \cup Q_{2k+1}$, say, contains exactly $2d - 3$ precolored edges. If there exist independent edges e_1' and e_2' in Q_1, as described in the preceding paragraph, then we may proceed as in that case, so suppose that there are no two such edges.

We first consider the case when we can choose exactly one such edge, that is, there is an edge $e_1' \in E(Q_1)$ adjacent to e_1 but not e_2, satisfying that e_1' and the corresponding edge of Q_2 are not precolored or adjacent to edges of $Q_1 \cup Q_2$ that are precolored c_1. Moreover, there is no edge e_2' adjacent to e_2 with analogous properties. Consider the precoloring φ' obtained from the restriction of φ to $Q_{2k+1} \cup Q_1 \cup Q_2$ by in addition coloring e_1' and also the corresponding edge of Q_2 by the color c_1. Now, since there is no uncolored edge e_2' as described above, it follows that all $d - 2$ edges $a_1, \dots, a_{d-2}$ adjacent to e_2 in Q_1 satisfies that either a_i, or the corresponding edge of Q_{2k+1}, is φ'-precolored or adjacent to an edge colored c_1 under φ'. Moreover, since $Q_{2k+1} \cup Q_1$ contains at most $2d - 2$ φ'-precolored edges, every precolored edge of $Q_{2k+1} \cup Q_1$ satisfies this condition. Thus by properly coloring the uncolored edges adjacent to e_2, except the ones that are adjacent to edges in $G - E(D_1)$ colored c_1, by colors from $\{1, \dots, 2d+1\} \setminus \{c_1, c_3, c_4\}$, we obtain a precoloring φ'' from φ'. Note that any extension of the restriction of φ'' to $Q_{2k+1} \cup Q_1$ using colors $\{1, \dots, 2d+1\} \setminus \{c_3, c_4\}$ does not use c_1 on an edge adjacent to e_2. Thus, since there is exactly one precolored edge in $G - E(D_1)$ that is not contained in $Q_{2k+1} \cup Q_1$, we may once again proceed as in Case 3 of Lemma 4.2 to obtain a proper $(2d+1)$-edge coloring of G that agrees with φ'', where no edge adjacent to e_2 is colored c_1, and where we color the edges of D_1 so that all edges between any two given planes are colored by a fixed color not appearing in these two planes. Thereafter we can swap colors on a bicolored 4-cycle and recolor e_2 to obtain an extension of φ.

Suppose now that neither an edge e_1', nor an edge e_2' as described above exist in Q_1. Then the endpoints u_1 and u_2 of e_1 and e_2 in Q_1, respectively, are adjacent via an uncolored edge e in Q_1, and the corresponding edges of Q_{2k+1} and Q_2 are uncolored. Moreover, since both $Q_{2k+1} \cup Q_1$ and $Q_1 \cup Q_2$ contains at most $2d-3$ precolored edges, it follows that there are $2d-4$ edges precolored c_1 in Q_1, the endpoints of which are adjacent to u_1 and u_2, respectively. Moreover, Q_{2k+1} contains exactly one precolored edge, and Q_2 contains exactly one precolored edge, so all precolored edges of $G - E(D_1)$ are contained in $Q_{2k+1} \cup Q_1 \cup Q_2$.

Now, from the restriction of φ to Q_{2k+1} we define a new precoloring φ' by coloring all edges of Q_{2k+1} corresponding to edges of Q_1 colored c_1, by the color c_1, and thereafter color every edge adjacent to e_2 in Q_{2k+1} that is neither precolored nor adjacent to any edge precolored c_1 by an arbitrary color in $\{1, \dots, 2d+1\} \setminus \{c_1, c_3, c_4\}$ so that the resulting coloring is proper. Next, we take an extension of φ' using colors $\{1, \dots, 2d+1\} \setminus \{c_3, c_4\}$. (Note that no edge adjacent to e_2 is colored c_1 in this extension.) Thereafter we color Q_1 correspondingly to how Q_{2k+1} is colored except that all edges colored c_1 that are not precolored c_1 under φ' are recolored c_3. Denote the obtained partial coloring of G by f.

Now, if the precolored edge b_2 of Q_2 is colored c_1 under φ, then we color Q_2 correspondingly to how Q_1 is colored under f, except that the edge φ-precolored c_1 is colored c_1. Therafter, we color $Q_3, \dots, Q_{2k}$ correspondingly to how Q_1 is colored. We may now apply Lemma 2.2 to color the edges of D_1 and thus obtain an extension of φ. (Since e_1 and e_2 are contained in different cycles of D_1, we can choose the coloring of D_1 so that it agrees with φ.) Otherwise, if b_2 is colored $c_5 \neq c_1$, then the corresponding edge of Q_1 is not colored c_1. We color Q_2 correspondingly to how Q_1 is colored except that we permute the colors c_5 and the color of the corresponding edge of Q_1 under f. Again, we color all the planes $Q_3, \dots, Q_{2k}$ correspondingly to how Q_1 is colored, and apply Lemma 2.2 to obtain an extension of φ.

Subcase 3.1.3 *e_1 and e_2 are not incident with any common plane:*

Suppose that e_1 is incident with Q_1 and Q_2 and e_2 is incident with Q_j and Q_{j+1}, $j > 2$. Since each Q_i is $(2d-2)$-regular and any pair of adjacent planes contain at most $2d-3$ precolored edges, it is straightforward that there are corresponding uncolored edges $e_1' \in E(Q_1)$, $e_2' \in E(Q_2)$ adjacent to e_1 but not to any other edge precolored c_1, and similarly for e_2. Hence, we may proceed as in Case 3 of Lemma 4.2 to obtain an extension of a precoloring φ' of $G - E(D_1)$ defined from φ by coloring the selected edges adjacent to e_1 and e_2, respectively, by the color c_1 and removing the color c_1 from e_1 and e_2. From the extension of φ', we obtain an extension of φ as before.

Case 3.2. *The precolored edges of D_1 have different colors under φ:*

Suppose that $\varphi(e_1) = c_1$ and $\varphi(e_2) = c_2$.

Subcase 3.2.1 *e_1 and e_2 are both incident with two common planes:*

Assume that e_1 and e_2 are both incident with the same pair of planes Q_1 and Q_2. Let u_1 and u_2 be the vertices of Q_1 that are incident with e_1 and e_2, respectively.

If $Q_1 \cup Q_2$ contains at most $2d-4$ precolored edges, then there are independent edges $e'_1 \in E(Q_1)$ and $e'_2 \in E(Q_1)$ that are incident with u_1 and u_2, respectively, such that neither e'_1 nor the corresponding edge e''_1 of Q_2 is precolored or adjacent to an edge precolored c_1 in $Q_1 \cup Q_2$, and similarly for e'_2, the corresponding edge e''_2 of Q_2 and c_2. Thus, from the restriction of φ to $Q_1 \cup Q_2$ we may define a new precoloring φ' by coloring these four edges by c_1 and c_2, respectively. We may now proceed as in Case 3 of Lemma 4.2 to obtain an extension of φ', and thereafter we can obtain an extension of φ by swapping colors on two bicolored 4-cycles.

Suppose now that $Q_1 \cup Q_2$ contains $2d-3$ precolored edges, so exactly one plane D_i, $i \neq 1, 2$ has exactly one precolored edge a_i; we assume $i \neq 3$. Then as in Claim 4.6, there is either

(i) an edge $e'_1 \in E(Q_1)$ incident with u_1 but not u_2, such that e'_1 and the corresponding edge e''_1 of Q_2 are uncolored, and not adjacent to any edge in $Q_1 \cup Q_2$ colored c_1, or

(ii) an edge $e'_2 \in E(Q_1)$ incident with u_2 but not u_1, such that e'_2 and the corresponding edge e''_2 of Q_2 are uncolored, and not adjacent to any edge in $Q_1 \cup Q_2$ colored c_2.

Suppose e.g. that (ii) holds. Then we define a new precoloring φ' from the restriction of φ to $Q_1 \cup Q_2$ by coloring e'_2 and e''_2 by the color c_2. By removing the color c_1 from any edge that is colored c_1 under φ', we obtain the precoloring φ'' of $Q_1 \cup Q_2$. Next, we take an extension of φ'' using colors $\{1, \dots, 2d+1\} \setminus \{c_1, c_3\}$, and then recolor all edges that are φ-colored c_1 by the color c_1 to obtain the coloring f which is an extension of φ'. We color all edges between Q_1 and Q_2 by the color c_3 except e_1 which is colored c_1, color Q_3 correspondingly to how Q_2 is colored, and color the edges between Q_2 and Q_3 by a color in $\{1, \dots, 2d+1\} \setminus \{c_1, c_3\}$ missing at its endpoints.

Next, consider the precolored edge a_i of Q_i, and the corresponding edge a_1 of Q_1. If $f(a_1) = \varphi(a_i)$, then we color all the planes $Q_4, \dots, Q_{2k+1}$ correspondingly to how Q_1 is colored. Thereafter, we color the edges between Q_3 and Q_4 similarly to how the edges between Q_1 and Q_2 are colored, and then color the remaining uncolored paths in D_1 using two colors not appearing at the endpoints of these paths. Finally, we swap colors on a bicolored 4-cycle containing e_2 to obtain an extension of φ.

Otherwise, if $f(a_1) \neq \varphi(a_i)$, then we color the planes $Q_4, \dots, Q_{2k+1}$ correspondingly to how Q_1 is colored, except that we permute the colors $f(a_1)$ and $\varphi(a_i)$. Then we color the edges between Q_3 and Q_4 with the color c_3, and consider the subgraph H consisting of the edges of D_1 with endpoints in two consecutive planes in the sequence $Q_4, \dots, Q_{2k+1}, Q_1$. If we define a list assignment for these edges by for every edge including the colors from $\{1, \dots, 2d+1\}$ that do not appear on any adjacent edges, then each edge, except the ones with endpoints in Q_1 and Q_{2k+1}, gets a list of size at least two. Hence, H is list edge colorable from these lists. This yields an edge coloring of G that agrees with φ except for e_2. Finally, we swap colors on a bicolored 4-cycle containing e_2 to obtain an extension of φ.

Subcase 3.2.2 *e_1 and e_2 are incident with exactly one common plane:*

Suppose now instead that e_1 and e_2 are incident to exactly one common plane, say Q_1, where e_1 in addition also is incident with Q_2. If there are uncolored corresponding edges $e'_1 \in E(Q_1)$ and $e'_{2k+1} \in E(Q_{2k+1})$ that are incident with e_2 but not to any other edge precolored c_2, and similar edges for e_1 and the color c_1 in Q_1 and Q_2, respectively, which are disjoint from e'_1, then we proceed as above: we can obtain an extension by coloring the edges adjacent to e_1 and e_2 by colors c_1 and c_2, respectively, and then proceed as in Case 3 of Lemma 4.2, as before.

Now, any two adjacent planes contain at most $2d-3$ precolored edges, so if there are no edges as described in the preceding paragraph, then all precolored edges are contained in $Q_{2k+1} \cup Q_1 \cup Q_2$, and e_1 and e_2 are adjacent to a common vertex $u_1 \in V(Q_1)$. Moreover, u_1 is incident with $2d-4$ edges colored by distinct colors from $\{1,\dots,2d+1\} \setminus \{c_1,c_2,c_3,c_4\}$, Q_{2k+1} contains exactly one precolored edge, and Q_2 contains exactly one precolored edge. Moreover, these precolored edges in $Q_{2k+1} \cup Q_2$ are either adjacent to vertices corresponding to u_1, or colored c_2 and c_1 respectively, and adjacent to edges that are incident with u_1. We consider some different cases, depending on the colors of the precolored edges of Q_1 and Q_2.

Suppose first that the precolored edge of Q_{2k+1} is colored c_2, and that Q_2 contains an edge precolored c_1. We color all edges of Q_{2k+1} adjacent to e_2 that are not precolored or adjacent to an edge precolored c_2 by arbitrary colors from $\{1,\dots,2d+1\} \setminus \{c_1,c_2,c_3\}$ so that the resulting precoloring is proper, and similarly for Q_2 but with c_1 in place of c_2. Next, we take an extension of the resulting precoloring φ' of $Q_{2k+1} \cup Q_1 \cup Q_2$, where we use colors $\{1,\dots,2d+1\} \setminus \{c_1,c_3\}$ for Q_{2k+1}, $\{1,\dots,2d+1\} \setminus \{c_1,c_2\}$ for Q_1, and $\{1,\dots,2d+1\} \setminus \{c_2,c_3\}$ for Q_2. We then color the edges between Q_{2k+1} and Q_1 by c_1 except e_2 which is colored c_2, the edges between Q_1 and Q_2 by c_2 except e_1 which is colored c_1. Next, we color the planes $Q_3,\dots,Q_{2k}$ correspondingly using colors $\{1,\dots,2d+1\} \setminus \{c_3,c_4\}$, and all remaining uncolored edges by c_3, c_4 alternately. This yields an extension of φ.

Now, if one of the colors c_1 and c_2 does not appear in $G - E(D_1)$, say c_2, then from the restriction of φ to $Q_1 \cup Q_2$, we define a new precoloring φ' by properly coloring all the edges adjacent to e_1 that are not precolored or adjacent to an edge colored c_1 by arbitrary colors in $\{1,\dots,2d+1\} \setminus \{c_1,c_2,c_3\}$ so that the resulting coloring is proper. We take an extension of φ' using colors $\{1,\dots,2d+1\} \setminus \{c_2,c_3\}$, and an extension of the restriction of φ to Q_{2k+1} using colors $\{1,\dots,2d+1\} \setminus \{c_2,c_3\}$. Thereafter, we color all edges between Q_{2k+1} and Q_1 by the color c_2, the edges between Q_1 and Q_2 by the color c_3 except that e_1 is colored c_1. Since no other planes in $G - E(D_1)$ contain precolored edges, it is now straightforward to construct an extension of φ from the obtained partial edge coloring of G.

Finally, if the precolored edge of Q_{2k+1} is colored c_1, and the edge of Q_2 is colored c_2, then we proceed similarly, but simply take extensions of the restriction of φ to Q_{2k+1} using colors $\{1,\dots,2d+1\} \setminus \{c_2,c_3\}$, of the restriction of φ to Q_1 using colors $\{1,\dots,2d+1\} \setminus \{c_1,c_2\}$ and of the restriction of φ to Q_2 using colors $\{1,\dots,2d+1\} \setminus \{c_1,c_3\}$.

Subcase 3.2.3 *e_1 and e_2 are not incident with any common plane:*

It remains to consider the case when e_1 and e_2 are not incident to a common plane. Here we may proceed precisely as in Subcase 3.1.3, so once again we omit the details. This concludes the proof of this lemma. □

5 Extending a precoloring of a distance-4 matching in C_{2k}^d

In this last section we consider the problem of extending a precoloring of C_{2k}^d where the precolored edges form a matching.

Theorem 5.1 *If φ is a $2d$-edge coloring of a distance-4 matching of $G = C_{2k}^d$, then φ can be extended to a proper $2d$-edge coloring of G.*

Proof: Let φ be a $2d$-edge precoloring of a distance-4 matching M of G, and let $D_1, \dots D_k$ be the dimensions of G. We define the edge coloring f of G by properly coloring all edges of D_j by $2j-1$ and $2j$, so that all corresponding edges have the same color. The resulting coloring satisfies that every 4-cycle in G is bicolored since corresponding edges have the same color.

We shall describe a procedure for obtaining a required coloring f' that agrees with φ. For all precolored edges we shall use transformations on some bicolored 4-cycles. As we shall see, if $e, e' \in M$, then the cycles used for transformations involving e will be edge-disjoint from cycles used for e'.

Consider an arbitrary precolored edge $e \in M$. We consider some different cases.

(i) If $f(e) = \varphi(e)$, then we are done;

(ii) If $f(e) \neq \varphi(e)$, and there is a bicolored 4-cycle containing e, and where color $\varphi(e)$ appears, then we interchange colors on this bicolored 4-cycle;

(iii) If none of the two previous conditions hold, then there are two edges e_1 and e_2, both of which are adjacent to e, and contained in the same dimension as e, such that $\varphi(e) = f(e_1) = f(e_2)$. By interchanging colors on two disjoint 4-cycles, containing e_1 and e_2 respectively, we obtain a coloring f_1, where e is contained in a bicolored 4-cycle with the color $\varphi(e)$. Thus by interchanging colors on this 4-cycle, we obtain a coloring f_2 satisfying that $f_2(e) = \varphi(e)$.

Note that all edges used in the transformations (i) - (iii) are at distance at most 1 from e. Thus if e and e' are distinct edges of M, and we perform one of the transformations (i)-(iii) for both edges, then the edges involved in the transformations concerning e will be edge disjoint from the ones used for e', since the precolored edges form a distance-4 matching.

Hence, we can repeat the above process for any precolored edge of G to obtain the required coloring f'. □

We believe that Proposition 5.1 might be true if we precolor a distance-3 instead of a distance-4 matching, but if e and e' are distinct edges of M, then the edges involved in the transformations for e may not necessarily be disjoint from the one used for e', and thus we cannot apply our technique here; we state the following conjecture.

Conjecture 5.2 *If φ is an edge precoloring of a distance-3 matching of C^d_{2k}, then φ can be extended to a proper 4-edge coloring of C^d_{2k}.*

Note that Proposition 5.1 becomes false if we precolor a distance-2 matching; for instance, consider a vertex v of degree $2d$ such that every edge incident with v is uncolored but there is a fixed color $c \in \{1, \dots, 2d\}$ satisfying that every edge incident with v is adjacent to another edge colored c. If f is an extension of φ, then since v has degree $2d$, exactly one edge incident with v is colored c, but such a coloring cannot be proper.

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References

[1] L.D. Andersen, A.J.W. Hilton, Thank Evans!, *Proc. London Math. Soc.* 47 (1983), 507–522.

[2] L.D. Andersen, A.J.W Hilton, Symmetric Latin square and complete graph analogues of the Evans conjecture, *Journal of Combinatorial Designs* 4 (1994), 197–252.

[3] A.S. Asratian, T.M.J. Denley, R. Häggkvist, *Bipartite graphs and their applications*, Cambridge University Press, Cambridge, 1998.

[4] C.J. Casselgren, F.B. Petros, S.A. Fufa, Extending partial edge colorings of cartesian products of graphs, submitted.

[5] C.J. Casselgren, F.B. Petros, Edge precoloring extension of trees, *Australasian Journal of Combinatorics* 81 (2021), 233–244.

[6] C.J. Casselgren, F.B. Petros, Edge precoloring extension of trees II, *Discussiones Mathematicae Graph Theory (in press).*

[7] C.J. Casselgren, K. Markström, L.A. Pham, Precoloring extension of hypercubes, *Journal of Graph Theory* 95 (2020), 410–444.

[8] T.Easton, R.Gary Parker, On completing latin squares, *Discrete Applied Mathematics* 15 (2001), 167–181.

[9] K. Edwards, A. Girao, J. van den Heuvel, R.J. Kang, G.J. Puleo, J.-S. Sereni, Extension from Precoloured Sets of Edges, *Electronic Journal of Combinatorics* 25 (2018), P3.1, 28 pp.

[10] T. Evans, Embedding incomplete latin squares, *American Mathematical Monthly* 67 (1960), 958–961.

[11] J. Fiala, NP-completeness of the edge precoloring extension problem on bipartite graphs, *Journal of Graph Theory* 43 (2003), pp. 156–160.

[12] A. Girao, R.J. Kang, Precolouring extension of Vizing's theorem, *Journal of Graph Theor* 92 (2019), 255–260.

[13] R. Häggkvist, A solution of the Evans conjecture for Latin squares of large size *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. I*, Colloq. Math. Soc. János Bolyai, 18, 495–513.

[14] O. Marcotte, P. Seymour, Extending an edge coloring *Journal of Graph Theory* 14 (1990), 565-–573.

[15] H.J. Ryser, A combinatorial theorem with an application to Latin squares, *Proceedings of the American Mathematical Society* 2 (1951), 550–552.

[16] B. Smetaniuk, A new construction for Latin squares I. Proof of the Evans conjecture, *Ars Combinatoria* 11 (1981), 155–172.