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A new sufficient condition for a 2-strong digraph to be Hamiltonian

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In this paper we prove the following new sufficient condition for a digraph to be Hamiltonian:

Let D be a 2-strong digraph of order $n \geq 9$. If $n-1$ vertices of D have degrees at least $n+k$ and the remaining vertex has degree at least $n-k-4$, where k is a non-negative integer, then D is Hamiltonian.

This is an extension of Ghouila-Houri's theorem for 2-strong digraphs and is a generalization of an early result of the author (DAN Arm. SSR (91(2):6-8, 1990). The obtained result is best possible in the sense that for $k = 0$ there is a digraph of order $n = 8$ (respectively, $n = 9$) with the minimum degree $n - 4 = 4$ (respectively, with the minimum degree $n - 5 = 4$) whose $n - 1$ vertices have degrees at least $n - 1$, but it is not Hamiltonian.

We also give a new sufficient condition for a 3-strong digraph to be Hamiltonian-connected.

Keywords: digraph, Hamiltonian cycle, Hamiltonian-connected, k-strong, degree

1 Introduction

In this paper, we consider finite digraphs without loops and multiple arcs. We shall assume that the reader is familiar with the standard terminology on digraphs and refer the reader to Bang-Jensen and Gutin (Springer-Verlag, London, 2000). Every cycle and path is assumed simple and directed. A cycle (a path) in a digraph D is called *Hamiltonian* (*Hamiltonian path*) if it includes all the vertices of D. A digraph D is *Hamiltonian* if it contains a Hamiltonian cycle. Hamiltonicity is one of the most central in graph theory, and it has been extensively studied by numerous researchers. The problem of deciding Hamiltonicity of a graph (digraph) is NP-complete, but there are numerous sufficient conditions which ensure the existence of a Hamiltonian cycle in a digraph (see Bang-Jensen and Gutin (Springer-Verlag, London, 2000); Bermond and Thomassen (1981); Gould (2014); Kühn and Osthus (2012)). Among them are the following classical sufficient conditions for a digraph to be Hamiltonian.

Theorem 1.1 (Nash-Williams (1969)). *Let D be a digraph of order $n \geq 2$. If for every vertex x of D, $d^+(x) \geq n/2$ and $d^-(x) \geq n/2$, then D is Hamiltonian.*

Theorem 1.2 (Ghouila-Houri (1960)). *Let D be a strong digraph of order $n \geq 2$. If for every vertex x of D, $d(x) \geq n$, then D is Hamiltonian.*

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Theorem 1.3 (Woodall (1972)). *Let D be digraph of order $n \geq 2$. If $d^+(x) + d^-(y) \geq n$ for all pairs of distinct vertices x and y of D such that there is no arc from x, to y, then D is Hamiltonian.*

Theorem 1.4 (Meyniel (1973)). Let D be a strong digraph of order $n \geq 2$. If $d(x) + d(y) \geq 2n - 1$ for all pairs of non-adjacent distinct vertices x and y of D, then D is Hamiltonian.

It is known that all the lower bounds in the above theorems are tight. Notice that for the strong digraphs Meyniel's theorem is a generalization of Nash-Williams, Ghouila-Houri's and Woodall's theorems. A beautiful short proof the later can found in the paper by Bondy and Thomassen (1977).

Nash-Williams (1969) suggested the problem of characterizing all the strong digraphs of order n and minimum degree $n-1$ that have no Hamiltonian cycle. As a partial solution of this problem, Thomassen (1981) in his excellent paper proved a structural theorem on the extremal digraphs. An analogous problem for the Meyniel theorem was considered by Darbinyan (1982), proving a structural theorem on the strong non-Hamiltonian digraphs D of order n, with the condition that $d(x) + d(y) \geq 2n - 2$ for every pair of non-adjacent distinct vertices x, y. This improves the corresponding structural theorem of Thomassen. Darbinyan (1982) also proved that: if m is the length of longest cycle in D, then D contains cycles of all lengths $k = 2, 3, \ldots, m$. Thomassen (1981) and Darbinyan (1986) described all the extremal digraphs for the Nash-Williams theorem, respectively, when the order of a digraph is odd and when the order of a digraph is even. Here we combine they in the following theorem .

Theorem 1.5 (Thomassen (1981) and Darbinyan (1986)). *Let D be a digraph of order $n \geq 4$ with minimum degree $n - 1$. If for every vertex x of D, $d^+(x) \geq n/2 - 1$ and $d^-(x) \geq n/2 - 1$, then D is Hamiltonian, unless some exceptions, which completely are characterized.*

Goldberg, Levitskaya and Satanovskyi (1971) relaxed the condition of the Ghouila-Houri theorem by proving the following theorem.

Theorem 1.6 (Goldberg, Levitskaya and Satanovskyi (1971)). *Let D be a strong digraph of order $n \geq 2$. If $n - 1$ vertices of D have degrees at least n and the remaining vertex has degree at least $n - 1$, then D is Hamiltonian.*

Note that Theorem 1.6 is an immediate consequence of Theorem 1.4. Goldberg, Levitskaya and Satanovskyi (1971) for any $n \geq 5$ presented two examples of non-Hamiltonian strong digraphs of order n such that: (i) In the first example, $n - 2$ vertices have degrees equal to $n + 1$ and the other two vertices have degrees equal to $n - 1$. (ii) In the second example, $n - 1$ vertices have degrees at least n and the remaining vertex has degree equal to $n - 2$.

Remark 1. It is worth to mention that Thomassen (1981) constructed a strong non-Hamiltonian digraph of order n with only two vertices of degree $n - 1$ and all other $n - 2$ vertices have degrees at least $(3n - 5)/2$.

Zhang, Zhang and Wen (2013) reduced the lower bound in Theorem 1.3 by 1, and proved that the conclusion still holds, with only a few exceptional cases that can be clearly characterized. Darbinyan (1990a) announced that the following theorem is holds.

Theorem 1.7 (Darbinyan (1990a)). *Let D be a 2-strong digraph of order $n \geq 9$ such that its $n-1$ vertices have degrees at least n and the remaining vertex has degree at least $n-4$. Then D is Hamiltonian.*

The proof of Theorem 1.7 has never been published. G. Gutin suggested me to publish the proof of this theorem anywhere. Recently, Darbinyan (2022) presented a new proof of the first part of Theorem 1.7, by proving the following:

Theorem 1.8 (Darbinyan (2022)). *Let D be a 2-strong digraph of order $n \geq 9$ such that its $n-1$ vertices have degrees at least n and the remaining vertex z has degree at least $n-4$. If D contains a cycle of length $n-2$ through z, then D is Hamiltonian.*

Darbinyan (2022) also proposed the following conjecture.

Conjecture 1. *Let D be a 2-strong digraph of order n. Suppose that $n-1$ vertices of D have degrees at least $n+k$ and the remaining vertex has degree at least $n-k-4$, where $k \geq 0$ is an integer. Then D is Hamiltonian.*

Note that, for $k = 0$ this conjecture is Theorem 1.7. By inspecting the proof of Theorem 1.8 and the handwritten proof of Theorem 1.7, by the similar arguments we settled Conjecture 1 by proving the following theorem.

Theorem 1.9. *Let D be a 2-strong digraph of order $n \geq 9$. If $n-1$ vertices of D have degrees at least $n+k$ and the remaining vertex z has degree at least $n-k-4$, where $k \geq 0$ is an integer, then D is Hamiltonian.*

Darbinyan (2024) presented the proof of the first part of Conjecture 1 for any $k \geq 1$, which we formulated as Theorem 3.6 (Section 3). The goal of this article to present the complete proof of the second part of the proof of Theorem 1.9 and show that this theorem is best possible in the sense that for $k = 0$ there is a 2-strong digraph of order $n = 8$ (respectively, $n = 9$) with the minimum degree $n-4 = 4$ (respectively, with the minimum degree $n-5 = 4$) whose $n-1$ vertices have degrees at least n, but it is not Hamiltonian. To see that the theorem is best possible, it suffices consider the digraphs defined in the Examples 1 and 2, see Figure 1. In figures an undirected edge represents two directed arcs of opposite directions.

Example 1. Let D_8 be a digraph of order 8 with vertex set $V(D_8) = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3, z\}$ and arc set $A(D_8)$, which satisfies the following conditions: $D_8\langle\{y_1, y_2, y_3\}\rangle$ is a complete digraph, $x_4 \to \{y_1, y_2, y_3\} \to x_1$, $x_2 \to \{y_1, y_2, y_3\} \to x_2$, D_8 contains the following 2-cycles and arcs $x_i \leftrightarrow x_{i+1}$ for all $i \in [1, 3]$, $x_1 \leftrightarrow x_3$, $x_3 \leftrightarrow z$, $x_4 \leftrightarrow x_2$, $x_4 \to x_1$, $x_4 \to z$ and $z \to x_1$. $A(D_8)$ contains no other arcs.

Example 2. Let D_9 be a digraph of order 9 with vertex set $V(D_9) = \{x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, z\}$ and arc set $A(D_9)$, which satisfies the following conditions: $D_9\langle\{y_1, y_2, y_3\}\rangle$ is a complete digraph, $x_5 \to \{y_1, y_2, y_3\} \to x_1$, $x_3 \to \{y_1, y_2, y_3\} \to \{x_1, x_2, x_3\}$, D_9 contains the following 2-cycles and arcs $x_i \leftrightarrow x_{i+1}$ for all $i \in [1, 4]$, $x_1 \leftrightarrow x_4$, $x_3 \leftrightarrow x_5$, $x_4 \leftrightarrow x_2$, $x_4 \leftrightarrow z$, $x_5 \to z$, $z \to x_1$ and $x_5 \to x_1$. $A(D_9)$ contains no other arcs.

Observe that every vertex other than z in D_8 (in D_9) has degree at least $|V(D_8)| = 8$ (at least $|V(D_9)| = 9$) and $d(z) = 4$ in both digraphs D_8 and D_9. It is not hard to check that for every $u \in V(D_8)$ ($u \in V(D_9)$), $D_8 - u$ ($D_9 - u$) is strong, i.e., D_8 and D_9 both are 2-strong. To see this, it suffices to consider a longest cycle in $D_8 - u$ (in $D_9 - u$) and apply the following well-known proposition.

Proposition 1 (see Exercise 7.26, Bang-Jensen and Gutin (Springer-Verlag, London, 2000)). Let D be a k-strong digraph with $k \geq 1$, let x be a new vertex and D' be a digraph obtained from D and x by adding k arcs from x to distinct vertices of D and k arcs from distinct vertices of D to x. Then D' also is k-strong.

Let D'_9 be the digraph obtained from D_9 by adding the arcs x_3x_1 and x_5x_2.

Now we will show that D'_9 is not Hamiltonian. Assume that this is not the case. Let R be an arbitrary Hamiltonian cycle in D'_9. Then R necessarily contains either the arc x_4z or the arc x_5z. If $x_4z \in A(R)$, then it is not difficult to see that either $R[x_4, y_i] = x_4zx_1x_2x_3y_i$ or $R[x_4, y_i] = x_4zx_1x_2x_3x_5y_i$, which is impossible since $N^+(y_i, \{x_1, x_2, \ldots, x_5\}) = \{x_1, x_2, x_3\}$. We may therefore assume that $x_5z \in A(R)$. Then necessarily R contains the arc x_3y_i and either the path x_5zx_1 or the path x_5zx_4. It is easy to check that either $x_2x_3 \in A(R)$ or $x_4x_3 \in A(R)$. If x_5zx_4 is in R, then $R[x_5, y_i] = x_5zx_4x_j \ldots x_3y_i$, where $j \in [1, 3]$, and if x_5zx_1 is in R, then $R[x_5, y_i]$ is one of the following pats: $x_5zx_1x_2x_3y_i$, $x_5zx_1x_2x_4x_3y_i$, $x_5zx_1x_4x_3y_i$ and $x_5zx_1x_4x_2x_3y_i$, which is impossible since $N^+(y_i, \{x_1, x_2, \ldots, x_5\}) = \{x_1, x_2, x_3\}$. So, in all cases we have a contradiction. Therefore, D'_9 is not Hamiltonian, which in turn implies that the digraphs D_9, $D_9 + \{(x_3x_1)\}$ and $D_9 + \{(x_5x_2)\}$ also are not Hamiltonian. By a similar argument we can show that D_8 also is not Hamiltonian.

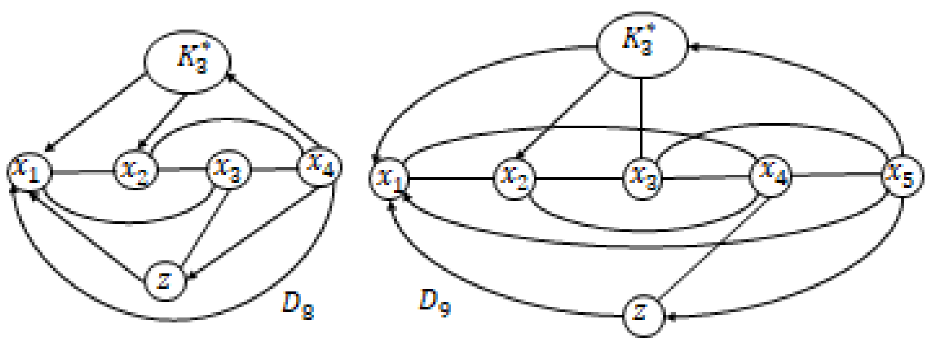

Fig. 1: The non-Hamiltonian 2-strong digraphs D_8 and D_9 of order 8 and 9.

A digraph D is *Hamiltonian-connected* if for any pair of distinct vertices x, y, D has a Hamiltonian path from x to y. Overbeck-Larisch (1976) proved the following sufficient condition for a digraph to be Hamiltonian-connected.

Theorem 1.10 (Overbeck-Larisch (1976)). *Let D be a 2-strong digraph of order $n \geq 3$ such that, for each two non-adjacent distinct vertices x, y we have $d(x) + d(y) \geq 2n + 1$. Then for each two distinct vertices u, v with $d^+(u) + d^-(v) \geq n + 1$ there is a Hamiltonian (u, v)-path.*

Let D be a digraph of order $n \geq 3$ and let u and v be two distinct vertices in $V(D)$. Following Overbeck-Larisch (1976), we define a new digraph $H_D(u, v)$ as follows:

$$V(H_D(u, v)) = V(D - \{u, v\}) \cup \{z\} \text{ (z a new vertex)},$$

$$A(H_D(u, v)) = A(D - \{u, v\}) \cup \{zy \mid y \in N^+_{D-v}(u)\} \cup \{yz \mid y \in N^-_{D-u}(v)\}.$$

Now, using Theorem 1.9, we will prove the following theorem, which is an analogue of the Overbeck-Larisch theorem.

Theorem 1.11. *Let D be a 3-strong digraph of order $n + 1 \geq 10$ and let u, v be arbitrary two distinct vertices in D. Suppose that $d^+_D(u) + d^-_D(v) \geq n - k - 2$ or $d^+_D(u) + d^-_D(v) \geq n - k - 4$ with $uv \notin A(D)$ and for every vertex $x \in V(D) \setminus \{u, v\}$, $d_D(x) \geq n + k + 2$. Then D has a Hamiltonian (u, v)-path.*

Proof: Let D be a 3-strong digraph of order $n + 1 \geq 10$ and let u, v be two distinct vertices in $V(D)$. Suppose that D and u, v satisfy the degree conditions of the theorem. Now we consider the digraph $H := H_D(u, v)$ of order $n \geq 9$. By an easy computation, we obtain that the minimum degree of H is at least $n - k - 4$, and H has $n - 1$ vertices of degrees at least $n + k$. Moreover, we know that H is 2-strong (see Darbinyan (1990)). Thus, the digraph H satisfies the conditions of Theorem 1.9. Therefore, H is Hamiltonian, which in turn implies that in D there is a Hamiltonian (u, v)-path. □

There are a number of sufficient conditions depending on degree or degree sum for Hamiltonicity of bipartite digraphs. Here we combine several of them in the following theorem.

Theorem 1.12. *Let D be a balanced bipartite strong digraph of order $2a \geq 6$. Then D is Hamiltonian provided one of the following holds:*

(a) (Adamus and Adamus (2012)). $d^+(x) + d^-(y) \geq a + 2$ for every pair of vertices x, y such that x, y belong to different partite sets and $xy \notin A(D)$.

(b) (Adamus, Adamus and Yeo (2014)). $d(x) + d(y) \geq 3a$ for every pair of non-adjacent distinct vertices x, y.

(c) (Adamus (2017)). $d(x) + d(y) \geq 3a$ for every pair of vertices x, y with a common in-neighbour or a common out-neighbour.

(d) (Adamus (2021)). $d(x) + d(y) \geq 3a + 1$ for every pair of vertices x, y with a common out-neighbour.

All the lower bounds in Theorem 1.12 are the best possible. However, Wang (2021) (respectively,Wang, Wu and Meng (2022); Wang and Wu (2021)) reduced the lower bound in Theorem 1.12(a) (respectively, Theorem 1.12(b); Theorem 1.12(c)) by one, and completely described all non-Hamiltonian bipartite digraphs, that is the extremal bipartite digraphs for Theorem 1.12(a) (respectively, Theorem 1.12(b); Theorem 1.12(c)). Wang (2022) reduced the bound by one in Theorem 1.12(d), but it is Hamiltonian whenever $d(x) + d(y) \geq 3a$ for every pair of distinct vertices x, y with a common out neighbour. Motivated by

Theorems 1.9, 1.12 and Remark 1, it is natural to suggest the following problems.

Problem 1. Suppose that D is a k-strong balanced bipartite digraph of order $2a \geq 6$. Let $\{x_0, y_0\}$ be a pair of distinct vertices in $V(D)$ such that $d(x_0) + d(y_0) \geq 3a - l$, where $l \geq 1$ is an integer. Find the minimum value of k and the maximum value of l such that D is Hamiltonian provided one of the following holds:

(i) x_0 and y_0 are not adjacent and $d(x) + d(y) \geq 3a$ for every pair $\{x, y\}$ of non-adjacent vertices x, y other than $\{x_0, y_0\}$.

(ii) $\{x_0, y_0\}$ is a pair with a common out-neighbour and $d(x) + d(y) \geq 3a$ for every pair $\{x, y\}$ of vertices x, y with a common out-neighbour such that $\{x, y\} \neq \{x_0, y_0\}$.

Problem 2. Suppose that D is a k-strong balanced bipartite digraph of order $2a \geq 6$. Let u_0 and v_0 be two vertices from different partite sets such that $u_0 \nrightarrow v_0$ and $d^+(u_0) + d^-(y_0) \geq a + 2 - l$, where $l \geq 2$ is an integer. Find the minimum value of k and the maximum value of l such that D is Hamiltonian provided that the following holds: $d^+(u) + d^-(v) \geq a + 2$ for all vertices u and v from different partite sets such that $\{u, v\} \neq \{u_0, v_0\}$ and $u \nrightarrow v$.

2 Terminology and notation

In this paper, we consider finite digraphs without loops and multiple arcs. For the terminology not defined in this paper, the reader is referred to the book Bang-Jensen and Gutin (Springer-Verlag, London, 2000). The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $A(D)$, respectively. The *order* of D is the number of its vertices. For any $x, y \in V(D)$, if $xy \in A(D)$, we also write $x \to y$, and say that x *dominates* y or y is *dominated* by x. The notion $xy \notin A(D)$ means that $xy \notin A(D)$. If $x \to y$ and $y \to x$ we shall use the notation $x \leftrightarrow y$ ($x \leftrightarrow y$ is called 2-*cycle*). If $x \to y$ and $y \to z$, we write $x \to y \to z$. Let A and B be two disjoint subsets of $V(D)$. The notation $A \to B$ means that every vertex of A dominates every vertex of B. We define $A_D(A \to B) = \{xy \in A(D) \,|\, x \in A, y \in B\}$ and $A_D(A, B) = A_D(A \to B) \cup A_D(B \to A)$. If $x \in V(D)$ and $A = \{x\}$ we sometimes write x instead of $\{x\}$. The *converse digraph* of D is the digraph obtained from D by reversing the direction of all arcs, and is denoted by D^{rev}. Let $N_D^+(x)$, $N_D^-(x)$ denote the set of out-neighbors, respectively the set of in-neighbors of a vertex x in a digraph D. If $A \subseteq V(D)$, then $N_D^+(x, A) = A \cap N_D^+(x)$ and $N_D^-(x, A) = A \cap N_D^-(x)$. The *out-degree* of x is $d_D^+(x) = |N_D^+(x)|$ and $d_D^-(x) = |N_D^-(x)|$ is the *in-degree* of x. Similarly, $d_D^+(x, A) = |N_D^+(x, A)|$ and $d_D^-(x, A) = |N_D^-(x, A)|$. The *degree* of the vertex x in D is defined as $d_D(x) = d_D^+(x) + d_D^-(x)$ (similarly, $d_D(x, A) = d_D^+(x, A) + d_D^-(x, A)$). We omit the subscript if the digraph is clear from the context. The subdigraph of D induced by a subset A of $V(D)$ is denoted by $D\langle A\rangle$ and $D - A$ is the subdigraph induced by $V(D) \setminus A$, i.e. $D - A = D\langle V(D) \setminus A\rangle$. For integers a and b, $a \leq b$, let $[a, b]$ denote the set $\{x_a, x_{a+1}, \ldots, x_b\}$. If $j < i$, then $\{x_i, \ldots, x_j\} = \emptyset$. A path is a digraph with vertex set $\{x_1, x_2, \ldots, x_k\}$ and arc set $\{x_1x_2, x_2x_3, \ldots, x_{k-1}x_k\}$, and is denoted by $x_1x_2 \ldots x_k$. This is also called an (x_1, x_k)–path or a path from x_1 to x_k. If we add the arc x_kx_1 to the above, we obtain a cycle $x_1x_2 \ldots x_kx_1$. The *length* of a cycle or a path is the number of its arcs. If a digraph D contains a path from a vertex x to a vertex y we say that y is *reachable* from x in D. In particular, x is reachable from itself. If P is a path containing a subpath from x to y, we let $P[x, y]$ denote that subpath. Similarly, if C is a cycle containing vertices x and y, $C[x,y]$ denotes the subpath of C from x to y. For a cycle C, a C-bypass is an (x,y)-path P of length at least two such that $V(P)\cap V(C)=\{x,y\}$. The *flight* of C-bypass P respect to C is $|V(C[x,y])|-2$.

For integers a and b, $a\leq b$, let $[a,b]$ denote the set of all integers, which are not less than a and are not greater than b.

The path (respectively, the cycle) consisting of the distinct vertices $x_1,x_2,\ldots,x_m$ ($m\geq 2$) and the arcs x_ix_{i+1}, $i\in[1,m-1]$ (respectively, x_ix_{i+1}, $i\in[1,m-1]$, and x_mx_1), is denoted by $x_1x_2\cdots x_m$ (respectively, $x_1x_2\cdots x_mx_1$). We say that $x_1x_2\cdots x_m$ is a path from x_1 to x_m or is an (x_1,x_m)*-path*. Let x and y be two distinct vertices of a digraph D. Cycle that passing through x and y in D, we denote by $C(x,y)$. By $C_m(x)$ (respectively, $C(x)$) we denote a cycle in D of length m through x (respectively, a cycle through x). Similarly, we denote by C_k a cycle of length k. By K_n^* is denoted the complete digraph of order n. Let D be a digraph of order n. If E is a set of arcs in K_n^*, then we denote by $D+E$ the digraph obtained from D by adding all arcs of E. A digraph D is *strongly connected* (or, just, *strong*) if there exists a path from x to y and a path from y to x for every pair of distinct vertices x,y. A digraph D is k*-strongly* connected (or k*-strong*), where $k\geq 1$, if $|V(D)|\geq k+1$ and $D-A$ is strongly connected for any subset $A\subset V(D)$ of at most $k-1$ vertices. Two distinct vertices x and y are *adjacent* if $xy\in A(D)$ or $yx\in A(D)$ (or both). We will use *the principle of digraph duality*: Let D be a digraph, then D contains a subdigraph H if and only if D^{rev} contains the subdigraph H^{rev}.

3 Preliminaries

In our proofs we extensively will use the following well-known simple lemmas.

Lemma 3.1 (Häggkvist and Thomassen (1976)). *Let D be a digraph of order $n\geq 3$ containing a cycle C_m, $m\in[2,n-1]$. Let x be a vertex not contained in this cycle. If $d(x,V(C))\geq m+1$, then D contains a cycle C_k for every $k\in[2,m+1]$.*

The next lemma is a slight modification of a lemma by Bondy and Thomassen (1977) it is very useful and will be used extensively throughout this paper.

Lemma 3.2. Let D be a digraph of order $n\geq 3$ containing a path $P:=x_1x_2\ldots x_m$, $m\in[1,n-1]$. Let x be a vertex not contained in this path. If one of the following condition holds: (i) $d(x,V(P))\geq m+2$, (ii) $d(x,V(P))\geq m+1$ and $x\nrightarrow x_1$ or $x_m\nrightarrow x$, (iii) $d(x,V(P))\geq m$, $x\nrightarrow x_1$ and $x_m\nrightarrow x$, then there is an $i\in[1,m-1]$ such that $x_i\rightarrow x\rightarrow x_{i+1}$, i.e., D contains a path $x_1x_2\ldots x_ixx_{i+1}\ldots x_m$ of length m (we say that x can be inserted into P).

We note that in the above Lemma 3.2 as well as throughout the whole paper we allow paths of length 0, i.e., paths that have exactly one vertex. Using Lemma 3.2, it is not difficult to prove the following lemma.

Lemma 3.3. *Let D be a digraph of order $n\geq 4$. Suppose that $P:=x_1x_2\ldots x_m$, $m\in[2,n-2]$, is a longest path from x_1 to x_m in D and $V(D)\setminus V(P)$ contains two distinct vertices y_1, y_2 such that $d(y_1,V(P))=d(y_2,V(P))=m+1$. If in subdigraph $D\langle V(D)\setminus V(P)\rangle$ there exists a path from y_1 to y_2 and a path from y_2 to y_1, then there is an integer $l\in[1,m]$ such that for every $i\in[1,2]$*

$$O(y_i,V(P))=\{x_1,x_2,\ldots,x_l\}\quad \textit{and}\quad I(y_i,V(P))=\{x_l,x_{l+1},\ldots,x_m\}.$$

Theorem 3.4 (Darbinyan (1990)). *Let D be a strong digraph of order $n \geq 2$. Suppose that $d(x) + d(y) \geq 2n - 1$ for all pairs of non-adjacent vertices $x, y \in V(D) \setminus \{z\}$, where z is an arbitrary fixed vertex in $V(D)$. Then D contains a cycle of length is at least $n - 1$.*

From Theorem 3.4 it follows that the following corollary is true.

Corollary 1. (Darbinyan (1990)). Let D be a strong digraph of order $n \geq 2$. Suppose that $n - 1$ vertices of D have degrees at least n. Then D either is Hamiltonian or contains a cycle of length $n - 1$ (in fact D has a cycle that contains all the vertices with degree at least n).

Lemma 3.5 (Darbinyan (2022)). *Let D be a digraph of order $n \geq 4$ such that for any vertex $x \in V(D) \setminus \{z\}$, $d(x) \geq n$, where z is an arbitrary fixed vertex in $V(D)$. Moreover, $d(z) \leq n - 2$. Suppose that $C_m(z) = x_1 x_2 \ldots x_m x_1$, $m \leq n - 1$, is a cycle of length m through z and $C_m(z)$ has an (x_i, x_j)-bypass such that $z \notin V(C_m(z)[x_{i+1}, x_{j-1}])$. Then D has a cycle, say Q, of length at least $m + 1$ such that $V(C_m(z)) \subset V(Q)$.*

Theorem 3.6 (Darbinyan (2024)). *Let D be a 2-strong digraph of order $n \geq 9$ such that $n - 1$ vertices of D have degrees at least $n + k$ and the remaining vertex z has degree at least $n - k - 4$, where $k \geq 0$ is an integer. If the length of a longest cycle through z is at least $n - k - 2$, then D is Hamiltonian.*

4 Proof of Theorem 1.9

Theorem 1.9. *Let D be a 2-strong digraph of order $n \geq 9$. If $n - 1$ vertices of D have degrees at least $n + k$ and the remaining vertex z has degree at least $n - k - 4$, where $k \geq 0$ is an integer, then D is Hamiltonian.*

Proof: By contradiction, suppose that D is not Hamiltonian. Then from Theorem 3.6 it follows that D has no $C(z)$-cycle of length greater than $n - k - 3$. By Corollary 1, D contains a cycle of length $n - 1$. Let $C_{n-1} := x_1 x_2 \ldots x_{n-1} x_1$ be an arbitrary cycle in D. By Lemma 3.1, $z \notin V(C_{n-1})$. Since D is 2-strong, there are two distinct vertices, say x_1 and x_{n-d-1}, such that $x_{n-d-1} \to z \to x_1$ and $d(z, \{x_{n-d}, x_{n-d+1}, \ldots, x_{n-1}\}) = 0$. Without loss of generality, assume that the flight $d := |\{x_{n-d}, x_{n-d+1}, \ldots, x_{n-1}\}|$ of z respect to C_{n-1} is smallest possible over all the cycles of length $n - 1$ in D.

For any $i \in [1, d]$, let $y_i = x_{n-d-1+i}$ and $Y = \{y_1, y_2, \ldots, y_d\}$. Note that $y_1 y_2 \ldots y_d$ is a path in $D\langle Y \rangle$. Since z cannot be inserted into C_{n-1}, using Lemma 3.2, we obtain $n - k - 4 \leq d(z) \leq n - d$. Hence, $d \leq k + 4$. On the other hand, $n - d \leq n - k - 3$, i.e., $d \geq k + 3$, since $z x_1 x_2 \ldots x_{n-d-1} z$ is a $C(z)$-cycle of length $n - d$. From now on, by P we denote the path $x_1 x_2 \ldots x_{n-d-1}$ (see Figure 2). In order to prove the theorem, it is convenient for the digraph D and the path P to prove the following Claims 1-4.

Claim 1. *Suppose that $D\langle Y \rangle$ is strong and each vertex y_j of Y cannot be inserted into P. If $d^+(x_i, Y) \geq 1$ with $i \in [1, n - d - 2]$, then $A(Y \to \{x_{i+1}, x_{i+2}, \ldots, x_{n-d-1}\}) = \emptyset$.*

Proof: By contradiction, suppose that there are vertices x_s, x_q with $1 \leq s < q \leq n - d - 1$ and

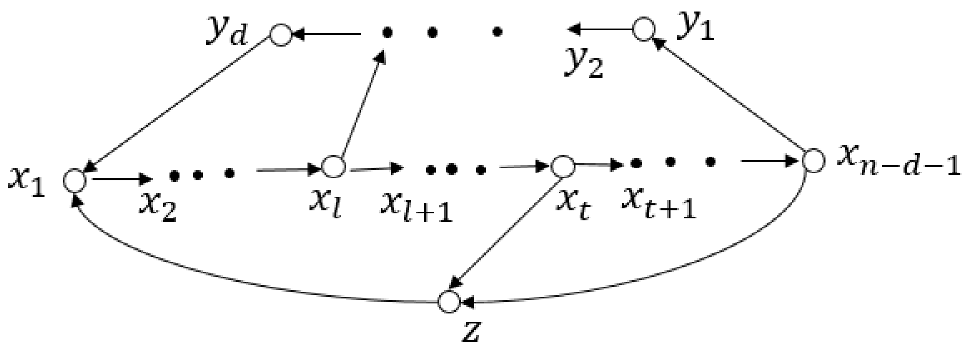

Fig. 2: The cycles $C_{n-1} = x_1x_2 \ldots x_{n-d-1}y_1y_2 \ldots y_dx_1$ and $C_{n-d}(z) = x_1x_2 \ldots x_{n-d-1}zx_1$ in D.

$u, v \in Y$ such that $x_s \to u$, $v \to x_q$. Since $D\langle Y\rangle$ is strong, it contains a (u, v)-path, and let Q be such a longest path. We may assume that $A(Y, \{x_{s+1}, \ldots, x_{q-1}\}) = \emptyset$. Since $D\langle Y\rangle$ is strong and every vertex y_j cannot be inserted into P, using the fact that D has no $C(z)$-cycle of length at least $n-k-2$, we obtain that $q-s \geq 2$. We now extend the path $x_qx_{q+1} \ldots x_{n-d-1}zx_1x_2 \ldots x_s$ with vertices $x_{s+1}, x_{s+2}, \ldots, x_{q-1}$ as much as possible. Then some vertices $z_1, z_2, \ldots, z_m \in \{x_{s+1}, x_{s+2}, \ldots, x_{q-1}\}$, where $0 \leq m \leq q - s - 1$, are not on the obtained extended path, say R. We consider the cases $m \geq 1$ and $m = 0$ separately.

Assume first that $m \geq 1$. Since every vertex y_j cannot be inserted into P and $d(y_j, \{z, x_{s+1}, x_{s+2}, \ldots, x_{q-1}\}) = 0$, using Lemma 3.2(i), we obtain

$$n + k \leq d(y_j) = d(y_j, Y) + d(y_j, \{x_1, x_2, \ldots, x_s\}) + d(y_j, \{x_q, x_{q+1}, \ldots, x_{n-d-1}\})$$
$$\leq 2d - 2 + (s + 1) + (n - d - 1 - q + 2) = n + s + d - q \quad \text{and}$$
$$n + k \leq d(z_i) = d(z_i, V(R)) + d(z_i, \{z_1, z_2, \ldots, z_m\}) \leq |V(R)| + 1 + 2m - 2$$
$$= n - d - m + 1 + 2m - 2 = n + m - d - 1.$$

Therefore,

$$2n + 2k \leq d(z_i) + d(y_j) \leq n + m - d - 1 + n + s + d - q = 2n + m - 1 + s - q$$
$$\leq 2n - 1 + q - s - 1 + s - q = 2n - 2,$$

which is a contradiction since $k \geq 0$.

Assume next that $m = 0$. This means that D contains an (x_q, x_s)-path with vertex set $\{z\} \cup V(P)$. This and the fact that D contains no cycle of length at least $n - k - 2$ through z imply that $d = k + 4$, $|V(Q)| = 1$, i.e., $u = v$, and $A(x_s \to Y \setminus \{u\}) = A(Y \setminus \{u\} \to x_q) = \emptyset$. Since any vertex of Y cannot be inserted into P, using Lemma 3.2(ii), for each $y \in Y \setminus \{u\}$ we obtain

$$n + k \leq d(y) = d(y, Y) + d(y, \{x_1, x_2, \ldots, x_s\}) + d(y, \{x_q, x_{q+1}, \ldots, x_{n-k-5}\})$$

$$\leq 2k+6+s+n-k-5-q+1 = n+k+2-(q-s).$$

This means that all the inequalities used in the last expression are actually equalities, i.e., $q-s=2$, $d(y,Y)=2k+6$, i.e., $D\langle Y\rangle$ is a complete digraph, and

$$d(y,\{x_1,x_2,\dots,x_s\})=s,\ d(y,\{x_q,x_{q+1},\dots,x_{n-k-5}\})=n-k-q-4.$$

Again using Lemma 3.2(ii), from the last two equalities and $A(x_s\to Y\setminus\{u\})=A(Y\setminus\{u\}\to x_q)=\emptyset$ we obtain $x_{n-k-5}\to Y\setminus\{u\}\to x_1$. We claim that x_{s+1} can be inserted into $x_1x_2\dots x_s$ or $x_qx_{q+1}\dots x_{n-k-5}$. Assume that this is not the case. Then by Lemma 3.2(i),

$$n+k\leq d(x_{s+1})=d(x_{s+1},\{x_1,x_2,\dots,x_s\})+d(x_{s+1},\{x_q=x_{s+2},x_{s+3},\dots,x_{n-k-5}\})$$

$$+d(x_{s+1},\{z\})\leq s+1+n-k-5-s-1+1+2=n-k-(q-s)=n-k-2,$$

which is a contradiction. This contradiction shows that there is either an (x_1,x_s)-path, say R_1, with vertex set $\{x_1,x_2,\dots,x_s,x_{s+1}\}$ or an (x_q,x_{n-k-5})-path, say R_2, with vertex set $\{x_{s+1},x_{s+2},\dots,x_{n-k-5}\}$. Let H be a Hamiltonian path in $D\langle Y\setminus\{u\}\rangle$. We know that $d(z,V(H))=0$, $|V(H)|=k+3$ and $x_{n-k-5}\to Y\setminus\{u\}\to x_1$. Therefore, $F_1:=x_1R_1ux_q\dots x_{n-k-5}Hx_1$ or $F_2:=x_1\dots x_suR_2x_{n-k-5}$ Hx_1, is a cycle of length $n-1$. We have that the flight of z respect to F_1 (or F_2) is equal to $k+3$, which contradicts the minimality of $d=k+4$ and the choice of the cycle C_{n-1} of length $n-1$. This completes the proof of the claim. □

Claim 2. *If $x_j\to z$ with $j\in[1,n-d-2]$, then $A(z\to\{x_{j+1},x_{j+2},\dots,x_{n-d-1}\})=\emptyset$.*

Proof: By contradiction, suppose that $x_j\to z$ with $j\in[1,n-d-2]$ and $z\to x_l$ with $l\in[j+1,n-d-1]$. We may assume that $d(z,\{x_{j+1},\dots,x_{l-1}\})=0$. Since D contains no $C(z)$-cycle of length at least $n-k-2$ and $C_{n-l+j+1}(z):=x_1x_2\dots x_jzx_l\dots x_{n-d-1}y_1y_2\dots y_dx_1$, it follows that $l\geq j+k+4$. Then, since z cannot be inserted into P, by Lemma 3.2(i), we have

$$\begin{aligned} n-k-4\leq d(z)&=d(z,\{x_1,x_2,\dots,x_j\})+d(z,\{x_l,x_{l+1},\dots,x_{n-d-1}\})\\ &\leq (j+1)+(n-d-1-l+2)=n+2+j-d-l\\ &\leq n+2+(l-k-4)-d-l=n-k-2-d,\end{aligned}$$

i.e., $d\leq 2$, which contradicts that $d\geq k+3$. Claim 2 is proved. □

Since D is 2-strong, we have $d^-(z)\geq 2$ and $d^+(z)\geq 2$. From this and Claim 2 it follows that there exists an integer $t\in[2,n-d-2]$ such that $x_t\to z$ and

$$d^-(z,\{x_1,x_2,\dots,x_{t-1}\})=d^+(z,\{x_{t+1},x_{t+2},\dots,x_{n-d-1}\})=0. \tag{1}$$

From (1) and $d(z)\geq n-k-4$ it follows that if $d=k+4$, then $n-d-1=n-k-5$ and

$$N^+(z)=\{x_1,x_2,\dots,x_t\}\quad\text{and}\quad N^-(z)=\{x_t,x_{t+1},\dots,x_{n-k-5}\}. \tag{2}$$

Claim 3. *Suppose that there is an integer $l\in[2,n-d-2]$ such that*

$$A(\{x_1,x_2,\dots,x_{l-1}\}\to Y)=A(Y\to\{x_{l+1},x_{l+2},\dots,x_{n-d-1}\})=\emptyset.$$

Then for every $j \in [2, n-d-2]$,

$$A(\{x_1, x_2, \dots, x_{j-1}\} \to \{x_{j+1}, x_{j+2}, \dots, x_{n-d-1}\}) \neq \emptyset.$$

Proof: Suppose, on the contrary, that for some $j \in [2, n-d-2]$, $A(\{x_1, x_2, \dots, x_{j-1}\} \to \{x_{j+1}, x_{j+2}, \dots, x_{n-d-1}\}) = \emptyset$. Without loss of generality, we may assume that $j \leq l$. If $d^-(z, \{x_1, x_2, \dots, x_{j-1}\}) = 0$, then by the suppositions of the claim, we have

$$A(\{x_1, x_2, \dots, x_{j-1}\} \to Y \cup \{z, x_{j+1}, x_{j+2}, \dots, x_{n-d-1}\}) = \emptyset.$$

If $d^-(z, \{x_1, x_2, \dots, x_{j-1}\}) \geq 1$, then by Claim 2, $d^+(z, \{x_{j+1}, x_{j+2}, \dots, x_{n-d-1}\}) = 0$. This together with the supposition of the claim implies that

$$A(\{z, x_1, x_2, \dots, x_{j-1}\} \to Y \cup \{x_{j+1}, x_{j+2}, \dots, x_{n-d-1}\}) = \emptyset.$$

Thus, in both cases, $D - x_j$ is not strong, which is a contradiction. Claim 3 is proved. □

Claim 4. *Any vertex* y_j *with* $j \in [1, d]$ *cannot be inserted into* P.

Proof: By contradiction, suppose that there is a vertex y_p with $p \in [1, d]$ and an integer $s \in [1, n-d-2]$ such that $x_s \to y_p \to x_{s+1}$. Then $R(z) := x_1 x_2 \dots x_s y_p x_{s+1} \dots x_{n-d-1} z x_1$ is a cycle of length $n-d+1$. Since D contains no $C(z)$-cycle of length at least $n-k-2$, it follows that $n-d+1 \leq n-k-3$, i.e., $d \geq k+4$. Therefore, $d = k+4$ since $d \leq k+4$. It is easy to see that any vertex y_i other than y_p cannot be inserted into P. Note that (2) holds since $d = k+4$. We will consider the cases $p \in [2, k+3]$ and $p = 1$ separately. Note that if $p = k+4$, then in the converse digraph of D we have case $p = 1$.

Case 1. $p \in [2, k+3]$.

If $y_{p-1} \to y_{p+1}$, then the cycle $x_1 x_2 \dots x_s y_p x_{s+1} \dots x_{n-k-5} y_1 \dots y_{p-1} y_{p+1} \dots y_{k+4} x_1$ is a cycle of length $n-1$ and the flight of z respect to this cycle is equal to $k+3$, which is a contradiction.

We may therefore assume that $y_{p-1} \nrightarrow y_{p+1}$. Since both y_{p-1} and y_{p+1} cannot be inserted into $R(z)$, using Lemma 3.2(i), we obtain $d(y_{p-1}, V(R(z))) \leq n-k-3$ and $d(y_{p+1}, V(R(z))) \leq n-k-3$. These together with $d(y_{p-1}) \geq n+k$ and $d(y_{p+1}) \geq n+k$ imply that $d(y_{p-1}, Y \setminus \{y_p\}) \geq 2k+3$ and $d(y_{p+1}, Y \setminus \{y_p\}) \geq 2k+3$. Hence, it is easy to see that $y_{p+1} \to y_{p-1}$ and $d^+(x_s, Y \setminus \{y_p\}) = d^-(x_{s+1}, Y \setminus \{y_p\}) = 0$ (for otherwise D contains a $C(z)$-cycle of length at least $n-k-2$, a contradiction). Since every vertex of $Y \setminus \{y_p\}$ cannot be extended into P, using Lemma 3.2 and the last equalities, we obtain that if $u \in \{y_{p-1}, y_{p-1}\}$, then

$$n+k \leq d(u) = d(u, Y) + d(u, \{x_1, x_2, \dots, x_s\}) + d(u, \{x_{s+1}, x_{s+2}, \dots, x_{n-k-5}\})$$

$$\leq 2k+5+s+(n-5-k-s) = n+k.$$

From this, in particular, we have $d(u, Y) = 2k+5$, $d(u, \{x_1, x_2, \dots, x_s\}) = s$ and $d(u, \{x_{s+1}, x_{s+2}, \dots, x_{n-k-5}\}) = n-k-5-s$. Again using Lemma 3.2(ii), we obtain that $x_{n-k-5} \to \{y_{p-1}, y_{p+1}\} \to x_1$. From $d(u, Y) = 2k+5$ it follows that $u \leftrightarrow Y \setminus \{y_{p-1}, y_{p+1}\}$ since $y_{p-1} y_{p+1} \notin A(D)$. Hence it is not difficult to see that in $D\langle Y \setminus \{y_p\}\rangle$ there is a (y_{p-1}, y_{k+4})- or (y_{p-1}, y_{p+1})-Hamiltonian path, say H. Thus $x_1 x_2 \dots x_s y_p x_{s+1} \dots x_{n-k-5} H x_1$ is a cycle of length $n-1$ and the flight of z respect to this cycle is equal to $k+3$, a contradiction.

Case 2. $p=1$, i.e., $x_s \to y_1 \to x_{s+1}$.

Observe that $d^-(x_{s+1}, \{y_2, y_3, \dots, y_{k+4}\}) = 0$ and $R(z)$ is a longest cycle through z in D, which has length $n-k-3$. For Case 2 we will prove the following proposition.

Proposition 2. Suppose that for j, $j \in [2, k+4]$, in $Q := D\langle\{y_2, y_3, \dots, y_{k+4}, x_1\}\rangle$ there is a Hamiltonian (y_j, x_1)-path, say H^j. Then $x_{n-k-5}y_j \notin A(D)$. In particular, $x_{n-k-5}y_2 \notin A(D)$.

Proof: Suppose that the claim is not true, that is $x_{n-k-5} \to y_j$ with $j \in [2, k+4]$ and Q has a Hamiltonian (y_j, x_1)-path, say H^j. Then $x_1 x_2 \dots x_s y_1 x_{s+1} \dots x_{n-k-5} H^j x_1$ is a cycle of length $n-1$ and the flight of z respect to this cycle is equal to $k+3$, a contradiction. Thus $x_{n-k-5} \not\to y_j$. It is easy to see that $H^2 = y_2 y_3 \dots y_{k+4} x_1$ is a Hamiltonian path in Q. Therefore by the first part of this proposition, $x_{n-k-5} \not\to y_2$. □

To complete the proof of Claim 4, we will consider the cases $x_s \not\to y_2$, $x_s \to y_2$ separately.

Subcase 2.1. $x_s y_2 \notin A(D)$.

We know that $y_2 x_{s+1} \notin A(D)$ and $x_{n-k-5} y_2 \notin A(D)$. Now, since y_2 cannot be inserted into P, using Lemmas 3.2(ii) and 3.2(iii), we obtain

$$n+k \le d(y_2) = d(y_2, Y) + d(y_2, \{x_1, x_2, \dots, x_s\})$$

$$+d(y_2, \{x_{s+1}, x_{s+2}, \dots, x_{n-k-5}\}) \le 2k+6+s+(n-k-s-6) = n+k.$$

This implies that $d(y_2, Y) = 2k+6$, i.e., $y_2 \leftrightarrow Y \setminus \{y_2\}$, in particular, $y_2 \leftrightarrow y_1$ and $D\langle Y\rangle$ is strong, and

$$d(y_2, \{x_1, x_2, \dots, x_s\}) = s \text{ and } d(y_2, \{x_{s+1}, x_{s+2}, \dots, x_{n-k-5}\}) = n-k-s-6. \tag{3}$$

Thus, for the longest cycle $R(z)$ we have that $V(D) \setminus V(R(z)) = \{y_2, y_3, \dots, y_{k+4}\}$, $D\langle V(D) \setminus V(R(z))\rangle$ is strong and $y_2 \leftrightarrow y_1$. Therefore by Lemma 3.5,

$$A(\{y_2, y_3, \dots, y_{k+4}\} \to \{x_{s+1}, x_{s+2}, \dots, x_{n-k-5}\}) = A(\{x_1, x_2, \dots, x_s\} \to \{y_2, y_3, \dots, y_{k+4}\}) = \emptyset. \tag{4}$$

This together with $x_{n-k-5} \not\to y_2$ and (3) implies that y_2 and x_{n-k-5} are not adjacent and

$$N^+(y_2, V(P)) = \{x_1, x_2, \dots, x_s\} \text{ and } N^-(y_2, V(P)) = \{x_{s+1}, x_{s+2}, \dots, x_{n-k-6}\}.$$

By the above arguments, we have that $H^3 = y_3 y_4 \dots y_{k+4} y_2 x_1$ is a (y_3, x_1)-Hamiltonian path in Q. Therefore by Proposition 1, $x_{n-k-5} \not\to y_3$. This together with (4) implies that x_{n-k-5} and y_3 are not adjacent. As for y_2, for y_3 we obtain that $y_3 \leftrightarrow Y \setminus \{y_3\}$ and

$$N^+(y_3, V(P)) = \{x_1, x_2, \dots, x_s\} \text{ and } N^-(y_3, V(P)) = \{x_{s+1}, x_{s+2}, \dots, x_{n-k-6}\}.$$

Proceeding in the same manner, we obtain that $d(x_{n-k-5}, \{y_2, y_3, \dots, y_{k+4}\}) = 0$, $D\langle Y\rangle$ is a complete digraph and

$$\{x_{s+1}, x_{s+2}, \dots, x_{n-k-6}\} \to Y \setminus \{y_1\} \to \{x_1, x_2, \dots, x_s\}. \tag{5}$$

If $s = n-k-6$, then from (4) and $d(x_{n-k-5}, \{y_2, y_3, \dots, y_{k+4}\}) = 0$ it follows that $A(V(P) \cup \{z\} \to Y \setminus \{y_1\}) = \emptyset$, i.e., $D - y_1$ is not strong, a contradiction. Therefore, we may assume that $s \le n-k-7$.

Let $s = 1$. Since $D\langle Y\rangle$ is strong, from (5) it follows that $A(Y \to \{x_3, x_4, \dots, x_{n-k-5}\}) = \emptyset$. (for otherwise, $y_1 \to x_i$ with $i \in [3, n-k-5]$ and $C_{n-k-2}(z) = x_1x_2 \dots x_{i-1}y_2y_1x_i \dots x_{n-k-5}zx_1$, a contradiction). If $d^+(x_1, \{x_3, x_4, \dots, x_{n-k-5}\}) = 0$, then $A(\{x_1\} \cup Y \to \{z, x_3, x_4, \dots, x_{n-k-5}\}) = \emptyset$, i.e., $D - x_2$ is not strong, a contradiction. So, we can assume that for some $b \in [3, n-k-5]$, $x_1 \to x_b$. By (5) and (2), respectively, we have $x_{b-1} \to y_2$ and $z \to x_2$. Therefore, $C_{n-1}(z) := x_1x_b \dots x_{n-k-5}zx_2 \dots x_{b-1}y_2y_3 \dots y_{k+4}x_1$, a contradiction. Let finally $2 \leq s \leq n-k-7$. It is easy to see that $A(\{x_1, x_2, \dots, x_{s-1}\} \to \{x_{s+1}, x_{s+2}, \dots, x_{n-k-5}\}) \neq \emptyset$ (for otherwise, using the fact that $A(\{x_1, x_2, \dots, x_{s-1}\} \to Y) = \emptyset$ (by (5)), Claim 2 and (2), it is not difficult to show that $D - x_s$ is not strong, a contradiction). Thus, there are integers $a \in [1, s-1]$ and $b \in [s+1, n-k-5]$ such that $x_a \to x_b$. Then by (4), $y_2 \to x_{a+1}$, and by (2), either $z \to x_{a+1}$ or $x_{b-1} \to z$. By (4), we also have that $x_{b-1} \to y_2$ or $x_{b-1} \to y_1$ when $b = s+1$. Therefore, $C(z) = x_1x_2 \dots x_ax_b \dots x_{n-k-5}zx_{a+1} \dots x_{b-1}(y_1 \, \mathit{or} \, y_2)y_2y_3 \dots y_{k+4}x_1$ is a cycle of length at least $n-1$ or $C_{n-k-2}(z) = x_1x_2 \dots x_ax_b \dots x_{n-k-5}y_1y_2x_{a+1} \dots x_{b-1}zx_1$, respectively, for $z \to x_{a+1}$ and for $x_{b-1} \to z$. Thus, for any possible case we have a contradiction. This completes the discussion of Subcase 2.1.

Subcase 2.2. $x_s \to y_2$.

Using Lemma 3.5 and the fact that $R(z)$ is a longest cycle of length $n-k-3$ through z, we obtain

$$A(Y \setminus \{y_1\} \to \{x_{s+1}, x_{s+2}, \dots, x_{n-k-5}\}) = \emptyset. \tag{6}$$

Since $x_s \to y_1 \to x_{s+1}$, it follows that in $D\langle Y\rangle$ there is no (y_2, y_1)-path, i.e., $d^-(y_1, \{y_2, y_3, \dots, y_{k+4}\}) = 0$ (for otherwise D has a cycle of length at least $n-k-2$ through z, which is a contradiction). This implies that for all $i \in [1, k+4]$, $d(y_i, Y) \leq 2k+5$. Recall that $x_{n-k-5} \nrightarrow y_2$ (Proposition 1). Therefore, since y_2 cannot be inserted into P and $y_2 \nrightarrow x_{s+1}$, using Lemma 3.2, we obtain

$$\begin{aligned} n+k \leq d(y_2) &= d(y_2, Y) + d(y_2, \{x_1, x_2, \dots, x_s\}) + d(y_2, \{x_{s+1}, x_{s+2}, \dots, x_{n-k-5}\}) \\ &\leq 2k+5+s+1+(n-k-s-6) = n+k. \end{aligned}$$

Therefore, $y_2 \leftrightarrow Y \setminus \{y_1, y_2\}$, in particular, $D\langle Y \setminus \{y_1\}\rangle$ is strong,

$$d(y_2, \{x_1, x_2, \dots, x_s\}) = s+1 \text{ and } d(y_2, \{x_{s+1}, x_{s+2}, \dots, x_{n-k-5}\}) = n-k-s-6. \tag{7}$$

From (6) and $x_{n-k-5}y_2 \notin A(D)$ it follows that y_2 and x_{n-k-5} are not adjacent. Therefore by (7) and (6), $\{x_{s+1}, x_{s+2}, \dots, x_{n-k-6}\} \to y_2$, and by Lemma 3.2, $y_2 \to x_1$. Note that $H^3 = y_3y_4 \dots y_{k+4}y_2x_1$ is a Hamiltonian (y_3, x_1)-path in Q. Therefore by Proposition 1, $x_{n-k-5}y_3 \notin A(D)$, which together with (6) implies that y_3 and x_{n-k-5} are not adjacent. Now by the same arguments, as for y_2, we obtain that $y_3 \leftrightarrow Y \setminus \{y_1, y_3\}$,

$$d(y_3, \{x_1, x_2, \dots, x_s\}) = s+1 \text{ and } d(y_3, \{x_{s+1}, x_{s+2}, \dots, x_{n-k-6}\}) = n-k-s-6. \tag{8}$$

Now by (8) and (6), $\{x_{s+1}, x_{s+2}, \dots, x_{n-k-6}\} \to y_3$. We know that $P_1 := x_1x_2 \dots x_s$ is a longest (x_1, x_s)-path in $D\langle V(P_1) \cup Y \setminus \{y_1\}\rangle$. Therefore, since $d(y_2, V(P_1)) = d(y_3, V(P_1)) = s+1$, by Lemma 3.3, there exists an integer $q \in [1, s]$ such that for every $j \in [2, 3]$

$$N^+(y_j, V(P_1)) = \{x_1, x_2, \dots, x_q\} \text{ and } N^-(y_j, V(P_1)) = \{x_q, x_{q+1}, \dots, x_s\}.$$

Proceeding in the same manner, we conclude that $\{x_{s+1}, x_{s+2}, \ldots, x_{n-k-6}\} \to Y \setminus \{y_1\}$, for all $j \in [2, k+d]$, the vertices y_j and x_{n-k-5} are not adjacent and

$$N^+(y_j, V(P_1)) = \{x_1, x_2, \ldots, x_q\} \text{ and } N^-(y_j, V(P_1)) = \{x_q, x_{q+1}, \ldots, x_s\}. \quad (9)$$

If $q = 1$, then $A(\{y_2, y_3, \ldots, y_{k+4}\} \to \{z, y_1, x_2, x_3, \ldots, x_{n-k-5}\}) = \emptyset$, which implies that $D - x_1$ is not strong, a contradiction. Therefore, we may assume that $q \geq 2$, i.e., $q \in [2, s]$. If $x_i \to y_1$ with $i \in [1, q-1]$ then by (9), $C_n(z) = x_1x_2 \ldots x_iy_1y_2 \ldots y_{k+4}x_{i+1}x_{i+2} \ldots x_{n-k-5}zx_1$, a contradiction. We may therefore assume that $d^-(y_1, \{x_1, x_2, \ldots, x_{q-1}\}) = 0$. This together with (9) implies that $A(\{x_1, x_2, \ldots, x_{q-1}\} \to Y) = \emptyset$. Since D is 2-strong, the last equality and (2) imply that there are integers $a \in [1, q-1]$ and $b \in [q+1, n-k-5]$ such that $x_a \to x_b$, for otherwise it is easy to see that $D - x_q$ is not strong. By (9) and (2), we have $y_{k+4} \to x_{a+1}$, $x_{b-1} \to y_2$ and $z \to x_{a+1}$ or $x_{b-1} \to z$. Therefore, if $z \to x_{a+1}$, then $C_{n-1}(z) = x_1x_2 \ldots x_ax_b \ldots x_{n-k-5}zx_{a+1} \ldots x_{b-1}y_2 \ldots y_{k+4}x_1$, and if $x_{b-1} \to z$, then $C_n(z) = x_1x_2 \ldots x_ax_b \ldots x_{n-k-5}y_1 \ldots y_{k+4}x_{a+1} \ldots x_{b-1}zx_1$. So, in any case we have a contradiction. Claim 4 is proved. □

For any $j \in [1, d]$, we have

$$n + k \leq d(y_j) = d(y_j, V(P)) + d(y_j, Y) \leq d(y_j, V(P)) + 2d - 2.$$

From this, $d(y_j, V(P)) \geq n + k - 2d + 2$. On the other hand, by Lemma 3.2 and Claim 4, $d(y_j, V(P)) \leq n - d$. Therefore,

$$n + k - 2d + 2 \leq d(y_j, V(P)) \leq n - d \quad \text{and} \quad d + k \leq d(y_j, Y) \leq 2d - 2. \quad (10)$$

We distinguish two cases according to the subdigraph $D\langle Y\rangle$ is strong or not.

Case A. $D\langle Y\rangle$ is strong.
In this case, by Claim 4, the suppositions of Claim 1 hold. Therefore, if for some

$$i \in [1, n-d-2] \text{ and } d^+(x_i, Y) \geq 1, \text{ then } A(Y \to \{x_{i+1}, x_{i+2}, \ldots, x_{n-d-1}\}) = \emptyset. \quad (11)$$

Since D is 2-strong, (11) implies that $d^+(x_1, Y) = d^-(x_{n-d-1}, Y) = 0$, there exists $l \in [2, n-d-2]$ such that $d^+(x_l, Y) \geq 1$ and

$$A(\{x_1, x_2, \ldots, x_{l-1}\} \to Y) = A(Y \to \{x_{l+1}, x_{l+2}, \ldots, x_{n-d-1}\}) = \emptyset. \quad (12)$$

From this we see that the supposition of Claim 3 holds. Therefore, for all $j \in [2, n-d-2]$,

$$A(\{x_1, x_2, \ldots, x_{j-1}\} \to \{x_{j+1}, x_{j+2}, \ldots, x_{n-d-1}\}) \neq \emptyset. \quad (13)$$

For Case A, we will prove the following two claims.

Claim 5. (i) *$A(D)$ contains every arc of the forms $z \to x_i$ and $x_j \to z$, where $i \in [1, t]$ and $j \in [t, n-d-1]$, maybe except one when $d = k+3$. (Recall that the definition of t is given immediately after the proof of Claim 2).*

(ii) For every $i \in [1, d]$, $A(D)$ contains every arc of the forms $y_i \to x_q$ and $x_j \to y_i$ where $q \in [1, l]$ and $j \in [l, n-d-1]$, maybe except one when $d = k+3$ or except two when $d = k+4$.

Proof: (i) If $d = k + 4$, then Claim 5(i) is an immediate consequence of (2). Assume that $d = k + 3$. Then by (1), we have

$$n - k - 4 \leq d(z) = d^+(z, \{x_1, x_2, \dots, x_{t-1}\}) + d(z, \{x_t\}) + d^-(z, \{x_{t+1}, x_{t+2}, \dots, x_{n-k-4}\})$$

$$\leq t - 1 + 2 + n - k - 4 - t = n - k - 3.$$

Now, it is easy to see that Claim 5(i) is true.

(ii) By (10) and (12) we have

$$n + k - 2d + 2 \leq d(y_i, V(P)) = d^+(y_i, \{x_1, x_2, \dots, x_{l-1}\}) + d(y_i, \{x_l\})$$

$$+d^-(y_i, \{x_{l+1}, x_{l+2}, \dots, x_{n-d-1}\}) \leq l - 1 + 2 + n - d - 1 - l = n - d.$$

Now, considering the cases $d = k + 3$ and $d = k + 4$ separately, it is not difficult to see that Claim 5(ii) also is true. Claim 5 is proved. □

Claim 6. *Suppose that for some integers a and b with $1 \leq a < b - 1 \leq n - d - 2$ we have $x_a \to x_b$. If $D\langle Y\rangle$ is strong and $z \to x_{a+1}$, then $d^+(x_{b-1}, Y) = 0$.*

Proof: Suppose, on the contrary, that is $D\langle Y\rangle$ is strong, $z \to x_{a+1}$ and $d^+(x_{b-1}, Y) \geq 1$. Let $x_{b-1} \to y_i$, where $i \in [1, d]$. Recall that $k + 3 \leq d \leq k + 4$. If $i \in [1, k + 3]$, then the cycle $C(z) = x_1 x_2 \dots x_a x_b \dots x_{n-d-1} z x_{a+1} \dots x_{b-1} y_i \dots y_d x_1$ has length at least $n - k - 2$, which is a contradiction. Therefore, we may assume that $d^+(x_{b-1}, \{y_1, y_2, \dots, y_{k+3}\}) = 0$. Then from $d^+(x_{b-1}, Y) \geq 1$ it follows that $d = k + 4$ and $x_{b-1} \to y_{k+4}$. Hence by (11), $A(Y \to \{x_b, x_{b+1}, \dots, x_{n-k-5}\}) = \emptyset$. Note that for each $i \in [1, k + 3]$, $D\langle Y\rangle$ contains a (y_{k+4}, y_i)-path since $D\langle Y\rangle$ is strong. Hence it is not difficult to see that if $d^-(x_1, \{y_1, y_2, \dots, y_{k+3}\}) \geq 1$, then D contains a $C(z)$-cycle of length at least $n - k - 2$, a contradiction. Therefore, we may assume that $d^-(x_1, \{y_1, y_2, \dots, y_{k+3}\}) = 0$. This together with $d^+(x_1, Y) = 0$ implies that $d(x_1, \{y_1, y_2, \dots, y_{k+3}\}) = 0$. Now using Lemma 3.2, Claim 4, $A(Y \to \{x_b, x_{b+1}, \dots, x_{n-k-5}\}) = \emptyset$ and $d^+(x_{b-1}, \{y_1, y_2, \dots, y_{k+3}\}) = 0$, for any $i \in [1, k + 3]$ we obtain,

$$n + k \leq d(y_i) = d(y_i, Y) + d(y_i, \{x_2, x_3, \dots, x_{b-1}\}) + d^-(y_i, \{x_b, x_{b+1}, \dots, x_{n-k-5}\})$$

$$\leq 2k + 6 + (b - 2) + (n - k - 5 - b + 1) = n + k.$$

This means that all inequalities which were used in the last expression in fact are equalities, i.e., for any $i \in [1, k+3]$, $d(y_i, Y) = 2k+6$ (i.e., $D\langle Y\rangle$ is a complete digraph), and $d(y_i, \{x_2, x_3, \dots, x_{b-1}\}) = b-2$. Therefore, since any vertex y_i with $i \in [1, k + 3]$ cannot be inserted into P (Claim 4), $d(y_i, \{x_2, x_3, \dots, x_{b-1}\}) = b - 2$ and $x_{b-1} \not\to y_i$, using Lemma 3.2, we obtain that $y_i \to x_2$. Hence, if $a \geq 2$, then $C_{n-1}(z) = x_2 \dots x_a x_b \dots x_{n-k-5} z x_{a+1} \dots x_{b-1} L x_2$, where L is a Hamiltonian (y_{k+4}, y_{k+3})-path in $D\langle Y\rangle$, a contradiction. Therefore, we may assume that $a = 1$. Then $z \to x_2$ and $C_{n-1} = x_1 x_a \dots x_{n-k-5} y_1 y_2 \dots y_{k+3} x_2 \dots x_{b-1} y_{k+4} x_1$ is a cycle of length $n - 1$ in D. We have $x_{n-k-5} \to z$ and $z \to x_2$, i.e., the flight of z respect to this cycle C_{n-1} is equal to $k + 3$, which contradicts that the minimal flight of z respect to on all cycles of length $n - 1$ is equal to $d = k + 4$. Claim 6 is proved. □

Now using the digraph duality, we prove that it suffices to consider only the case $t \geq l$.

Indeed, assume that $l \geq t+1$ and consider the converse digraph D^{rev} of D. Let $V(D^{rev}) = \{u_1, u_2, \dots, u_{n-d-1}, v_1, v_2, \dots, v_d, z\}$, where $u_i := x_{n-d-i}$ and $v_j := y_{d+1-j}$ for all $i \in [1, n-d-1]$ and $j \in [1, d]$, in particular, $x_l = u_{n-d-l}$ and $x_t = u_{n-d-t}$. Let $p := n-d-l$ and $q := n-d-t$. Note that $q \geq p+1$ and $\{v_1, v_2, \dots, v_d\} = Y$.

Observe that from the definitions of l, t, p and q it follows that $d^-_{D^{rev}}(u_p, Y) \geq 1$ $zu_q \in A(D^{rev})$, $d^-_{D^{rev}}(z, \{u_1, u_2, \dots, u_{q-1}\}) = 0$ and $A_{D^{rev}}(\{u_1, u_2, \dots, u_{p-1}\} \to Y) = \emptyset$. Now using Claim 5(i), we obtain that $d^-_{D^{rev}}(z, \{u_q, u_{q+1}\}) \geq 1$ and $A_{D^{rev}}(\{u_p, u_{p+1}\} \to Y) \neq \emptyset$ when $d = k+3$ and $A_{D^{rev}}(\{u_p, u_{p+1}, u_{p+2}\} \to Y) \neq \emptyset$ when $d = k+4$. Let $u_{t'}z \in A(D^{rev})$, $d^+_{D^{rev}}(u_{l'}, Y) \geq 1$ and t', l' are minimal with these properties. It is clear that $t' \in [q, q+1]$ and $l' \in [p, p+2]$. We claim that $t' \geq l'$. Assume that this is not the case, i.e., $t' \leq l'-1$. Then it is not difficult to see that $t' \leq l'-1$ is possible when $l' = p+2$ and $t' = p+1 = q$. By Claim 5(ii), $d = k+4$ and $2 \leq p = q-1 \leq n-k-7$. Therefore, in D^{rev} the following hold:

$$d_{D^{rev}}(u_{p+1}, Y) = 0, \ \{u_{q+1}, u_{q+2}, \dots, u_{n-k-5}\} \to Y \to \{u_1, u_2, \dots, u_p\},$$

$$N^+_{D^{rev}}(z) = \{u_1, u_2, \dots, u_q\} \text{ and } N^-_{D^{rev}}(z) = \{u_q, u_{q+1}, \dots, u_{n-k-5}\}.$$

Since D^{rev} is 2-strong and $A_{D^{rev}}(\{u_1, u_2, \dots, u_p\} \to \{z\} \cup Y) = \emptyset$, it follows that there are $r \in [1, p]$ and $s \in [p+2, n-k-5]$ such that $u_r u_s \in A(D^{rev})$. Taking into account the above observation, it is not difficult to show that if $r \leq p-1$, then $C_n(z) = u_1 u_2 \dots u_r u_s \dots u_{n-k-5} v_1 v_2 \dots v_{k+4} u_{r+1} \dots u_{s-1} z u_1$ is a Hamiltonian cycle in D^{rev}, and if $s \geq p+3 = q+2$, then $C_n(z) = u_1 u_2 \dots u_r u_s \dots u_{n-k-5} z u_{r+1} \dots u_{s-1} v_1 v_2 \dots v_{k+4} u_1$ is a Hamiltonian cycle in D^{rev}, which contradicts that D is not Hamiltonian. We may therefore assume that $r = p$ and $s = p+2$. This means that $A_{D^{rev}}(\{u_1, u_2, \dots, u_{p-1}\} \to \{u_{p+2}, u_{p+3}, \dots, u_{n-k-5}\}) = \emptyset$. Therefore, since D^{rev} is 2-strong, for some $i \in [1, p-1]$, $u_i u_{p+1} \in A(D^{rev})$. Hence, $u_1 u_2 \dots u_i u_{p+1} z x_{i+1} \dots u_p u_{p+2} \dots u_{n-k-5} v_1 v_2 \dots v_{k+4} u_1$ is a Hamiltonian cycle in D^{rev}, a contradiction. Therefore, the case $t \leq l-1$ is equivalent to the case $t \geq l$.

Using Lemma 3.1, it is easy to see that the following proposition holds.

Proposition 3. If $k = 0$ and a longest $C(z)$-cycle in D has length $n-3$, then $D\langle V(D) \setminus V(C(z))\rangle$ is strong.

From now on, we assume that $l \leq t$. Note that from (13) it follows that there are $a \in [1, t-1]$ and $b \in [t+1, n-d-1]$ such that $x_a \to x_b$.

Subcase A.1. $z \to x_{a+1}$.

Recall that $a \in [1, t-1]$ and $b \in [t+1, n-d-1]$. By Claim 6, we have that $d^+(x_{b-1}, Y) = 0$.

Subcase A.1.1. $z \to x_{a+1}$ and $b \geq t+2$.

Then $b-2 \geq t \geq l$. If $x_{b-2} \to y_i$ with $i \in [1, 2]$, then the cycle $C(z) = x_1 x_2 \dots x_a x_b \dots x_{n-d-1} z x_{a+1} \dots x_{b-2} y_i \dots y_d x_1$ has length at least $n-2$, a contradiction. We may therefore assume that $d^+(x_{b-2}, \{y_1, y_2\}) = 0$. This together with Claim 6 implies that $A(\{x_{b-2}, x_{b-1}\} \to \{y_1, y_2\}) = \emptyset$. Therefore by Claim 5(ii) and $l \leq t$, we have that $d = k+4$, in particular, (2) holds. If $b \geq t+3$, then from $d^-(y_1, \{x_{b-2}, x_{b-1}\}) = 0$ and Claim 5(ii) it follows that $x_{b-3} \to y_1$ and $C_{n-2}(z) = x_1 x_2 \dots x_a x_b \dots x_{n-k-5} z x_{a+1} \dots x_{b-3} y_1 y_2 \dots y_{k+4} x_1$, a contradiction. Therefore, we may assume that $b = t+2$. If $x_t \to y_3$, then $C_{n-3}(z) = x_1 x_2 \dots x_a x_{t+2} \dots x_{n-k-5} z x_{a+1} \dots x_t y_3 \dots y_{k+4} x_1$ and the subdigraph $D\langle V(D)\setminus V(C_{n-3}(z))\rangle = D\langle\{x_{t+1}, y_1, y_2\}\rangle$ is not strong since $d^+(x_{t+1}, \{y_1, y_2\}) = 0$. This implies that $n-3 \leq n-k-3$ (i.e., $k = 0$) since the length of a longest $C(z)$-cycle is at most $n-k-3$. So, we have a contradiction to Proposition 3. Therefore, $d^-(y_3, \{x_t, x_{t+1}\}) = 0$. Hence, if $l = t$, then $y_3 \to x_{a+1}$ (Claim 5(ii)) and $C_{n-k-2}(z) = x_1x_2 \ldots x_ax_{t+2} \ldots x_{n-k-5}y_1y_2y_3x_{a+1} \ldots x_tzx_1$, a contradiction. We may assume that $l \leq t-1$. If $a \leq t-2$, then from $d^-(y_1, \{x_t, x_{t+1}\}) = 0$ and Claim 5(ii), we have $x_{t-1} \to y_1$ and $C_{n-2}(z) = x_1x_2 \ldots x_ax_{t+2} \ldots x_{n-k-5}zx_{a+1} \ldots x_{t-1}y_1y_2 \ldots y_{k+4}x_1$, a contradiction. We may therefore assume that $a = t-1$. From $l \leq t-1$ and $l \geq 2$ it follows that $t \geq 3$. Thus we have that $a = t-1 \geq 2$ and $b = t+2$, which mean that $A(\{x_1, x_2, \ldots, x_{a-1} = x_{t-2}\} \to \{x_{t+2}, x_{t+3}, \ldots, x_{n-k-5}\}) = \emptyset$. This together with (13) implies that for some $i \in [1, t-2]$ and $j \in [t, t+1]$, $x_i \to x_j$. Recall that $z \to x_{i+1}$ and $x_{t+1} \to z$ because of (2). Therefore, $C(z) = x_1x_2 \ldots x_ix_jx_{t+1}zx_{i+1} \ldots x_{t-1}x_{t+2} \ldots x_{n-k-5}y_1y_2 \ldots y_{k+4}x_1$ is cycle of length at least $n-1$, a contradiction. This completes the discussion of Sabcase A.1.1.

Subcase A.1.2. $z \to x_{a+1}$ and $b = t+1$. Since $b-1 = t$, $d^+(x_{b-1}, Y) = 0$ (Claim 6) and $d^+(x_l, Y) \geq 1$, we have $d^+(x_t, Y) = 0$, $t-1 \geq l \geq 2$.

Assume first that $t+1 \leq n-d-2$. Taking into account Subcase A.1.1 and $b = t+1$, we may assume that $A(\{x_1, x_2, \ldots, x_{t-1}\} \to \{x_{t+2}, x_{t+3}, \ldots, x_{n-d-1}\}) = \emptyset$. This together with (13) implies that there is $j \in [t+2, n-d-1]$ such that $x_t \to x_j$. If $x_{j-1} \to z$, then $C_n(z) = x_1x_2 \ldots x_ax_{t+1} \ldots x_{j-1}zx_{a+1} \ldots x_t x_j \ldots x_{n-d-1}y_1y_2 \ldots y_dx_1$, a contradiction. Therefore, we may assume that $x_{j-1} \nrightarrow z$. This together with (2) implies that $d = k+3$. If $j \geq t+3$, then $x_{j-2} \to z$ (Claim 5(i)) and $C_{n-1}(z) = x_1x_2 \ldots x_ax_{t+1} \ldots x_{j-2}zx_{a+1} \ldots x_tx_j \ldots x_{n-k-4}y_1y_2 \ldots y_{k+3}x_1$, a contradiction. Assume that $j = t+2$. Since $d^+(x_t, Y) = 0$, $d^+(x_l, Y) \geq 1$, $d = k+3$ and $l \leq t-1$, by Claim 5(ii) we have $\{x_l, x_{l+1}, \ldots, x_{t-1}\} \to y_1$. If $a \leq t-2$, then $x_{t-1} \to y_1$ and $C_{n-1}(z) = x_1x_2 \ldots x_ax_{t+1} \ldots x_{n-k-4}z x_{a+1} \ldots x_{t-1}y_1y_2 \ldots y_{k+3}x_1$, a contradiction. Assume that $a = t-1$. From $t \geq 3$ and $a = t-1$ it follows that $A(\{x_1, x_2, \ldots, x_{t-2}\} \to \{x_{t+1}, x_{t+2}, \ldots, x_{n-k-4}\}) = \emptyset$. This together with (13) implies that for some $s \in [1, t-2]$, $x_s \to x_t$. Since $d = k+3$ and $x_{j-1} \nrightarrow z$, from Claim 5(i) it follows that $z \to x_{s+1}$. Therefore, $C(z) = x_1x_2 \ldots x_sx_tzx_{s+1} \ldots x_{t-1}x_{t+1} \ldots x_{n-k-4}y_1y_2 \ldots y_{k+3}x_1$ is a Hamiltonian cycle in D, a contradiction.

Assume next that $t+1 = n-d-1$. Recall that $b = t+1$ and $d^+(x_t, Y) = 0$.

Let $a \leq t-2$. If $x_{t-1} \to y_i$ with $i \in [1, 2]$, then $C(z) = x_1x_2 \ldots x_ax_{n-d-1}zx_{a+1} \ldots x_{t-1}y_iy_2 \ldots y_dx_1$ is a cycle of length at least $n-2$, a contradiction. Therefore, we may assume that for every $i \in [1, 2]$, $d^-(y_i, \{x_{t-1}, x_t\}) = 0$. This together with Claim 5(ii) implies that $d = k+4$, which in turn implies that (2) holds, in particular, $z \to x_t$. If $l = t-1$, then from $d^-(y_2, \{x_{t-1}, x_t\}) = 0$ and Claim 5(ii) it follows that $y_2 \to x_{a+1}$ and $C_{n-k-2}(z) = x_1x_2 \ldots x_ax_{n-k-5}y_1y_2x_{a+1} \ldots x_tzx_1$, a contradiction. Therefore, we may assume that $l \leq t-2$. It is easy to see that $a = t-2$ (for otherwise $a \leq t-3$, $x_{t-2} \to y_1$ and $C_{n-2}(z) = x_1x_2 \ldots x_ax_{n-k-5}zx_{a+1} \ldots x_{t-2}y_1y_2 \ldots y_{k+4}x_1$, a contradiction). Using Claim 5(ii) and the facts that $a = t-2 \geq l \geq 2$, $d^-(y_1, \{x_{t-1}, x_t\}) = 0$, it is easy to see that $x_a \to y_1$. From $a = t-2 \geq 2$ it follows that $d^-(x_{n-k-5}, \{x_1, x_2, \ldots, x_{a-1}\}) = 0$. Therefore by (13), there exist $s \in [1, a-1]$ and $j \in [t-1, t]$ such that $x_s \to x_j$. Then by (2), $z \to x_{s+1}$ and $C(z) = x_1x_2 \ldots x_sx_jx_tx_{n-k-5}zx_{s+1} \ldots x_ay_1y_2 \ldots y_{k+4}x_1$ is a cycle of length at least $n-1$, a contradiction. Let now $a = t-1$. Recall that $b = t+1 = n-d-1$. Then from $a = t-1 \geq 2$ we have that $d^-(x_{n-d-1}, \{x_1, x_2, \ldots, x_{a-1}\}) = 0$. This together with (13) implies that for some $s \in [1, t-2]$, $x_s \to x_t$. It is easy to see that $z \nrightarrow x_{s+1}$ (for otherwise, $z \to x_{s+1}$ and $C_n(z) = x_1x_2 \ldots x_sx_tzx_{s+1} \ldots x_{t-1}x_{t+1}y_1y_2 \ldots y_dx_1$, a contradiction). From (2), Claim 5(ii) and $z \nrightarrow x_{s+1}$ it follows that $d = k+3$ and $x_{t-1} \to y_1$. If $s \leq t-3$, then $z \to x_{s+2}$ and $C_{n-1}(z) = x_1x_2 \ldots x_s x_t z x_{s+2} \ldots x_{t-1}x_{t+1}y_1y_2 \ldots y_d x_1$, a contradiction. Thus, we may assume that $s = t-2$. If $t-2 \geq 2$, then we have that $A(\{x_1, x_2, \ldots, x_{t-2}\} \to \{x_t, x_{t+1} = x_{n-k-4}\}) = \emptyset$. Therefore by (13), there is $p \in [1, t-3]$ such that $x_p \to x_{t-1}$ and $z \to x_{p+1}$. If $l \leq t-2$, then $x_{t-2} \to y_1$ and $C_n(z) = x_1x_2 \ldots x_p x_{t-1}x_t x_{t+1} z x_{p+1} \ldots x_{t-2}y_1y_2 \ldots y_{k+3}x_1$, a contradiction. Assume that $l = t-1$. Then $y_1 \to x_{p+1}$ and $C_{n-k-2} = x_1x_2 \ldots x_p x_{t-1}x_{t+1}y_1x_{p+1} \ldots x_{t-2}x_t z x_1$, a contradiction. Finally assume that $t-2 = 1$. Then $n-k-4 = 4$ and $d(x_3, Y) = 0$. Therefore, $n+k \leq d(x_3) \leq 8$ and $n \leq 8$, which contradicts that $n \geq 9$. This completes the discussion of Subcase A.1.2.

Subcase A.2. $z \nrightarrow x_{a+1}$.

From $z \nrightarrow x_{a+1}$, Claim 5(i), (1) and (2) it follows that $d = k+3$, $x_{b-1} \to z$ and

$$\{x_t, x_{t+1}, \ldots, x_{n-k-4}\} \to z \to \{x_1, x_2, \ldots, x_a, x_{a+2}, x_{a+3}, \ldots, x_t\}. \tag{14}$$

Assume first that

$$A(\{x_1, x_2, \ldots, x_{t-2}\} \to \{x_{t+1}, x_{t+2}, \ldots, x_{n-k-4}\}) = \emptyset. \tag{15}$$

Then $a = t-1$, i.e., $x_{t-1} \to x_b$. Using (14), $d^+(z) \geq 2$ and Claim 2, we obtain that $t-1 \geq 2$. From (13) and (15) it follows that there exists $s \in [1, t-2]$ such that $x_s \to x_t$. Then, since $z \to x_{s+1}$, $C_n(z) = x_1x_2 \ldots x_s x_t \ldots x_{b-1} z x_{s+1} \ldots x_{t-1}x_b \ldots x_{n-k-4}y_1y_2 \ldots y_{k+3}x_1$, a contradiction.

Assume next that (15) is not true. Then we may assume that $a \leq t-2$. Note that $z \to \{x_{a+2}, \ldots, x_t\}$ (by (14)). If $y_i \to x_{a+1}$ with $i \in [1, k+3]$, then the cycle $C(z) = x_1x_2 \ldots x_a x_b \ldots x_{n-k-4}y_1 \ldots y_i x_{a+1} \ldots x_{b-1} z x_1$ has length at least $n-k-2$, a contradiction. We may therefore assume that $d^-(x_{a+1}, Y) = 0$. Let $b \geq t+2$. Then, since $d = k+3$ and $t \geq l$, from Claim 5(ii) it follows that for some $j \in [b-2, b-1]$, $x_j \to y_1$. Then the cycle $C(z) = x_1x_2 \ldots x_a x_b \ldots x_{n-k-4} z x_{a+2} \ldots x_j y_1 y_2 \ldots y_{k+3}x_1$ has length at least $n-2$, a contradiction. Let now $b = t+1$. We claim that $l \leq t-1$. Assume that this is not the case, i.e., $l = t$. Then using Claim 5(ii) and the facts that $d = k+3$, $d^-(x_{a+1}, Y) = 0$, we obtain that $x_t \to y_1$. Therefore, $C_{n-1}(z) = x_1x_2 \ldots x_a x_{t+1} \ldots x_{n-k-4} z x_{a+2} \ldots x_t y_1 \ldots y_{k+3}x_1$, a contradiction. This shows that $l \leq t-1$. From $a \leq t-2$, $l \leq t-1$, $x_t \nrightarrow y_1$ and Claim 5(ii) it follows that $x_{t-1} \to y_1$. Therefore, if $a \leq t-3$, then $C_{n-2}(z) = x_1x_2 \ldots x_a x_{t+1} \ldots x_{n-k-4} z x_{a+2} \ldots x_{t-1}y_1y_2 \ldots y_{k+3}x_1$, a contradiction. We may therefore assume that $a = t-2$. Assume first that $a \geq 2$. Since $a = t-2$, we have

$$A(\{x_1, x_2, \ldots, x_{t-3}\} \to \{x_{t+1}, x_{t+2}, \ldots, x_{n-k-4}\}) = \emptyset.$$

This together with (13) implies that there exist $s \in [1, a-1 = t-3]$ and $p \in [t-1, t]$ such that $x_s \to x_p$. Then by (14), the cycle $C(z) = x_1x_2 \ldots x_s x_p x_t z x_{s+1} \ldots x_{t-2}x_{t+1} \ldots x_{n-k-4}y_1y_2 \ldots y_{k+3}x_1$ has length at least $n-1$, a contradiction.

Assume next that $a = 1$. Then $t = 3$. Let $t+1 \leq n-k-5$. Since $b = t+1$, we have $d^+(x_1, \{x_{t+2}, x_{t+3}, \ldots, x_{n-k-4}\}) = 0$. Again using (13), we obtain that there exist $p \in [t-1, t]$ and $q \in [t+2, n-k-4]$ such that $x_p \to x_q$. Recall that $z \to x_t$ and $x_{q-1} \to \{z, y_1\}$. Therefore, if $p = t$, then $C_{n-1}(z) = x_1x_{t+1} \ldots x_{q-1} z x_t x_q \ldots x_{n-k-4}y_1y_2 \ldots y_{k+3}x_1$, and if $p = t-1$, then $C_n(z) = x_1x_2 \ldots x_{t-1}x_q \ldots x_{n-k-4} z x_t \ldots x_{q-1}y_1y_2 \ldots y_{k+3}x_1$. Thus, in both cases, we have a contradiction. This completes the discussion of Subcase A.2, and also completes the proof of the theorem when $D\langle Y \rangle$ is strong.

Case B. $D\langle Y\rangle$ is not strong.

Since $y_1y_2\dots y_d$ is a path in $D\langle Y\rangle$ and $k+3\leq d\leq k+4$, using the fact that every vertex y_i with $i\in[1,d]$ cannot be inserted into P (Claim 4) and Lemma 3.2, we obtain $d(y_i,V(P))\leq n-d$, $d(y_i,Y)\geq d+k$. Now, we claim that $d=k+4$ and $k=0$. Indeed, since $D\langle Y\rangle$ is not strong, $y_1y_2\dots y_d$ is a Hamiltonian path in $D\langle Y\rangle$ and $d(y_i)\geq n+k$, it follows that for some $l\in[2,d-1]$, $y_l\to y_1$ and $d^-(y_1,\{y_{l+1},y_{l+2},\dots,y_d\})=d^+(y_d,\{y_1,y_2\dots,y_l\})=0$. From this we have $k+d\leq d(y_1,Y)\leq d-l+2(l-1)=d+l-2$ and $k+d\leq d(y_d,Y)\leq l+2(d-l-1)=2d-l-2$. Therefore, $k\leq l-2$ and $d\geq k+l+2$. From the last two inequalities and the facts that $d\leq k+4$, $l\geq 2$ it follows that $d=k+4$ and $k=0$. Therefore, $d(y_i,V(P))\leq n-4$ and $d(y_i,Y)\geq 4$. Since $D\langle Y\rangle$ is not strong and $y_1y_2y_3y_4$ is a path in $D\langle Y\rangle$, it is not difficult to check that for all $i\in[1,4]$, $d(y_i,Y)=4$, $d(y_i,V(P))=n-4$, the arcs $y_1y_3,y_1y_4,y_2y_1,y_2y_4,y_4y_3$ also are in $A(D)$ and $A(\{y_3,y_4\}\to\{y_1,y_2\})=\emptyset$. Since D has no $C(z)$-cycle of length at least $n-2$ and any vertex y_i with $i\in[1,4]$ cannot be inserted into $P=x_1x_2\dots x_{n-5}$, using Lemma 3.3 and Proposition 3, it is not difficult to show that there are two integers l_1 and l_2 with $2\leq l_1,l_2\leq n-6$ such that

$$\begin{cases}\{x_{l_1},\dots,x_{n-5}\}\to\{y_1,y_2\}\to\{x_1,\dots,x_{l_1}\},\\ \{x_{l_2},\dots,x_{n-5}\}\to\{y_3,y_4\}\to\{x_1,\dots,x_{l_2}\}.\end{cases}\tag{16}$$

It is easy to see that $l_1\geq l_2$. Indeed, if $l_1\leq l_2-1$, then from (16) it follows that $x_{l_1}\to y_1$, $y_4\to x_{l_1+1}$ and hence, $C_n(z)=x_1\dots x_{l_1}y_1y_2y_3y_4x_{l_1+1}\dots x_{n-5}zx_1$, a contradiction. Since D is 2-strong, (16) together with (2) implies that there are two integers $p\in[1,l_2-1]$ and $q\in[l_2+1,n-5]$ such that $x_p\to x_q$ (for otherwise $D-x_{l_2}$ is not strong). Assume first that $l_2\leq t$. Then from (2) and (16), respectively, we have $z\to x_{p+1}$ and $x_{q-1}\to y_3$. Therefore, $C_{n-2}(z)=x_1\dots x_px_q\dots x_{n-5}zx_{p+1}\dots x_{q-1}y_3y_4x_1$, a contradiction. Assume next that $l_2\geq t+1$. Then by (16), $y_4\to x_{p+1}$, and by (2), $x_{q-1}\to z$. Therefore, $C(z)=x_1x_2\dots x_px_q\dots x_{n-5}y_1\dots y_4x_{p+1}\dots x_{q-1}zx_1$ is a Hamiltonian cycle in D, which is contradiction. This completes the discussion of Case B. The theorem is proved. □

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