A new sufficient condition for a 2-strong digraph to be Hamiltonian

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In this paper we prove the following new sufficient condition for a digraph to be Hamiltonian:

Let \( D \) be a 2-strong digraph of order \( n \geq 9 \). If \( n - 1 \) vertices of \( D \) have degrees at least \( n + k \) and the remaining vertex has degree at least \( n - k - 4 \), where \( k \) is a non-negative integer, then \( D \) is Hamiltonian.

This is an extension of Ghouila-Houri’s theorem for 2-strong digraphs and is a generalization of an early result of the author (DAN Arm. SSR (91(2):6-8, 1990). The obtained result is best possible in the sense that for \( k = 0 \) there is a digraph of order \( n = 8 \) (respectively, \( n = 9 \)) with the minimum degree \( n - 4 = 4 \) (respectively, with the minimum degree \( n - 5 = 4 \)) whose \( n - 1 \) vertices have degrees at least \( n - 1 \), but it is not Hamiltonian.

We also give a new sufficient condition for a 3-strong digraph to be Hamiltonian-connected.

Keywords: digraph, Hamiltonian cycle, Hamiltonian-connected, k-strong, degree

1 Introduction

In this paper, we consider finite digraphs without loops and multiple arcs. We shall assume that the reader is familiar with the standard terminology on digraphs and refer the reader to Bang-Jensen and Gutin (Springer-Verlag, London, 2000). Every cycle and path is assumed simple and directed. A cycle (a path) in a digraph \( D \) is called Hamiltonian (Hamiltonian path) if it includes all the vertices of \( D \). A digraph \( D \) is Hamiltonian if it contains a Hamiltonian cycle. Hamiltonicity is one of the most central in graph theory, and it has been extensively studied by numerous researchers. The problem of deciding Hamiltonicity of a graph (digraph) is \( NP \)-complete, but there are numerous sufficient conditions which ensure the existence of a Hamiltonian cycle in a digraph (see Bang-Jensen and Gutin (Springer-Verlag, London, 2000), Bermond and Thomassen (1981), Gould (2014), Kühn and Osthus (2012)). Among them are the following classical sufficient conditions for a digraph to be Hamiltonian.

**Theorem 1.1** (Nash-Williams (1969)). Let \( D \) be a digraph of order \( n \geq 2 \). If for every vertex \( x \) of \( D \), \( d^+(x) \geq n/2 \) and \( d^-(x) \geq n/2 \), then \( D \) is Hamiltonian.

**Theorem 1.2** (Ghouila-Houri (1960)). Let \( D \) be a strong digraph of order \( n \geq 2 \). If for every vertex \( x \) of \( D \), \( d(x) \geq n \), then \( D \) is Hamiltonian.
Theorem 1.3 (Woodall (1972)). Let $D$ be digraph of order $n \geq 2$. If $d^+(x) + d^-(y) \geq n$ for all pairs of distinct vertices $x$ and $y$ of $D$ such that there is no arc from $x$, to $y$, then $D$ is Hamiltonian.

Theorem 1.4 (Meyniel (1973)). Let $D$ be a strong digraph of order $n \geq 2$. If $d(x) + d(y) \geq 2n - 1$ for all pairs of non-adjacent distinct vertices $x$ and $y$ of $D$, then $D$ is Hamiltonian.

It is known that all the lower bounds in the above theorems are tight. Notice that for the strong digraphs Meyniel’s theorem is a generalization of Nash-Williams, Ghouila-Houri’s and Woodall’s theorems. A beautiful short proof the later can found in the paper by Bondy and Thomassen (1977).

Nash-Williams (1969) suggested the problem of characterizing all the strong digraphs of order $n$ and minimum degree $n - 1$ that have no Hamiltonian cycle. As a partial solution of this problem, Thomassen (1981) in his excellent paper proved a structural theorem on the extremal digraphs. An analogous problem for the Meyniel theorem was considered by Darbinyan (1982), proving a structural theorem on the strong non-Hamiltonian digraphs $D$ of order $n$, with the condition that $d(x) + d(y) \geq 2n - 2$ for every pair of non-adjacent distinct vertices $x$, $y$. This improves the corresponding structural theorem of Thomassen.

Darbinyan (1982) also proved that: if $m$ is the length of longest cycle in $D$, then $D$ contains cycles of all lengths $k = 2, 3, \ldots, m$. Thomassen (1981) and Darbinyan (1986) described all the extremal digraphs for the Nash-Williams theorem, respectively, when the order of a digraph is odd and when the order of a digraph is even. Here we combine they in the following theorem.

Theorem 1.5 (Thomassen (1981) and Darbinyan (1986)). Let $D$ be a digraph of order $n \geq 4$ with minimum degree $n - 1$. If for every vertex $x$ of $D$, $d^+(x) \geq n/2 - 1$ and $d^-(x) \geq n/2 - 1$, then $D$ is Hamiltonian, unless some exceptions, which completely are characterized.

Goldberg, Levitskaya and Satanovskyi (1971) relaxed the condition of the Ghouila-Houri theorem by proving the following theorem.

Theorem 1.6 (Goldberg, Levitskaya and Satanovskyi (1971)). Let $D$ be a strong digraph of order $n \geq 2$. If $n - 1$ vertices of $D$ have degrees at least $n$ and the remaining vertex has degree at least $n - 1$, then $D$ is Hamiltonian.

Note that Theorem 1.6 is an immediate consequence of Theorem 1.4. Goldberg, Levitskaya and Satanovskyi (1971) for any $n \geq 5$ presented two examples of non-Hamiltonian strong digraphs of order $n$ such that: (i) In the first example, $n - 2$ vertices have degrees equal to $n + 1$ and the other two vertices have degrees equal to $n - 1$. (ii) In the second example, $n - 1$ vertices have degrees at least $n$ and the remaining vertex has degree equal to $n - 2$.

Remark 1. It is worth to mention that Thomassen (1981) constructed a strong non-Hamiltonian digraph of order $n$ with only two vertices of degree $n - 1$ and all other $n - 2$ vertices have degrees at least $(3n - 5)/2$.

Zhang, Zhang and Wen (2013) reduced the lower bound in Theorem 1.3 by 1, and proved that the conclusion still holds, with only a few exceptional cases that can be clearly characterized. Darbinyan (1990a) announced that the following theorem is holds.
**Theorem 1.7** ([Darbinyan 1990a]). Let $D$ be a 2-strong digraph of order $n \geq 9$ such that its $n-1$ vertices have degrees at least $n$ and the remaining vertex has degree at least $n-4$. Then $D$ is Hamiltonian.

The proof of Theorem 1.7 has never been published. G. Gutin suggested me to publish the proof of this theorem anywhere. Recently, [Darbinyan 2022](Darbinyan 2022) presented a new proof of the first part of Theorem 1.7, by proving the following:

**Theorem 1.8** ([Darbinyan 2022]). Let $D$ be a 2-strong digraph of order $n \geq 9$ such that its $n-1$ vertices have degrees at least $n$ and the remaining vertex $z$ has degree at least $n-4$. If $D$ contains a cycle of length $n-2$ through $z$, then $D$ is Hamiltonian.

[Darbinyan 2022](Darbinyan 2022) also proposed the following conjecture.

**Conjecture 1.** Let $D$ be a 2-strong digraph of order $n$. Suppose that $n-1$ vertices of $D$ have degrees at least $n+k$ and the remaining vertex has degree at least $n-k-4$, where $k \geq 0$ is an integer. Then $D$ is Hamiltonian.

Note that, for $k = 0$ this conjecture is Theorem 1.7. By inspecting the proof of Theorem 1.8 and the handwritten proof of Theorem 1.7, by the similar arguments we settled Conjecture 1 by proving the following theorem.

**Theorem 1.9.** Let $D$ be a 2-strong digraph of order $n \geq 9$. If $n-1$ vertices of $D$ have degrees at least $n+k$ and the remaining vertex $z$ has degree at least $n-k-4$, where $k \geq 0$ is an integer, then $D$ is Hamiltonian.

[Darbinyan 2022](Darbinyan 2022) presented the proof of the first part of Conjecture 1 for any $k \geq 1$, which we formulated as Theorem 3.6 (Section 3). The goal of this article to present the complete proof of the second part of the proof of Theorem 1.9 and show that this theorem is best possible in the sense that for $k = 0$ there is a 2-strong digraph of order $n = 8$ (respectively, $n = 9$) with the minimum degree $n-4 = 4$ (respectively, with the minimum degree $n-5 = 4$) whose $n-1$ vertices have degrees at least $n$, but it is not Hamiltonian. To see that the theorem is best possible, it suffices consider the digraphs defined in the Examples 1 and 2, see Figure 1. In figures an undirected edge represents two directed arcs of opposite directions.

**Example 1.** Let $D_8$ be a digraph of order 8 with vertex set $V(D_8) = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3, z\}$ and arc set $A(D_8)$, which satisfies the following conditions: $D_8(\{y_1, y_2, y_3\})$ is a complete digraph, $x_4 \rightarrow \{y_1, y_2, y_3\} \rightarrow x_1$, $x_2 \rightarrow \{y_1, y_2, y_3\} \rightarrow x_2$, $D_8$ contains the following 2-cycles and arcs $x_4 \leftrightarrow x_{i+1}$ for all $i \in \{1, 3\}$, $x_1 \leftrightarrow x_3$, $x_3 \leftrightarrow z$, $x_4 \leftrightarrow x_2$, $x_4 \rightarrow x_1$, $x_4 \rightarrow z$ and $z \rightarrow x_1$. $A(D_8)$ contains no other arcs.

**Example 2.** Let $D_9$ be a digraph of order 9 with vertex set $V(D_9) = \{x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, z\}$ and arc set $A(D_9)$, which satisfies the following conditions: $D_9(\{y_1, y_2, y_3\})$ is a complete digraph, $x_5 \rightarrow \{y_1, y_2, y_3\} \rightarrow x_1$, $x_3 \rightarrow \{y_1, y_2, y_3\} \rightarrow \{x_1, x_2, x_3\}$, $D_9$ contains the following 2-cycles and arcs $x_1 \leftrightarrow x_{i+1}$ for all $i \in \{1, 4\}$, $x_1 \leftrightarrow x_4$, $x_3 \leftrightarrow x_5$, $x_4 \leftrightarrow x_2$, $x_4 \leftrightarrow z$, $x_5 \rightarrow z$, $z \rightarrow x_1$ and $x_5 \rightarrow x_1$. $A(D_9)$ contains no other arcs.
Observe that every vertex other than \( z \) in \( D_8 \) (in \( D_9 \)) has degree at least \(|V(D_8)| = 8\) (at least \(|V(D_9)| = 9\)) and \( d(z) = 4 \) in both digraphs \( D_8 \) and \( D_9 \). It is not hard to check that for every \( u \in V(D_8) \) (\( u \in V(D_9) \)), \( D_8 - u \) (\( D_9 - u \)) is strong, i.e., \( D_8 \) and \( D_9 \) both are 2-strong. To see this, it suffices to consider a longest cycle in \( D_8 - u \) (in \( D_9 - u \)) and apply the following well-known proposition.

**Proposition 1** (see Exercise 7.26, Bang-Jensen and Gutin (Springer-Verlag, London, 2000)). Let \( D \) be a \( k \)-strong digraph with \( k \geq 1 \), let \( x \) be a new vertex and \( D' \) be a digraph obtained from \( D \) and \( x \) by adding \( k \) arcs from \( x \) to distinct vertices of \( D \) and \( k \) arcs from distinct vertices of \( D \) to \( x \). Then \( D' \) also is \( k \)-strong.

Let \( D'_9 \) be the digraph obtained from \( D_9 \) by adding the arcs \( x_3x_1 \) and \( x_5x_2 \).

Now we will show that \( D'_9 \) is not Hamiltonian. Assume that this is not the case. Let \( R \) be an arbitrary Hamiltonian cycle in \( D'_9 \). Then \( R \) necessarily contains either the arc \( x_4z \) or the arc \( x_5z \). If \( x_4z \in A(R) \), then it is not difficult to see that either \( R[x_4, y_i] = x_4zx_1x_2x_3y_i \) or \( R[x_4, y_i] = x_4zx_1x_2x_3x_5y_i \), which is impossible since \( N^+(y_i, \{x_1, x_2, \ldots, x_5\}) = \{x_1, x_2, x_3\} \). We may therefore assume that \( x_5z \in A(R) \). Then necessarily \( R \) contains the arc \( x_3y_i \) and either the path \( x_5zx_1 \) or the path \( x_5zx_4 \). It is easy to check that either \( x_5zx_3 \in A(R) \) or \( x_4x_3 \in A(R) \). If \( x_5zx_3 \in A(R) \), then \( R[x_5, y_i] = x_5zx_4x_j \ldots x_3y_i \), where \( j \in [1, 3] \), and if \( x_5zx_1 \) is in \( R \), then \( R[x_5, y_i] \) is one of the following paths: \( x_5zx_1x_2x_3y_i, x_5zx_1x_2x_4x_3y_i, x_5zx_1x_4x_3y_i \), which is impossible since \( N^+(y_i, \{x_1, x_2, \ldots, x_5\}) = \{x_1, x_2, x_3\} \).

So, in all cases we have a contradiction. Therefore, \( D'_9 \) is not Hamiltonian, which in turn implies that the digraphs \( D_8 \) and \( D_9 + \{(x_3x_1)\} \) also are not Hamiltonian. By a similar argument we can show that \( D_8 \) also is not Hamiltonian.

![Fig. 1: The non-Hamiltonian 2-strong digraphs \( D_8 \) and \( D_9 \) of order 8 and 9.](image)

A digraph \( D \) is **Hamiltonian-connected** if for any pair of distinct vertices \( x, y \), \( D \) has a Hamiltonian path from \( x \) to \( y \). Overbeck-Larisch (1976) proved the following sufficient condition for a digraph to be Hamiltonian-connected.
A new sufficient condition for a 2-strong digraph to be Hamiltonian

**Theorem 1.10** ([Overbeck-Larisch](1976)). Let $D$ be a 2-strong digraph of order $n \geq 3$ such that, for each two non-adjacent distinct vertices $x, y$ we have $d(x) + d(y) \geq 2n + 1$. Then for each two distinct vertices $u, v$ with $d^+(u) + d^-(v) \geq n + 1$ there is a Hamiltonian $(u, v)$-path.

Let $D$ be a digraph of order $n \geq 3$ and let $u$ and $v$ be two distinct vertices in $V(D)$. Following [Overbeck-Larisch](1976), we define a new digraph $H_D(u, v)$ as follows:

$$V(H_D(u, v)) = V(D - \{u, v\}) \cup \{z\} \text{ (z a new vertex)},$$

$$A(H_D(u, v)) = A(D - \{u, v\}) \cup \{zy | y \in N^+_D(u)\} \cup \{yz | y \in N^-_{D-u}(v)\}.$$

Now, using Theorem 1.9, we will prove the following theorem, which is an analogue of the Overbeck-Larisch theorem.

**Theorem 1.11.** Let $D$ be a 3-strong digraph of order $n + 1 \geq 10$ and let $u, v$ be arbitrary two distinct vertices in $D$. Suppose that $d^+_D(u) + d^-_D(v) \geq n - k - 2$ or $d^+_D(v) + d^-_D(u) \geq n - k - 4$ with $uv \notin A(D)$ and for every vertex $x \in V(D) \setminus \{u, v\}$, $d_D(x) \geq n + k + 2$. Then $D$ has a Hamiltonian $(u, v)$-path.

**Proof:** Let $D$ be a 3-strong digraph of order $n + 1 \geq 10$ and let $u, v$ be two distinct vertices in $V(D)$. Suppose that $D$ and $u, v$ satisfy the degree conditions of the theorem. Now we consider the digraph $H := H_D(u, v)$ of order $n \geq 9$. By an easy computation, we obtain that the minimum degree of $H$ is at least $n - k - 4$, and $H$ has $n - 1$ vertices of degrees at least $n + k$. Moreover, we know that $H$ is 2-strong (see [Darbinyan](1990)). Thus, the digraph $H$ satisfies the conditions of Theorem 1.9. Therefore, $H$ is Hamiltonian, which in turn implies that in $D$ there is a Hamiltonian $(u, v)$-path. \qed

There are a number of sufficient conditions depending on degree or degree sum for Hamiltonicity of bipartite digraphs. Here we combine several of them in the following theorem.

**Theorem 1.12.** Let $D$ be a balanced bipartite strong digraph of order $2a \geq 6$. Then $D$ is Hamiltonian provided one of the following holds:

(a) ([Adamus and Adamus](2012)). $d^+(x) + d^-(y) \geq a + 2$ for every pair of vertices $x, y$ such that $x, y$ belong to different partite sets and $xy \notin A(D)$.

(b) ([Adamus, Adamus and Yeo](2014)). $d(x) + d(y) \geq 3a$ for every pair of non-adjacent distinct vertices $x, y$.

(c) ([Adamus](2017)). $d(x) + d(y) \geq 3a$ for every pair of vertices $x, y$ with a common in-neighbour or a common out-neighbour.

(d) ([Adamus](2021)). $d(x) + d(y) \geq 3a + 1$ for every pair of vertices $x, y$ with a common out-neighbour.

All the lower bounds in Theorem 1.12 are the best possible. However, [Wang](2021) (respectively [Wang, Wu and Meng](2022), [Wang and Wu](2021)) reduced the lower bound in Theorem 1.12(a) (respectively, Theorem 1.12(b); Theorem 1.12(c)) by one, and completely described all non-Hamiltonian bipartite digraphs, that is the extremal bipartite digraphs for Theorem 1.12(a) (respectively, Theorem 1.12(b); Theorem 1.12(c)). [Wang](2022) reduced the bound by one in Theorem 1.12(d), but it is Hamiltonian whenever $d(x) + d(y) \geq 3a$ for every pair of distinct vertices $x, y$ with a common out neighbour. Motivated by
Theorems 1.9, 1.12 and Remark 1, it is natural to suggest the following problems.

**Problem 1.** Suppose that $D$ is a $k$-strong balanced bipartite digraph of order $2a \geq 6$. Let $\{x_0, y_0\}$ be a pair of distinct vertices in $V(D)$ such that $d(x_0) + d(y_0) \geq 3a - l$, where $l \geq 1$ is an integer. Find the minimum value of $k$ and the maximum value of $l$ such that $D$ is Hamiltonian provided one of the following holds:

1. $x_0$ and $y_0$ are not adjacent and $d(x) + d(y) \geq 3a$ for every pair $\{x, y\}$ of non-adjacent vertices $x, y$ other than $\{x_0, y_0\}$.
2. $\{x_0, y_0\}$ is a pair with a common out-neighbour and $d(x) + d(y) \geq 3a$ for every pair $\{x, y\}$ of vertices $x, y$ with a common out-neighbour such that $\{x, y\} \neq \{x_0, y_0\}$.

**Problem 2.** Suppose that $D$ is a $k$-strong balanced bipartite digraph of order $2a \geq 6$. Let $u_0$ and $v_0$ be two vertices from different partite sets such that $u_0 \rightarrow v_0$ and $d^+(u_0) + d^-(v_0) \geq a + 2 - l$, where $l \geq 2$ is an integer. Find the minimum value of $k$ and the maximum value of $l$ such that $D$ is Hamiltonian provided that the following holds: $d^+(u) + d^-(v) \geq a + 2$ for all vertices $u$ and $v$ from different partite sets such that $\{u, v\} \neq \{u_0, v_0\}$ and $u \rightarrow v$.

2 Terminology and notation

In this paper, we consider finite digraphs without loops and multiple arcs. For the terminology not defined in this paper, the reader is referred to the book [Bang-Jensen and Gutin (Springer-Verlag, London, 2000)](http://example.com). The vertex set and the arc set of a digraph $D$ are denoted by $V(D)$ and $A(D)$, respectively. The order of $D$ is the number of its vertices. For any $x, y \in V(D)$, if $xy \in A(D)$, we also write $x \rightarrow y$, and say that $x$ dominates $y$ or $y$ is dominated by $x$. The notion $xy \notin A(D)$ means that $xy \notin A(D)$. If $x \rightarrow y$ and $y \rightarrow x$ we shall use the notation $x \leftrightarrow y$ ($x \leftrightarrow y$ is called 2-cycle). If $x \rightarrow y$ and $y \rightarrow z$, we write $x \rightarrow y \rightarrow z$. Let $A$ and $B$ be two disjoint subsets of $V(D)$. The notation $A \rightarrow B$ means that every vertex of $A$ dominates every vertex of $B$. We define $A_D(A \rightarrow B) = \{xy \in A(D) \mid x \in A, y \in B\}$ and $A_D(A, B) = A_D(A \rightarrow B) \cup A_D(B \rightarrow A)$. If $x \in V(D)$ and $A = \{x\}$ we sometimes write $x$ instead of $\{x\}$. The converse digraph of $D$ is the digraph obtained from $D$ by reversing the direction of all arcs, and is denoted by $D^{rev}$. Let $N^+_D(x), N^-_D(x)$ denote the set of out-neighbors, respectively the set of in-neighbors of a vertex $x$ in a digraph $D$. If $A \subseteq V(D)$, then $N^+_D(x, A) = A \cap N^+_D(x)$ and $N^-_D(x, A) = A \cap N^-_D(x)$. The out-degree of $x$ is $d^+_D(x) = |N^+_D(x)|$ and $d^-_D(x) = |N^-_D(x)|$ is the in-degree of $x$. Similarly, $d^+_D(x, A) = |N^+_D(x, A)|$ and $d^-_D(x, A) = |N^-_D(x, A)|$. The degree of the vertex $x$ in $D$ is defined as $d_D(x) = d^+_D(x) + d^-_D(x)$ (similarly, $d_D(x, A) = d^+_D(x, A) + d^-_D(x, A)$). We omit the subscript if the digraph is clear from the context. The subdigraph of $D$ induced by a subset $A$ of $V(D)$ is denoted by $D(A)$ and $D - A$ is the subdigraph induced by $V(D) \setminus A$, i.e. $D - A = D(V(D) \setminus A)$. For integers $a$ and $b$, $a \leq b$, let $[a, b]$ denote the set $\{x_0, x_{a+1}, \ldots, x_b\}$. If $j < i$, then $\{x_1, \ldots, x_j\} = \emptyset$. A path is a digraph with vertex set $\{x_1, x_2, \ldots, x_k\}$ and arc set $\{x_1x_2, x_2x_3, \ldots, x_{k-1}x_k\}$, and is denoted by $x_1x_2 \ldots x_k$. This is also called an $(x_1, x_k)$-path or a path from $x_1$ to $x_k$. If we add the arc $x_kx_1$ to the above, we obtain a cycle $x_1x_2 \ldots x_kx_1$. The length of a cycle or a path is the number of its arcs. If a digraph $D$ contains a path from a vertex $x$ to a vertex $y$ we say that $y$ is reachable from $x$ in $D$. In particular, $x$ is reachable from itself. If $P$ is a path containing a subpath from $x$ to $y$, we let $P|x, y|$ denote...
that subpath. Similarly, if $C$ is a cycle containing vertices $x$ and $y$, $C[x, y]$ denotes the subpath of $C$ from $x$ to $y$. For a cycle $C$, a $C$-bypass is an $(x, y)$-path $P$ of length at least two such that $V(P) \cap V(C) = \{x, y\}$. The flight of $C$-bypass $P$ with respect to $C$ is $|V(C[x, y])| - 2$.

For integers $a$ and $b$, $a \leq b$, let $[a, b]$ denote the set of all integers, which are not less than $a$ and are not greater than $b$.

The path (respectively, the cycle) consisting of the distinct vertices $x_1, x_2, \ldots, x_m$ $(m \geq 2)$ and the arcs $x_i x_{i+1}$, $i \in [1, m - 1]$ (respectively, $x_i x_{i+1}$, $i \in [1, m - 1]$, and $x_m x_1$), is denoted by $x_1 x_2 \cdots x_m$ (respectively, $x_1 x_2 \cdots x_m x_1$). We say that $x_1 x_2 \cdots x_m$ is a path from $x_1$ to $x_m$ or is an $(x_1, x_m)$-path.

Let $x$ and $y$ be two distinct vertices of a digraph $D$. A digraph through $x$ and $y$ in $D$, we denote by $C(x, y)$. By $C_m(x)$ (respectively, $C(x)$) we denote a cycle in $D$ of length $m$ through $x$ (respectively, a cycle through $x$). Similarly, we denote by $C_k$ a cycle of length $k$. By $K_n^*$ is denoted the complete digraph of order $n$. Let $D$ be a digraph of order $n$. If $E$ is a set of arcs in $K_n^*$, then we denote by $D + E$ the digraph obtained from $D$ by adding all arcs of $E$. A digraph $D$ is strongly connected (or, just, strong) if there exists a path from $x$ to $y$ and a path from $y$ to $x$ for every pair of distinct vertices $x$ and $y$. A digraph $D$ is $k$-strongly connected (or $k$-strong), where $k \geq 1$, if $|V(D)| \geq k + 1$ and $D - A$ is strongly connected for any subset $A \subseteq V(D)$ of at most $k - 1$ vertices. Two distinct vertices $x$ and $y$ are adjacent if $xy \in E(D)$ or $yx \in E(D)$ (or both). We will use the principle of digraph duality: Let $D$ be a digraph, then $D$ contains a subdigraph $H$ if and only if $D^{rev}$ contains the subdigraph $H^{rev}$.

### 3 Preliminaries

In our proofs we extensively will use the following well-known simple lemmas.

**Lemma 3.1** (**Häggkvist and Thomassen** (1976)). Let $D$ be a digraph of order $n \geq 3$ containing a cycle $C_m$, $m \in [2, n - 1]$. Let $x$ be a vertex not contained in this cycle. If $d(x, V(C)) \geq m + 1$, then $D$ contains a cycle $C_k$ for every $k \in [2, m + 1]$.

The next lemma is a slight modification of a lemma by **Bondy and Thomassen** (1977) it is very useful and will be used extensively throughout this paper.

**Lemma 3.2.** Let $D$ be a digraph of order $n \geq 3$ containing a path $P := x_1 x_2 \cdots x_m$, $m \in [1, n - 1]$. Let $x$ be a vertex not contained in this path. If one of the following condition holds: (i) $d(x, V(P)) \geq m + 2$, (ii) $d(x, V(P)) \geq m + 1$ and $x \not\rightarrow x_1$ or $x_m \not\rightarrow x$, (iii) $d(x, V(P)) \geq m$, $x \not\rightarrow x_1$ and $x_m \not\rightarrow x$, then there is an $i \in [1, m - 1]$ such that $x_i \rightarrow x \rightarrow x_{i+1}$, i.e., $D$ contains a path $x_1 x_2 x_{i+1} \cdots x_m$ of length $m$ (we say that $x$ can be inserted into $P$).

We note that in the above Lemma 3.2 as well as throughout the whole paper we allow paths of length $0$, i.e., paths that have exactly one vertex. Using Lemma 3.2, it is not difficult to prove the following lemma.

**Lemma 3.3.** Let $D$ be a digraph of order $n \geq 4$. Suppose that $P := x_1 x_2 \cdots x_m$, $m \in [2, n - 2]$, is a longest path from $x_1$ to $x_m$ in $D$ and $V(D) \setminus V(P)$ contains two distinct vertices $y_1$, $y_2$ such that $d(y_1, V(P)) = d(y_2, V(P)) = m + 1$. If in subdigraph $D(V(D) \setminus V(P))$ there exists a path from $y_1$ to $y_2$ and a path from $y_2$ to $y_1$, then there is an integer $l \in [1, m]$ such that for every $i \in [1, 2]$ $O(y_i, V(P)) = \{x_1, x_2, \ldots, x_l\}$ and $I(y_i, V(P)) = \{x_1, x_{l+1}, \ldots, x_m\}$. 


Theorem 3.4 (Darbinyan [1990]). Let \( D \) be a strong digraph of order \( n \geq 2 \). Suppose that \( d(x) + d(y) \geq 2n - 1 \) for all pairs of non-adjacent vertices \( x, y \in V(D) \setminus \{z\} \), where \( z \) is an arbitrary fixed vertex in \( V(D) \). Then \( D \) contains a cycle of length at least \( n - 1 \).

From Theorem 3.4 it follows that the following corollary is true.

Corollary 1. (Darbinyan [1990]). Let \( D \) be a strong digraph of order \( n \geq 2 \). Suppose that \( n - 1 \) vertices of \( D \) have degrees at least \( n \). Then \( D \) either is Hamiltonian or contains a cycle of length \( n - 1 \) (in fact \( D \) has a cycle that contains all the vertices with degree at least \( n \)).

Lemma 3.5 (Darbinyan [2022]). Let \( D \) be a digraph of order \( n \geq 4 \) such that for any vertex \( x \in V(D) \setminus \{z\} \), \( d(x) \geq n \), where \( z \) is an arbitrary fixed vertex in \( V(D) \). Moreover, \( d(z) \leq n - 2 \). Suppose that \( C_m(z) = x_1x_2\ldots x_mx_1 \), \( m \leq n - 1 \), is a cycle of length \( m \) through \( z \) and \( C_m(z) \) has an \((x_i, x_j)\)-bypass such that \( z \notin V(C_m(z)[x_{i+1}, x_{j-1}]) \). Then \( D \) has a cycle, say \( Q \), of length at least \( m + 1 \) such that \( V(C_m(z)) \subset V(Q) \).

Theorem 3.6 (Darbinyan [2024]). Let \( D \) be a 2-strong digraph of order \( n \geq 9 \) such that \( n - 1 \) vertices of \( D \) have degrees at least \( n + k \) and the remaining vertex \( z \) has degree at least \( n - k - 4 \), where \( k \geq 0 \) is an integer. If the length of a longest cycle through \( z \) is at least \( n - k - 2 \), then \( D \) is Hamiltonian.

4 Proof of Theorem 1.9

Theorem 1.9. Let \( D \) be a 2-strong digraph of order \( n \geq 9 \). If \( n - 1 \) vertices of \( D \) have degrees at least \( n + k \) and the remaining vertex \( z \) has degree at least \( n - k - 4 \), where \( k \geq 0 \) is an integer, then \( D \) is Hamiltonian.

Proof: By contradiction, suppose that \( D \) is not Hamiltonian. Then from Theorem 3.6 it follows that \( D \) has no \( C(z) \)-cycle of length greater than \( n - k - 3 \). By Corollary 1, \( D \) contains a cycle of length \( n - 1 \). Let \( C_{n-1} := x_1x_2\ldots x_{n-1}x_1 \) be an arbitrary cycle in \( D \). By Lemma 3.1, \( z \notin V(C_{n-1}) \).

Since \( D \) is 2-strong, there are two distinct vertices, say \( x_1 \) and \( x_{n-d-1} \), such that \( x_{n-d-1} \rightarrow z \rightarrow x_1 \) and \( d(z, \{x_{n-d}, x_{n-d+1}, \ldots , x_{n-1}\}) = 0 \). Without loss of generality, assume that the flight \( d := \left|\{x_{n-d}, x_{n-d+1}, \ldots , x_{n-1}\}\right| \) of \( z \) respect to \( C_{n-1} \) is smallest possible over all the cycles of length \( n - 1 \) in \( D \).

For any \( i \in [1, d] \), let \( y_i := x_{n-d-1+i} \) and \( Y = \{y_1, y_2, \ldots , y_d\} \). Note that \( y_1y_2\ldots y_d \) is a path in \( D \backslash Y \). Since \( z \) cannot be inserted into \( C_{n-1} \), using Lemma 3.2, we obtain \( n - d - 4 \leq d(z) \leq n - d \).

Hence, \( d \leq k + 4 \). On the other hand, \( n - d \leq n - k - 3 \), i.e., \( d \geq k + 3 \), since \( zx_1x_2\ldots x_{n-d-1}z \) is a \( C(z) \)-cycle of length \( n - d \). From now on, by \( P \) we denote the path \( x_1x_2\ldots x_{n-d-1} \) (see Figure 2). In order to prove the theorem, it is convenient for the digraph \( D \) and the path \( P \) to prove the following Claims 1-4.

Claim 1. Suppose that \( D(Y) \) is strong and each vertex \( y_j \) of \( Y \) cannot be inserted into \( P \).

If \( d^+(x_i, Y) \geq 1 \) with \( i \in [1, n - d - 2] \), then \( A(Y \rightarrow \{x_{i+1}, x_{i+2}, \ldots , x_{n-d-1}\}) = \emptyset \).

Proof: By contradiction, suppose that there are vertices \( x_s, x_q \) with \( 1 \leq s < q \leq n - d - 1 \) and
A new sufficient condition for a 2-strong digraph to be Hamiltonian

Fig. 2: The cycles \( C_{n-1} = x_1x_2 \ldots x_{n-d-1}y_1y_2 \ldots y_dx_1 \) and \( C_{n-d}(z) = x_1x_2 \ldots x_{n-d-1}zx_1 \) in \( D \).

\( u, v \in Y \) such that \( x_s \to u, v \to x_q \). Since \( D(Y) \) is strong, it contains a \((u, v)\)-path, and let \( Q \) be such a longest path. We may assume that \( A(Y, \{x_{s+1}, \ldots, x_{q-1}\}) = \emptyset \). Since \( D(Y) \) is strong and every vertex \( y_j \) cannot be inserted into \( P \), using the fact that \( D \) has no \( C(z) \)-cycle of length at least \( n-k-2 \), we obtain that \( q-s \geq 2 \). We now extend the path \( x_qx_{q+1} \ldots x_{n-d-1}zx_1x_2 \ldots x_s \) with vertices \( x_{s+1}, x_{s+2}, \ldots, x_{q-1} \) as much as possible. Then some vertices \( z_1, z_2, \ldots, z_m \in \{x_{s+1}, x_{s+2}, \ldots, x_{q-1}\} \), where \( 0 \leq m \leq q-s-1 \), are not on the obtained extended path, say \( R \). We consider the cases \( m \geq 1 \) and \( m = 0 \) separately.

Assume first that \( m \geq 1 \). Since every vertex \( y_j \) cannot be inserted into \( P \) and \( d(y_j, \{z, x_{s+1}, x_{s+2}, \ldots, x_{q-1}\}) = 0 \), using Lemma 3.2(i), we obtain

\[
\begin{align*}
n + k &\leq d(y_j) = d(y_j, Y) + d(y_j, \{x_1, x_2, \ldots, x_s\}) + d(y_j, \{x_q, x_{q+1}, \ldots, x_{n-d-1}\}) \\
&\leq 2d - 2 + (s + 1) + (n - d - 1 - q + 2) = n + s + d - q
\end{align*}
\]

and

\[
\begin{align*}
n + k &\leq d(z_i) = d(z_i, V(R)) + d(z_i, \{z_1, z_2, \ldots, z_m\}) \leq |V(R)| + 1 + 2m - 2 \\
&= n - d - m + 1 + 2m - 2 = n + m - d - 1.
\end{align*}
\]

Therefore,

\[
2n + 2k \leq d(z_i) + d(y_j) \leq n + m - d + 1 + n + s + d - q = 2n + m - 1 + s - q
\]

\[
\leq 2n - 1 + q - s - 1 + s - q = 2n - 2,
\]

which is a contradiction since \( k \geq 0 \).

Assume next that \( m = 0 \). This means that \( D \) contains an \((x_q, x_s)\)-path with vertex set \( \{z\} \cup V(P) \). This and the fact that \( D \) contains no cycle of length at least \( n-k-2 \) through \( z \) imply that \( d = k + 4 \), \( |V(Q)| = 1 \), i.e., \( u = v \), and \( A(x_s \to Y \setminus \{u\}) = A(Y \setminus \{u\} \to x_q) = 0 \). Since any vertex of \( Y \) cannot be inserted into \( P \), using Lemma 3.2(ii), for each \( y \in Y \setminus \{u\} \) we obtain

\[
\begin{align*}
n + k &\leq d(y) = d(y, Y) + d(y, \{x_1, x_2, \ldots, x_s\}) + d(y, \{x_q, x_{q+1}, \ldots, x_{n-k-5}\})
\end{align*}
\]
\[
\leq 2k + 6 + s + n - k - 5 - q + 1 = n + k + 2 - (q - s).
\]

This means that all the inequalities used in the last expression are actually equalities, i.e., \( q - s = 2 \), \( d(y, Y) = 2k + 6 \), i.e., \( D(Y) \) is a complete digraph, and

\[
d(y, \{x_1, \ldots, x_s\}) = s, \ d(y, \{x_1, x_2, \ldots, x_{n-k-5}\}) = n - k - 4.
\]

Again using Lemma 3.2(ii), from the last two equalities and \( A(x_s \to Y \setminus \{u\}) = A(Y \setminus \{u\} \to x_q) = \emptyset \) we obtain \( x_{n-k-5} \to Y \setminus \{u\} \to x_1 \). We claim that \( x_{s+1} \) can be inserted into \( x_1 x_2 \ldots x_s \) or \( x_q x_{q+1} \ldots x_{n-k-5} \). Assume that this is not the case. Then by Lemma 3.2(i),

\[
n + k \leq d(x_{s+1}) = d(x_{s+1}, \{x_1, x_2, \ldots, x_s\}) + d(x_{s+1}, \{x_q = x_{s+2}, x_{s+3}, \ldots, x_{n-k-5}\})
\]

\[
+ d(x_{s+1}, \{z\}) \leq s + 1 + n - k - 5 - s - 1 + 2 = n - k - (q - s) = n - k - 2,
\]

which is a contradiction. This contradiction shows that there is either an \((x_1, x_s)\)-path, say \( R_1 \), with vertex set \( \{x_1, x_2, \ldots, x_s, x_{s+1}\} \) or an \((x_q, x_{n-k-5})\)-path, say \( R_2 \), with vertex set \( \{x_{s+1}, x_{s+2}, \ldots, x_{n-k-5}\} \). Let \( H \) be a Hamiltonian path in \( D(Y \setminus \{u\}) \). We know that \( d(z, V(H)) = 0, |V(H)| = k + 3 \) and \( x_{n-k-5} \to Y \setminus \{u\} \to x_1 \). Therefore, \( F_1 := x_1 R_1 u x_q \ldots x_{n-k-5} H x_1 \) or \( F_2 := x_1 x_{s+1} u R_2 x_{n-k-5} H x_1 \), is a cycle of length \( n - 1 \). We have that the flight of \( z \) respect to \( F_1 \) (or \( F_2 \)) is equal to \( k + 3 \), which contradicts the minimality of \( d = k + 4 \) and the choice of the cycle \( C_{n-1} \) of length \( n - 1 \). This completes the proof of the claim.

**Claim 2.** If \( x_j \to z \) with \( j \in [1, n - d - 2] \), then \( A(z \to \{x_{j+1}, x_{j+2}, \ldots, x_{n-d-1}\}) = \emptyset \).

**Proof:** By contradiction, suppose that \( x_j \to z \) with \( j \in [1, n - d - 2] \) and \( z \to x_l \) with \( l \in [j + 1, n - d - 1] \). We may assume that \( d(z, \{x_{j+1}, \ldots, x_{l-1}\}) = 0 \). Since \( D \) contains no \( C(z) \)-cycle of length at least \( n - k - 2 \) and \( C_{n-l+j+1}(z) := x_1 x_2 \ldots x_l x_{l+1} \ldots x_{n-d-y} y_2 \ldots y_d x_1 \), it follows that \( l \geq j + k + 4 \). Then, since \( z \) cannot be inserted into \( P \), by Lemma 3.2(i), we have

\[
n - k - 4 \leq d(z) = d(z, \{x_1, x_2, \ldots, x_j\}) + d(z, \{x_l, x_{l+1}, \ldots, x_{n-d-1}\})
\]

\[
\leq (j + 1) + (n - d - 1 - l + 2) = n + 2 - j - d - l
\]

\[
\leq n + 2 + (l - k - 4) - d - l = n - k - 2 - d,
\]

i.e., \( d \leq 2 \), which contradicts that \( d \geq k + 3 \). Claim 2 is proved.

Since \( D \) is 2-strong, we have \( d^-(z) \geq 2 \) and \( d^+(z) \geq 2 \). From this and Claim 2 it follows that there exists an integer \( t \in [2, n - d - 2] \) such that \( x_t \to z \) and

\[
d^-(z, \{x_1, x_2, \ldots, x_t\}) = d^+(z, \{x_{t+1}, x_{t+2}, \ldots, x_{n-d-1}\}) = 0.
\]

From (1) and \( d(z) \geq n - k - 4 \) it follows that if \( d = k + 4 \), then \( n - d - 1 = n - k - 5 \) and

\[
N^+(z) = \{x_1, x_2, \ldots, x_l\} \quad \text{and} \quad N^-(z) = \{x_{l+1}, \ldots, x_{n-k-5}\}.
\]

**Claim 3.** Suppose that there is an integer \( l \in [2, n - d - 2] \) such that

\[
A(\{x_1, x_2, \ldots, x_{l-1}\} \to Y) = A(Y \to \{x_{l+1}, x_{l+2}, \ldots, x_{n-d-1}\}) = \emptyset.
\]
A new sufficient condition for a 2-strong digraph to be Hamiltonian

Then for every $j \in [2, n - d - 2]$,

$$A(\{x_1, x_2, \ldots, x_{j-1}\} \rightarrow \{x_{j+1}, x_{j+2}, \ldots, x_{n-d-1}\}) \neq \emptyset.$$

**Proof:** Suppose, on the contrary, that for some $j \in [2, n - d - 2], A(\{x_1, x_2, \ldots, x_{j-1}\} \rightarrow \{x_{j+1}, x_{j+2}, \ldots, x_{n-d-1}\}) = \emptyset$. Without loss of generality, we may assume that $j \leq l$. If $d^-(z, \{x_1, x_2, \ldots, x_{j-1}\}) = 0$, then by the suppositions of the claim, we have

$$A(\{x_1, x_2, \ldots, x_{j-1}\} \rightarrow Y \cup \{z, x_{j+1}, x_{j+2}, \ldots, x_{n-d-1}\}) = \emptyset.$$

If $d^-(z, \{x_1, x_2, \ldots, x_{j-1}\}) \geq 1$, then by Claim 2, $d^+(z, \{x_{j+1}, x_{j+2}, \ldots, x_{n-d-1}\}) = 0$. This together with the supposition of the claim implies that

$$A(\{z, x_1, x_2, \ldots, x_{j-1}\} \rightarrow Y \cup \{x_{j+1}, x_{j+2}, \ldots, x_{n-d-1}\}) = \emptyset.$$

Thus, in both cases, $D - x_j$ is not strong, which is a contradiction. Claim 3 is proved.

**Claim 4.** Any vertex $y_j$ with $j \in [1, d]$ cannot be inserted into $P$.

**Proof:** By contradiction, suppose that there is a vertex $y_p$ with $p \in [1, d]$ and an integer $s \in [1, n - d - 2]$ such that $x_s \rightarrow y_p \rightarrow x_{s+1}$. Then $R(z) := x_1x_2 \ldots x_s y_p x_{s+1} \ldots x_{n-d-1} z x_1$ is a cycle of length $n - d + 1$. Since $D$ contains no $(z)$-cycle of length at least $n - k - 2$, it follows that $n - d + 1 \leq n - k - 3$, i.e., $d \geq k + 4$. Therefore, $d = k + 4$ since $d \leq k + 4$. It is easy to see that any vertex $y_i$ other than $y_p$ cannot be inserted into $P$. Note that (2) holds since $d = k + 4$. We will consider the cases $p \in [2, k + 3]$ and $p = 1$ separately. Note that if $p = k + 4$, then in the converse digraph of $D$ we have case $p = 1$.

**Case 1.** $p \in [2, k + 3]$.

If $y_{p-1} \rightarrow y_{p+1}$, then the cycle $x_1 x_2 \ldots x_s y_p x_{s+1} \ldots x_{n-k-5} y_1 \ldots y_{p-1} y_{p+1} \ldots y_{k+4} x_1$ is a cycle of length $n - 1$ and the flight of $z$ respect to this cycle is equal to $k + 3$, which is a contradiction.

We may therefore assume that $y_{p-1} \not\rightarrow y_{p+1}$. Since both $y_{p-1}$ and $y_{p+1}$ cannot be inserted into $R(z)$, using Lemma 3.2(i), we obtain $d(y_{p-1}, V(R(z))) \leq n - k - 3$ and $d(y_{p+1}, V(R(z))) \leq n - k - 3$. These together with $d(y_{p-1}) \geq n + k$ and $d(y_{p+1}) \geq n + k$ imply that $d(y_{p-1}, Y \backslash \{y_p\}) \geq 2k + 3$ and $d(y_{p+1}, Y \backslash \{y_p\}) \geq 2k + 3$. Hence, it is easy to see that $y_{p+1} \rightarrow y_{p-1}$ and $d^+(x_s, Y \backslash \{y_p\}) = d^-(x_{s+1}, Y \backslash \{y_p\}) = 0$ (for otherwise $D$ contains a $(z)$-cycle of length at least $n - k - 2$, a contradiction). Since every vertex of $Y \backslash \{y_p\}$ cannot be extended into $P$, using Lemma 3.2 and the last equalities, we obtain that if $u \in \{y_{p-1}, y_{p+1}\}$, then

$$n + k \leq d(u) = d(u, Y) + d(u, \{x_1, x_2, \ldots, x_s\}) + d(u, \{x_{s+1}, x_{s+2}, \ldots, x_{n-k-5}\})$$

$$\leq 2k + 5 + s + (n - 5 - k - s) = n + k.$$

From this, in particular, we have $d(u, Y) = 2k + 5$, $d(u, \{x_1, x_2, \ldots, x_s\}) = s$ and $d(u, \{x_{s+1}, x_{s+2}, \ldots, x_{n-k-5}\}) = n - k - 5$. Again using Lemma 3.2(ii), we obtain that $x_{n-k-5} \rightarrow \{y_{p-1}, y_{p+1}\} \rightarrow x_1$. From $d(u, Y) = 2k + 5$ it follows that $u \leftrightarrow Y \backslash \{y_{p-1}, y_{p+1}\}$ since $y_{p-1} y_{p+1} \notin A(D)$. Hence it is not difficult to see that in $D(Y \backslash \{y_p\})$ there is a $(y_{p-1}, y_{k+4})$- or $(y_{p-1}, y_{p+1})$-Hamiltonian path, say $H$. Thus $x_1 x_2 \ldots x_s y_p x_{s+1} \ldots x_{n-k-5} H x_1$ is a cycle of length $n - 1$ and the flight of $z$ respect to this cycle
is equal to \( k + 3 \), a contradiction.

**Case 2.** \( p = 1 \), i.e., \( x_s \to y_1 \to x_{s+1} \).

Observe that \( d^-(x_{s+1}, \{y_2, y_3, \ldots, y_{k+4}\}) = 0 \) and \( R(z) \) is a longest cycle through \( z \) in \( D \), which has length \( n - k - 3 \). For Case 2 we will prove the following proposition.

**Proposition 2.** Suppose that for \( j, j' \in [2, k+4] \), in \( Q := D\langle \{y_2, y_3, \ldots, y_{k+4}, x_1\} \rangle \) there is a Hamiltonian \((y_j, x_1)\)-path, say \( H_1 \). Then \( x_{n-k-5}y_j \notin A(D) \). In particular, \( x_{n-k-5}y_2 \notin A(D) \).

**Proof:** Suppose that the claim is not true, that is \( x_{n-k-5} \to y_1 \) with \( j \in [2, k+4] \) and \( Q \) has a Hamiltonian \((y_j, x_1)\)-path, say \( H_1 \). Then \( x_1x_2 \ldots x_s y_1 x_{s+1} \ldots x_{n-k-5}H_1 x_1 \) is a cycle of length \( n - 1 \) and the flight of \( y_1 \) respect to this cycle is equal to \( k + 3 \), a contradiction. Thus \( x_{n-k-5} \not\to y_j \). It is easy to see that \( H_2^\prime = y_2y_3 \ldots y_{k+4}x_1 \) is a Hamiltonian path in \( Q \). Therefore by the first part of this proposition, \( x_{n-k-5} \not\to y_2 \).

To complete the proof of Claim 4, we will consider the cases \( x_s \not\to y_2 \), \( x_s \not\to y_1 \) separately.

**Subcase 2.1.** \( x_s y_2 \notin A(D) \).

We know that \( y_2x_{s+1} \notin A(D) \) and \( x_{n-k-5}y_2 \notin A(D) \). Now, since \( y_2 \) cannot be inserted into \( P \), using Lemmas 3.2(ii) and 3.2(iii), we obtain

\[
n + k \leq d(y_2) = d(y_2, Y) + d(y_2, \{x_1, x_2, \ldots, x_s\}) + d(y_2, \{x_{s+1}, x_{s+2}, \ldots, x_{n-k-5}\}) \leq 2k + 6 + s + (n - k - s - 6) = n + k.
\]

This implies that \( d(y_2, Y) = 2k + 6 \), i.e., \( y_2 \not\leftrightarrow Y \setminus \{y_2\} \), in particular, \( y_2 \not\leftrightarrow y_1 \) and \( D\langle Y \rangle \) is strong, and

\[
d(y_2, \{x_1, x_2, \ldots, x_s\}) = s \text{ and } d(y_2, \{x_{s+1}, x_{s+2}, \ldots, x_{n-k-5}\}) = n - k - s - 6. \tag{3}
\]

Thus, for the longest cycle \( R(z) \) we have that \( V(D \setminus V(R(z))) = \{y_2, y_3, \ldots, y_{k+4}\} \). \( D\langle V(D \setminus V(R(z))) \rangle \) is strong and \( y_2 \leftrightarrow y_1 \). Therefore by Lemma 3.5,

\[
A(\{y_2, y_3, \ldots, y_{k+4}\} \to \{x_{s+1}, x_{s+2}, \ldots, x_{n-k-5}\}) = A(\{x_1, x_2, \ldots, x_s\} \to \{y_2, y_3, y_4, \ldots, y_{k+4}\}) = 0. \tag{4}
\]

This together with \( x_{n-k-5} \not\to y_2 \) and (3) implies that \( y_2 \) and \( x_{n-k-5} \) are not adjacent and

\[
N^+(y_2, V(P)) = \{x_1, x_2, \ldots, x_s\} \text{ and } N^-(y_2, V(P)) = \{x_{s+1}, x_{s+2}, \ldots, x_{n-k-6}\}.
\]

By the above arguments, we have that \( H_3^\prime = y_3y_4 \ldots y_{k+4}y_2x_1 \) is a \((y_3, x_1)\)-Hamiltonian path in \( Q \). Therefore by Proposition 1, \( x_{n-k-5} \not\to y_3 \). This together with (4) implies that \( x_{n-k-5} \) and \( y_3 \) are not adjacent. As for \( y_2 \), for \( y_3 \) we obtain that \( y_3 \not\leftrightarrow Y \setminus \{y_3\} \) and

\[
N^+(y_3, V(P)) = \{x_1, x_2, \ldots, x_s\} \text{ and } N^-(y_3, V(P)) = \{x_{s+1}, x_{s+2}, \ldots, x_{n-k-6}\}.
\]

Proceeding in the same manner, we obtain that \( d(x_{n-k-5}, \{y_2, y_3, \ldots, y_{k+4}\}) = 0 \), \( D\langle Y \rangle \) is a complete digraph and

\[
\{x_{s+1}, x_{s+2}, \ldots, x_{n-k-6}\} \to Y \setminus \{y_1\} \to \{x_1, x_2, \ldots, x_s\}. \tag{5}
\]

If \( s = n - k - 6 \), then from (4) and \( d(x_{n-k-5}, \{y_2, y_3, \ldots, y_{k+4}\}) = 0 \) it follows that \( A(V(P) \cup \{z\} \to Y \setminus \{y_1\}) = 0 \), i.e., \( D - y_1 \) is not strong, a contradiction. Therefore, we may assume that \( s \leq n - k - 7 \).
A new sufficient condition for a 2-strong digraph to be Hamiltonian

Let $s = 1$. Since $D(Y)$ is strong, from (5) it follows that $A(Y \to \{x_3, x_4, \ldots, x_{n-k-5}\}) = \emptyset$. (for otherwise, $y_1 \to x_i$ with $i \in [3, n-k-5]$ and $C_{n-k-2}(z) = x_1x_2\ldots x_{i-1}y_1x_i\ldots x_{n-k-5}z$, a contradiction). If $d^+(x_1, \{x_3, x_4, \ldots, x_{n-k-5}\}) = 0$, then $A(\{x_1\} \cup Y \to \{z, x_3, x_4, \ldots, x_{n-k-5}\}) = \emptyset$, i.e., $D - x_2$ is not strong, a contradiction. So, we can assume that for some $b \in [3, n-k-5]$, $x_1 \to x_b$. By (5) and (2), respectively, we have $x_{b-1} \to y_2$ and $z \to x_2$. Therefore, $C_{n-1}(z) := x_1x_2\ldots x_{n-k-5}z x_2\ldots x_{b-1}y_2y_3\ldots y_{b+1}x_1$, a contradiction. Let finally $2 \leq s \leq n-k-7$. It is easy to see that $A(\{x_1, x_2, \ldots, x_{s-1}\} \to \{x_{s+1}, x_{s+2}, \ldots, x_{n-k-5}\}) \neq \emptyset$ (for otherwise, using the fact that $A(\{x_1, x_2, \ldots, x_{s-1}\} \to Y) = \emptyset$ (by (5)), Claim 2 and (2), it is not difficult to show that $D - x_s$ is not strong, a contradiction). Thus, there are integers $a \in [1, s-1]$ and $b \in [s + 1, n-k-5]$ such that $x_a \to x_b$. Then by (4), $y_2 \to x_{a+1}$, and by (2), either $z \to x_{a+1}$ or $x_{b-1} \to z$. By (4), we also have that $x_{b-1} \to y_2$ or $x_{b-1} \to y_1$ when $b = s + 1$. Therefore, $C(z) = x_1x_2\ldots x_a x_b\ldots x_{n-k-5}z x_{a+1}\ldots x_{b-1}(y_1 \text{ or } y_2)y_3\ldots y_{b+4}x_1$ is a cycle of length at least $n-1$ or $C_{n-k-2}(z) = x_1x_2\ldots x_a x_b\ldots x_{n-k-5} z y_1x_2a_1\ldots x_{b-1}z x_1$, respectively, for $z \to x_{a+1}$ and for $x_{b-1} \to z$. Thus, for any possible case we have a contradiction. This completes the discussion of Subcase 2.1.

**Subcase 2.2.** $x_s \to y_2$.

Using Lemma 3.5 and the fact that $R(z)$ is a longest cycle of length $n-k-3$ through $z$, we obtain

$$A(Y \setminus \{y_1\} \to \{x_{s+1}, x_{s+2}, \ldots, x_{n-k-5}\}) = \emptyset.$$  
(6)

Since $x_s \to y_1 \to x_{s+1}$, it follows that in $D(Y)$ there is no $(y_2, y_1)$-path, i.e., $d^-(y_1, \{y_2, y_3, \ldots, y_{k+4}\}) = 0$ (for otherwise $D$ has a cycle of length at least $n-k-2$ through $z$, which is a contradiction). This implies that for all $i \in [1, k+4]$, $d(y_i, Y) \leq 2k + 5$. Recall that $x_{n-k-5} \to y_2$ (Proposition 1). Therefore, since $y_2$ cannot be inserted into $P$ and $y_2 \not\rightarrow x_{s+1}$, using Lemma 3.2, we obtain

$$n + k \leq d(y_2) = d(y_2, Y) + d(y_2, \{x_1, x_2, \ldots, x_s\}) + d(y_2, \{x_{s+1}, x_{s+2}, \ldots, x_{n-k-5}\})$$

$$\leq 2k + 5 + s + 1 + (n-k-s-6) = n+k.$$  

Therefore, $y_2 \leftrightarrow Y \setminus \{y_1, y_2\}$, in particular, $D(Y \setminus \{y_1\})$ is strong.

$$d(y_2, \{x_1, x_2, \ldots, x_s\}) = s + 1 \text{ and } d(y_2, \{x_{s+1}, x_{s+2}, \ldots, x_{n-k-5}\}) = n-k-s-6.$$  
(7)

From (6) and $x_{n-k-5}y_2 \not\in A(D)$ it follows that $y_2$ and $x_{n-k-5}$ are not adjacent. Therefore by (7) and (6), $\{x_{s+1}, x_{s+2}, \ldots, x_{n-k-6}\} \to y_2$, and by Lemma 3.2, $y_2 \to x_1$. Note that $H^3 = y_3y_4 \ldots y_{k+4}y_2x_1$ is a Hamiltonian $(y_3, x_1)$-path in $Q$. Therefore by Proposition 1, $x_{n-k-5}y_3 \not\in A(D)$, which together with (6) implies that $y_3$ and $x_{n-k-5}$ are not adjacent. Now by the same arguments, as for $y_2$, we obtain that $y_3 \leftrightarrow Y \setminus \{y_1, y_3\}$,

$$d(y_3, \{x_1, x_2, \ldots, x_s\}) = s + 1 \text{ and } d(y_3, \{x_{s+1}, x_{s+2}, \ldots, x_{n-k-6}\}) = n-k-s-6.$$  
(8)

Now by (8) and (6), $\{x_{s+1}, x_{s+2}, \ldots, x_{n-k-6}\} \to y_3$. We know that $P_1 := x_1x_2\ldots x_s$ is a longest $(x_1, x_s)$-path in $D(V(P_1)) \cup Y \setminus \{y_1\}$. Therefore, since $d(y_2, V(P_1)) = d(y_3, V(P_1)) = s + 1$, by Lemma 3.3, there exists an integer $q \in [1, s]$ such that for every $j \in [2, 3]$

$$N^+(y_j, V(P_1)) = \{x_1, x_2, \ldots, x_q\} \text{ and } N^-(y_j, V(P_1)) = \{x_q, x_{q+1}, \ldots, x_s\}.$$
Proceeding in the same manner, we conclude that \( \{x_{s+1}, x_{s+2}, \ldots, x_{n-k-6}\} \to Y \setminus \{y_1\} \), for all \( j \in [2, k + d] \), the vertices \( y_j \) and \( x_{n-k-5} \) are not adjacent and

\[
N^+(y_j, V(P_1)) = \{x_1, x_2, \ldots, x_q\} \quad \text{and} \quad N^-(y_j, V(P_1)) = \{x_q, x_{q+1}, \ldots, x_s\}. \tag{9}
\]

If \( q = 1 \), then \( A(\{y_2, y_3, \ldots, y_{k+4}\} \to \{z, y_1, x_2, x_3, \ldots, x_{n-k-5}\}) = \emptyset \), which implies that \( D - x_1 \) is not strong, a contradiction. Therefore, we may assume that \( q \geq 2 \), i.e., \( q \in [2, s] \). If \( x_i \to y_1 \) with \( i \in [1, q - 1] \) then by (9), \( C_n(z) = x_1 x_2 \ldots x_i y_1 y_2 \ldots y_{k+4} x_{i+1} x_{i+2} \ldots x_{n-k-5} x_1 \), a contradiction. We may therefore assume that \( d^-(y_1, \{x_1, x_2, \ldots, x_{q-1}\}) = 0 \). This together with (9) implies that \( A(\{x_1, x_2, \ldots, x_{q-1}\} \to Y) = \emptyset \). Since \( D \) is 2-strong, the last equality and (2) imply that there are integers \( a \in [1, q - 1] \) and \( b \in [q + 1, n - k - 5] \) such that \( x_a \to x_b \), for otherwise it is easy to see that \( D - x_q \) is not strong. By (9) and (2), we have \( y_{k+4} \to x_{a+1}, x_{b-1} \to y_2 \) and \( z \to x_{a+1} \) or \( x_{b-1} \to z \). Therefore, if \( z \to x_{a+1} \), then \( C_{n-1}(z) = x_1 x_2 \ldots x_a x_b \ldots x_{n-k-5} x_{a+1} x_{b-1} y_2 \ldots y_{k+4} x_1 \), and if \( x_{b-1} \to z \), then \( C_n(z) = x_1 x_2 \ldots x_a x_b \ldots x_{n-k-5} y_1 \ldots y_{k+4} x_{a+1} \ldots x_{b-1} x_1 \). So, in any case we have a contradiction. Claim 4 is proved. \( \square \)

For any \( j \in [1, d] \), we have

\[
n + k \leq d(y_j) = d(y_j, V(P)) + d(y_j, Y) \leq d(y_j, V(P)) + 2d - 2.
\]

From this, \( d(y_j, V(P)) \geq n + k - 2d + 2 \). On the other hand, by Lemma 3.2 and Claim 4, \( d(y_j, V(P)) \leq n - d \). Therefore,

\[
n + k - 2d + 2 \leq d(y_j, V(P)) \leq n - d \quad \text{and} \quad d + k \leq d(y_j, Y) \leq 2d - 2. \tag{10}
\]

We distinguish two cases according to the subdigraph \( D(Y) \) is strong or not.

**Case A.** \( D(Y) \) is strong.

In this case, by Claim 4, the suppositions of Claim 1 hold. Therefore, if for some

\[
i \in [1, n - d - 2] \quad \text{and} \quad d^+(x_i, Y) \geq 1, \quad \text{then} \quad A(Y \to \{x_{i+1}, x_{i+2}, \ldots, x_{n-d-1}\}) = \emptyset. \tag{11}
\]

Since \( D \) is 2-strong, (11) implies that \( d^+(x_1, Y) = d^-(x_{n-d-1}, Y) = 0 \), there exists \( l \in [2, n - d - 2] \) such that \( d^+(x_l, Y) \geq 1 \) and

\[
A(\{x_1, x_2, \ldots, x_{l-1}\} \to Y) = A(Y \to \{x_{l+1}, x_{l+2}, \ldots, x_{n-d-1}\}) = \emptyset. \tag{12}
\]

From this we see that the supposition of Claim 3 holds. Therefore, for all \( j \in [2, n - d - 2] \),

\[
A(\{x_1, x_2, \ldots, x_{j-1}\} \to \{x_{j+1}, x_{j+2}, \ldots, x_{n-d-1}\}) \neq \emptyset. \tag{13}
\]

For Case A, we will prove the following two claims.

**Claim 5.** (i) \( A(D) \) contains every arc of the forms \( z \to x_i \) and \( x_j \to z \), where \( i \in [1, t] \) and \( j \in [t, n - d - 1] \), maybe except one when \( d = k + 3 \). (Recall that the definition of \( t \) is given immediately after the proof of Claim 2).

(ii) For every \( i \in [1, d] \), \( A(D) \) contains every arc of the forms \( y_i \to x_q \) and \( x_j \to y_i \), where \( q \in [1, t] \) and \( j \in [t, n - d - 1] \), maybe except one when \( d = k + 3 \) or except two when \( d = k + 4 \).
A new sufficient condition for a 2-strong digraph to be Hamiltonian

Proof: (i) If \( d = k + 4 \), then Claim 5(i) is an immediate consequence of (2). Assume that \( d = k + 3 \). Then by (1), we have

\[
 n - k - 4 \leq d(z) = d^+(z, \{x_1, x_2, \ldots, x_{t-1}\}) + d(z, \{x_t\}) + d^-(z, \{x_{t+1}, x_{t+2}, \ldots, x_{n-k-4}\}) \\
 \leq t - 1 + 2 + n - k - 4 - t = n - k - 3.
\]

Now, it is easy to see that Claim 5(i) is true.

(ii) By (10) and (12) we have

\[
 n + k - 2d + 2 \leq d(y_i, V(P)) = d^+(y_i, \{x_1, x_2, \ldots, x_{t-1}\}) + d(y_i, \{x_t\}) \\
 + d^-(y_i, \{x_{t+1}, x_{t+2}, \ldots, x_{n-d-1}\}) \leq l - 1 + 2 + n - d - 1 - l = n - d.
\]

Now, considering the cases \( d = k + 3 \) and \( d = k + 4 \) separately, it is not difficult to see that Claim 5(ii) also is true. Claim 5 is proved.

Claim 6. Suppose that for some integers \( a \) and \( b \) with \( 1 \leq a < b - 1 \leq n - d - 2 \) we have \( x_a \to x_b \).
If \( D(Y) \) is strong and \( z \to x_a+1 \), then \( d^+(x_{b-1}, Y) = 0 \).

Proof: Suppose, on the contrary, that is \( D(Y) \) is strong, \( z \to x_{a+1} \) and \( d^+(x_{b-1}, Y) \geq 1 \). Let \( x_{b-1} \to y_i \), where \( i \in [1, d] \). Recall that \( k + 3 \leq d \leq k + 4 \). If \( i \in [1, k + 3] \), then the cycle \( C(z) = x_{1}x_{2}\ldots x_{a}x_{b}\ldots x_{n-d-2}x_{a+1}\ldots x_{b-1}y_{i}\ldots y_{d}x_{1} \) has length at least \( n - k - 2 \), which is a contradiction. Therefore, we may assume that \( d^+(x_{b-1}, \{y_{1}, y_{2}, \ldots, y_{k+3}\}) = 0 \). Then from \( d^+(x_{b-1}, Y) \geq 1 \) it follows that \( d = k + 4 \) and \( x_{b-1} \to y_{k+4} \). Hence by (11), \( A(Y) \to \{x_{b}, x_{b+1}, \ldots, x_{n-k-5}\} = 0 \). Note that for each \( i \in [1, k + 3] \), \( D(Y) \) contains a \((y_{k+4}, y_{i})\)-path since \( D(Y) \) is strong. Hence it is not difficult to see that if \( d^-(x_{1}, \{y_{1}, y_{2}, \ldots, y_{k+3}\}) \geq 1 \), then \( D \) contains a \( C(z)\)-cycle of length at least \( n - k - 2 \), a contradiction. Therefore, we may assume that \( d^-(x_{1}, \{y_{1}, y_{2}, \ldots, y_{k+3}\}) = 0 \). This together with \( d^+(x_{1}, Y) = 0 \) implies that \( d(x_{1}, \{y_{1}, y_{2}, \ldots, y_{k+3}\}) = 0 \). Now using Lemma 3.2, Claim 4, \( A(Y) \to \{x_{b}, x_{b+1}, \ldots, x_{n-k-5}\} = 0 \) and \( d^+(x_{b-1}, \{y_{1}, y_{2}, \ldots, y_{k+3}\}) = 0 \), for any \( i \in [1, k + 3] \) we obtain,

\[
 n + k \leq d(y_i) = d(y_i, Y) + d(y_i, \{x_2, x_3, \ldots, x_{b-1}\}) + d^-(y_i, \{x_b, x_{b+1}, \ldots, x_{n-k-5}\}) \\
 \leq 2k + 6 + (b - 2) + (n - k - 5 - b + 1) = n + k.
\]

This means that all inequalities which were used in the last expression in fact are equalities, i.e., for any \( i \in [1, k + 3] \), \( d(y_i, Y) = 2k+6 \) (i.e., \( D(Y) \) is a complete digraph), and \( d(y_i, \{x_2, x_3, \ldots, x_{b-1}\}) = b - 2 \).

Therefore, since any vertex \( y_i \) with \( i \in [1, k + 3] \) cannot be inserted into \( P \) (Claim 4), \( d(y_i, \{x_2, x_3, \ldots, x_{b-1}\}) = b - 2 \) and \( x_{b-1} \to y_i \), using Lemma 3.2, we obtain that \( y_i \to x_2 \).

Hence, if \( a \geq 2 \), then \( C_{n-1}(z) = x_2 \ldots x_ax_{b}\ldots x_{n-k-5}z x_{a+1}\ldots x_{b-1}Lx_2 \), where \( L \) is a Hamiltonian \((y_{k+4}, y_{k+3})\)-path in \( D(Y) \), a contradiction. Therefore, we may assume that \( a = 1 \). Then \( z \to x_2 \) and \( C_{n-1} = x_1x_2\ldots x_{n-k-5}y_1y_2\ldots y_{k+3}x_2\ldots x_{b-1}y_{k+4}x_1 \) is a cycle of length \( n - 1 \) in \( D \). We have \( x_{n-k-5} \to z \) and \( z \to x_2 \); i.e., the flight of \( z \) respect to this cycle \( C_{n-1} \) is equal to \( k + 3 \), which contradicts that the minimal flight of \( z \) respect to all cycles of length \( n - 1 \) is equal to \( d = k + 4 \). Claim 6 is proved. \( \square \)

Now using the digraph duality, we prove that it suffices to consider only the case \( t \geq l \).
Indeed, assume that \( l \geq t + 1 \) and consider the converse digraph \( D^{rev} \) of \( D \). Let \( V(D^{rev}) = \{u_1, u_2, \ldots, u_{n-d-1}, v_1, v_2, \ldots, v_d\} \), where \( u_i := x_{n-d-i} \) and \( v_j := y_{d+1-j} \) for all \( i \in [1, n-d-1] \) and \( j \in [1, d] \), in particular, \( x_l = u_{n-d-l} \) and \( x_t = u_{n-d-t} \). Let \( p := n - d - l \) and \( q := n - d - t \). Note that \( q \geq p + 1 \) and \( \{v_1, v_2, \ldots, v_d\} = Y \).

Observe that from the definitions of \( l, p \) and \( q \) it follows that \( d_+^{D^{rev}}(u_p, Y) \geq 1 \) and \( A^{D^{rev}}(\{u_p, \ldots, u_{p-1}\} \rightarrow Y) = 0 \). Now using Claim 5(i), we obtain that \( d^{D^{rev}}(z, \{u_q, u_{q+1}\}) \geq 1 \) and \( A^{D^{rev}}(\{u_p, u_{p+1}\} \rightarrow Y) \neq 0 \) when \( d = k + 3 \) and \( A^{D^{rev}}(\{u_p, u_{p+1}, u_{p+2}\} \rightarrow Y) \neq 0 \) when \( d = k + 4 \). Let \( u_{z+1} \in A(D^{rev}), d_+^{D^{rev}}(u_t, Y) \geq 1 \) and \( t' \) are minimal with these properties. It is clear that \( t' \in [q, q + 1] \) and \( t' \in [p, p + 2] \). We claim that \( t' \geq t' \). Assume that this is not the case, i.e., \( t' \leq t' - 1 \). Then it is not difficult to see that \( t' \leq t' - 1 \) is possible when \( t' = p + 2 \) and \( t' = p + 1 = q \). By Claim 5(ii), \( d = k + 4 \) and \( 2 \leq p = q - 1 \leq n - k - 7 \). Therefore, in \( D^{rev} \) the following hold:

\[
d_+^{D^{rev}}(u_{p+1}, Y) = 0, \quad \{u_{q+1}, u_{q+2}, \ldots, u_{n-k-5}\} \rightarrow Y \rightarrow \{u_1, u_2, \ldots, u_p\},
\]

\[
N_+^{D^{rev}}(z) = \{u_1, u_2, \ldots, u_q\} \quad \text{and} \quad N^-^{D^{rev}}(z) = \{u_q, u_{q+1}, \ldots, u_{n-k-5}\}.
\]

Since \( D^{rev} \) is 2-strong and \( A^{D^{rev}}(\{u_1, u_2, \ldots, u_p\} \rightarrow \{z\} \cup Y) = \emptyset \), it follows that there are \( r \in [1, p] \) and \( s \in [p + 2, n - k - 5] \) such that \( u_r, u_s \in A(D^{rev}) \). Taking into account the above observation, it is not difficult to show that if \( r \leq p - 1 \), then \( C_n(z) = u_1 u_2 u_r u_s u_{n-k-5} v_1 v_2 \ldots v_{k+4} u_{p+1} \ldots u_{n-k-5} u_{p+1} \ldots u_{n-k-5} v_1 v_2 \ldots v_{k+4} u_1 \) is a Hamiltonian cycle in \( D^{rev} \), and if \( s \geq p + 3 = q + 2 \), then \( C_n(z) = u_1 u_2 u_r u_s u_{n-k-5} v_1 v_2 \ldots v_{k+4} u_1 \) is a Hamiltonian cycle in \( D^{rev} \), which contradicts that \( D \) is not Hamiltonian. We may therefore assume that \( r = p \) and \( s = p + 2 \). This means that \( A^{D^{rev}}(\{u_1, u_2, \ldots, u_{p-1}\} \rightarrow \{u_p, u_{p+1}, \ldots, u_{n-k-5}\}) = 0 \). Therefore, since \( D^{rev} \) is 2-strong, for some \( i \in [1, p - 1] \), \( u_i u_{p+1} \in A(D^{rev}) \). Hence, \( u_i u_2 u_1 u_i u_{p+1} v_i \ldots u_p u_{p+2} \ldots u_{n-k-5} v_1 v_2 \ldots v_{k+4} u_1 \) is a Hamiltonian cycle in \( D^{rev} \), a contradiction. Therefore, the case \( l \leq t - 1 \) is equivalent to the case \( t \leq l \).

Using Lemma 3.1, it is easy to see that the following proposition holds.

**Proposition 3.** If \( k = 0 \) and a longest \( C(z) \)-cycle in \( D \) has length \( n - 3 \), then \( D(V(D) \setminus V(C(z))) \) is strong.

From now on, we assume that \( l \leq t \). Note that from (13) it follows that there are \( a \in [1, t - 1] \) and \( b \in [t + 1, n - d - 1] \) such that \( x_a \rightarrow x_b \).

**Subcase A.1.** \( z \rightarrow x_{a+1} \).
Recall that \( a \in [1, t - 1] \) and \( b \in [t + 1, n - d - 1] \). By Claim 6, we have that \( d^+(x_{b-1}, Y) = 0 \).

**Subcase A.1.1.** \( z \rightarrow x_{a+1} \) and \( b \geq t + 2 \).

Then \( b - 2 \geq t \geq l \). If \( x_{b-2} \rightarrow y_i \) with \( i \in [1, 2] \), then the cycle \( C(z) = x_1 x_2 \ldots x_a x_{b} \ldots x_{n-d-1} \)
\[\cdots x_{b-2} y_i \ldots y_d x_1\]
has length at least \( n - 2 \), a contradiction. We may therefore assume that \( d^+(x_{b-2}, \{y_1, y_2\}) = 0 \). This together with Claim 6 implies that \( A(x_{b-2}, x_{b-1}) \rightarrow \{y_1, y_2\} = 0 \). Therefore by Claim 5(ii) and \( l \leq t \), we have that \( d = k + 4 \), in particular, (2) holds. If \( b \geq t + 3 \), then from \( d^-(y_1, \{x_{b-2} \rightarrow x_{b-1}\}) = 0 \) and Claim 5(ii) it follows that \( x_{b-3} \rightarrow y_1 \) and \( C_{n-2}(z) = x_1 x_2 \ldots x_a x_2 \ldots x_{n-k-5} x_{a+1} \ldots x_{b-3} y_1 y_2 \ldots y_{k+4} x_1 \), a contradiction. Therefore, we may assume that \( b = t + 2 \). If \( x_t \rightarrow y_3 \), then \( C_{n-3}(z) = x_1 x_2 \ldots x_a x_{t+2} \ldots x_{n-k-5} x_{a+1} \ldots x_t y_3 \ldots y_{k+4} x_1 \) and the
A new sufficient condition for a 2-strong digraph to be Hamiltonian

Subcase A.1.2. \( z \to x_{a+1} \) and \( b = t+1 \). Since \( b-1 = t \), \( d^+(x_{b-1}, Y) = 0 \) (Claim 6) and \( d^+(x_{1}, Y) \geq 1 \), we have \( d^+(x_{1}, Y) = 0 \). Assure first that \( t+1 \leq n-d-2 \). Taking into account Subcase A.1.1 and \( b = t+1 \), we may assume that \( A\{x_{1}, x_{2}, \ldots, x_{t-1}\} \to \{x_{t+2}, x_{t+3}, \ldots, x_{n-d-1}\} = \emptyset \). This together with (13) implies that there is \( j \in \{t+2, n-d-1\} \) such that \( x_{j} \to x_{j} \). If \( x_{j} = x_{t} \), then \( C_{n-k-2} = x_{1}x_{2}\ldots x_{a}x_{a+1}\ldots x_{t}x_{t+1}x_{t+2}\ldots x_{n-k-5}y_{1}y_{2}y_{k+1}x_{k+2}x_{k+3}x_{k+4}x_{k+5} \) is cycle of length at least \( n-1 \), a contradiction. This completes the discussion of Subcase A.1.1.
and $z \mapsto x_{s+1}$ it follows that $d = k + 3$ and $x_{t-1} \rightarrow y_1$. If $s \leq t - 3$, then $z \rightarrow x_{s+2}$ and $C_n(z) = x_1x_2 \ldots x_s x_{s+2} \ldots x_{t-1} x_{t+1} y_1 y_2 \ldots y_k x_1$, a contradiction. Thus, we may assume that $s = t - 2$. If $t - 2 \geq 2$, then we have that $A(x_1, x_2, \ldots, x_{t-2}) \rightarrow \{x_t, x_{t+1} = x_{n-k-4}\} = \emptyset$. Therefore by (13), there is $p \in [1, t - 3]$ such that $x_p \rightarrow x_{t-1}$ and $z \rightarrow x_{p+1}$. If $l \leq t - 2$, then $x_{t-2} \rightarrow y_1$ and $C_n(z) = x_1 x_2 \ldots x_p x_{t-1} x_t x_{t+1} y_p \ldots y_k x_1$, a contradiction. Assume that $l = t - 1$. Then $y_1 \rightarrow x_{p+1}$ and $C_n = x_1 x_2 \ldots x_p x_{t-1} x_{t+1} y_1 y_p \ldots y_k x_1$, a contradiction. Finally assume that $t - 2 = 1$. Then $n - k - 4 = 4$ and $d(x_3, Y) = 0$. Therefore, $n + k \leq d(x_3) \leq 8$ and $n \leq 8$, which contradicts that $n \geq 9$. This completes the discussion of Subcase A.1.2.

**Subcase A.2.** $z \mapsto x_{a+1}$.

From $z \mapsto x_{a+1}$, Claim 5(i), (1) and (2) it follows that $d = k + 3, x_{b-1} \rightarrow z$ and

$$
\{x_t, x_{t+1}, \ldots, x_{n-k-4}\} \rightarrow z \rightarrow \{x_1, x_2, \ldots, x_a, x_{a+2}, x_{a+3}, \ldots, x_t\}. \tag{14}
$$

Assume first that

$$A(\{x_1, x_2, \ldots, x_{t-2}\} \rightarrow \{x_{t+1}, x_{t+2}, \ldots, x_{n-k-4}\}) = \emptyset. \tag{15}
$$

Then $a = t - 1$, i.e., $x_{t-1} \rightarrow x_b$. Using (14), $d^+(z) \geq 2$ and Claim 2, we obtain that $t - 1 \geq 2$. From (13) and (15) it follows that there exists $s \in [1, t - 2]$ such that $x_s \rightarrow x_t$. Then, since $z \rightarrow x_{a+1}$, $C_n(z) = x_1 x_2 \ldots x_s x_{b-1} x_{b+1} \ldots x_{t-1} x_b \ldots x_{n-k-4} y_1 y_2 \ldots y_k x_1$, a contradiction.

Assume next that (15) is not true. Then we may assume that $a \leq t - 2$. Note that $z \rightarrow \{x_{a+2}, \ldots, x_t\}$ (by (14)). If $y_i \rightarrow x_{a+1}$ with $i \in [1, k + 3]$, then the cycle $C(z) = x_1 x_2 \ldots x_a x_b \ldots x_{n-k-4} y_1 \ldots y_k x_{a+1} \ldots x_{b-1} x_1$ has length at least $n - k - 2$, a contradiction. We may therefore assume that $d^-(x_{a+1}, Y) = 0$. Let $b \geq t + 2$. Then, since $d = k + 3$ and $t \geq 1$, from Claim 5(ii) it follows that for some $j \in [b - 2, b - 1]$, $x_j \rightarrow y_1$. Then the cycle $C(z) = x_1 x_2 \ldots x_a x_{t+1} \ldots x_{n-k-4} x_{a+2} \ldots x_j y_1 y_2 \ldots y_k x_1$ has length at least $n - 2$, a contradiction. Let now $b = t + 1$. We claim that $l \leq t - 1$. Assume that this is not the case, i.e., $l = t$. Then using Claim 5(ii) and the facts that $d = k + 3, d^-(x_{a+1}, Y) = 0$, we obtain that $x_t \rightarrow y_1$. Therefore, $C_n(z) = x_1 x_2 \ldots x_a x_{t+1} \ldots x_{n-k-4} x_{a+2} \ldots x_j y_1 \ldots y_k x_1$, a contradiction. This shows that $l \leq t - 1$. From $a \leq t - 2, l \leq t - 1, x_t \rightarrow y_1$ and Claim 5(ii) it follows that $x_{t-1} \rightarrow y_1$. Therefore, if $a \leq t - 3$, then $C_{n-1}(z) = x_1 x_2 \ldots x_a x_{t+1} \ldots x_{n-k-4} x_{a+2} \ldots x_t y_1 y_2 \ldots y_k x_1$, a contradiction. We may therefore assume that $a = t - 2$. Assume first that $a \geq 2$. Since $a = t - 2$, we have

$$A(\{x_1, x_2, \ldots, x_{t-3}\} \rightarrow \{x_{t+1}, x_{t+2}, \ldots, x_{n-k-4}\}) = \emptyset.
$$

This together with (13) implies that there exist $s \in [1, a - 1 = t - 3]$ and $p \in [t - 1, t]$ such that $x_s \rightarrow x_p$. Then by (14), the cycle $C(z) = x_1 x_2 \ldots x_s x_p x_t x_{t+1} \ldots x_{t-2} x_{t+1} \ldots x_{n-k-4} y_1 y_2 \ldots y_k x_1$ has length at least $n - 1$, a contradiction.

Assume next that $a = 1$. Then $t = 3$. Let $t + 1 \leq n - k - 5$. Since $b = t + 1$, we have $d^+(x_1, \{x_{t+2}, x_{t+3}, \ldots, x_{n-k-4}\}) = 0$. Again using (13), we obtain that there exist $p \in [t - 1, t]$ and $q \in [t + 2, n - k - 4]$ such that $x_p \rightarrow x_q$. Recall that $z \rightarrow x_t$ and $x_{q-1} \rightarrow \{z, y_1\}$. Therefore, if $p = t$, then $C_n(z) = x_1 x_{t+1} \ldots x_{q-1} x_{t+1} \ldots x_{n-k-4} y_1 y_2 \ldots y_k x_1$, and if $p = t - 1$, then $C_n(z) = x_1 x_2 \ldots x_{t-2} x_{t+1} \ldots x_{n-k-4} x_{t+1} \ldots x_{q-1} y_1 y_2 \ldots y_k x_1$. Thus, in both cases, we have a contradiction. This completes the discussion of Subcase A.2, and also completes the proof of the theorem when $D(Y)$ is strong.
Case B. $D(Y)$ is not strong.

Since $y_1 y_2 \ldots y_d$ is a path in $D(Y)$ and $k + 3 \leq d \leq k + 4$, using the fact that every vertex $y_i$ with $i \in [1, d]$ cannot be inserted into $P$ (Claim 4) and Lemma 3.2, we obtain $d(y_i, V(P)) \leq n - d$, $d(y_i, Y) \geq d + k$. Now, we claim that $d = k + 4$ and $k = 0$. Indeed, since $D(Y)$ is not strong, $y_1 y_2 \ldots y_d$ is a Hamiltonian path in $D(Y)$ and $d(y_i) \geq n + k$, it follows that for some $l \in [2, d - 1]$, $y_l \rightarrow y_1$ and $\mathcal{d}^- (\{ y_1, y_1, y_2, \ldots, y_d \}) = \mathcal{d}^+ (\{ y_d, \{ y_1, y_2, \ldots, y_d \} \}) = 0$. From this we have $k + d \leq d(\{ y_1, Y \}) \leq d - l + 2(l - 1) = d - l - 2$ and $k + d \leq d(\{ y_d, Y \}) \leq l + 2(d - l - 1) = 2d - l - 2$. Therefore, $k \leq l - 2$ and $d \geq k + l + 2$. From the last two inequalities and the facts that $d \leq k + 4$, $l \geq 2$ it follows that $d = k + 4$ and $k = 0$. Therefore, $d(y_i, V(P)) \leq n - 4$ and $d(y_i, Y) \geq 4$. Since $D(Y)$ is not strong and $y_1 y_2 y_3 y_4$ is a path in $D(Y)$, it is not difficult to check that for all $i \in [1, 4]$, $d(y_i, Y) = 4$, $d(y_i, V(P)) = n - 4$, the arcs $y_1 y_3, y_1 y_4, y_2 y_1, y_2 y_4, y_3 y_3$ also are in $A(D)$ and $A(\{ y_3, y_4 \} \rightarrow \{ y_1, y_2 \}) = \emptyset$. Since $D$ has no $C(z)$-cycle of length at least $n - 2$ and any vertex $y_i$ with $i \in [1, 4]$ cannot be inserted into $P = x_1 x_2 \ldots x_{n - 5}$, using Lemma 3.3 and Proposition 3, it is not difficult to show that there are two integers $l_1$ and $l_2$ with $2 \leq l_1, l_2 \leq n - 6$ such that

$$\begin{align*}
\{ x_{i_1}, \ldots, x_{n - 5} \} &\rightarrow \{ y_1, y_2 \} \rightarrow \{ x_1, \ldots, x_{l_1} \}, \\
\{ x_{l_2}, \ldots, x_{n - 5} \} &\rightarrow \{ y_3, y_4 \} \rightarrow \{ x_1, \ldots, x_{l_2} \}.
\end{align*}$$

(16)

It is easy to see that $l_1 \geq l_2$. Indeed, if $l_1 \leq l_2 - 1$, then from (16) it follows that $x_{l_1} \rightarrow y_1, y_4 \rightarrow x_{l_1 + 1}$ and hence, $C_n(z) = x_1, \ldots, x_{l_1} y_1 y_2 y_3 y_4 x_1, \ldots, x_{n - 5} z x_1$, a contradiction. Since $D$ is 2-strong, (16) together with (2) implies that there are two integers $p \in [1, l_2 - 1]$ and $q \in [l_2 + 1, n - 5]$ such that $x_p \rightarrow x_q$ (for otherwise $D - x_2$ is not strong). Assume first that $l_2 \leq t$. Then from (2) and (16), respectively, we have $z \rightarrow x_{p + 1}$ and $x_{q - 1} \rightarrow y_3$. Therefore, $C_{n - 2}(z) = x_1, \ldots, x_p x_q, \ldots, x_{n - 5} z x_{p + 1}, \ldots, x_{q - 1} y_3 y_4 x_1$, a contradiction. Assume now that $l_2 \geq t + 1$. Then by (16), $y_4 \rightarrow x_{p + 1}$, and by (2), $x_{q - 1} \rightarrow z$. Therefore, $C(z) = x_1, x_2, \ldots, x_p x_q, \ldots, x_{n - 5} y_1, \ldots, y_4 x_{p + 1}, \ldots, x_{q - 1} z x_1$ is a Hamiltonian cycle in $D$, which is contradiction. This completes the discussion of Case B. The theorem is proved.

\[\square\]

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