On harmonious coloring of hypergraphs

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A harmonious coloring of a $k$-uniform hypergraph $H$ is a vertex coloring such that no two vertices in the same edge have the same color, and each $k$-element subset of colors appears on at most one edge. The harmonious number $h(H)$ is the least number of colors needed for such a coloring.

The paper contains a new proof of the upper bound $h(H) = O(\sqrt[k]{k!m})$ on the harmonious number of hypergraphs of maximum degree $\Delta$ with $m$ edges. We use the local cut lemma of A. Bernshteyn.

Keywords: Harmonious coloring, Hypergraphs

1 Introduction

Let $H = (V, E)$ be a $k$-uniform hypergraph with the set of vertices $V$ and the set of edges $E$. The set of edges is a family of $k$-element sets of $V$, where $k \geq 2$.

A rainbow coloring $c$ of a hypergraph $H$ is a map $c : V \mapsto \{1, \ldots, r\}$ in which no two vertices in the same edge have the same color. If two vertices are in the same edge $e$ with the same color, we say that the edge $e$ is bad.

A coloring $c$ is called harmonious if $c(e) \neq c(f)$ for every pair of distinct edges $e, f \in E$ and $c$ is a rainbow coloring.

We say that distinct edges $e$ and $f$ have the same pattern of colors if $c(e \setminus f) = c(f \setminus e)$ and there is no uncolored vertex in the set $e \setminus f$.

Let $h(H)$ be the least number of colors needed for a harmonious coloring of $H$. In Bosek et al. (2016), the following result is proved

**Theorem 1** (Bosek et al. (2016)). For every $\varepsilon > 0$ and every $\Delta > 0$ there exist integers $k_0$ and $m_0$ such that every $k$-uniform hypergraph $H$ with $m$ edges (where $m \geq m_0$ and $k \geq k_0$) and maximum degree $\Delta$ satisfies

$$h(H) \leq (1 + \varepsilon)\frac{k}{k - 1}\sqrt[k]{\Delta(k - 1)k!m}. $$

**Remark 1.** The paper Bosek et al. (2016) contains the following upper bound on the harmonious number

$$h(H) \leq \frac{k}{k - 1}\sqrt[k]{\Delta(k - 1)k!m} + 1 + \Delta^2 + (k - 1)\Delta + \sum_{i=2}^{k-1} \frac{i}{i-1}\sqrt{(i-1)\frac{(k-1)\Delta^2}{k-i}}.$$
The proof of this theorem is based on the entropy compression method, see Grytczuk et al. (2013); Esperet and Parreau (2013).

Because a number \( r \) of used colors must satisfy the inequality \( \binom{r}{k} \leq m \), we get the lower bound \( \Omega(\sqrt{k!m}) \). By these observations, it is conjectured by Bosek et al. (2016) that

**Conjecture 1.** For each \( k, \Delta \geq 2 \) there exist a constant \( c = c(k, \Delta) \) such that every \( k \)-uniform hypergraph \( H \) with \( m \) edges and maximum degree \( \Delta \) satisfies

\[
h(H) \leq \frac{k}{\sqrt{k!m}} + c.
\]

A similar conjecture was posed by Edwards (1997b) for simple graphs \( G \). He prove there that

\[
h(G) \leq (1 + o(1))\sqrt{2m}.
\]

There are many results about the harmonious number of particular classes of graphs, see Aflaki et al. (2012); Akbari et al. (2012); Edwards (1997a); Edwards and McDiarmid (1994); Edwards (1996); Krasikov and Roditty (1994) or Aigner et al. (1992); Balister et al. (2002, 2003); Bazgan et al. (1999); Burris and Schelp (1997).

The present paper new contains proof of the theorem of Bosek et al. We use a different method, the local cut lemma of Bernshteyn (2017, 2016). The proof is simpler and shorter than the original proof of Bosek et al.

## 2 A special version of the local cut lemma

Let \( A \) be a family of subsets of a powerset \( \text{Pow}(I) \), where \( I \) is a finite set. We say that it is *downwards-closed* if for each \( S \in A \), \( \text{Pow}(S) \subseteq (A) \). A subset \( \partial A \) of \( I \) is called *boundary* of a downwards-closed family \( A \) if

\[
\partial A := \{i \in I : S \in A \text{ and } S \cup \{i\} \not\in A \text{ for some } S \subseteq I \setminus \{i\}\}.
\]

Let \( \tau : T \mapsto [1; +\infty) \) be a function, then for every \( X \subseteq I \) we denote by \( \tau(X) \) a number

\[
\tau(X) := \prod_{x \in X} \tau(x).
\]

Let \( B \) be a random event, \( X \subseteq I \) and \( i \in I \). We introduce two quantities:

\[
\sigma^A_i(B, X) := \max_{Z \subseteq I \setminus X} \Pr(B \text{ and } Z \cup X \not\in A | Z \in A) \cdot \tau(X)
\]

and

\[
\sigma_i^A(B, i) := \min_{X \subseteq I} \sigma^A_i(B, X).
\]

If \( \Pr(Z \in A) = 0 \), then \( \Pr(P | Z \in A) = 0 \), for all events \( P \).

**Theorem 2** (Bernshteyn (2017)). Let \( I \) be a finite set. Let \( \Omega \) be a probability space and let \( A : \Omega \mapsto \text{Pow(Pow}(I)) \) be a random variable such that with probability 1, \( A \) is a nonempty downwards-closed family of subsets of \( I \). For each \( i \in I \), Let \( B(i) \) be a finite collection of random events such that whenever
On harmonious coloring of hypergraphs

3

$i \in \partial A$, at least one of the events in $B(i)$ holds. Suppose that there is a function $\tau : I \mapsto [1, +\infty)$ such that for all $i \in I$ we have

$$
\tau(i) \geq 1 + \sum_{B \in B(i)} \sigma^A_\tau(B, i).
$$

Then $\Pr(I \in A) \geq 1/\tau(I) > 0$.

3 Proof of Theorem 1

We choose a coloring $f : V \mapsto \{1, \ldots, t\}$ uniformly at random. Let $A$ be a subset of the power set of $V$ given by

$$
A := \{S \subseteq V : c \text{ is a harmonious coloring of } H(V)\}.
$$

It is a nonempty downwards-closed family with probability 1 (the empty set is an element of $A$).

By a set $\partial A$, we denote the set of all vertices $v$ such that there is an element $X$ of $A$ such that the coloring $c$ is not a harmonious coloring of $X \cup \{v\}$. If the coloring $c$ is not a harmonious coloring, then there is a bad edge or there are two different edges with the same pattern of colors. So, we define for every $v \in V$, a collection $B(v)$ as union of sets:

$$
B^1(v) := \{B_e : v \in e \in E(H) \text{ and } e \text{ is not proper colored}\},
$$

and for every $i \in \{0, 1, \ldots, k - 1\}$,

$$
B^2_i(v) := \{B_{e,f} : v \in e, f \in E(H) \text{ and } c(e) = c(f), |e \setminus f| = i\}.
$$

That is $B(v) = B^1(v) \cup \bigcup_{i=1}^{k-1} B^2_i(v)$.

We assume that the event $B_e$ happens if and only if the edge $e$ is the bad edge and the event $B_{e,f}$ happens if and only if the edges $e$ and $f$ have the same pattern of colors.

We also assume that a function $\tau$ is a constant function, that is, $\tau(v) = \tau \in [1, +\infty)$. This implies that for any subset $S$ of $V$, we have $\tau(S) = \tau^{|S|}$.

Now, we must find an upper bound on

$$
\sigma^A_\tau(B, v) = \min_{X \subseteq V : v \in X} \max_{Z \subseteq V \setminus X} \Pr(B \land Z \cup X \not\in A | Z \in A) \tau(X),
$$

where $v \in V$ and $B \in B(v)$. We will use an estimation

$$
\sigma^A_\tau(B, v) \leq \max_{Z \subseteq V \setminus X} \Pr(B | Z \in A) \tau(X).
$$

Now, we consider two cases.

Case 1: $B \in B^1$, i.e. $B = B_e$.

We choose as $X$ the set $\{e\}$. Because the colors of distinct vertices are independent, we get an upper bound $\sigma^A_\tau(B_e, v) \leq \Pr(B_e) \tau^k$ (events $B_e$ and $Z \in A^c$ are independent). The probability $\Pr(B_e)$, opposite to $\Pr(\overline{B_e})$, fulfills

$$
\Pr(B_e) = 1 - \frac{t}{t} \cdot \frac{t - 1}{t} \cdot \ldots \cdot \frac{t - k + 1}{t} \geq 1 - (1 - \frac{k-1}{t})^{k-1}.
$$
By Bernoulli’s inequality, we get

\[ \Pr(B_e) \geq 1 - \left(1 - \frac{k-1}{t}\right) \cdot (k-1) = \frac{(k-1)^2}{t}. \]

So, \( \Pr(B_e) \leq \frac{k^2}{t} \).

Case 2: \( B \in \mathcal{B}_2^i \), i.e. \( B = B_{e,f} \) and \( |e \setminus f| = i \).

Now, we set \( X = e \setminus f \). The probability \( \Pr(B_{e,f}) \) is bounded from above by \( \frac{t}{k^2} \). So, we get

\[ \sigma_i^A(B_{e,f}, v) \leq \Pr(B_{e,f}) \tau^i \leq \frac{i!}{t^i} \tau^i. \]

To end the proof we must find an upper bound on sizes of sets \( \mathcal{B}_1^1(v) \), \( \mathcal{B}_0^2(v) \) and \( \mathcal{B}_i^2(v) \), where \( i > 0 \). Because the degree of a vertex is bounded by \( \Delta \) and the number of edges is \( m \) we get that

\[ |\mathcal{B}_1^1(v)| \leq \Delta \text{ and } |\mathcal{B}_0^2(v)| \leq \Delta m. \]

The hardest part is an upper bound on \( \mathcal{B}_i^2(v) \), \( i > 0 \). The number of edges \( f \) such that \( e \setminus f = i \) is bounded by \( \frac{k \Delta}{k-i} \). There are at most \( k \Delta \) edges with a nonempty intersection with the edge \( e \), and the edge \( f \) has exactly \( k - i \) common elements with \( e \). So, we have \( |\mathcal{B}_i^2(v)| \leq \Delta \frac{k \Delta}{k-i} \). To apply Theorem 2 we must find \( \tau \in [1, +\infty) \) and \( c \in \mathbb{N} \) such that, for all \( v \in V \), the inequality below holds

\[ \tau \geq 1 + \frac{k}{\Delta} \frac{k^2}{t} \tau^k + \Delta m \frac{k!}{t^k} \tau^k + \sum_{i=1}^{k-1} \Delta \frac{k \Delta}{k-i} \frac{i!}{t^i} \tau^i. \]

If we choose \( \tau = \frac{k}{\Delta} \frac{k^2}{t} \) and \( t = \frac{k}{\Delta} \sqrt{\Delta (k-1)} m (1 + \varepsilon) \), it is easy to see that the inequality holds for sufficiently large \( m \).

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References


On harmonious coloring of hypergraphs


