Discrete Mathematics and Theoretical Computer Science ....
.... **vol. 26:3 #1 (2024)**

A note on graph burning of path forests

Ta Sheng Tan[1] Wen Chean Teh[2*]

[1] *Institute of Mathematical Sciences, Faculty of Science, University Malaya, Malaysia*
[2] *School of Mathematical Sciences, Universiti Sains Malaysia, Malaysia*

revisions 19th Dec. 2023, 21st Mar. 2024; accepted 12th July 2024.

Graph burning is a natural discrete graph algorithm inspired by the spread of social contagion. Despite its simplicity, some open problems remain steadfastly unsolved, notably the burning number conjecture, which says that every connected graph of order m^2 has burning number at most m. Earlier, we showed that the conjecture also holds for a path forest, which is disconnected, provided each of its paths is sufficiently long. However, finding the least sufficient length for this to hold turns out to be nontrivial. In this note, we present our initial findings and conjectures that associate the problem to some naturally impossibly burnable path forests. It is noteworthy that our problem can be reformulated as a topic concerning sumset partition of integers.

Keywords: discrete graph algorithm, burning number conjecture, spread of social contagion, sumset partition of integers, well-burnable

1 Introduction

Graph burning is a discrete-time process introduced by Bonato et al. (2016) that can be viewed as a simplified model for the spread of contagion in a network. Given a simple finite graph G, each vertex of the graph is either *burned* or *unburned* throughout the process. Initially, every vertex of G is unburned. At the beginning of every round $t \geq 1$, a *burning source* is placed at an unburned vertex to burn it. If a vertex is burned in round $t-1$, then in round t, each of its unburned neighbours becomes burned. A burned vertex will remain burned throughout the process. The burning process ends when all vertices of G are burned, in which case we say the graph G is *burned*. The *burning number of* G is the least number of rounds needed for the burning process to be completed.

The study of graph burning is extensive, with the main open problem being the burning number conjecture by Bonato et al. (2016).

Burning number conjecture. *(Bonato et al., 2016) The burning number of every connected graph of order* N *is at most* $\lceil \sqrt{N} \rceil$.

In the literature of graph burning, a graph is said to be m*-burnable* if its burning number is at most m, and a graph (including a disconnected graph) is said to be *well-burnable* if it satisfies the burning number conjecture. Many classes of graphs have been verified to be well-burnable, including hamiltonian graphs

*Corresponding author

ISSN 1365–8050

(Bonato et al., 2016), spiders (Bonato and Lidbetter, 2019; Das et al., 2018), and caterpillars (Hiller et al., 2021; Liu et al., 2020). Recently, the burning number conjecture was shown to hold asymptotically by Norin and Turcotte (2024).

The reader is referred to Bonato (2021) for a survey on graph burning. In this short note, we are interested in the graph burning of path forests. Here, a *path forest* is a disjoint union of paths. While not all path forests are well-burnable, it was shown in Tan and Teh (2023) that a path forest with a sufficiently long shortest path is well-burnable.

Theorem 1.1. *(Tan and Teh, 2023) For every $n \in \mathbb{N}$, there exists a smallest $L_n \in \mathbb{N}$ such that if T is a path forest with n paths and the shortest path of T has order at least L_n, then T is well-burnable.*

When determining whether a path forest is well-burnable, we can always extend some of its paths so that the order of the graph is m^2 for some $m \in \mathbb{N}$. So from here onwards, we will assume that the order of a path forest is always an integer squared. We say a path forest is *deficient* if it is not well-burnable, and by an *n-path forest*, we mean a path forest with n paths.

The main purpose of this note is to provide insights on L_n, and some conjectures related to L_n. We clearly have $L_1 = 1$ as every path is well-burnable, and it is straightforward that $L_2 = 3$, since a 2-path forest is deficient if and only if its path orders are $m^2 - 2$ and 2 for some $m \geq 2$ (see Tan and Teh, 2020, Lemma 3.1). The study of the values of L_n was posed as an open problem in Tan and Teh (2023), where it was mentioned that $L_3 = 18$ and $L_4 = 26$ (determined with careful analysis and the help of a computer).

Note that for a path forest of order m^2 to be well-burnable, a burning process of m rounds must have its ith burning source burning exactly $2m - 2i + 1$ vertices, and every vertex is burned by exactly one burning source. So, each path in the path forest is burned exactly, and the number of burning sources used on the path has the same parity as its order. While investigating deficient n-path forests for some small values of n, we notice that when their shortest paths have orders slightly smaller than L_n, they are all trivially deficient, in the following sense. For such a deficient path forest, even if we pretend that for each of its paths, the ith burning source used on it (not on the path forest) would burn $2m - 2i + 1$ vertices, m burning sources are not enough to completely burn the path forest, provided we insist that the number of burning sources used on each path has the same parity as its order. We will call these trivially deficient path forests *impossibly burnable* (this will be made more precise in Section 2).

Question 1.2. *Is it true that if T is a deficient n-path forest such that its shortest path has order $L_n - 1$, then T is impossibly burnable?*

As mentioned earlier, the answer to the above question is affirmative for small values of n (up to $n = 7$). However, what about the remaining values of n? While we are unable to verify for any $n \geq 8$ (due to computational limitations), we believe that the answer to the above question remains affirmative (see Conjecture 2.7). This belief has motivated us to study impossibly burnable path forests, which would lead to the computation of the exact values of L_n.

For $n \geq 2$, let M_n be the smallest positive integer such that if T is an impossibly burnable n-path forest, then its shortest path has order at most M_n. So assuming the answer to Question 1.2 is affirmative, it follows that $L_n = M_n + 1$ for all $n \geq 2$. It is straightforward that $M_2 = 2$, and for the main result of this note, we determine the exact values of M_n.

Theorem 1.3. *For $n \geq 3$, M_n is the largest odd number smaller than or equal to*

$$12n - 2\sqrt{18n - 12} - 6.$$

We digress briefly to mention another formulation of the graph burning of path forests, presented in the form of a sumset partition problem. The problem of determining whether an n-path forest $(l_1, l_2, \ldots, l_n)$ of order m^2 is well-burnable is equivalent to deciding whether the set of the first m odd positive integers can be partitioned into n subsets $S_1, S_2, \ldots, S_n$ such that for every $i \in [n]$, the sum of the numbers in S_i is equal to l_i. Interested readers may refer to Ando et al. (1990), Chen et al. (2005), Enomoto and Kano (1995), Fu and Hu (1994), Lladó and Moragas (2012), and Ma et al. (1994) for some related studies on this formulation.

2 Impossibly burnable path forests

In this section, we will first describe our observations on deficient path forests that lead us to Question 1.2, and then we will proceed to prove Theorem 1.3. For a path forest, its *path orders* indicate the respective order of each of its paths. We may represent an n-path forest by an n-tuple $(l_1, l_2, \cdots, l_n)$ of its path orders. Often, we assume $l_1 \leq l_2 \leq \cdots \leq l_n$, and if T is a path forest, we may write $l_1(T)$ (or just l_1) for the order of its shortest path.

In Bessy et al. (2017), the graph burning problem was shown to be NP-complete for general path forests, and a polynomial time algorithm for the problem was constructed when the number of paths is fixed. Based on this algorithm, given the number of paths n and a positive integer $M \geq n$, we are able to construct a complete list of well-burnable n-path forests of order m^2 for all $m \leq M$ using Matlab subject to computational limitations. Note that $(1, 3, 5, \ldots, 2n-1)$ is the unique well-burnable n-path forest of order n^2. Our lists are constructed recursively based on the following strategy: to obtain a well-burnable n-path forest of order $(m+1)^2$,

1. add a path of order $2m+1$ to any well-burnable $(n-1)$-path forest T such that $|T| = m^2$; or
2. add $2m+1$ vertices to any one of the paths of any well-burnable n-path forest T such that $|T| = m^2$.

Suppose T is a 3-path forest of order m^2 and $l_1 \geq 8$. From the complete list of well-burnable 3-path forests for $m \leq 9$, we observe that if T is deficient, then T is one of the following six possibilities. Furthermore, if $m = 9$, then T is well-burnable. We can then deduce by induction that T is well-burnable for any $m \geq 9$ by considering the 3-path forest $(l_1, l_2, l_3 - 2m + 1)$.

Observation 2.1. *Every 3-path forest with $l_1 \geq 8$ is well-burnable, unless it is one of the following exceptional cases:*

$$(8, 13, 15), (8, 15, 26), (10, 13, 13), (15, 15, 19), (15, 17, 17), (17, 17, 30).$$

Similarly, we have the following observation for 4-path forests.

Observation 2.2. *Every 4-path forest with $l_1 \geq 25$ is well-burnable, unless it is one of the following exceptional cases:*

$$(25, 25, 25, 25), (25, 25, 25, 46), (25, 25, 27, 44), (25, 25, 29, 42), (25, 27, 27, 42), (25, 25, 25, 69),$$

$$(25, 25, 27, 67), (25, 25, 29, 65), (25, 27, 27, 65), (25, 25, 46, 48), (25, 27, 46, 46).$$

It follows from Observations 2.1 and 2.2 that $L_3 = 18$ and $L_4 = 26$. Furthermore, as mentioned in the Introduction, we observed that all the deficient path forests in Observations 2.1 and 2.2 are impossibly burnable. We now give the precise definition of an impossibly burnable path forest.

Definition 2.3. For $m \in \mathbb{N}$ and $1 \leq l \leq m^2$, let $B_m(l)$ be the least $t \in \mathbb{N}$ having the same parity as l such that $l \leq \sum_{i=1}^{t}[2m-(2i-1)] = 2mt - t^2$.

Example 2.4. If l is odd and $2m-1 < l \leq (2m-1)+(2m-3)+(2m-5)$, then $B_m(l) = 3$. Meanwhile, if l is even and $(2m-1)+(2m-3) < l \leq (2m-1)+(2m-3)+(2m-5)+(2m-7)$, then $B_m(l) = 4$.

Definition 2.5. Suppose $T = (l_1, l_2, \ldots, l_n)$ is an n-path forest of order m^2. We say that T is *impossibly burnable* if $\sum_{i=1}^{n} B_m(l_i) > m$.

Remark 2.6.

(i) An impossibly burnable path forest is clearly deficient. Indeed, for a path of order l in a path forest of order m^2, at least $B_m(l)$ burning sources are required to burn the path completely in m rounds, regardless of the parity of l.

(ii) Not all deficient path forests are impossibly burnable. For example, the path forest $(2, 7, 7)$ is deficient but not impossibly burnable.

(iii) The parities of $\sum_{i=1}^{n} B_m(l_i)$ and m are equal, and thus if T is impossibly burnable, then $m \leq \sum_{i=1}^{n} B_m(l_i) - 2$.

Based on Observation 2.1 and the subsequent discussion, any deficient 3-path forest with $l_1 \geq 8$ is impossibly burnable. For the case of 4-path forests, a Matlab search reveals that there are exactly 47 deficient 4-path forests with $l_1 \geq 18$, all of which are impossibly burnable. Note, however, that the path forest $(17, 17, 17, 30)$ is deficient, but it is not impossibly burnable. Analysing the cases for $5 \leq n \leq 7$ gives similar results, leading us to the following conjecture.

Conjecture 2.7. *Let $n \geq 4$. If T is a deficient n-path forest with $l_1 \geq L_{n-1}$, then T is impossibly burnable.*

As mentioned earlier, Conjecture 2.7 is true for $n = 4$, but it is not true for $n = 3$ as $L_2 = 3$. The path forest $(3, 3, 3)$ is deficient but not impossibly burnable. Through an extensive Matlab search, we have determined that $L_5 = 36$, $L_6 = 46$, and $L_7 = 56$. Here, we briefly mention our computational validation of Conjecture 2.7.

Analysing the list of well-burnable 5-path forests order m^2 for $m \leq 18$, we observe that there are exactly 608 deficient 5-path forests with $l_1 \geq 26$, all of which are impossibly burnable. Furthermore, all 5-path forests of order 18^2 with $l_1 \geq 26$ are well burnable. Hence, we can again deduce by induction that for $m \geq 18$, every 5-path forest of order m^2 with $l_1 \geq 26$ is well-burnable. Similarly, Conjecture 2.7 holds true for $n = 6$. Specifically, all the 5185 deficient 6-path forests with $l_1 \geq 36$ are impossibly burnable. We have also managed to verify Conjecture 2.7 for $n = 7$, with a more significant effort due to computational limitations. (See Appendix A for a brief account of this verification.)

As M_n increases as n grows by Theorem 1.3, we remark that Conjecture 2.7 implies a strongly affirmative answer to Question 1.2, resulting in $L_n = M_n + 1$. We are now ready to determine the exact values of M_n.

Proof of Theorem 1.3: Suppose T is an impossibly burnable n-path forest of order m^2 with path orders $l_1 \leq l_2 \leq \cdots \leq l_n$. Writing $t_i = B_m(l_i)$ for each $i \in [n]$ and recalling that $t_i \equiv l_i \pmod 2$ for every $i \in [n]$, we have that $\sum_{i=1}^{n} t_i \geq m + 2$. Consider the partition of $[n]$ into

$$A = \{i \in [n] : t_i \geq 4\} \text{ and } B = \{i \in [n] : t_i \leq 3\}.$$

For convenience, we let $s_i = t_i - 2 \geq 2$ for each $i \in A$. So for every $i \in A$, we have

$$l_i \geq (2m-1) + (2m-3) + \cdots + (2m - 2s_i + 1) + 2 = 2ms_i - s_i^2 + 2.$$

Let $s = \sum_{i \in A} s_i = \left(\sum_{i \in A} t_i\right) - 2|A|$ and note that $s < m$, as otherwise, $\sum_{i \in A} l_i > m^2$. Observe now that

$$\begin{aligned}\sum_{i \in A} l_i &\geq \sum_{i \in A}(2ms_i - s_i^2 + 2) \\ &= 2ms - \sum_{i \in A} s_i^2 + 2|A| \\ &= 2ms - s^2 + \left(\sum_{i,j \in A, i \neq j} s_i s_j\right) + 2|A|.\end{aligned}$$

It follows that $m^2 = \sum_{i=1}^n l_i \geq l_1|B| + 2ms - s^2 + \left(\sum_{i,j \in A, i \neq j} s_i s_j\right) + 2|A|$, implying that

$$(m-s)^2 \geq l_1|B| + \left(\sum_{i,j \in A, i \neq j} s_i s_j\right) + 2|A|.$$

On the other hand, note that $m + 2 \leq \sum_{i=1}^n t_i \leq 3|B| + s + 2|A|$, or in other words,

$$0 < m - s \leq 3|B| + 2|A| - 2.$$

Putting these two inequalities together, we get

$$(3|B| + 2|A| - 2)^2 \geq l_1|B| + \left(\sum_{i,j \in A, i \neq j} s_i s_j\right) + 2|A|. \tag{1}$$

To bound l_1 from above, we consider a few cases. If $|B| = 0$, we have $(2n-2)^2 \geq \sum_{i,j \in [n], i \neq j} s_i s_j + 2n \geq 4n(n-1) + 2n$, which is impossible, and so we must have $|B| > 0$. If $|A| = 0$, we have $(3n-2)^2 \geq nl_1$, and so $l_1 \leq 9n - 12 + \frac{4}{n}$. If $|A| = 1$, we have $(3n-3)^2 \geq (n-1)l_1 + 2$, and so $l_1 \leq 9n - 9 - \frac{2}{n-1}$.

For the final case where $|A| \geq 2$, we first observe that

$$\begin{aligned}\sum_{i,j \in A, i \neq j} s_i s_j &= \sum_{i \in A} s_i \left(\sum_{j \in A, j \neq i} s_j\right) = \sum_{i \in A} s_i(s - s_i) \\ &\geq \sum_{i \in A} 2(s-2) \qquad \text{(as } 2 \leq s_i \leq s-2 \text{ for every } i \in A\text{)} \\ &= 2|A|(s-2),\end{aligned}$$

Letting $s = 2|A| + k$ for some $k \geq 0$, we see that

$$\sum_{i,j \in A, i \neq j} s_i s_j \geq 2|A|(2|A| + k - 2) = 4|A|^2 + 2k|A| - 4|A|.$$

Together with Inequality (1), we have

$$\begin{aligned} (3|B| + 2|A| - 2)^2 &\geq l_1|B| + 4|A|^2 + 2k|A| - 2|A| \\ 9|B|^2 + 12|A||B| - 12|B| - 6|A| + 4 &\geq l_1|B| + 2k|A| \\ \implies \quad \frac{2k|A|}{|B|} + l_1 &\leq 9|B| + 12|A| - 6 - \frac{6(|A| + |B|) - 4}{|B|} \\ \implies \quad l_1 &\leq 9n - 6 + 3|A| - \frac{6n - 4}{n - |A|}. \end{aligned}$$

It is straightforward that in the range of $0 < x < n$, the function $3x - \frac{6n-4}{n-x}$ is maximised when $x = n - \sqrt{\frac{6n-4}{3}}$, with the maximum value being $3n - 2\sqrt{18n - 12}$. Therefore, $l_1 \leq 12n - 2\sqrt{18n - 12} - 6$.

We now see that for $n \geq 3$,

$$\begin{aligned} l_1 &\leq \max\left\{9n - 12 + \frac{4}{n}, 9n - 9 - \frac{2}{n-1}, 12n - 2\sqrt{18n - 12} - 6\right\} \\ &= 12n - 2\sqrt{18n - 12} - 6. \end{aligned}$$

Before we proceed, we make a relevant observation. Pick

$$x_0 \in \left\{ \left\lfloor n - \sqrt{\frac{6n-4}{3}} \right\rfloor, \left\lceil n - \sqrt{\frac{6n-4}{3}} \right\rceil \right\}$$

such that $3x - \frac{6n-4}{n-x}$ attains the larger value. It can be verified carefully but elementarily that the largest odd integer smaller than or equal to $9n - 6 + 3x_0 - \frac{6n-4}{n-x_0}$ coincides with that of $12n - 2\sqrt{18n - 12} - 6$ for every $n \geq 3$.

Now, consider the n-path forest T' with $m = 3n + x_0 - 2$ and path orders as follows:

1. $l'_i = 4m - 2$ (and so $B_m(l'_i) = 4$) for each $i > n - x_0$;

2. $l'_1, l'_2, \ldots, l'_{n-x_0}$ are odd and any two of them are equal or differ by two.

Such a path forest exists as the second requirement can be satisfied because m is odd if and only if $n - x_0$ is odd. Note that $\sum_{i=1}^{n-x_0} l'_i$ is equal to

$$\begin{aligned} m^2 - x_0(4m - 2) = m(m - 4x_0) + 2x_0 &= m(3n - 3x_0 - 2) + 2x_0 \\ &= 3m(n - x_0) - 2m + 2x_0 = (9n - 6 + 3x_0)(n - x_0) - 6n + 4. \end{aligned}$$

Hence, l'_1 must be the largest odd integer smaller than or equal to $9n - 6 + 3x_0 - \frac{6n-4}{n-x_0}$. It is straightforward to see that $B_m(l'_i) = 3$ for $1 \leq i \leq n - x_0$, and thus T' is impossibly burnable. From our earlier observation, it follows that M_n is bounded below by the largest odd integer smaller than or equal to $12n - 2\sqrt{18n-12} - 6$.

Therefore, to complete our proof, we shall show that l_1 is odd for any optimal impossibly burnable T with the length of its shortest path maximised. Indeed, with a more careful analysis, such T would have $t_i = 3$ for all $i \in B$, and furthermore, $t_i = 4$ for all $i \in A$, assuming $n \geq 8$ for the latter as our previous observations and discussions have shown that M_n is as claimed in the theorem for $3 \leq n \leq 7$. (See Appendix B for details.) Hence, assuming l_1 is not odd, it implies $l_1 \geq 4m - 2$. Noting that $m \leq 4n - 2$,

$$m^2 - \sum_{i=1}^{n} l_i \leq m^2 - n(4m-2) = m(m-4n) + 2n \leq 2n - 2m < 0,$$

which is a contradiction. □

3 Conclusion

For every $n \geq 4$, let Δ_n denote the least integer with the property that whenever T is a deficient n-path forest with $l_1 \geq \Delta_n$, then T is impossibly burnable. In our verification of Conjecture 2.7 for small values of n, we have observed that $\Delta_n = L_{n-1}$ for $n \in \{4, 5, 6, 7\}$. In fact, our conjectures propose that Δ_n exists and $\Delta_n \leq L_{n-1}$ for all $n \geq 4$. Upon further analysis using Matlab, we have found that there is only one deficient 7-path forest with $l_1 = 45$ that is not impossibly burnable, namely, the path forest $(45, 45, 45, 45, 72, 74, 74)$, confirming $\Delta_7 = L_6 = 46$. The scarceness of such deficient path forests has led us to anticipate the likelihood of $\Delta_n < L_{n-1}$ for larger n. The study of the values of Δ_n potentially poses another challenging open problem.

Theorem 1.3 implies that the values of L_n are known if the answer to Question 1.2 is affirmative. However, although impossibly burnability is a simpler concept, Conjecture 2.7 is surprisingly nontrivial. Furthermore, as pointed out above, L_{n-1} is not necessarily the tight lower bound on the order of the shortest path for the conclusion to be true, and thus its essentiality in a possible proof by induction is doubtful. As an alternative approach in light of Theorem 1.3, we now propose another conjecture, the truth of which implies a good asymptotic approximation to the values of L_n.

Conjecture 3.1. *$L_n \leq 12n$ for all $n \geq 2$.*

Note that $L_n \geq M_n + 1$ for all $n \geq 2$. By Theorem 1.3, we have $M_n \sim 12n$. Therefore, assuming Conjecture 3.1 holds, it follows that $L_n \sim 12n$, that is, $\frac{L_n}{12n} \to 1$ as $n \to \infty$.

Acknowledgements

The second author acknowledges the support for this research by the Malaysian Ministry of Higher Education for Fundamental Research Grant Scheme with Project Code: FRGS/1/2023/STG06/USM/02/7.

References

K. Ando, S. Gervacio, and M. Kano. Disjoint subsets of integers having a constant sum. *Discrete Math.*, 82(1):7–11, 1990. doi: 10.1016/0012-365X(90)90040-O.

S. Bessy, A. Bonato, J. Janssen, D. Rautenbach, and E. Roshanbin. Burning a graph is hard. *Discrete Appl. Math.*, 232:73–87, 2017. doi: 10.1016/j.dam.2017.07.016.

A. Bonato. A survey of graph burning. *Contrib. Discrete Math.*, 16(1):185–197, 2021.

A. Bonato and T. Lidbetter. Bounds on the burning numbers of spiders and path-forests. *Theoret. Comput. Sci.*, 794:12–19, 2019. doi: 10.1016/j.tcs.2018.05.035.

A. Bonato, J. Janssen, and E. Roshanbin. How to burn a graph. *Internet Math.*, 12(1-2):85–100, 2016. doi: 10.1080/15427951.2015.1103339.

F.-L. Chen, H.-L. Fu, Y. Wang, and J. Zhou. Partition of a set of integers into subsets with prescribed sums. *Taiwanese J. Math.*, 9(4):629–638, 2005. doi: 10.11650/twjm/1500407887.

S. Das, S. Ranjan Dev, A. Sadhukhan, U. k. Sahoo, and S. Sen. Burning spiders. In *Algorithms and discrete applied mathematics*, volume 10743 of *Lecture Notes in Comput. Sci.*, pages 155–163. Springer, Cham, 2018. doi: 10.1007/978-3-319-74180-2_13.

H. Enomoto and M. Kano. Disjoint odd integer subsets having a constant even sum. *Discrete Math.*, 137 (1-3):189–193, 1995. doi: 10.1016/0012-365X(93)E0128-Q.

H.-L. Fu and W. H. Hu. Disjoint odd integer subsets having a constant odd sum. *Discrete Math.*, 128 (1-3):143–150, 1994. doi: 10.1016/0012-365X(94)90108-2.

M. Hiller, A. M. C. A. Koster, and E. Triesch. On the burning number of p-caterpillars. In *Graphs and combinatorial optimization: from theory to applications—CTW2020 proceedings*, volume 5 of *AIRO Springer Ser.*, pages 145–156. Springer, Cham, 2021. doi: 10.1007/978-3-030-63072-0_12.

H. Liu, X. Hu, and X. Hu. Burning number of caterpillars. *Discrete Appl. Math.*, 284:332–340, 2020. doi: 10.1016/j.dam.2020.03.062.

A. Lladó and J. Moragas. On the modular sumset partition problem. *European J. Combin.*, 33(4):427–434, 2012. doi: 10.1016/j.ejc.2011.09.001.

K. J. Ma, H. S. Zhou, and J. Q. Zhou. On the ascending star subgraph decomposition of star forests. *Combinatorica*, 14(3):307–320, 1994. doi: 10.1007/BF01212979.

S. Norin and J. Turcotte. The burning number conjecture holds asymptotically. *J. Combin. Theory Ser. B*, 168:208–235, 2024. doi: 10.1016/j.jctb.2024.05.003.

T. S. Tan and W. C. Teh. Graph burning: tight bounds on the burning numbers of path forests and spiders. *Appl. Math. Comput.*, 385:Paper No. 125447, 9, 2020. doi: 10.1016/j.amc.2020.125447.

T. S. Tan and W. C. Teh. Burnability of double spiders and path forests. *Appl. Math. Comput.*, 438:Paper No. 127574, 12, 2023. doi: 10.1016/j.amc.2022.127574.

Appendix A

Henceforth, unless stated otherwise, T is a 7-path forest of order m^2 for some m. Note that when $l_1 \geq 46$, m is at least 18. For m up to 22, we obtained the complete list of all well-burnable 7-path forests of order m^2 and thus the corresponding list of deficient 7-path forests thereafter. From here, it is easy to filter out those with $l_1 \geq 46$. As a matter of fact, we saved the lists of well-burnable 7-path forests for $m = 21$ and $m = 22$ in many parts, as the memory required for them to be saved as a single array is too large. Hence, we managed to verify Conjecture 2.7 (henceforth, our conjecture) for the case of seven paths for m up to 22 this way. However, we can no longer proceed in this manner for larger m. Therefore, in this appendix, we give a brief account on how we go around it. Table 1 gives some statistics from our Matlab search.

m	# well-burnable 7-path forests with $l_1 \geq 46$	# deficient (impossibly burnable) 7-path forests with $l_1 \geq 46$
18	2	0
19	5553	178
20	162074	1588
21	1504741	5460
22	8134818	9536
23	31981775	9572
24	101854804	1294
25	279148714	79
26	683537772	4
27	1532853276	0

Tab. 1: Verification of Conjecture 2.7 for the case of seven paths

For convenience, we use the following definition.

Definition 3.2. Let $n \geq 2$. Suppose $T = (l_1, l_2, \ldots, l_n)$ and $T' = (l'_1, l'_2, \ldots, l'_n)$ are path forests of orders m^2 and $(m+1)^2$, respectively. If there exists $1 \leq i \leq n$ such that $l'_i = l_i + (2m+1)$ and $l'_j = l_j$ for all $j \neq i$, then we say that T' is an *extension* of T (or T is a *reduction* of T') at the ith component.

Suppose $|T| = 23^2$ and $l_1 \geq 46$. Note that $l_7 \geq 76$ and thus $l_7 - 45 \geq 31$. If T is deficient, then $T' = (l_1, l_2, \ldots, l_6, l_7 - 45)$ is deficient. Hence, we first identified all such path forests T that are potentially deficient. Such T can be obtained from a deficient 7-path forest T' of order 22^2 with $31 \leq l'_1 \leq 45$ and $l'_2 \geq 46$ by extension at the first component or from a deficient 7-path forest T' of order 22^2 with $l'_1 \geq 46$ by extension at any of the seven components. This way, we obtained 36529 potentially deficient 7-path forests of order 23^2 with $l_1 \geq 46$. From here, we noticed immediately that 9572 among them are impossibly burnable. Hence, to verify our conjecture for $m = 23$, it suffices to check that the remaining 26957 path forests are all well-burnable. To show this, we first extracted from the list of all deficient 7-path forests with $m = 22$, a sublist of those with $l_1 \geq 1$ and $l_2 \geq 46$. We exhaustively checked and found that for each of the 26957 path forests T, at least one of its seven reductions is not in the said sublist and thus T is well-burnable.

To deal with larger m, we obtained the following two lists.

List A. All 9612 deficient 7-path forests of order 23^2 with $l_1 \geq 42$.

List B. All 9931471 deficient 7-path forests of order 23^2 with $l_1 \geq 1$ and $l_2 \geq 49$.

There are three ways to obtain a well-burnable path forest of order 23^2 with its shortest path having order at least 42:

1. adding a path of order 45 to any well-burnable 6-path forest T such that $|T| = 22^2$ and $l_1 \geq 42$;
2. adding 45 vertices to any of the paths of any well-burnable 7-path forest T such that $|T| = 22^2$ and $l_1 \geq 42$;
3. adding 45 vertices to the first path of any well-burnable 7-path forest T such that $|T| = 22^2$, $l_1 \leq 41$, and $l_2 \geq 42$.

This way, we obtained the complete list of 65485064 well-burnable 7-path forests of order 23^2 with $l_1 \geq 42$. From here, List A was obtained.

Before we proceed, an observation about List A will be useful later. Among the 9612 members of List A, there are only 40 of them with $42 \leq l_1 \leq 44$ and none with $l_1 = 45$. Furthermore, $l_7 \geq 207$ for each of the 40 path forests. (Note that $B_{23}(205) = 5$ while $B_{23}(207) = 7$.)

Similarly, we obtained the complete list of 311596739 well-burnable 7-path forests of order 23^2 with $l_2 \geq 49$. From here, List B was obtained.

From both List A and List B, we extracted the sublist of deficient 7-path forests of order 23^2 with $l_1 \geq 42$ and $l_2 \geq 49$ and they coincide. Hence, this gives an assurance that our lists are correct. Furthermore, for our purposes, some sublists from the lists we obtained are probably sufficient, but as things developed, we ended up with those lists, as well as other lists that are not reported here.

Now, suppose $|T| = 24^2$ and $l_1 \geq 46$. Note that $l_7 - 47 \geq 36$. If T is deficient, then $T' = (l_1, l_2, \ldots, l_6, l_7 - 47)$ is deficient. Hence, similarly, we first identified all such path forests T that are potentially deficient. Such T can be obtained from a deficient 7-path forest T' of order 23^2 with $36 \leq l_1' \leq 45$ and $l_2' \geq 46$ by extension at the first component or from a deficient 7-path forest T' of order 23^2 with $l_1' \geq 46$ by extension at any of the seven components. This way, we obtained 34959 potentially deficient 7-path forests of order 24^2 with $l_1 \geq 46$. From here, we noticed immediately that 1294 among them are impossibly burnable. Hence, to verify our conjecture for $m = 24$, it suffices to check that the remaining 33665 path forests are all well-burnable.

Note that if one of the paths of T has order 47, then deleting this path would result in a 6-path forest that is well-burnable because $L_6 = 46$. Furthermore, only 34 among those 33665 path forests have $l_1 = 46$ and only one among these 34 path forests has $l_2 \neq 47$, namely the path forest $(46, 49, 49, 49, 49, 92, 242)$, which is 24-burnable as $(46, 49, 49, 49, 49, 242)$ is 22-burnable. Hence, filtering out those with $l_1 \in \{46, 47\}$, we are left with 10712 path forests to be checked. If T is any of the 10712 path forests, we observed that $l_1 \geq 49$ and found that at least one of its seven reductions is outside List B, and thus T is well-burnable.

To deal with the case $m = 25$, we consider the following three lists separately.

List C. All deficient 7-path forests of order 25^2 with $l_1 \geq 46$ and $l_7 \geq 95$.

List D. All deficient 7-path forests of order 25^2 with $l_1 \geq 46$ and $91 \leq l_7 \leq 94$.

List E. All deficient 7-path forests of order 25^2 with $l_1 \geq 46$ and $l_7 = 90$.

Suppose $|T| = 25^2$ and $l_1 \geq 46$. If $l_7 = 90$, then $l_6 = 90$ and thus T is well-burnable because $(l_1, \ldots, l_4, l_5 - 45)$ is 20-burnable as $L_5 = 36$. Hence, List E is empty.

Now, suppose $91 \leq l_7 \leq 94$ and thus $l_6 \geq 89$. Consider the path forest $T'' = (l_1, \ldots, l_5, l_6 - 47, l_7 - 49)$. If T were to be deficient, then T'' would be among the 40 members in List A specifically mentioned earlier as $42 \leq l_1'' \leq 45$. However, from our earlier observation, that would force $l_5 \geq 207$, which is impossible as $l_5 \leq l_7$. Hence, List D is empty as well.

To deal with List C, we first identified all 7-path forests T of order 25^2 with $l_1 \geq 46$ and $l_7 \geq 95$ that are potentially deficient. Such T can be obtained from a deficient (and impossibly burnable) 7-path forest T' of order 24^2 with $l_1' \geq 46$ by extension at any of the seven components. This way, we obtained 5042 path forests and 79 among them are impossibly burnable. Hence, to verify our conjecture for $m = 25$, it suffices to check that the remaining 4963 path forests are all well-burnable.

Suppose T is any of the 4963 potentially deficient 7-path forests. We noticed that the first path of T is either $49, 51, 53, 55$, or 57. If the first path has order 49, then deleting this path will result in a well burnable 6-path forest as $L_6 = 46$. Hence, filtering out those where $l_1 = 49$, we are left with 751 path forests. If any reduction T' of T has $l_1' \geq 46$ but is not impossibly burnable, then T' is well-burnable and thus is T. Filtering further based on this, we are left with a much shorter list of 45 path forests. Suppose T is any of the 45 path forests. We noticed that one of the paths of T has order 94. Let T' be the 6-path forest obtained from T by deleting this path and deleting 47 vertices from the longest path. Then T' is well-burnable as $L_6 = 46$ and thus T is well-burnable. Therefore, the 79 impossibly burnable 7-path forests are the only deficient 7-path forests of order 25^2 with $l_1 \geq 46$.

Before we proceed to $m = 26$, we make a note that among the 79 impossibly burnable path forests, there are exactly four with $l_1 = 53$, namely:

$$(53, 53, 53, 53, 53, 53, 307), (53, 53, 53, 53, 53, 55, 305),$$

$$(53, 53, 53, 53, 53, 57, 303), (53, 53, 53, 53, 55, 55, 303).$$

Suppose $|T| = 26^2$ and $l_1 \geq 46$. If T is deficient, then $T' = (l_1, \ldots, l_6, l_7 - 51)$ is deficient and $l_7 - 51 \geq 46$. Hence, we first obtained the list of 7-path forests T of order 26^2 with $l_1 \geq 46$ that are potentially deficient. There are 292 members in this list of potentially deficient path forests and among them, only four are impossibly burnable and they are extensions of the four above, namely:

$$(53, 53, 53, 53, 53, 53, 358), (53, 53, 53, 53, 53, 55, 356),$$

$$(53, 53, 53, 53, 53, 57, 354), (53, 53, 53, 53, 55, 55, 354).$$

Suppose T is any of the remaining 288 potentially deficient path forests. If the first path of T has order 51, then deleting this path will result in a well-burnable 6-path forest as $L_6 = 46$ and thus T is well-burnable. Hence, filtering out those where $l_1 = 51$, we are left with 14 path forests where $l_1 = 53$ for each of them coincidently. However, the reduction of each of the 14 path forests at the last component can no longer be one of the four impossibly burnable path forests for $m = 25$ with $l_1 = 53$. It follows that T is well-burnable.

Finally, we can see that no path forest of order 27^2 with $l_1 \geq 46$ is deficient. Suppose T is one such path forest and is deficient. Then $T' = (l_1, \ldots, l_6, l_7 - 53)$ is deficient and thus T' is one of the four impossibly burnable path forests for $m = 26$ above. However, this implies that $l_1 = 53$ (so is $l_2 = l_3 = 53$). Since $(l_2, l_3, \ldots, l_7)$ is well-burnable as $L_6 = 46$, it follows that T is well-burnable, which gives a contradiction. By induction, no path forest of order m^2 with $l_1 \geq 46$ for $m \geq 27$ is deficient.

Appendix B

In this appendix, we provide some details about the careful analysis referred to in the proof of Theorem 1.3.

Suppose there is an $i \in B$ such that $t_i \leq 2$. Then we would have

$$m + 2 \leq \sum_{i=1}^{n} t_i \leq 3(|B| - 1) + 2 + s + 2|A|$$

and so Inequality (1) would become

$$(3|B| + 2|A| - 3)^2 \geq l_1|B| + \left(\sum_{i,j \in A, i \neq j} s_i s_j\right) + 2|A|.$$

As before, it can be shown that $|B| > 0$; if $|A| = 0$, then $l_1 \leq 9n - 18 + \frac{9}{n}$; and if $|A| = 1$, then $l_1 \leq 9n - 15 - \frac{1}{n-1}$.

If $|A| \geq 2$, letting $s = 2|A| + k$, following the same exact analysis would lead to

$$\begin{aligned} (3|B| + 2|A| - 3)^2 &\geq l_1|B| + 4|A|^2 + 2k|A| - 2|A| \\ \implies \quad l_1 &\leq 9n - 8 + 3|A| - \frac{10n - 9}{n - |A|}. \end{aligned}$$

It is straightforward that in the range of $0 < x < n$, the function $3x - \frac{10n-9}{n-x}$ is maximised when $x = n - \sqrt{\frac{10n-9}{3}}$, with the maximum value being $3n - 2\sqrt{30n - 27}$. Therefore, $l_1 \leq 12n - 2\sqrt{30n - 27} - 8$.

We now see that for $n \geq 3$,

$$\begin{aligned} l_1 &\leq \max\left\{9n - 18 + \frac{9}{n}, 9n - 15 - \frac{1}{n-1}, 12n - 2\sqrt{30n - 27} - 8\right\} \\ &= 12n - 2\sqrt{30n - 27} - 8. \end{aligned}$$

However, $(12n - 2\sqrt{18n - 12} - 6) - (12n - 2\sqrt{30n - 27} - 8) > 4$ for $n \geq 3$. Hence, l_1 is not maximised. Therefore, for an impossibly burnable n-path forest to be optimal such that l_1 is maximised, we shall need $t_i = 3$ for all $i \in B$.

Now, suppose there is an $i \in A$ such that $t_i \geq 5$. We proceed from Inequality (1), namely,

$$(3|B| + 2|A| - 2)^2 \geq l_1|B| + \left(\sum_{i,j \in A, i \neq j} s_i s_j\right) + 2|A|.$$

We have seen that $|B| > 0$; if $|A| = 0$, then $l_1 \leq 9n - 12 + \frac{4}{n}$; and if $|A| = 1$, then $l_1 \leq 9n - 9 - \frac{2}{n-1}$. Also, if $|A| = 2$, then $l_1 \leq 9n - 6 + 3(2) - \frac{6n-4}{n-2} = 9n - 6 - \frac{8}{n-2}$.

For the final case where $|A| \geq 3$, say $s_{i_0} \geq 3$, we first observe that

$$\begin{aligned}
\sum_{i,j\in A, i\neq j} s_i s_j &= \sum_{i\in A} s_i \left(\sum_{j\in A, j\neq i} s_j \right) = \sum_{i\in A} s_i(s-s_i) \\
&\geq \sum_{i\in A\setminus\{i_0\}} 2(s-2) + s_{i_0}(s-s_{i_0}) \\
&\geq 2(|A|-1)(s-2) + 3(s-3) \qquad (\text{as } 3 \leq s_{i_0} \leq s-4\ ) \\
&= (2|A|+1)(s-2) - 3.
\end{aligned}$$

Letting $s = 2|A|+1+k$ for some $k \geq 0$, we see that

$$\sum_{i,j\in A, i\neq j} s_i s_j \geq (2|A|+1)(2|A|-1+k) - 3 = 4|A|^2 - 4 + k(2|A|+1).$$

Together with Inequality (1), we have

$$\begin{aligned}
9|B|^2 + 12|A||B| - 12|B| - 10|A| + 8 &\geq l_1|B| + k(2|A|+1) \\
\implies \qquad l_1 &\leq 9n - 2 + 3|A| - \frac{10n-8}{n-|A|}.
\end{aligned}$$

It is straightforward that in the range of $0 < x < n$, the function $3x - \frac{10n-8}{n-x}$ is maximised when $x = n - \sqrt{\frac{10n-8}{3}}$, with the maximum value being $3n - 2\sqrt{30n-24}$. Therefore, $l_1 \leq 12n - 2\sqrt{30n-24} - 2$.

We now see that for $n \geq 8$,

$$\begin{aligned}
l_1 &\leq \max\left\{ 9n - 12 + \frac{4}{n}, 9n - 9 - \frac{2}{n-1}, 9n - 6 - \frac{8}{n-2}, 12n - 2\sqrt{30n-24} - 2 \right\} \\
&= \begin{cases} 12n - 2\sqrt{30n-24} - 2 & \text{if } n > 8 \\ 9n - 6 - \frac{8}{n-2} & \text{if } n = 8. \end{cases}
\end{aligned}$$

However, $(12n - 2\sqrt{18n-12} - 6) - (12n - 2\sqrt{30n-24} - 2) > 2$ for $n > 8$ and $(12n - 2\sqrt{18n-12} - 6) - (9n - 6 - \frac{8}{n-2}) > 2$ for $n = 8$. Hence, l_1 is not maximised. Therefore, for an impossibly burnable n-path forest to be optimal such that l_1 is maximised when $n \geq 8$, we shall need $t_i = 4$ for all $i \in A$. (Note that $(45, 45, 45, 45, 74, 107)$ is an impossibly burnable 6-path forest such that $l_1 = M_6 = 45$ is maximised, but $B_m(l_6) = 5$ where $m = 19$.)