

An improved algorithm for the vertex cover P_3 problem on graphs of bounded treewidth

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Given a graph $G = (V, E)$ and a positive integer $t \geq 2$, the task in the vertex cover P_t (VCP_t) problem is to find a minimum subset of vertices $F \subseteq V$ such that every path of order t in G contains at least one vertex from F . The VCP_t problem is NP-complete for any integer $t \geq 2$. Recently, the authors presented a dynamic programming algorithm with runtime $4^p \cdot n^{O(1)}$ that can solve the VCP_3 problem in any n -vertex graph given together with its tree decomposition of width at most p . In this paper, we propose an improvement of it and improved the time-complexity to $3^p \cdot n^{O(1)}$.

The connected vertex cover P_3 ($CVCP_3$) problem is the connected variation of the VCP_3 problem where $G[F]$ is required to be connected. Using the Cut&Count technique, we give a randomized algorithm with runtime $4^p \cdot n^{O(1)}$ that can solve the $CVCP_3$ problem in any n -vertex graph given together with its tree decomposition of width at most p .

Keywords: Combinatorial optimization, Vertex cover P_3 problem, Connected vertex cover P_3 problem, Treewidth, Dynamic programming

1 Introduction

In this paper, we consider only finite, simple and undirected graphs G . For a graph G , we denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. Unless stated otherwise, let $n := |V(G)|$ and $m := |E(G)|$. As usual, P_t denotes the path on t vertices. Given a graph G and a positive integer $t \geq 2$, a subset of vertices F is called a vertex cover P_t (VCP_t) set if every path of order t contains at least one vertex from F . The task in the VCP_t problem is to find a minimum VCP_t set in G .

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The VCP_t problem is NP-complete for any integer $t \geq 2$ Lewis and Yannakakis (1980). Clearly, the VCP_2 problem is the well-known vertex cover problem. Thus, the VCP_t problem is a natural extension of the vertex cover problem.

In this paper, we restrict our attention to the VCP_3 problem. In the literature, the VCP_3 problem is also known as the Bounded-Degree-one Deletion (1-BDD) problem. Given a graph $G = (V, E)$, the task in the 1-BDD problem is to find a minimum subset of vertices F such that the graph $G[V \setminus F]$ has maximum degree at most one. For the VCP_3 problem, Kardoš et al. Kardoš et al. (2011) gave an exact algorithm with runtime $1.5171^n \cdot n^{O(1)}$ and a randomized approximation algorithm with an expected approximation ratio $23/11$. Chang et al. Chang et al. (2014) presented an improved exact algorithm with runtime $1.4658^n \cdot n^{O(1)}$. For weighted version of the VCP_3 problem, Tu and Zhou Tu and Zhou (2011a,b) presented two 2-approximation algorithms using the primal-dual method and the local-ratio method.

On the other hand, many efforts have also been made in study of parameterized complexity of the VCP_3 problem with the size k of the solution as the parameter. Using the iterative compression technique, a parameterized algorithm running in time $2^k \cdot n^{O(1)}$ was given independently by Fellows et al. Fellows et al. (2011) and Moser et al. Moser et al. (2012). Wu Wu (2015) gave an improved algorithm with runtime $1.882^k \cdot n^{O(1)}$ using the Measure & Conquer approach. Katrenič Katrenič (2016) gave further improvement so that running time reaches $1.8172^k \cdot n^{O(1)}$.

In this paper, we consider the VCP_3 problem on graphs of bounded treewidth. The treewidth (tw) is a graph parameter that plays a fundamental role in various graph algorithms. Very roughly, treewidth captures how similar a graph is to a tree. It is well-known that many NP-hard graph problem can be solved efficiently if the input graph G has bounded treewidth. For example, an algorithm to solve the vertex cover problem running in time $4^{tw} \cdot n^{O(1)}$ was given in Kleinberg and Tardos (2005), while the book Niedermeier (2006) presented an algorithm running in time $2^{tw} \cdot n^{O(1)}$. For one more example, Alber et al. Alber and Niedermeier (2002) gave a $4^{tw} \cdot n^{O(1)}$ time algorithm for the dominating set problem, improving over the natural $9^{tw} \cdot n^{O(1)}$ algorithm of Telle and Proskurowski Telle and Proskurowski (1993). Using fast subset convolution Björklund et al. (2007); Cygan et al. (2015), the running time of the algorithm of Alber et al. can be improved and reaches $3^{tw} \cdot n^{O(1)}$.

Recently, Tu et al. Tu et al. (2017) presented a dynamic programming algorithm running in time $4^{tw} \cdot n^{O(1)}$ for the VCP_3 problem on graphs of bounded treewidth.

Theorem 1 *Tu et al. (2017) Let G be an n -vertex graph given together with its tree decomposition of width at most p . Then the VCP_3 problem in G can be solved in time $4^p \cdot n^{O(1)}$.*

Some families of graphs with bounded treewidth include the cactus graphs, pseudoforests, series-parallel graphs, outerplanar graphs, Halin graphs, Apollonian networks, and so on. Thus, the VCP_3 problem on these families of graphs can be solved in polynomial time. Moreover, in order to solve the VCP_3 problem on these families of graphs more efficiently, it is necessary to obtain an improved algorithm on graphs of bounded treewidth.

We improve the result in Tu et al. (2017) in the following ways.

1. Using a refined variant of nice tree decompositions, we present a new dynamic programming algorithm running in time $3^{tw} \cdot n^{O(1)}$ for the VCP_3 problem.

2. Consider the connected vertex cover P_3 ($CVCP_3$) problem, a variation of the VCP_3 problem where it is required that the graph induced by the VCP_3 set is connected. Using the Cut&Count technique, we give a randomized algorithm with runtime $4^{tw} \cdot n^{O(1)}$ for the $CVCP_3$ problem.

It should be pointed out that the dynamic programming and the Cut&Count technique are two widely used methods. The dynamic programming can give exact algorithms running in time $2^{O(tw)} \cdot n^{O(1)}$ for many NP-hard problems on graphs with a given tree decomposition of width tw Cygan et al. (2015), while the Cut&Count method can give randomized algorithms for a wide range of connectivity problems running in time $c^{O(tw)} \cdot n^{O(1)}$ for a small constant c Cygan et al. (2011). The main work of this paper is to give related results for the VCP_3 problem and its connected variation using the two methods.

In Pilipczuk (1994), Pilipczuk introduced the existential counting modal logic (ECML), then extended ECML by adding connectivity requirements and proved that problems expressible in the extension are tractable in single exponential time when parameterized by treewidth; however, using randomization. Thus, the work by Pilipczuk already shows that $CVCP_3$ can be solved in single-exponential time. Our result provides a better exponential dependence than the very general result in Pilipczuk (1994).

The remaining part of this paper is organized as follows. In Section 2, we give some notation and introduce formally the concepts of tree decomposition and treewidth. In Section 3, we present a new dynamic programming algorithm for the VCP_3 problem on graphs of bounded treewidth. In Section 4, a randomized algorithm for the $CVCP_3$ problem is given.

2 Preliminaries

In this section we give some notation which we make use of in the paper. Let $G = (V, E)$ be a graph. For $v \in V$, denote by $N_G(v)$ the set of neighbors of v in G . Let $d_G(v) := |N_G(v)|$. For a subset $V' \subseteq V$, we denote by $G[V']$ the subgraph induced by V' and write $G - V'$ as an abbreviation for the induced subgraph $G[V \setminus V']$. For all terminology and notation not defined here, we refer the reader to Bondy and Murty (2008).

Definition 1 A tree decomposition of a graph $G = (V, E)$ is a pair $\mathcal{T} = (T, \{X_i : i \in I\})$, where each $X_i \subseteq V$ is called a bag, and T is a rooted tree with the elements of I as nodes. The following three conditions must hold:

1. $\bigcup_{i \in I} X_i = V$,
2. for all edges $uv \in E$ there exists an $i \in I$ with $u \in X_i$ and $v \in X_i$, and
3. for all $i, j, k \in I$, if j lies on the path from i to k in T then $X_i \cap X_k \subseteq X_j$.

The width of a tree decomposition $\mathcal{T} = (T, \{X_i : i \in I\})$ is $\max_{i \in I} |X_i| - 1$. The treewidth $tw(G)$ of a graph G is defined as the minimum width over all tree decompositions of G . A tree has treewidth one, many well-studied graph families also have bounded treewidth.

Given a graph $G = (V, E)$, if p is a fixed constant, the problem to determine whether the treewidth of G is at most p can be decided in linear time and a corresponding tree decomposition can be constructed in linear time with a high constant factor Bodlaender (1996). There are also several good heuristic algorithms which can construct tree decompositions of small width, if existing, and often work well in practice (see Bodlaender and Koster (2010)).

Given an n -vertex graph $G = (V, E)$, a tree decomposition $\mathcal{T} = (T, \{X_i : i \in I\})$ can be converted (in polynomial time) in a nice tree decomposition of the same width p and with $O(pn)$ nodes Kloks (1994): here the tree T is rooted and binary, and the following conditions are satisfied.

- All the leaves as well as the root of T contain empty bags.

- All non-leaf nodes of T are of three types:
 - **Introduce node:** a node t with exactly one child t' such that $|X_t| = |X_{t'}| + 1$ and $X_t = X_{t'} \cup \{v\}$ for some vertex $v \notin X_{t'}$.
 - **Forget node:** a node t with exactly one child t' such that $|X_t| = |X_{t'}| - 1$ and $X_t = X_{t'} \setminus \{v\}$ for some vertex $v \in X_{t'}$.
 - **Join node:** a node t with two children t_1 and t_2 such that $X_t = X_{t_1} = X_{t_2}$.

3 An improved algorithm for the VCP_3 problem on graphs of bounded treewidth

In this section, we will use a refined variant of nice tree decompositions to obtain an improved dynamic programming algorithm for solving the VCP_3 problem in graphs of bounded treewidth. Given a graph $G = (V, E)$ and its nice tree decomposition $\mathcal{T} = (T, \{X_i : i \in I\})$, we will add a new type of a node called an *introduce edge node* in the refined variant of nice decomposition, which is defined as follows.

- **Introduce edge node:** an internal node t of T , labeled with an edge $uv \in E(G)$ with exactly one child t' for which $u, v \in X_t = X_{t'}$. We say that edge uv is introduced at t .

We additionally require that every edge in E is introduced exactly once. The introduce edge node enables us to add edges one by one. A standard nice tree decomposition of width p can be transformed to the refined variant with the same width p and with $O(pn)$ nodes in polynomial time Cygan et al. (2015).

Let $\mathcal{T} = (T, \{X_i : i \in I\})$ be a refined variant of nice tree decomposition. Recall that then T is rooted at some node r . For each node t of T , we denote by V_t the union of all the bags present in the subtree of T rooted at t . Associate a subgraph G_t of G with each node t of T defined as follows.

$$G_t = (V_t, E_t = \{e : e \text{ is introduced in the subtree rooted at } t\})$$

Now, we present an improved dynamic programming algorithm on the refined variant of nice tree decomposition for solving the VCP_3 problem and give the following theorem.

Theorem 2 *Let G be an n -vertex graph given together with its refined variant of nice tree decomposition of width at most p . Then the VCP_3 problem in G can be solved in time $O(3^p \cdot pn^6)$.*

Proof: Let $\mathcal{T} = (T, \{X_i\}_{i \in I})$ be a refined variant of nice tree decomposition of G . Assume that r is the root node of T .

For each node t of T , a coloring of bag X_t is a mapping $f: X_t \rightarrow \{1, 0_0, 0_1\}$ assigning three different colors to vertices of the bag. Clearly, there exist $3^{|X_t|}$ colorings of X_t . For a coloring f of X_t , denote by $c[t, f]$ the minimum size of a VCP_3 set $F \subseteq V_t$ in G_t so that

- $F \cap X_t = f^{-1}(1)$. The meaning is that all vertices of X_t colored 1 have to be contained in F ,
- each vertex colored 0_0 is an isolated vertex in $G_t - F$,
- each vertex colored 0_1 has degree 1 in $G_t - F$.

We put $c[t, f] = +\infty$ if no such VCP_3 set F for t and f exists. Note that because $G_r = G$ and $X_r = \emptyset$, $c[r, \emptyset]$ is exactly the minimum size of a VCP_3 set in G .

Now, we give the recursive formulas for the values of $c[t, f]$. For each node t and a coloring f of X_t , we compute the value of $c[t, f]$ based on the values computed for the children of t . By applying the formulas in a bottom-up manner on T , we can obtain the value of $c[r, \emptyset]$, which is exactly the minimum size of a VCP_3 set in G .

Leaf node.

For a leaf node t , $X_t = \emptyset$. Hence, there is only one empty coloring for X_t and $c[t, \emptyset] = 0$.

Introduce node.

Let t be an introduce node with a child t' so that $X_t = X_{t'} \cup \{v\}$ for some $v \notin X_{t'}$. Note that $V_t = V_{t'} \cup \{v\}$ and $E_t = E_{t'}$, i.e., v is an isolated vertex in G_t . Hence, for a coloring f of X_t , we just need to be sure that if F is a VCP_3 set for f and t and v is not contained in F , then v must be an isolated vertex in $G_t - F$.

For a coloring f of X_t , we denote by $f|_{X_{t'}}$, the restriction of f to $X_{t'}$. It is easy to see that the following formula holds.

$$c[t, f] = \begin{cases} c[t', f|_{X_{t'}}] + 1, & \text{when } f(v) = 1, \\ c[t', f|_{X_{t'}}] & \text{when } f(v) = 0_0, \\ +\infty, & \text{when } f(v) = 0_1. \end{cases}$$

Introduce edge node.

Let t be an introduce edge node labeled with an edge $uv \in E(G)$ and t' be the child of t . Note that $V_t = V_{t'}$, $E_t = E_{t'} \cup \{uv\}$ and $uv \notin E_{t'}$. In other words, the only difference between the graph G_t and the graph $G_{t'}$ is that $uv \in E(G_t)$ and $uv \notin E(G_{t'})$. Hence, for a coloring f of X_t , we just need to be sure that if F is a VCP_3 set for f and t and neither u nor v is contained in F , then u and v are vertices of degree 1 in $G_t - F$. Thus, for a coloring f we can obtain the following formula.

$$c[t, f] = \begin{cases} c[t', f], & \text{when } f(u) = 1 \text{ or } f(v) = 1, \\ c[t', f'] & \text{when } (f(u), f(v)) = (0_1, 0_1), \\ +\infty, & \text{otherwise,} \end{cases}$$

where f' is a coloring of $X_{t'}$ and $f'(x) = \begin{cases} f(x), & \text{when } x \neq u \text{ and } x \neq v, \\ 0_0, & \text{when } x = u \text{ or } x = v. \end{cases}$

Forget node.

Let t be a forget node with a child t' so that $X_t = X_{t'} \setminus \{v\}$ for some $v \in X_{t'}$. Note that $V_t = V_{t'}$ and $G_t = G_{t'}$. For a coloring f of X_t and a color $\alpha \in \{1, 0_0, 0_1\}$, we define a coloring $f_{v \rightarrow \alpha}$ of $X_{t'}$ as follows:

$$f_{v \rightarrow \alpha}(x) = \begin{cases} f(x), & \text{when } x \neq v, \\ \alpha, & \text{when } x = v. \end{cases}$$

We claim that the following formula holds.

$$c[t, f] = \min\{c[t', f_{v \rightarrow 1}], c[t', f_{v \rightarrow 0_0}], c[t', f_{v \rightarrow 0_1}]\}$$

Suppose that F is a VCP_3 set for which the minimum is attained in the definition of $c[t, f]$. Thus, for any vertex $u \in V_t$, either $u \in F$ or $d_{G_t-F}(u) \leq 1$. If $v \in F$, then F is one of the sets considered in the definition of $c[t', f_{v \rightarrow 1}]$, and hence $c[t', f_{v \rightarrow 1}] \leq c[t, f]$. For $k \in \{0, 1\}$, if $v \notin F$ and v is a vertex of degree k in $G_t - F$, then F is one of the sets considered in the definition of $c[t', f_{v \rightarrow 0_k}]$, and hence $c[t', f_{v \rightarrow 0_k}] \leq c[t, f]$. Thus $\min\{c[t', f_{v \rightarrow 1}], c[t', f_{v \rightarrow 0_0}], c[t', f_{v \rightarrow 0_1}]\} \leq c[t, f]$.

On the other hand, each set that is considered in the definition of $c[t', f_{v \rightarrow 1}]$ is also considered in the definition of $c[t, f]$, and the same holds also for $c[t', f_{v \rightarrow 0_0}]$ and $c[t', f_{v \rightarrow 0_1}]$. This means $c[t, f] \leq c[t', f_{v \rightarrow 1}]$, $c[t, f] \leq c[t', f_{v \rightarrow 0_0}]$ and $c[t, f] \leq c[t', f_{v \rightarrow 0_1}]$.

In conclusion, $c[t, f] = \min\{c[t', f_{v \rightarrow 1}], c[t', f_{v \rightarrow 0_0}], c[t', f_{v \rightarrow 0_1}]\}$.

Join node.

Suppose that t is a join node with children t_1, t_2 so that $X_t = X_{t_1} = X_{t_2}$. Note that $V_t = V_{t_1} \cup V_{t_2}$, $V_{t_1} \cap V_{t_2} = X_t$ and $E_t = E_{t_1} \cup E_{t_2}$. It is important to point out that because every edge in $E(G)$ is introduced exactly once in the refined variant of nice tree decomposition, $E_{t_1} \cap E_{t_2} = \emptyset$.

We say that a pair of colorings f_1 of X_{t_1} , f_2 of X_{t_2} is consistent with a coloring f of X_t if for each vertex $v \in X_t$, the following conditions hold.

- (1) $f(v) = 1$ if and only if $f_1(v) = f_2(v) = 1$,
- (2) $f(v) = 0_0$ if and only if $f_1(v) = f_2(v) = 0_0$,
- (3) $f(v) = 0_1$ if and only if $(f_1(v), f_2(v)) \in \{(0_1, 0_0), (0_0, 0_1)\}$.

We will show that the following recursive formula holds:

$$c[t, f] = \min_{f_1, f_2} \{c[t_1, f_1] + c[t_2, f_2]\} - |f^{-1}(1)|,$$

where the minimum is taken over all pairs of colorings f_1 (of X_{t_1}), f_2 (of X_{t_2}) consistent with f .

On one hand, if F is a VCP_3 set for t and f , then $F \cap V_{t_1}$ and $F \cap V_{t_2}$ are VCP_3 sets for t_1 and f_1 , and t_2 and f_2 , for some pairs of colorings f_1, f_2 that are consistent with f . The reason is as follows: For a vertex $v \in X_t$, if $v \in F \cap X_t$, then $v \in F \cap X_{t_1}$ and $v \in F \cap X_{t_2}$; if $v \notin F$ and $d_{G_t-F}(v) = 0$, then $d_{G_{t_1}-F \cap V_{t_1}}(v) = 0$ and $d_{G_{t_2}-F \cap V_{t_2}}(v) = 0$; if $v \notin F$ and $d_{G_t-F}(v) = 1$, then v is a 1-degree vertex in $G_{t_1} - F \cap V_{t_1}$ and an isolated vertex in $G_{t_2} - F \cap V_{t_2}$, or v is an isolated vertex in $G_{t_1} - F \cap V_{t_1}$ and a 1-degree vertex in $G_{t_2} - F \cap V_{t_2}$, because there exist no edges between $V_{t_1} \setminus X_t$ and $V_{t_2} \setminus X_t$, $d_{G_t-F}(v) \leq 1$ for any vertex $v \in V_t \setminus (X_t \cup F)$. Hence,

$$\min_{f_1, f_2} \{c[t_1, f_1] + c[t_2, f_2]\} \leq |F \cap V_{t_1}| + |F \cap V_{t_2}| = |F| + |F \cap X_t| = c[t, f] + |f^{-1}(1)|,$$

where the minimum is taken over all pairs of colorings f_1, f_2 consistent with f .

On the other hand, given a coloring f of X_t , consider any pair of colorings f_1, f_2 that is consistent with f . Suppose that F_1 and F_2 are VCP_3 sets for t_1 and f_1 , t_2 and f_2 . Let $F := F_1 \cup F_2$. Since $f_1^{-1}(1) = f_2^{-1}(1) = f^{-1}(1)$, $F_1 \cap X_{t_1} = F_2 \cap X_{t_2} = F \cap X_t$ and $|F| = |F_1| + |F_2| - |f^{-1}(1)|$. Recall that the pair of colorings f_1, f_2 is consistent with f . For every vertex $v \in X_t \setminus F$, $d_{G_t-F}(v) \leq 1$, because there exist no edges between $V_{t_1} \setminus X_t$ and $V_{t_2} \setminus X_t$, $d_{G_t-F}(v) \leq 1$ for any vertex $v \in V_t \setminus (X_t \cup F)$. Thus, F is a VCP_3 set for t and f . Hence, $c[t, f] \leq |F| = |F_1| + |F_2| - |f^{-1}(1)|$. Since the pair of colorings can be chosen arbitrarily, $c[t, f] \leq \min_{f_1, f_2} \{c[t_1, f_1] + c[t_2, f_2]\} - |f^{-1}(1)|$, where the minimum is taken over all pairs of colorings f_1, f_2 consistent with f . We thus have proved the above recursive formula.

Now, we have finished the description of the recursive formulas for the values of $c[t, f]$, i.e., we have presented an improved dynamic programming algorithm for the VCP_3 problem on graphs of bounded treewidth. The value of $c[t, \emptyset]$ is exactly the minimum size of a VCP_3 set in G . Moreover, when backtracking how the value of $c[t, \emptyset]$ is obtained, we can construct a minimum VCP_3 set F in G .

Next, we analyze the running time of the algorithm. Clearly, for each leaf node, introduce node, introduce edge node, forget node, the time needed to compute all values of $c[t, f]$ is 3^p . We will apply fast algorithms for subset convolution to compute the time needed to process each join node.

Given a set S and two functions $g, h: 2^S \rightarrow \mathbb{Z}$, the *subset convolution* of g and h is a function $(g * h): 2^S \rightarrow \mathbb{Z}$ so that for every $Y \subseteq S$,

$$(g * h)(Y) = \min_{\substack{A \cup B = Y \\ A \cap B = \emptyset}} (g(A) + h(B)).$$

Lemma 1 *Björklund et al. (2007); van Rooij et al. (2009)* Let S be a set with n elements and M be a positive integer. For two functions $g, h: 2^S \rightarrow \{-M, \dots, M\} \cup \{+\infty\}$, if all the values of g and h are given, then all the 2^n values of the subset convolution of g and h can be computed in $2^n \cdot n^3 \cdot O(M \log(Mn) \log \log(Mn))$ time.

Let t be a join node with the children t_1 and t_2 . For a coloring f of X_t , if colorings f_1 of X_{t_1} and f_2 of X_{t_2} are consistent with f , then the following conditions hold.

- $f^{-1}(1) = f_1^{-1}(1) = f_2^{-1}(1)$,
- $f^{-1}(0_1) = f_1^{-1}(0_1) \cup f_2^{-1}(0_1)$,
- $f_1^{-1}(0_1) \cap f_2^{-1}(0_1) = \emptyset$.

Fix a set $R \subseteq X_t$, let \mathcal{F}_R denote the set of all functions $f: X_t \rightarrow \{1, 0_0, 0_1\}$ so that $f^{-1}(1) = R$. Observe that every coloring $f \in \mathcal{F}_R$ can be represented by a set $Y \subseteq X_t \setminus R$ as follows.

$$g_Y(x) = \begin{cases} 1, & \text{if } x \in R; \\ 0_1, & \text{if } x \in Y; \\ 0_0, & \text{if } x \in X_t \setminus (R \cup Y). \end{cases}$$

Then for every $f(= g_Y) \in \mathcal{F}_R$, the following formula holds.

$$c[t, f] = c[t, g_Y] = \min_{\substack{A \cup B = Y \\ A \cap B = \emptyset}} (c[t_1, g_A] + c[t_2, g_B]) - |R|.$$

Recall that n is the number of vertices in G . We can define two functions $h_1, h_2: 2^{X_t \setminus R} \rightarrow \{0, \dots, n\} \cup \{+\infty\}$ so that for every set $T \subseteq X_t \setminus R$ we have $h_1(T) = c[t_1, g_T]$ and $h_2(T) = c[t_2, g_T]$. Thus,

$$c[t, f] = c[t, g_Y] = \min_{\substack{A \cup B = Y \\ A \cap B = \emptyset}} (h_1(A) + h_2(B)) - |R| = (h_1 * h_2)(Y) - |R|,$$

where $h_1 * h_2$ is the subset convolution of h_1 and h_2 . By Lemma 1, we can compute all the $2^{|X_t \setminus R|}$ values of $c[t, f]$ for all $f \in \mathcal{F}_R$ in time $2^{|X_t \setminus R|} \cdot O(n^5)$. Since $\sum_{R \subseteq X_t} 2^{|X_t \setminus R|} = 3^{|X_t|} \leq 3^{p+1}$, the time needed to process each joint node is $O(3^p \cdot n^5)$.

Recall that the number of nodes of T in the tree decomposition is $O(pn)$, the running time of the algorithm presented here is $O(3^p \cdot pn^6)$. \blacksquare

4 A randomized algorithm for the connected vertex cover P_3 problem

In this section we consider the connected vertex cover P_3 ($CVCP_3$) problem, a variation of the VCP_3 problem where it is required that the graph induced by the VCP_3 set is connected. We present a randomized algorithm with runtime $4^{tw} \cdot n^{O(1)}$ for the $CVCP_3$ problem on n -vertex graphs with treewidth tw . The main method used here is the Cut&Count technique. The technique was introduced by Cygan et al. (2011) and can give randomized algorithms for many problems with certain connectivity requirement running in time $c^{tw} n^{O(1)}$ for a small constant c . We show that for the $CVCP_3$ problem the constant c can be 4.

We will solve more general versions of the $CVCP_3$ problem where additionally as a part of the input we are given a set $S \subseteq V$ which contains vertices that must belong to a solution.

CONSTRAINED CONNECTED VERTEX COVER P_3 (Constrained $CVCP_3$)

Input: A graph $G = (V, E)$, a subset $S \subseteq V$ and an integer k .

Question: Does there exist a VCP_3 -set F of size at most k in G such that $S \subseteq F$ and $G[F]$ is connected?

Remark. We assume that the set $S \subseteq V$ in Constrained $CVCP_3$ is nonempty. To solve the problem where $S = \emptyset$, we simply iterate over all possible choices of $v_1 \in V$ and put $S = \{v_1\}$.

Definition 2 Given a graph G , assume F is a VCP_3 set of G . A cut of F is a pair (F^1, F^2) with $F^1 \cap F^2 = \emptyset$, $F^1 \cup F^2 = F$. We refer to F^1 and F^2 as to the sides of the cut. A cut (F^1, F^2) of F is called a consistent cut of F if $u \in F^1$ and $v \in F^2$ imply $uv \notin E(G[F])$.

Since no edge of $G[F]$ can go across the cut, each of its connected components has to be either entirely in F^1 , or entirely in F^2 . But we can freely choose the side of the cut for every connected component of $G[F]$ independently. Thus, if the induced subgraph $G[F]$ has c connected components, then the VCP_3 set is consistent with 2^c cuts. Unfortunately if $G[F]$ is connected, F is consistent with two cuts, (F, \emptyset) and (\emptyset, F) . What we do want is that if $G[F]$ is connected, F is consistent with only one cut. Thus, we will select an arbitrary vertex v_1 and fix it permanently into the F^1 side of the cut.

Theorem 3 There exists a randomized algorithm that given an n -vertex graph $G = (V, E)$ with a tree decomposition of width p solves Constrained $CVCP_3$ in $O(4^p \cdot pn^6)$ time. The algorithm cannot give false positives and may give false negatives with probability at most $1/2$.

Proof: We use the Cut&Count technique Cygan et al. (2011) which can be applied to problems with certain connectivity requirements. Let $\mathcal{F} \subseteq 2^V$ be a set of solutions of Constrained $CVCP_3$; we aim to decide whether it is empty. Cut&Count is split in two parts: (1) The Cut part: Relax the connectivity requirement by considering the set $\mathcal{R} \supseteq \mathcal{F}$ of all VCP_3 sets containing S . Furthermore, consider the set \mathcal{C} of pairs (X, C) where $X \in \mathcal{R}$ and C is a consistent cut of F ; (2) The Count part: Compute $|\mathcal{C}|$ modulo 2 using a sub-procedure. Non-connected candidate solutions $X \in \mathcal{R} \setminus \mathcal{F}$ cancel since they are consistent with an even number of cuts. Connected candidates in \mathcal{F} remain.

The Cut part.

Suppose we are given a weight function $w: V \rightarrow \{1, 2, \dots, N\}$, taking $N = 2n$. For a subset $X \subseteq V$, $w(X) = \sum_{v \in X} w(v)$. Recall that we may assume that $S \neq \emptyset$ and we choose one fixed vertex $v_1 \in S$. For any integer W , we define:

$$\begin{aligned}\mathcal{R}_W &= \{F \subseteq V(G) : F \text{ is a } VCP_3 \text{ set of } G, |F| = k, S \subseteq F \text{ and } w(F) = W\}, \\ \mathcal{F}_W &= \{F \subseteq V(G) : F \in \mathcal{R}_W \text{ and } G[F] \text{ is connected}\}, \\ \mathcal{C}_W &= \{(F, (F^1, F^2)) : F \in \mathcal{R}_W, (F^1, F^2) \text{ is a consistent cut of } F, \text{ and } v_1 \in F^1\}.\end{aligned}$$

The set \mathcal{R}_W is the set of candidate solutions, where we relax the connectivity requirement. The set $\bigcup_W \mathcal{F}_W$ is our set of solutions. If for any W this set is nonempty, our problem has a positive answer. By the following Lemma we show that instead of calculating the parity of \mathcal{F}_W directly, we can count pairs consisting of a set from \mathcal{R}_W and a cut consistent with it, i.e., \mathcal{C}_W .

Lemma 2 *Let $w, \mathcal{F}_W, \mathcal{C}_W$ be as defined above. The parity of $|\mathcal{C}_W|$ is the same as the parity of $|\mathcal{F}_W|$, in other words, $|\mathcal{C}_W| \equiv |\mathcal{F}_W| \pmod{2}$.*

Proof of Lemma 2. For any VCP_3 set F from \mathcal{R}_W , each connected component of $G[F]$ has to be contained either in F^1 or in F^2 . However, the connected component of $G[F]$ containing v_1 has to be contained in the F^1 side of the cut. Thus, if $G[F]$ has c components, then F is consistent with 2^{c-1} cuts, which is an odd number for $F \in \mathcal{F}_W$ and an even number for $F \in \mathcal{R}_W \setminus \mathcal{F}_W$. ■

The Count part.

Next we describe a procedure $CountC(w, W, \mathcal{T})$ that given a refined variant of nice tree decomposition \mathcal{T} , weight function w and an integer W , computes $|\mathcal{C}_W|$ modulo 2.

Recall that for each node t of T we denote by V_t the union of all the bags present in the subtree of T rooted at t and

$$G_t = (V_t, E_t = \{e : e \text{ is introduced in the subtree rooted at } t\}).$$

For every node $t \in T$ of the tree decomposition, integers $0 \leq i \leq |V| = n$, $0 \leq w \leq N|V| = 2n^2$ and colorings (or, mappings) $f: X_t \rightarrow \{1_1, 1_2, 0_0, 0_1\}$, define

$$\begin{aligned}\mathcal{R}_t(i, w) &= \{F \subseteq V_t : (F \text{ is a } VCP_3 \text{ set of } G_t) \wedge (|F| = i) \wedge (w(F) = w) \\ &\quad \wedge (S \cap V_t \subseteq F)\}, \\ \mathcal{C}_t(i, w) &= \{(F, (F^1, F^2)) : (F \in \mathcal{R}_t(i, w)) \wedge ((F^1, F^2) \text{ is a consistent cut of } F) \\ &\quad \wedge (v_1 \in V_t \Rightarrow v_1 \in F^1)\}, \\ A_t(i, w, f) &= |\{(F, (F^1, F^2)) \in \mathcal{C}_t(i, w) : (f(v) = 1_1 \Rightarrow v \in F^1) \wedge (f(v) = 1_2 \Rightarrow v \in F^2) \\ &\quad \wedge (f(v) = 0_0 \Rightarrow v \text{ is an isolated vertex in } G_t - F) \\ &\quad \wedge (f(v) = 0_1 \Rightarrow v \text{ is a 1-degree vertex in } G_t - F)\}|.\end{aligned}$$

$A_t(i, w, f)$ is the number of pairs from \mathcal{C} of candidate solutions and consistent cuts on G_t , with fixed size, weight and interface on vertices from V_t . And the number $A_t(i, w, f)$ counts those elements of

$\mathcal{C}_t(i, w)$ which additionally behave on vertices of V_t in a fashion prescribed by the coloring f . In particular note that

$$\sum_f A_t(i, w, f) = |\mathcal{C}_t(i, w)|,$$

where the sum is taken over all colorings from X_t to $\{1_1, 1_2, 0_0, 0_1\}$. For a vertex $x \in X_t$, $f(v) = 1_j$ means $v \in F$ and $v \in F^j$ for $j = 1, 2$, $f(v) = 0_0$ means v is an isolated vertex in $G_t - F$, $f(v) = 0_1$ means v is a 1-degree vertex in $G_t - F$.

Recall that r is the root of the tree decomposition, we have that $|\mathcal{C}_W| = |\mathcal{C}_r(k, W)|$. Since we are interested in values \mathcal{C}_W modulo 2, it suffices to compute values $A_r(k, W, \emptyset)$ for all W .

We present a dynamic programming algorithm that computes $A_t(i, w, f)$ for all nodes $t \in T$ in a bottom-up fashion for all reasonable values of i, w and f .

Leaf node.

For a leaf node t , $X_t = \emptyset$. $A_t(0, 0, \emptyset) = 1$ and all other values of $A_t(i, w, f)$ are zeros.

Introduce node.

Let t be an introduce node with a child t' so that $X_t = X_{t'} \cup \{v\}$ for some $v \notin X_{t'}$. Note that $V_t = V_{t'} \cup \{v\}$ and $E_t = E_{t'}$, i.e., v is an isolated vertex in G_t .

$$A_t(i, w, f) = \begin{cases} A_{t'}(i-1, w-w(v), f|_{X_{t'}}), & \text{when } f(v) = 1_1; \\ A_{t'}(i-1, w-w(v), f|_{X_{t'}}), & \text{when } f(v) = 1_2 \text{ and } v \neq v_1; \\ 0, & \text{when } f(v) = 1_2 \text{ and } v = v_1; \\ A_{t'}(i, w, f|_{X_{t'}}), & \text{when } f(v) = 0_0; \\ 0, & \text{when } f(v) = 0_1, \end{cases}$$

where $f|_{X_{t'}}$ is the restriction of f to $X_{t'}$.

Introduce edge node.

Let t be an introduce edge node labeled with an edge $uv \in E(G)$ and t' be the child of t . Note that $V_t = V_{t'}$, $E_t = E_{t'} \cup \{uv\}$ and $uv \notin E_{t'}$.

$$A_t(i, w, f) = \begin{cases} 0, & \text{when } (f(u), f(v)) \in \{(1_1, 1_2), (1_2, 1_1)\}; \\ 0, & \text{when } (f(u), f(v)) = (1_2, 1_2) \text{ and } v_1 \in \{u, v\}; \\ 0, & \text{when } (f(u), f(v)) \in \{(0_0, 0_0), (0_0, 0_1), (0_1, 0_0)\}; \\ A_{t'}(i, w, f'), & \text{when } (f(u), f(v)) = (0_1, 0_1); \\ A_{t'}(i, w, f), & \text{otherwise,} \end{cases}$$

where f' is a coloring of $X_t = X_{t'}$, and $f'(w) = \begin{cases} f(w), & \text{when } w \neq u \text{ and } w \neq v; \\ 0_0, & \text{when } w = u \text{ or } w = v. \end{cases}$

Forget node.

Let t be a forget node with a child t' so that $X_t = X_{t'} \setminus \{v\}$ for some $v \in X_{t'}$. Note that $V_t = V_{t'}$ and $G_t = G_{t'}$. For a coloring f of X_t and a color $\alpha \in \{1_1, 1_2, 0_0, 0_1\}$, we define a coloring $f_{v \rightarrow \alpha}$ of $X_{t'}$ as follows:

$$f_{v \rightarrow \alpha}(x) = \begin{cases} f(x), & \text{when } x \neq v, \\ \alpha, & \text{when } x = v. \end{cases}$$

We have

$$A_t(i, w, f) = \sum_{\alpha \in \{1_1, 1_2, 0_0, 0_1\}} A_{t'}(i, w, f_{v \rightarrow \alpha}).$$

Join node.

Suppose that t is a join node with children t_1, t_2 so that $X_t = X_{t_1} = X_{t_2}$. Note that $V_t = V_{t_1} \cup V_{t_2}$, $V_{t_1} \cap V_{t_2} = X_t$ and $E_t = E_{t_1} \cup E_{t_2}$.

We say that a pair of colorings f_1 of X_{t_1} , f_2 of X_{t_2} is consistent with a coloring f of X_t if for each vertex $v \in X_t$, the following conditions hold.

- (1) $f(v) = 1_1$ if and only if $f_1(v) = f_2(v) = 1_1$,
- (2) $f(v) = 1_2$ if and only if $f_1(v) = f_2(v) = 1_2$,
- (3) $f(v) = 0_0$ if and only if $f_1(v) = f_2(v) = 0_0$,
- (4) $f(v) = 0_1$ if and only if $(f_1(v), f_2(v)) \in \{(0_1, 0_0), (0_0, 0_1)\}$.

We can now write a recursion formula for join nodes.

$$A_t(i, w, f) = \sum_{i_1+i_2=i+|f^{-1}(\{1_1, 1_2\})|} \sum_{w_1+w_2=w+w(f^{-1}(\{1_1, 1_2\}))} \sum_{f_1, f_2} A_{t_1}(i_1, w_1, f_1) A_{t_2}(i_2, w_2, f_2),$$

where the third sum is taken over all pairs of colorings f_1, f_2 consistent with f . To achieve the coloring f , the colorings of children must be a pair of colorings consistent with f .

On one hand, if $(F, (F^1, F^2)) \in \mathcal{C}_t(i, w)$ is a pair of candidate solutions for coloring f and consistent cuts on G_t , with fixed size i , weight w and interface on vertices from V_t , then $F_1 = F \cap V_{t_1}$ and $F_2 = F \cap V_{t_2}$ are VCP_3 sets for t_1 and f_1 , and t_2 and f_2 , for some pairs of colorings f_1, f_2 that are consistent with f . Accordingly, $(F_k, (F_k^1, F_k^2))$ ($k = 1, 2$) is a pair of candidate solutions and consistent cuts on G_{t_k} , with fixed size i_k , weight w_k and interface on vertices from V_{t_k} , where $F_k^j = \{v | v \in F_k \wedge f_k(v) = 1_j\}$ for $j = 1, 2$. Since vertices coloured 1_j for $j = 1, 2$ in X_t are accounted for both numbers $A_t(i, w, f)$ of the children, we add their contribution to the accumulators.

On the other hand, given a coloring f of X_t , consider any pair of colorings f_1, f_2 that is consistent with f . Suppose $(F_k, (F_k^1, F_k^2))$ ($k = 1, 2$) is a pair of candidate solutions for coloring f_k and consistent cuts on G_{t_k} , with fixed size i_k , weight w_k and interface on vertices from V_{t_k} . Then $(F_1 \cup F_2, (F_1^1 \cup F_2^1, F_1^2 \cup F_2^2))$ is pair of candidate solutions for coloring f and consistent cuts on G_t , with fixed size $i_1 + i_2 - |f^{-1}(\{1_1, 1_2\})|$, weight $w_1 + w_2 - w(f^{-1}(\{1_1, 1_2\}))$ and interface on vertices from V_t .

So, the above recursive formula holds.

Similarly, we can show that the above recurrences lead to a dynamic programming algorithm that computes the parity of $|\mathcal{F}_W|$ for all values of W in $O(4^p \cdot pn^6)$ time, since $|\mathcal{C}_W| = A_r(k, W, \emptyset)$ and $|\mathcal{F}_W| \equiv |\mathcal{C}_W|$.

It remains to show why solving the parity version of the problem is helpful when facing the decision version. We use the following isolation lemma.

Definition 3 A function $w: U \rightarrow \mathbb{Z}$ isolates a set family $\mathcal{F} \subseteq 2^U$ if there is a unique $S' \in \mathcal{F}$ with $w(S') = \min_{S \in \mathcal{F}} w(S)$.

Lemma 3 *Cygan et al. (2015, 2011) (Isolation Lemma).* Let $\mathcal{F} \subseteq 2^U$ be a set family over a universe U with $|\mathcal{F}| > 0$. For each $u \in U$, choose a weight $w(u) \in \{1, 2, \dots, N\}$ uniformly and independently at random. Then

$$\Pr(w \text{ isolates } \mathcal{F}) \geq 1 - \frac{|U|}{N}.$$

The Isolation Lemma gives a reduction from the decision version of the problem to the parity version as follows. Our universe is $U = V(G)$, and we put

$$\mathcal{F} = \{F \subseteq V(G) : F \text{ is a } VCP_3 \text{ set of } G, |F| \leq k, S \subseteq F \text{ and } G[F] \text{ is connected.}\}.$$

We sample an integral weight $w: V \rightarrow \{1, 2, \dots, N = 2|U|\}$. For each W between 1 and $2nk$ we compute the parity of the number of pairs in \mathcal{C}_W . Since $\mathcal{C}_W \equiv \mathcal{F}_W$, if for some W the parity is odd, we answer YES, otherwise we answer NO.

Even though there might be exponentially many solutions, that is elements of \mathcal{F} , with probability at least $1/2$ for some weight W between 1 and $2kn$ there is exactly one solution from \mathcal{F} of weight W with respect to w . Thus we can obtain a Monte Carlo algorithm where the probability of error is at most $1/2$.

We have completed the proof of the theorem. ■

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