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A positional statistic for 1324*-avoiding permutations*

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We consider the class $\mathcal{S}_n(1324)$ of permutations of size n that avoid the pattern 1324 and examine the subset $\mathcal{S}_n^{a\prec n}(1324)$ of elements for which $a \prec n \prec [a-1]$, $a \geq 1$. This notation means that, when written in one-line notation, such a permutation must have a to the left of n, and the elements of $\{1,\dots,a-1\}$ must all be to the right of n. For $n \geq 2$, we establish a connection between the subset of permutations in $\mathcal{S}_n^{1\prec n}(1324)$ having the 1 adjacent to the n (called *primitives*), and the set of 1324-avoiding dominoes with $n-2$ points. For $a \in \{1,2\}$, we introduce constructive algorithms and give formulas for the enumeration of $\mathcal{S}_n^{a\prec n}(1324)$ by the position of a relative to the position of n. For $a \geq 3$, we formulate some conjectures for the corresponding generating functions.

Keywords: pattern-avoiding permutations, enumerative combinatorics

1 Introduction

Finding a closed formula for the enumeration of the class $\mathcal{S}_n(1324)$ remains an open problem that we do NOT solve in this paper. However, in our attempt to understand what makes this class so difficult to handle, we came across a simple but interesting type of statistics: distance between the smallest and largest element of a permutation. We found that permutations of size n having the 1 adjacent to the n are manageable and can be used to enumerate certain related subsets of $\mathcal{S}_n(1324)$.

The goal of this paper is to present our findings and formulate a conjecture for the enumeration of other subsets of $\mathcal{S}_n(1324)$ according to similar positional statistics.

For $a, k \geq 1$, let $\mathcal{S}_{n,k}^{a\prec n}(1324)$ be the set of permutations $\sigma \in \mathcal{S}_n(1324)$ such that:

- $\sigma^{-1}(n) - \sigma^{-1}(a) = k$,
- $\sigma^{-1}(b) - \sigma^{-1}(n) > 0$ for every $b \in \{1,\dots,a-1\}$.

Let $\mathcal{S}_n^{a\prec n}(1324) = \bigcup\limits_{k\geq 1} \mathcal{S}_{n,k}^{a\prec n}(1324)$ and observe that, in one-line notation, permutations in the set $\mathcal{S}_n^{a\prec n}(1324)$ have the entry a to the left of n and all the elements of $\{1,\dots,a-1\}$ to the right of n. We also let

$$T_{a,k}(x) = \sum_{n=k+1}^{\infty} |\mathcal{S}_{n,k}^{a\prec n}(1324)| x^n \quad\text{and}\quad g_a(x,t) = \sum_{k=1}^{\infty} t^k T_{a,k}(x).$$

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In Section 2, we discuss $\mathcal{S}_{n,k}^{1\prec n}(1324)$ and give formulas for $T_{1,k}(x)$ and $g_1(x,t)$. First, we give a bijection between $\mathcal{S}_{n,1}^{1\prec n}(1324)$ and the set of 1324-avoiding dominoes with $n-2$ points, which is known to be counted by the OEIS sequence [3, A000139]. We then introduce a product $\sigma_1 \odot \sigma_2$ on $\mathcal{S}_{n,1}^{1\prec n}(1324)$ that we use to construct and enumerate the elements of $\mathcal{S}_{n,k}^{1\prec n}(1324)$. In Section 3, we examine the set $\mathcal{S}_{n,k}^{2\prec n}(1324)$ and give explicit formulas for $T_{2,k}(x)$ and $g_2(x,t)$. Finally, in the last section of the paper, we conjecture a formula for $T_{a,k}(x)$ for $3 \le a \le k$.

The significance of $g_a(x,t)$ lies in the fact that if $G(x) = \sum_{n=1}^{\infty} |\mathcal{S}_n(1324)|x^n$, then

$$G(x) = \frac{1}{1-x}\bigg(x + \sum_{a=1}^{\infty} g_a(x,1)\bigg).$$

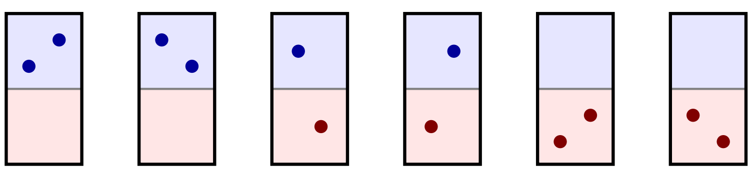
Note that for $n \ge 2$, permutations in $\mathcal{S}_n(1324)$ that start with n are counted by the function $xG(x)$. Thus $G(x) = x + xG(x) + \sum_{a=1}^{\infty} g_a(x,1)$.

Again, we are still far from giving a formula for $G(x)$, but the breakdown using the sets $\mathcal{S}_{n,k}^{a\prec n}(1324)$ provides a different viewpoint that we believe is worth pursuing further.

2 Enumeration of $\mathcal{S}_n^{1\prec n}(1324)$

We start by establishing a bijection between $\mathcal{S}_{n,1}^{1\prec n}(1324)$ and the set of 1324-avoiding dominoes. As studied by D. Bevan, R. Brignall, A. Elvey Price, and J. Pantone [1], a 1324-avoiding vertical *domino* is a two-cell gridded permutation in $\mathrm{Grid}^{\#}\binom{\mathrm{Av}(213)}{\mathrm{Av}(132)}$ whose underlying permutation avoids 1324. These dominoes are counted by the sequence $1, 2, 6, 22, 91, 408, 1938, 9614, \ldots$, cf. [3, A000139].

For example, the six distinct 1324-avoiding dominoes with two points are



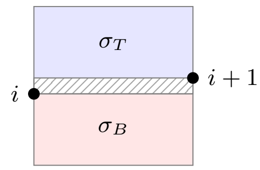
Proposition 2.1 *For $n \ge 2$, there is a one-to-one correspondence between $\mathcal{S}_{n,1}^{1\prec n}(1324)$ and the set of 1324-avoiding dominoes with $n-2$ points.*

Proof: The bijection relies on the inverse map. Every $\sigma \in \mathcal{S}_{n,1}^{1\prec n}(1324)$ is of the form

$$\sigma = \sigma_L\, 1n\, \sigma_R,$$

where σ_L and σ_R are words (possibly empty) such that $|\sigma_L| + |\sigma_R| = n-2$, and their reduced permutations[(i)] $\mathrm{red}(\sigma_L)$ and $\mathrm{red}(\sigma_R)$ avoid 132 and 213, respectively. Clearly, since 1324 is an involution, σ^{-1} also avoids 1324. Moreover, if $i = \sigma^{-1}(1)$, then σ^{-1} is of the form

[(i)] The permutation $\mathrm{red}(\sigma)$ is obtained by replacing the ith smallest letter of σ by i, for $i = 1, \ldots, |\sigma|$.



where $\mathrm{red}(\sigma_T) = \mathrm{red}(\sigma_R)^{-1}$ avoids 213 and $\mathrm{red}(\sigma_B) = \mathrm{red}(\sigma_L)^{-1}$ avoids 132. Finally, merging the lines through i and $i+1$ as a separator, we get a 1324-avoiding domino. □

As a consequence ([1, Theorem 2] with n replaced by $n-2$), we have

$$|\mathcal{S}_{n,1}^{1\prec n}(1324)| = \frac{2\,(3n-3)!}{(2n-1)!n!} \quad \text{for } n \geq 2.$$

The elements of $\mathcal{S}_{n,1}^{1\prec n}(1324)$ serve as primitives to factor and enumerate the elements of $\mathcal{S}_{n,k}^{1\prec n}(1324)$ for every $k \geq 2$.

We start with a definition.

Definition 2.2 *A permutation $\sigma \in \mathcal{S}_{n,1}^{1\prec n}(1324)$ will be called a* primitive. *For $n \geq 2$, such a permutation must be of the form $\sigma = \pi\, 1 n\, \tau$ with $\mathrm{red}(\pi) \in \mathcal{S}_k(132)$, $\mathrm{red}(\tau) \in \mathcal{S}_\ell(213)$, and $k+\ell = n-2$. Given a primitive $\sigma_1 = \pi_1\, 1m\, \tau_1 \in \mathcal{S}_m(1324)$ and a permutation $\sigma_2 \in \mathcal{S}_\ell(1324)$ of the form $\sigma_2 = \pi_2\, 1\, \theta_2\, \ell\, \tau_2$ with $|\theta_2| \geq 0$, we define the product*

$$\sigma_1 \odot \sigma_2 = \widehat{\pi}_2 \pi_1\, 1m\, \widehat{\theta}_2\, n\, \widehat{\tau}_2 \tau_1 \in \mathcal{S}_n, \tag{2.1}$$

where $n = \ell + m - 1$, and $\widehat{\pi}_2$, $\widehat{\theta}_2$, and $\widehat{\tau}_2$ are obtained from π_2, θ_2, and τ_2, by increasing all of their entries by $m-1$ (see Figure 1).

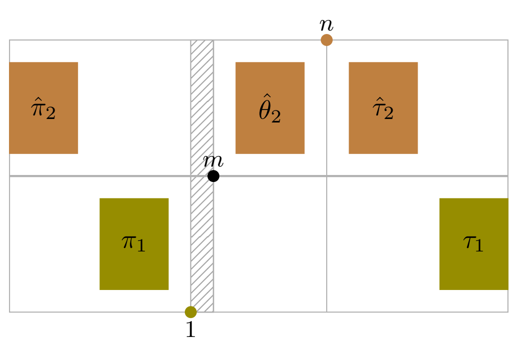


Fig. 1: Visualization of $\sigma_1 \odot \sigma_2$.

For instance, $2143 \odot 41253 = 7\,21\,4\,586\,3$. Also,

$$213 \odot 3142 = 5\,21\,364 \quad \text{and} \quad 3142 \odot 213 = 5\,31\,46\,2.$$

In particular, $\odot$ is not commutative. This is true even if σ_1 and σ_2 are both primitives. For example, $213 \odot 12 = 2134$ but $12 \odot 213 = 3124$.

Proposition 2.3 *If* $\sigma_1 \in \mathcal{S}_{m,1}^{1\prec m}(1324)$ *and* $\sigma_2 \in \mathcal{S}_{\ell,k}^{1\prec \ell}(1324)$*, then*

$$\sigma_1 \odot \sigma_2 \in \mathcal{S}_{n,k+1}^{1\prec n}(1324) \text{ with } n = \ell + m - 1.$$

Proof: The permutations σ_1 and σ_2 must be of the form

$$\sigma_1 = \pi_1\, 1m\, \tau_1 \ \text{ and } \ \sigma_2 = \pi_2\, 1\theta_2\, \ell\, \tau_2,$$

where $\mathrm{red}(\pi_1)$ and $\mathrm{red}(\pi_2)$ both avoid 132, $\mathrm{red}(\tau_1)$ and $\mathrm{red}(\tau_2)$ both avoid 213, $\mathrm{red}(\theta_2)$ avoids 21, and $|\theta_2| = k-1$. By (2.1), we have $\sigma_1 \odot \sigma_2 \in \mathcal{S}_{n,k+1}^{1\prec n}$, so we only need to show that $\sigma_1 \odot \sigma_2$ avoids 1324.

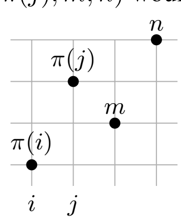
The graph of $\sigma_1 \odot \sigma_2$ (see Figure 1) makes it clear that, since both σ_1 and σ_2 avoid 1324, the parts of the permutation below and above the horizontal line $y = m$ cannot have a 1324 pattern. Thus, if there is a 1324 pattern in $\sigma_1 \odot \sigma_2$, the 1 will have to be in $\pi_1\, 1$, and since π_1 avoids 132, the 3, 2, and 4 of the pattern will have to be above the line $y = m$. But this is not possible since $\theta_2\, \ell\, \tau_2$ avoids 213. □

Proposition 2.4 *Every non-primitive* $\sigma \in \mathcal{S}_n^{1\prec n}(1324)$ *admits a unique decomposition*

$$\sigma = \sigma_1 \odot \sigma_2,$$

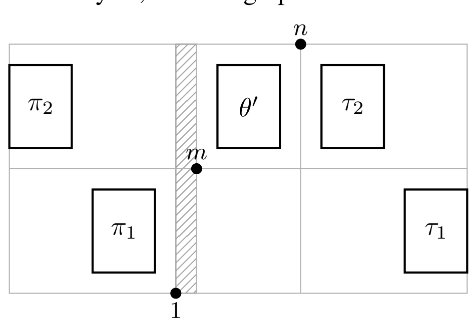
where σ_1 *is a primitive in* $\mathcal{S}_{m,1}^{1\prec m}(1324)$ *and* $\sigma_2 \in \mathcal{S}_\ell^{1\prec \ell}(1324)$ *with* $\ell = n - m + 1$.

Proof: If σ is not primitive, then it must be of the form $\sigma = \pi\, 1\theta n\, \tau$, where $\mathrm{red}(\pi)$ avoids 132, $\mathrm{red}(\tau)$ avoids 213, and θ is increasing (i.e., avoids 21). Let $m = \min(\theta)$. If there were indices $i < j$ such that $\pi(i) < m < \pi(j)$, then the sequence $(\pi(i), \pi(j), m, n)$ would make a forbidden 1324 pattern:



Thus, if π is not empty, it must be of the form $\pi = \pi_2\pi_1$ where π_2 consists of the values of π that are larger than m, and π_2 contains the values of π smaller than m, if any. Similarly, if $\tau(i) < m < \tau(j)$ for some $i < j$, then the sequence $(1, m, \tau(i), \tau(j))$ would make a forbidden 1324 pattern. Thus, if τ is not empty, it must be of the form $\tau = \tau_2\tau_1$ where the values of τ_2 (if not empty) are larger than m and the values of τ_1 are smaller than m.

Now, if θ' denotes θ without the entry m, then the graph of σ takes the form



where any box could be empty. If we let $\sigma_1 = \pi_1\, 1m\, \tau_1$ and $\sigma_2 = \mathrm{red}(\pi_2\, m\theta' n\, \tau_2)$, then σ_1 is primitive, $\sigma_2 \in \mathcal{S}_\ell^{1\prec\ell}(1324)$ with $\ell = n - m + 1$, and it is easy to see that $\sigma = \sigma_1 \odot \sigma_2$.

Now suppose $\sigma = \rho_1 \odot \rho_2$ with a primitive ρ_1 of the form $\rho_1 = \alpha_1\, 1m'\, \beta_1$. The definition of $\rho_1 \odot \rho_2$ implies $m' = m$ and all the values of σ less than or equal to m must coincide with the values of ρ_1. Hence $\rho_1 = \sigma_1$ and the factorization is unique. □

Corollary 2.5 *Every permutation in* $\mathcal{S}_{n,k}^{1\prec n}(1324)$ *can be uniquely decomposed as a product of* k *primitive permutations.*

$(k=3)$	$1234 = 12 \odot 12 \odot 12$
$(k=2)$	$1243 = 12 \odot 132$
	$1342 = 132 \odot 12$
	$2134 = 213 \odot 12$
	$3124 = 12 \odot 213$
$(k=1)$	$1423, 1432, 2143, 3142, 2314, 3214$ (primitives)

Tab. 1: Elements of $\mathcal{S}_{4,k}^{1\prec 4}(1324)$ and their primitive decomposition.

In general, every permutation $\sigma \in \mathcal{S}_{n,k}^{1\prec n}(1324)$ with k primitive components is of the form depicted in Figure 2, where Θ is increasing (possibly empty), and each primitive component can be read from the horizontal regions determined by the values of σ to the right of 1 and to the left of n.

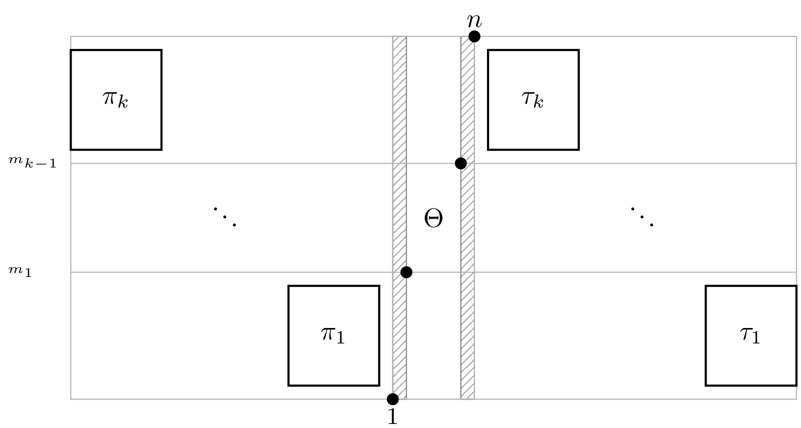


Fig. 2: $\sigma = \sigma_1 \odot \cdots \odot \sigma_k$.

As mentioned at the beginning of this section, the elements of $\mathcal{S}_{n,1}^{1\prec n}(1324)$ are counted by the OEIS sequence [3, A000139]. Let $f(x)$ denote the generating function for A000139 without the constant term, that is,

$$f(x) = x + 2x^2 + 6x^3 + 22x^4 + 91x^5 + 408x^6 + 1938x^7 + 9614x^8 + \cdots, \tag{2.2}$$

and recall the notation (given in the introduction)

$$T_{1,k}(x) = \sum_{n=k+1}^{\infty} |\mathcal{S}_{n,k}^{1\prec n}(1324)|x^n \ \text{ and } \ g_1(x,t) = \sum_{k=1}^{\infty} t^k T_{1,k}(x).$$

By Proposition 2.1, we have $T_{1,1}(x) = xf(x)$. More generally:

Theorem 2.6 *Let $a_{n,k} = |\mathcal{S}_{n,k}^{1\prec n}(1324)|$. For $n \geq 3$ and $2 \leq k \leq n-1$, we have*

$$a_{n,k} = \sum_{m=2}^{n-k+1} a_{m,1} \cdot a_{n-m+1,k-1}.$$

Consequently, $T_{1,k}(x) = xf(x)^k$ and thus $g_1(x,t) = \dfrac{xtf(x)}{1-tf(x)}$.

Proof: By the above propositions, the permutations in $\mathcal{S}_{n,k}^{1\prec n}(1324)$ can be built by taking all possible products $\sigma_1 \odot \sigma_2$, where σ_1 is a primitive of size m for some $m \geq 2$, and σ_2 is a 1324-avoiding permutation of size $n-m+1$ with $\sigma_2^{-1}(n-m+1) - \sigma_2^{-1}(1) = k-1$. There are $a_{m,1}$ permutations of the first kind, and $a_{n-m+1,k-1}$ of the latter. This leads to the claimed formula for $a_{n,k}$. Observe that $a_{n-m+1,k-1}$ only makes sense whenever $n-m+1 \geq k$, so we need $m \leq n-k+1$.

The formula for $T_{1,k}(x)$ can be shown by induction in k. As mentioned above, $T_{1,1}(x) = xf(x)$. Suppose $T_{1,k-1}(x) = xf(x)^{k-1}$. Then,

$$\begin{aligned} xT_{1,k} = \sum_{n=k+1}^{\infty} a_{n,k}x^{n+1} &= \sum_{n=k+1}^{\infty} \Big(\sum_{m=2}^{n-k+1} a_{m,1} \cdot a_{n-m+1,k-1} \Big) x^{n+1} \\ &= \sum_{n=k+2}^{\infty} \Big(\sum_{m=2}^{n-k} a_{m,1} \cdot a_{n-m,k-1} \Big) x^{n} \\ &= \Big(\sum_{m=2}^{\infty} a_{m,1}x^m \Big) \Big(\sum_{\ell=k}^{\infty} a_{\ell,k-1}x^\ell \Big). \end{aligned}$$

In other words, $xT_{1,k} = T_{1,1}(x)T_{1,k-1}(x)$ and therefore $T_{1,k}(x) = xf(x)^k$, as claimed. □

3 Enumeration of $\mathcal{S}_n^{2\prec n}(1324)$

In this section, we focus on the set $\mathcal{S}_{n,k}^{2\prec n}(1324)$ and give formulas for the functions

$$T_{2,k}(x) = \sum_{n=k+1}^{\infty} |\mathcal{S}_{n,k}^{2\prec n}(1324)|x^n \ \text{ and } \ g_2(x,t) = \sum_{k=1}^{\infty} t^k T_{2,k}(x)$$

in terms of the functions $f(x)$ and $g_1(x,t)$ from the previous section. Recall that $\mathcal{S}_{n,k}^{2\prec n}(1324)$ is the set of permutations $\sigma\in\mathcal{S}_n(1324)$ having (in one-line notation) the 2 to the left of n at distance k, and the 1 to the right of n. Note that if the 1 is removed from such a permutation, the reduced permutation is an element of $\mathcal{S}_{n-1,k}^{1\prec n-1}(1324)$. This basic observation is used as guideline for most of our combinatorial arguments.

As before, we let $a_{n,k}=|\mathcal{S}_{n,k}^{1\prec n}(1324)|$, so $g_1(x,t)=\sum\limits_{k=1}^{\infty}\sum\limits_{n=k+1}^{\infty}a_{n,k}\,t^k x^n$.

Theorem 3.1 *For $n\geq 3$ and $2\leq k\leq n-1$, we have*

$$|\mathcal{S}_{n,k}^{2\prec n}(1324)|=\tfrac{n-k}{2}a_{n-1,k}. \tag{3.1}$$

Moreover, the corresponding generating function satisfies

$$g_2(x,t)=\frac{1}{2}\Big(x^2\frac{\partial g_1}{\partial x}(x,t)-g_1(x,t)^2\Big).$$

Proof: Let $\mathcal{A}(n,k)=\mathcal{S}_{n,k}^{1\prec n}(1324)$. For every $\sigma\in\mathcal{A}(n-1,k)$, we let

$$i=\sigma^{-1}(1)-1\ \text{ and }\ j=n-1-\sigma^{-1}(n-1).$$

Thus, there are i entries to the left of 1, j entries to the right of $n-1$, and $i+j+k+1=n-1$.

By inserting 1 at any of the the $j+1$ positions to the right of $n-1$, the permutation σ gives rise to $j+1$ permutations in $\mathcal{S}_{n,k}^{2\prec n}(1324)$, and with a similar process, the reverse complement σ^{rc} leads to $i+1$ such permutations. In other words, the pair (σ,σ^{rc}) produces a total of $i+1+j+1=n-k$ permutations in $\mathcal{S}_{n,k}^{2\prec n}(1324)$, but so does the pair (σ^{rc},σ). Therefore, as we go over all permutations $\sigma\in\mathcal{A}(n-1,k)$, the above process generates all of the elements of $\mathcal{S}_{n,k}^{2\prec n}(1324)$ twice. Thus there are $\frac{n-k}{2}a_{n-1,k}$ permutations in $\mathcal{S}_{n,k}^{2\prec n}(1324)$.

Now, given that $g_1(x,t)=\sum\limits_{k=1}^{\infty}\sum\limits_{n=k+1}^{\infty}a_{n,k}\,t^k x^n$, formula (3.1) implies

$$\begin{aligned} g_2(x,t)&=\frac{1}{2}\bigg(x\frac{\partial}{\partial x}\Big(xg_1(x,t)\Big)-tx\frac{\partial}{\partial t}g_1(x,t)\bigg)\\ &=\frac{1}{2}\Big(x^2\frac{\partial}{\partial x}g_1(x,t)+xg_1(x,t)-tx\frac{\partial}{\partial t}g_1(x,t)\Big).\end{aligned} \tag{3.2}$$

Finally, since $g_1(x,t)=\frac{xtf(x)}{1-tf(x)}$ by Theorem 2.6, we get $\frac{\partial}{\partial t}g_1(x,t)=\frac{xf(x)}{(1-tf(x))^2}$, and it follows that $xg_1(x,t)-tx\frac{\partial}{\partial t}g_1(x,t)=-g_1(x,t)^2$. □

There is a formula for $T_{2,k}(x)$ that seems more suitable for generalizations. We start by letting $T_{2,0}(x)=x^2$ (to account for the permutation 21, where the 2 is at distance zero from the maximal element). Moreover, since $T_{1,1}(x)=\sum\limits_{n=2}^{\infty}a_{n,1}x^n$ and $|\mathcal{S}_{n,1}^{2\prec n}(1324)|=\frac{n-1}{2}a_{n-1,1}$ by (3.1), we have

$$T_{2,1}(x)=\sum_{n=3}^{\infty}\frac{n-1}{2}a_{n-1,1}\,x^n=\frac{1}{2}x^2\frac{d}{dx}T_{1,1}(x).$$

Note that by Theorem 2.6, this identity can be written as $T_{2,1}(x)=\frac{1}{2}x^2\frac{d}{dx}(xf(x))$.

Theorem 3.2 *For every $k \geq 2$, we have*

$$T_{2,k}(x) = f(x)^k T_{2,0}(x) + kf(x)^{k-1}\big(T_{2,1}(x) - f(x)T_{2,0}(x)\big).$$

Proof: Every permutation $\sigma \in \mathcal{S}_{n,k}^{2\prec n}(1324)$ that ends with 1 is of the form $\sigma = \sigma_1 \ominus 1$, where $\sigma_1 \in \mathcal{S}_{n-1,k}^{1\prec n-1}(1324)$. Thus, this subset (of permutations in $\mathcal{S}_{n,k}^{2\prec n}(1324)$ ending with 1) is counted by the function $xT_{1,k}(x)$, which by Theorem 2.6 equals $x\big(xf(x)^k\big) = f(x)^k T_{2,0}(x)$.

Let $\mathcal{A}_{2,1}(n)$ be the set of permutations in $\mathcal{S}_{n,1}^{2\prec n}(1324)$ not ending with 1. This set is counted by the generating function $T_{2,1}(x) - f(x)T_{2,0}(x)$ (all permutations in $\mathcal{S}_{n,1}^{2\prec n}(1324)$ minus those ending with 1).

We will show that every element $\sigma \in \mathcal{S}_{n,k}^{2\prec n}(1324)$ with $\sigma(n) > 1$ can be built from (and uniquely corresponds to) a k-tuple of permutations $(\sigma_1, \dots, \sigma_k)$ such that:

- $\sigma_\ell \in \mathcal{A}_{2,1}(m_\ell)$ for some $\ell \in \{1, \dots, k\}$, $m_\ell > 3$,
- $\sigma_j \in \mathcal{S}_{m_j,1}^{1\prec m_j}(1324)$ for every $j \neq \ell$, $m_j \geq 2$,
- $|\sigma_1| + \cdots + |\sigma_k| = n + k - 1$.

The construction goes as follows. Given such a k-tuple, identify σ_ℓ, mark the entry adjacent to the right of 1 (always possible since σ_ℓ does not end with 1), and let σ_ℓ' be the reduced permutation obtained from σ_ℓ by removing the 1. For example, given $(12, 2413, 132)$, we have $\sigma_1 = 12$, $\sigma_2 = 241\underline{3}$, $\sigma_3 = 132$, and so $\sigma_2' = 13\underline{2}$.

Let $\sigma' = \sigma_1 \odot \cdots \odot \sigma_\ell' \odot \cdots \odot \sigma_k$, and let i be the position of the transformed (via the dot product) marked entry. Note that σ' belongs to $\mathcal{S}_{n-1,k}^{1\prec n-1}(1324)$, and the position of $n-1$ in σ' is less than i. For the above example, we get

$$\sigma' = 12 \odot 13\underline{2} \odot 132 = 12 \odot 1354\underline{2} = 12465\underline{3}$$

Next, we let σ be the permutation obtained by inserting 1 into σ' at position i. By our construction, $\sigma \in \mathcal{S}_{n,k}^{2\prec n}(1324)$ and $\sigma(n) > 1$. For example, the tuple $(12, 2413, 132)$ yields the marked permutation $\sigma' = 12465\underline{3}$, and so $\sigma = 2357614$. The permutations in $\mathcal{S}_{7,3}^{2\prec 7}(1324)$ not ending with 1 are all listed in Table 2 together with their corresponding 3-tuples.

Conversely, given $\sigma \in \mathcal{S}_{n,k}^{2\prec n}(1324)$ not ending with 1, mark the entry adjacent to the right of 1, remove the 1, and decompose the reduced permutation $\sigma' \in \mathcal{S}_{n-1,k}^{1\prec n-1}(1324)$ as a product of k primitives. Use the factors to make a k-tuple. Identify the component with the marked entry, call it σ_ℓ', and insert 1 at the position of the mark to obtain an element σ_ℓ of $\mathcal{A}_{2,1}$. The resulting k-tuple is of the form described above.

Finally, since the generating function for $|\mathcal{A}_{2,1}(n)|$ is $T_{2,1}(x) - f(x)T_{2,0}(x)$, the set of k-tuples described above, and thus the set of permutations in $\mathcal{S}_{n,1}^{2\prec n}(1324)$ not ending with 1, are enumerated by the generating function $kf(x)^{k-1}\big(T_{2,1}(x) - f(x)T_{2,0}(x)\big)$. □

Remark 1 *It is worth noting that the above formula for $T_{2,k}(x)$ can also be directly obtained from* (3.2) *together with the fact that $T_{1,k}(x) = xf(x)^k$ and $T_{2,1}(x) = \frac{1}{2}x^2\frac{d}{dx}(xf(x))$.*

(25134, 12, 12) $\rightsquigarrow$	2567134	(12, 25314, 12) $\rightsquigarrow$	2367415	(132, 2413, 12) $\rightsquigarrow$	2467153
(12, 25134, 12) $\rightsquigarrow$	2367145	(12, 12, 25314) $\rightsquigarrow$	2347516	(12, 2413, 132) $\rightsquigarrow$	2357614
(12, 12, 25134) $\rightsquigarrow$	2347156	(32514, 12, 12) $\rightsquigarrow$	3256714	(132, 12, 2413) $\rightsquigarrow$	2457163
(25143, 12, 12) $\rightsquigarrow$	2567143	(12, 32514, 12) $\rightsquigarrow$	4236715	(12, 132, 2413) $\rightsquigarrow$	2357164
(12, 25143, 12) $\rightsquigarrow$	2367154	(12, 12, 32514) $\rightsquigarrow$	5234716	(2413, 213, 12) $\rightsquigarrow$	5246713
(12, 12, 25143) $\rightsquigarrow$	2347165	(42513, 12, 12) $\rightsquigarrow$	4256713	(2413, 12, 213) $\rightsquigarrow$	6245713
(25413, 12, 12) $\rightsquigarrow$	2567413	(12, 42513, 12) $\rightsquigarrow$	5236714	(213, 2413, 12) $\rightsquigarrow$	3246715
(12, 25413, 12) $\rightsquigarrow$	2367514	(12, 12, 42513) $\rightsquigarrow$	6234715	(12, 2413, 213) $\rightsquigarrow$	6235714
(12, 12, 25413) $\rightsquigarrow$	2347615	(2413, 132, 12) $\rightsquigarrow$	2467513	(213, 12, 2413) $\rightsquigarrow$	3245716
(25314, 12, 12) $\rightsquigarrow$	2567314	(2413, 12, 132) $\rightsquigarrow$	2457613	(12, 213, 2413) $\rightsquigarrow$	4235716

Tab. 2: The 30 elements of $\mathcal{S}_{7,3}^{2\prec 7}(1324)$ not ending with 1.

4 Conjecture and final remarks

At the beginning of the paper, we introduced the notation

$$T_{a,k}(x) = \sum_{n=k+1}^{\infty} |\mathcal{S}_{n,k}^{a\prec n}(1324)|x^n \quad \text{and} \quad g_a(x,t) = \sum_{k=1}^{\infty} t^k T_{a,k}(x).$$

Let $T_{1,0}(x) = x$ and $T_{a,0}(x) = |\mathcal{S}_{a-1}(1324)|x^a$. These functions account for the permutations of size a that start with a.

Recall that, in one-line notation, a permutation $\sigma \in \mathcal{S}_{n,k}^{a\prec n}(1324)$ has all of its entries less than a to the right of n, and if they are removed, we are left with a reduced 1324-avoiding permutation of size $n-a+1$ having the 1 to the left of the maximal element. Thus, it is not unreasonable to expect a connection between $T_{a,k}(x)$ and $T_{1,k}(x)$. With that in mind, and based on how we proved Theorem 3.2, we spent some time looking for an expansion of $T_{a,k}(x)$ in terms of powers of $f(x)$ and the functions $T_{a,j}(x)$ for $j = 1, \ldots, a-1$. Our search lead to the following conjecture.

Conjecture 1 *For $k \geq a$, we have the following equivalent formulas:*

(i) $\sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x)^j T_{a,k-j} = 0.$

(ii) $T_{a,k}(x) = \sum_{j=0}^{a-1} \binom{k}{j} f(x)^{k-j} \sum_{i=0}^{j} (-1)^i \binom{j}{i} f(x)^i T_{a,j-i}(x).$

(iii) $T_{a,k}(x) = \sum_{j=0}^{a-1} (-1)^{a-j-1} \binom{k}{j} \binom{k-j-1}{a-j-1} f(x)^{k-j} T_{a,j}(x).$

For $a = 1$ the statements are trivial, and for $a = 2$, (ii) becomes

$$T_{2,k}(x) = f(x)^k T_{2,0}(x) + k f(x)^{k-1}\big(T_{2,1}(x) - f(x)T_{2,0}(x)\big),$$

as claimed and proved in Theorem 3.2.

For $a = 3$ the conjectured formula (ii) becomes

$$\begin{aligned} T_{3,k}(x) = f(x)^k T_{3,0}(x) &\\ + k f(x)^{k-1}\big(T_{3,1}(x) - f(x)T_{3,0}(x)\big) &\\ + \tbinom{k}{2} f(x)^{k-2}\big(T_{3,2}(x) - 2f(x)T_{3,1}(x) + f(x)^2 T_{3,0}(x)\big), & \end{aligned}$$

where $T_{3,0}(x) = 2x^3$ (counting the permutations 312 and 321). Similar to what we did for $a = 2$, we can interpret the term $f(x)^k T_{3,0}(x)$ as counting the permutations in $\mathcal{S}^{3\prec n}_{n,k}(1324)$ that end with 12 or 21. Moreover, the function $kf(x)^{k-1}\big(T_{3,1}(x) - f(x)T_{3,0}(x)\big)$ can be interpreted as counting k-tuples $(\sigma_1, \dots, \sigma_k)$, where $\sigma_j \in \mathcal{S}^{1\prec m_j}_{m_j,1}(1324)$, $m_j \ge 2$, and one of these permutations, say σ_ℓ, is marked in such a way that it corresponds to an element of $\mathcal{S}^{3\prec m_\ell}_{m_\ell,1}(1324)$ that does not end with 12 or 21.

Finally, rewriting the third component of $T_{3,k}(x)$ as

$$\tbinom{k}{2} f(x)^{k-2}\big(T_{3,2}(x) - 2f(x)(T_{3,1}(x) - f(x)T_{3,0}(x)) - f(x)^2 T_{3,0}(x)\big),$$

it can be argued that this function counts k-tuples $(\sigma_1, \dots, \sigma_k)$ of primitives, where two of them, say σ_{ℓ_1} and σ_{ℓ_2}, are marked in such a way that the pair $(\sigma_{\ell_1}, \sigma_{\ell_2})$ corresponds to a permutation in $\mathcal{S}^{3\prec m}_{m,2}(1324)$, $m = m_{\ell_1} + m_{\ell_2} + 1$, that does not end with 12 or 21.

Final remarks

In this paper, we have introduced a notion of positional statistics that seems particularly suited to and provides a new way to think about 1324-avoiding permutations. As we did for $a = 1$ in Theorem 2.6 and for $a = 2$ in Theorem 3.1, the ultimate goal is to find an expression for $g_a(x,t)$ in terms of known functions.

Proving the above conjecture would be a significant step in that direction, but it is not the whole story. For instance, while the conjecture is true for $a = 3$, we still need to find $T_{3,1}(x)$ and $T_{3,2}(x)$ in order to have a full expression for $g_3(x,t)$. For an arbitrary $a > 3$, our conjecture would reduce the problem to finding $T_{a,j}$ for $j = 1, \dots, a-1$.

We have observed interesting properties for several patterns and hope that our work motivates the community to explore positional statistics for patterns other than 1324.

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