2-polarity and algorithmic aspects of polarity variants on cograph superclasses*

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A graph G is said to be an (s, k)-polar graph if its vertex set admits a partition (A, B) such that A and B induce, respectively, a complete s-partite graph and the disjoint union of at most k complete graphs. Polar graphs and monopolar graphs are defined as (∞, ∞) - and $(1, \infty)$ -polar graphs, respectively, and unipolar graphs are those graphs with a polar partition (A, B) such that A is a clique.

The problems of deciding whether an arbitrary graph is a polar graph or a monopolar graph are known to be NPcomplete. In contrast, deciding whether a graph is a unipolar graph can be done in polynomial time. In this work we prove that the three previous problems can be solved in linear time on the classes of P_4 -sparse and P_4 -extendible graphs, generalizing analogous results previously known for cographs.

Additionally, we provide finite forbidden subgraph characterizations for (2, 2)-polar graphs on P_4 -sparse and P_4 -extendible graphs, also generalizing analogous results recently obtained for the class of cographs.

Keywords: P₄-sparse graph, P₄-extendible graph, cograph, polar graph, 2-polar graph, graph algorithms

1 Introduction

All graphs in this paper are finite and simple; for basic terminology not defined here we refer the reader to the beautiful book of Bondy and Murty [1]. For graphs G and H, we denote that H is an induced subgraph of G by $H \leq G$. Given a family of graphs \mathcal{H} , we say that G is \mathcal{H} -free if G does not have induced subgraphs isomorphic to any graph $H \in \mathcal{H}$; accordingly, we say that G is an H-free graph if it is $\{H\}$ -free. A property of graphs is *hereditary* if it is closed under taking induced subgraphs. Given a hereditary property \mathcal{P} of graphs, a *minimal* \mathcal{P} -obstruction is a graph G that does not have the property \mathcal{P} but such that any vertex-deleted subgraph of G does.

A *k*-cluster is the disjoint union of at most *k* complete graphs; a *cluster* is a *k*-cluster for some positive integer *k*. It is easy to verify that *k*-clusters coincide with $\{\overline{K_{k+1}}, P_3\}$ -free graphs, while clusters are precisely P_3 -free graphs. A *complete k-partite graph* is the complement of a *k*-cluster, or equivalently, a $\{\overline{K_{k+1}}, \overline{P_3}\}$ -free graph; a *complete multipartite graph* is the complement of a cluster, i.e., a $\overline{P_3}$ -free graph.

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An (s,k)-polar partition of a graph G is a partition of V_G in two possible empty sets A and B such that G[A] is a complete s-partite graph, and G[B] is a k-cluster. If G admits an (s,k)-polar partition we say that it is an (s,k)-polar graph. A (k,k)-polar partition is simply referred as a k-polar partition, and a graph which admits such partition is a k-polar graph. A 1-polar graph is commonly called a split graph; in the classic article of Foldes and Hammer [9], split graphs were characterized to be $\{2K_2, C_4, C_5\}$ -free graphs. If we replace s or k by ∞ , it means that the number of components of $\overline{G[A]}$ or G[B], respectively, is unbounded. An (∞, ∞) -polar partition of a graph is simply called a polar partition, and a graph with such partition is a polar graph. A graph with polar partition (A, B) such that A is an independent set (respectively, a clique) is called a monopolar graph (resp. a unipolar graph). Naturally, the polar partitions associated to monopolar and unipolar graphs are referred as monopolar and unipolar partitions, respectively.

Graphs without induced paths on four vertices are known as *cographs*. A graph such that any set of five vertices induces at most one P_4 is called a P_4 -sparse graph, and a graph such that, for any vertex subset W inducing a P_4 there exists at most one vertex $v \notin W$ belonging to a P_4 which shares vertices with W, is a P_4 -extendible graph.

In [3] it was proved that any hereditary property of graphs restricted to P_4 -sparse graphs and P_4 extendible graphs can be characterized by a finite set of forbidden induced subgraphs. In the same paper, such characterizations for the properties of having a polar partition, a monopolar partition, a unipolar partition, and an (s, 1)-polar partition for any fixed positive integer s were given. In this paper, we continue with the work started in [3], establishing linear-time algorithms to find maximum subgraphs associated with properties related to polarity in P_4 -sparse and P_4 -extendible graphs, and giving forbidden subgraph characterizations for P_4 -sparse and P_4 -extendible graph which admit a 2-polar partition. For the sake of length we invite the reader to read [3] where a discussion on the relevance of the topic of this paper can be found.

The rest of the paper is organized as follows. Section 2 is devoted to a brief introduction of P_4 -sparse and P_4 -extendible graphs. In Section 3 we give complete lists of minimal P_4 -sparse and P_4 -extendible 2-polar obstructions, while in Section 4 we provide algorithms for finding maximum polar, unipolar, and monopolar subgraphs in both P_4 -sparse and P_4 -extendible graphs. Conclusions and some open problems are given in Section 5.

2 Cograph generalizations

We use G + H to denote the disjoint union of the graphs G and H; accordingly, we denote by nG the disjoint union of n copies of the graph G. The join of G and H, defined as the graph $\overline{G} + \overline{H}$, will be denoted by $G \oplus H$. We say that two vertex subsets are *completely adjacent* if every vertex of one of them is adjacent to any vertex of the other. Similarly, if no vertex of one of them is adjacent to a vertex subsets are *completely nonadjacent*. The following proposition include some characterizations for cographs which are particularly relevant for this work.

Theorem 1. [4] Let G be a graph. The following statements are equivalent.

- 1. G is a P_4 -free graph (i.e. a cograph).
- 2. G can be constructed from trivial graphs by means of join and disjoint union operations.
- 3. For any nontrivial induced subgraph H of G, either H or \overline{H} is disconnected.

It follows from item 3 of the previous theorem that cographs can be uniquely represented by a rooted labeled tree, its cotree, introduced by Corneil, Lerchs and Stewart Burlingham in [4]. In [2], Bretscher, Corneil, Habib and Paul, showed that cographs can be recognized, and its associated cotree can be constructed, in linear time by an algorithm based on LexBFS. From here, using bottom-up algorithms on their cotrees, many algorithmic problems which are difficult in general graphs can be efficiently solved on cographs.

Much of the relevance of cographs comes from real-life applications involving graph models with just a few induced paths of length three, as discussed by Corneil, Perl and Stewart Burlingham in [5]. Evidently, P_4 -free graphs (cographs) are the most restrictive graph class in this way, so it becomes important to ask whether a cograph superclass with less restrictions on the amount of allowed induced P_4 's has a behavior similar to cographs, particularly, whether it allows us to develop efficient algorithms for solving problems by using a unique tree representation. Next, we briefly introduce two graph classes which are unlikely to have many induced paths on four vertices. Such families are known to have unique tree representations analogous to the cotree, which can be computed in linear time and can be used to solve some problems in linear time.

2.1 P_4 -sparse

The P_4 -sparse graphs are defined as the graphs such that the subgraphs induced by any five vertices have at most one induced copy of P_4 . Clearly, P_4 -sparse graphs are precisely the $\{C_5, P_5, \overline{P_5}, P, \overline{P}, F, \overline{F}\}$ -free graphs (see Figure 1). Additionally, Jamison and Olariu [14] provided a connectedness characterization of P_4 -sparse graphs based on some special graphs called spiders, which we now introduce.

A graph G is said to be an *spider* if its vertex set admits a partition (S, K, R) such that S is an independent set with at least two vertices, K is a clique, R is completely adjacent to K but completely nonadjacent to S, and there is a bijection $f: S \to K$ such that either $N(s) = \{f(s)\}$ for each $s \in S$ or $N(s) = K - \{f(s)\}$ for each $s \in S$. For a spider G = (S, K, R) we will say that S is its *legs set*, K is its *body*, and R is its *head*. A *headless spider* is a spider with empty head. An spider will be called *thin* (respectively *thick*) if d(s) = 1 (respectively d(s) = |K| - 1) for any $s \in S$. Observe that the complement of a thin spider is a thick spider and vice versa.

Theorem 2. [14] A graph G is a P_4 -sparse graph if and only if for every nontrivial induced subgraph H of G, exactly one of the following statements is satisfied

- 1. H is disconnected.
- 2. \overline{H} is disconnected.
- 3. H is an spider.

The next observation about spiders will be important in Section 3. It follows from the fact that a graph is $(0, \infty)$ -polar (i.e. a cluster) if and only if it is a P_3 -free graph and, complementarily, that a graph is $(\infty, 0)$ -polar if and only if it is a $\overline{P_3}$ -free graph.

Remark 3. Let G be a spider. If G is a headless spider or the head of G induces a split graph, then G is a split graph that has both, P_3 and its complement, as proper induced subgraphs. Hence, G is not a minimal (s, k)-polar obstruction for any election of s and k.

2.2 P_4 -extendible graphs

Given a graph G and a vertex subset W, we denote by S(W) the set of vertices $x \in V_G - W$ such that x belongs to a P_4 sharing vertices with W. If a vertex subset W inducing P_4 is such that S(W) has at most one vertex, we say that $W \cup S(W)$ is an *extension set*. In [13], P_4 -*extendible graphs* were introduced by Jamison and Olariu as the graphs G such that, for every set W inducing a P_4 , $W \cup S(W)$ is an extension set.

An extension set D is *separable* if no vertex of D is both an endpoint of some P_4 and a midpoint of some P_4 in G[D]. Notice that any extension set must induce one of the eight graphs depicted in Figure 1; we call these graphs *extension graphs*. In addition, separable extension sets must induce one of P_4 , P, F or their complements; these graphs are called *separable extension graphs*.

For a separable extension graph X with midpoints set K and endpoints set S, a graph H is said to be an X-spider if H is an induced supergraph of X such that $R := V_H \setminus V_X$ is completely adjacent to K but completely nonadjacent to S. If H is an X-spider, we say that (S, K, R) is an X-spider partition of H, and we refer to S, K and R as the legs set, the body, and the head of H, respectively. From now on, every time we use the term X-spider, we are assuming that X is a separable extension graph.



Fig. 1: The eight extension graphs. Black vertices are the midpoints of separable extension graphs.

Jamison and Olariu also gave in [13] the following connectedness characterization for the class of P_4 -extendible graphs.

Theorem 4. [13] A graph G is a P_4 -extendible graph if and only if, for every nontrivial induced subgraph H of G, precisely one of the following conditions holds

- 1. H is disconnected.
- 2. \overline{H} is disconnected.
- 3. H is an extension graph.
- 4. There is a unique separable extension graph X such that H is an X-spider with nonempty head.

Notice that any extension graph, but P_4 , is a P_4 -extendible graph which is not a P_4 -sparse graph. In addition, any headless spider of order at least six is a P_4 -sparse graph which is not a P_4 -extendible graph. Thus, P_4 -sparse and P_4 -extendible graphs are two cograph superclasses which are incomparable to each other.

3 Minimal 2-polar obstructions

Throughout this section we give complete lists of minimal P_4 -sparse and minimal P_4 -extendible 2-polar obstructions, obtaining in this way characterizations for the P_4 -sparse and P_4 -extendible graphs which admit a 2-polar partition. These characterizations generalize analogous results given for cographs by Hell, Hernández-Cruz and Linhares-Sales in [11]. In fact, we base our characterizations in the following propositions, most of them taken from the mentioned paper.

We start with two lemmas which provide some useful general structural properties about minimal k-polar obstructions.

Lemma 5. [11] Let H be a minimal k-polar obstruction. The following statements are true

- 1. *H* has at most k + 2 components.
- 2. H has at least one nontrivial component.
- *3. H* has at most k + 1 trivial components.
- 4. If H has at least one trivial component, H has at most one noncomplete component.
- 5. If $H \not\cong (k+1)K_{k+1}$, every complete component of H is isomorphic to K_1 or K_2 .

Lemma 6. [11] Let H be a minimal 2-polar obstruction.

- 1. H has at least seven vertices.
- 2. If *H* has seven vertices and three connected components, then at least one of them is an isolated vertex.

Next, we give a slight correction to Lemma 2 in [11], which characterize the minimal k-polar obstructions with the maximum possible number of components; it is worth noticing that it does not affect the main results in such paper.

Lemma 7. Let k be an integer, $k \ge 2$, and let G be graph. Then, G is a minimal k-polar obstruction with exactly k + 2 connected components if and only if $G \cong \ell K_1 + (k - \ell + 1)K_2 + G'$, where ℓ is an integer in the set $\{1, \ldots, k+1\}$ and G' is a connected complete k-partite graph which is a minimal $(1, \ell - 1)$ -polar obstruction and such that, if $\ell \le k$, G' is a $(1, \ell)$ -polar graph.

Proof: Suppose $G \cong \ell K_1 + (k - \ell + 1)K_2 + G'$, where ℓ is an integer in the set $\{1, \ldots, k + 1\}$ and G' is a connected complete k-partite graph which is a minimal $(1, \ell - 1)$ -polar obstruction such that, if $\ell \leq k$, it is a $(1, \ell)$ -polar graph. If G is a (1, k)-polar graph, then G' is a $(1, \ell - 1)$ -polar graph, but it is not. Thus, since G is not (1, k)-polar, if its admits a k-polar partition (A, B), the subgraph G[A] is a connected graph and hence it is completely contained in some component of G. But then, G would have at most k + 1 connected components, which is not the case. Hence, G is not a k-polar graph.

Let v be an isolated vertex of G. Then G - v is the disjoint union of a k-cluster with G', and since G' is a complete k-partite graph, then G - v is a k-polar graph. Now, since G' is a minimal $(1, \ell - 1)$ -polar obstruction, for any vertex w of G', G' - w can be partitioned into an stable set and an $(\ell - 1)$ -cluster, so G - w is a (1, k)-polar graph, and then a k-polar graph. Finally, if at least one component of G is a copy of K_2 , then $\ell \leq k$ and we have that G' is a $(1, \ell)$ -polar graph. Thus, for any vertex u in a K_2 -component of G, G - u is a (1, k)-polar graph. Hence, G is a minimal k-polar obstruction which evidently has exactly k + 2 connected components.

For the converse implication, assume that G is a minimal k-polar obstruction with precisely k + 2 components and ℓ isolated vertices. If $\ell = 0$ then G properly contains $K_1 + (k + 1)K_2$ as an induced subgraph, but that is impossible because both, G and $K_1 + (k + 1)K_2$ are minimal k-polar obstructions. Then, G has at least one isolated vertex, and evidently G is not an empty graph, so $\ell \le k + 1$. We know by Lemma 5 that G has at most one noncomplete connected component and that any complete component of G has at most two vertices, so $G \cong \ell K_1 + (k - \ell + 1)K_2 + G'$ where $\ell \in \{1, \ldots, k + 1\}$ and G' is a connected graph.

Notice that G' is not a $(1, \ell - 1)$ -polar graph, otherwise G would be a (1, k)-polar graph, and hence a k-polar graph. Let u be a vertex of G'. By the minimality of G, we have that G - u is a k-polar graph. Moreover, since G - u has at least k + 2 connected components, any k-polar partition of G - uis necessarily a (1, k)-polar partition, which implies that G' - u is a $(1, \ell - 1)$ -polar graph. Then G' is a minimal $(1, \ell - 1)$ -polar obstruction. Now, let v be an isolated vertex of G. By the minimality of G, G - uhas a k-polar partition (A, B), but it cannot be a (1, k)-polar partition or G would be a (1, k)-polar graph. Thus, either $G' \cong K_2$ and $\ell = 1$, or A = V(G') and hence G' is a complete k-partite graph. Finally, if $l \leq k$, G has at least one K_2 -component. Let w be a vertex in one of such components. Then G - w is a k-polar graph with k + 2 connected components, which implies that in fact G - w is a (1, k)-polar graph, and hence G' is a $(1, \ell)$ -polar graph.

A partial complement of a graph H is either the usual complement of H, or a graph $\overline{H_1} + \overline{H_2}$, where H_1 and H_2 are subgraphs of H obtained by splitting the components of H into two parts, H_1 and H_2 . The next result shows how the partial complement operation preserves 2-polarity, which will be useful for giving compact lists of minimal 2-polar obstructions on P_4 -sparse and P_4 -extendible graphs. Remarkably, this lemma was originally proven for the special class of cographs, but the same proof works for any hereditary class of graphs closed under complement and disjoint union operations, particularly, it works for the classes of P_4 -sparse and P_4 -extendible graphs.

Lemma 8. [11] Let \mathcal{G} be a hereditary class of graphs closed under complement and disjoint union operations, and let $G \in \mathcal{G}$ be a 2-polar graph. Then, any partial complement of G is a 2-polar graph belonging to \mathcal{G} .

Based on the previous propositions, we can easily check that the graphs in Figures 5 to 11, as well as their complements, are all of them minimal 2-polar obstructions: it is enough to verify that, for each of the mentioned figures, one of its graphs, let say F, is a 2-polar obstruction, and that the closure of $\{F\}$ under partial complements is precisely the set of all graphs in the same figure and their complements. In the following sections we prove that any minimal 2-polar obstruction that is a P_4 -sparse or a P_4 -extendible graph is either a graph depicted in Figures 5 to 11 of the complement of one of them.

3.1 P₄-sparse minimal 2-polar obstructions

Throughout this section we characterize P_4 -sparse graphs admitting a 2-polar partition by means of its family of minimal obstructions. At the end of the section we conclude that any P_4 -sparse minimal 2-polar obstruction is in fact a cograph, which is interesting since also any known P_4 -sparse minimal (s, k)-polar obstruction is a cograph. We start by proving that the complement of any connected P_4 -sparse minimal 2-polar obstruction is a disconnected graph.

Proposition 9. If G is a spider, then G is 2-polar if and only if G is a split graph.

Proof: Let (S, K, R) be the spider partition of G. We only need to prove that any 2-polar spider is, in fact, a split graph. Since k-polar graphs are closed under complements, and headless spiders trivially are split graphs, we can assume that G is a thin spider with nonempty head. Let (V_1, V_2, V_3, V_4) be a 2-polar partition of G, and for any $i \in \{1, 2, 3, 4\}$, let $R_i = V_i \cap R$. Notice that, since K is completely adjacent to $R, R_i = \emptyset$ for some $i \in \{1, \ldots, 4\}$.

First, suppose that (R_1, R_3, R_4) is a (1, 2)-polar partition of G[R]. Again, some of R_1, R_3 and R_4 must be empty because K and R are completely adjacent and K has at least two vertices. Thus, either (R_1, R_3) is a split partition of G[R], or (R_3, R_4) is a (0, 2)-polar partition of G[R]. But the second case is not possible since then, $S \cup K \subseteq V_1 \cup V_2$, which is impossible since $G[S \cup K]$ is not a complete multipartite graph. Hence, G[R] is a split graph and, by Remark 3, also is G. The case in which (R_1, R_2, R_3) is a (2, 1)-polar partition of G[R] can be treated in a similar way.

Corollary 10. If G is a spider, then G is not a minimal 2-polar obstruction. In consequence, for any P_4 -sparse minimal 2-polar obstruction H, either H or its complement is disconnected.

Proof: Let (S, K, R) be the spider partition of G. As in the lemma above, we can suppose that G is a thin spider. Assume for a contradiction that G is a minimal 2-polar obstruction so, by the previous lemma and Remark 3, we have that G[R] is not a split graph. Then, for any $r \in R$, G - r is a spider which is 2-polar, so G[R] - r is a split graph. Thus G[R] is a P_4 -sparse minimal split obstruction, that is to say, G[R] is isomorphic to either $2K_2$ or C_4 . From here is easy to prove that deleting either one leg or one vertex of the body of G the resulting graph is not a 2-polar graph, contradicting the minimality of G. Hence, a P_4 -sparse minimal 2-polar obstruction is not a spider, and the result directly follows from Theorem 2.

By Lemma 5, any P_4 -sparse minimal 2-polar obstruction has at most four connected components. With the purpose of giving the complete list of such obstructions, we mention first some useful propositions on minimal (s, 1)-polar obstructions.

Theorem 11. [3] Let s be an integer, $s \ge 2$. If G is a disconnected minimal (s, 1)-polar obstruction, then G satisfies one of the following assertions:

- 1. G is isomorphic to one of the graphs depicted in Figure 2.
- 2. $G \cong 2K_{s+1}$.
- 3. $G \cong K_2 + (2K_1 \oplus K_s).$
- 4. $G \cong K_1 + (C_4 \oplus K_{s-1}).$



Fig. 2: Some minimal $(\infty, 1)$ -polar obstructions.

Proposition 12. [3] Let s be a positive integer. Any P_4 -sparse minimal (s, 1)-polar obstruction G is a cograph. In consequence, either G or its complement is disconnected.

Proposition 13. [3] There are exactly nine P_4 -sparse minimal (2,1)-polar obstructions; they are the graphs E_1, \ldots, E_9 depicted in Figures 2 and 3.



Fig. 3: Some minimal (2, 1)-polar obstructions.

Now we have the necessary tools to prove that there are exactly three P_4 -sparse minimal 2-polar obstructions with four connected components.

Proposition 14. Let ℓ be a positive integer. If G is a connected P_4 -sparse minimal $(1, \ell - 1)$ -polar obstruction which is a complete multipartite graph, then G is isomorphic to either $K_{\ell,\ell}$ or $K_1 \oplus C_4$.

Proof: Clearly, if $\ell = 1$, $G \cong K_2$, while if $\ell = 2$, $G \cong C_4$. For $\ell \ge 3$, we have from Proposition 12 that \overline{G} is a disconnected graph, and it follows from Theorem 11 that G is isomorphic to either $K_{\ell,\ell}$ or $K_1 \oplus C_4$.

Corollary 15. If G is a P_4 -sparse graph, then G is a minimal 2-polar obstruction with exactly 4 connected components if and only if $G \cong \ell K_1 + (3 - \ell)K_2 + K_{\ell,\ell}$ for some integer $\ell \in \{1, 2, 3\}$.

Proof: Let G be a P_4 -sparse graph. By Lemma 7 we have that G is a minimal 2-polar obstruction with precisely four connected components if and only if $G \cong \ell K_1 + (3 - \ell)K_2 + G'$, where $\ell \in \{1, 2, 3\}$, and G' is a connected complete bipartite graph which is a minimal $(1, \ell - 1)$ -polar obstruction such that, if $\ell \neq 3$, G' is a $(1, \ell)$ -polar graph. In addition, we have from Proposition 14 that the only connected P_4 -sparse minimal $(1, \ell - 1)$ -polar obstruction which is a complete bipartite graph is $K_{\ell,\ell}$. The result follows since $K_{\ell,\ell}$ trivially is a $(1, \ell)$ -polar graph.

Hannebauer [10] proved that, for any nonnegative integers s and k, any P_4 -sparse minimal (s, k)-polar obstruction has at most (s + 1)(k + 1) vertices. Thus, we have by Lemma 6 that any P_4 -sparse minimal 2-polar obstruction has at least seven and at most nine vertices. The following three lemmas completely characterize such minimal obstructions depending on their order; the proofs are simple generalizations of the analogous proofs given in [11] for cographs.

Lemma 16. The disconnected P_4 -sparse minimal 2-polar obstructions on 7 vertices are exactly the graphs F_1, \ldots, F_5 depicted in Figure 4.



Fig. 4: P₄-sparse minimal 2-polar obstructions on 7 vertices.

Proof: Let H be a disconnected P_4 -sparse minimal 2-polar obstruction on seven vertices. By the observation after Lemma 8 it is enough to prove that $H \cong F_i$ for some $i \in \{1, ..., 5\}$. If H has four connected components or it can be transformed by a sequence of partial complementations into a graph with four components, it follows from Corollary 15 and Lemma 8 that H is isomorphic to F_i for some $i \in \{1, ..., 5\}$. Thus, we can assume that any graph obtained from H by partial complementations has at most three components; from here we can replicate the argument in Lemma 7 of [11] to assume that H is a graph with precisely two connected components, one of them being a trivial graph.

Since *H* is not a 2-polar graph, its nontrivial component must contain a minimal (2, 1)-polar obstruction *H'* as an induced subgraph. Moreover, *H'* cannot be a disconnected graph on six vertices, so we have from Proposition 13 that $H \in \{K_1 + 2K_2, \overline{3K_2}, 2K_2 \oplus 2K_1\}$. If $H' \cong \overline{3K_2}$, *H* is the graph F_5 in Figure 4. If $H' \cong 2K_2 \oplus 2K_1$, is straightforward to verify that *H* is a (1, 2)-polar graph, which cannot occur. Otherwise, if $H' \cong K_1 + 2K_2$, we have that $H \cong F_3$, because P_4 -sparse graphs are $\{\overline{P}, P_5\}$ -free and *H'* is contained in a connected component of *H* on six vertices.

Lemma 17. The disconnected P_4 -sparse minimal 2-polar obstructions on 9 vertices are the graphs F_{21}, \ldots, F_{24} depicted in Figure 5.



Fig. 5: P₄-sparse minimal 2-polar obstructions on 9 vertices.

Proof: Almost all the arguments used in the proof of Lemma 8 in [11] are still valid for P_4 -sparse graphs. We only have to care about the case when H is a P_4 -sparse minimal 2-polar obstruction on 9 vertices with three connected components and precisely two isolated vertices. In such a case the nontrivial connected component of H, B_3 , is either a spider or the join of two smaller P_4 -sparse graphs T_1 and T_2 . In the former case, since the head of B_3 has at most three vertices, B_3 is a split graph, so H is too. The latter case follows as in the original proof.



Fig. 6: Family A of P₄-sparse minimal 2-polar obstructions on 8 vertices.

Lemma 18. The disconnected P_4 -sparse minimal 2-polar obstructions on 8 vertices are the graphs F_6, \ldots, F_{20} and F_{25} , depicted in Figures 6 and 7.

Proof: The proof of Lemma 9 in [11] is still valid for P_4 -sparse graphs with the only addition of the graph F_{25} as a partial complement of the graph F_{19} , which was omitted by mistake in [11]. The main arguments are similar to those used in the proof of Lemma 17.

We summarize the results of this section in the following theorem.



Fig. 7: Family B of P₄-sparse minimal 2-polar obstructions on 8 vertices.

Theorem 19. There are exactly 50 P_4 -sparse minimal 2-polar obstructions, and each of them is a cograph. The disconnected P_4 -sparse minimal 2-polar obstructions are the graphs F_1, \ldots, F_{25} depicted in Figures 4 to 7.

*3.2 P*₄-extendible minimal 2-polar obstructions

In [3] it was observed that the set of cograph minimal (s, k)-polar obstructions is a proper subset of the set of P_4 -extendible minimal (s, k)-polar obstructions for the cases min $\{s, k\} = 1$ and $s = k = \infty$. In the present section we give the complete family of P_4 -extendible minimal 2-polar obstructions, and show that also in the case s = k = 2 there are P_4 -extendible minimal (s, k)-polar obstructions which are not cographs. Indeed, each graph depicted in Figures 8 to 11 is a P_4 -extendible minimal 2-polar obstruction which is not a cograph.

We start by proving that there exists only one P_4 -extendible connected minimal 2-polar obstruction whose complement is also a connected graph.

Lemma 20. If G = (S, K, R) is a \overline{P} -spider and H = G[R], then G is a minimal 2-polar obstruction if and only if $H \cong P_3$, that is, if G is isomorphic to the graph F_{26} in Figure 8.

Proof: If $H \cong P_3$, then $G \cong F_{26}$, so G is a minimal 2-polar obstruction. Suppose for a contradiction that G is another \overline{P} -spider minimal 2-polar obstruction. Being P_3 -free, H is a cluster. Moreover, if H is not a complete multipartite graph, then G properly contains F_3 as an induced subgraph, which is impossible. Then H is a cluster which is a complete multipartite graph, so it is either a complete or an empty graph. However, it is easy to check that in both cases G is a 2-polar graph, contradicting our original assumption.



Fig. 8: A connected P₄-extendible minimal 2-polar obstruction with connected complement.

The proofs of the next proposition and its corollaries are very similar to the proofs of Proposition 9 and its corollaries, so we only sketch them without going into details.

Proposition 21. Let $X \in \{P_4, F\}$. If G is an X-spider, then G is a 2-polar graph if and only if R induces a split graph.

Proof: Let (S, K, R) be the spider partition of G. First, assume that (A, B) is a split partition of G[R]. Then, $(A \cup S, B \cup K)$ is a split partition of G, so G is a split graph, and hence a 2-polar graph. Now, suppose that G has a 2-polar partition (V_1, V_2, V_3, V_4) , and let $R_i = V_i \cap R$ for each $i \in \{1, \ldots, 4\}$. Notice that, if R_1 and R_2 are both nonempty, then $S \cup K \subseteq V_3 \cup V_4$, which is impossible since X is not a cluster. Analogously, since X is not a complete multipartite graph, R_3 and R_4 cannot be both nonempty. Therefore G[R] is a split graph.

Corollary 22. Let $X \in \{P_4, F\}$. If G is an X-spider, then it is not a minimal 2-polar obstruction.

Proof: Let (S, K, R) be the spider partition of G. In order to reach a contradiction, suppose that G is a minimal 2-polar obstruction. By Proposition 21, G[R] is not a split graph, but for any vertex $v \in R$, G[R] - v is. Hence, G[R] is a minimal split obstruction, i.e., G[R] is isomorphic to some of $2K_2, C_4$ or C_5 . But then, G contains $F_3, \overline{F_3}$ or F_{27} , respectively, as a proper induced subgraph, contradicting the minimality of G.

Corollary 23. If G is a P_4 -extendible minimal 2-polar obstruction different from F_{26} and its complement, then G or its complement is disconnected.

Proof: It is a simple exercise to verify that any extension graph is a 2-polar graph. In addition, by Lemma 20 and Proposition 21, the only X-spiders that are minimal 2-polar obstructions are F_{26} and its complement. Therefore, by Theorem 4, any other P_4 -extendible minimal 2-polar obstruction is disconnected or has a disconnected complement.

As we did in the case of P_4 -sparse graphs, now we characterize the P_4 -extendible minimal 2-polar obstructions with the maximum possible number of connected components. We start by quoting two useful results of P_4 -extendible minimal (s, 1)-polar obstructions.

Theorem 24. [3] Let s be an integer, $s \ge 2$. If G is a P_4 -extendible graph, then G is a minimal (s, 1)-polar obstruction if and only if G satisfies exactly one of the following assertions:

1. *G* is isomorphic to one of the seven graphs depicted in Figure 2.

- 2. *G* is isomorphic to some of $2K_{s+1}$, $K_2 + (K_s \oplus 2K_1)$ or $K_1 + (K_{s-1} \oplus C_4)$.
- 3. For some nonnegative integers s_1, s_2, \ldots, s_t such that $s = t 1 + \sum_{i=1}^t s_i$, the complement of G is a disconnected graph with components G_1, \ldots, G_t , where each G_i is a minimal $(1, s_i)$ -polar obstruction whose complement is different from the graphs in Figure 2.

Corollary 25. There are exactly 13 P_4 -extendible minimal (2, 1)-polar obstructions; they are the graphs E_1, \ldots, E_{13} depicted in Figures 2 and 3.

As the reader can check, the proofs of the next proposition and its corollary are analogous to those of Proposition 14 and Corollary 15.

Proposition 26. Let ℓ be a positive integer. If G is a connected P_4 -extendible minimal $(1, \ell - 1)$ -polar obstruction which is a complete multipartite graph, then G is isomorphic to either $K_{\ell,\ell}$ or $K_1 \oplus C_4$.

Proof: Clearly, if $\ell = 1$, then $G \cong K_2$, while if $\ell = 2$, we have $G \cong C_4$. By Theorem 24, if $\ell \ge 3$, G is isomorphic to either $K_{\ell,\ell}$ or $K_1 \oplus C_4$.

Corollary 27. If G is a P_4 -extendible graph, then G is a minimal 2-polar obstruction with exactly 4 connected components if and only if $G \cong \ell K_1 + (3 - \ell)K_2 + K_{\ell,\ell}$ for some integer $\ell \in \{1, 2, 3\}$.

Proof: This result represents to P_4 -extendible graphs the same as Corollary 15 is to P_4 -sparse graphs. In fact, the proof of this result is basically the same as that of Corollary 15, but using instead Proposition 26, which is to P_4 -extendible graphs as Proposition 14 is to P_4 -sparse graphs.

By Lemma 6, we have that no P_4 -extendible minimal 2-polar obstruction has less than seven vertices. In the rest of the section we give the complete list of such obstructions, obtaining as a consequence that they have at most 9 vertices, as in the case of P_4 -sparse graphs. We remark that these proofs are very similar in flavor to the analogous proofs for P_4 -sparse graphs.

Lemma 28. The disconnected P_4 -extendible minimal 2-polar obstructions on 7 vertices are exactly the graphs F_1, \ldots, F_5 depicted in Figure 4.

Proof: Let H be a disconnected P_4 -extendible minimal 2-polar obstruction on 7 vertices. By the observation after Lemma 8, it is enough to prove that $H \cong F_i$ for some $i \in \{1, \ldots, 5\}$. It follows from Corollary 27 that, if H has four components, or it can be transformed into a graph with four components through a sequence of partial complementations, then it is one of F_1, \ldots, F_5 .

So, assume that none of the graphs that can be obtained from H by means of partial complements has more than three connected components. Notice that any P_4 -extendible graph H on seven vertices with exactly two components, can be transformed by partial complementation into a graph with at least three components, one of which is an isolated vertex, except in the case that H is the disjoint union of K_1 with an X-spider on 6 vertices, in which case it can be checked that H is a (1, 2)-polar graph. Taking a partial complementation separating one isolated vertex of H from the rest of the graph, we obtain a graph with two components, one of them being an isolated vertex. Let us suppose without loss of generality that Hhas this form.

Since H is not 2-polar, its nontrivial component must contain a P_4 -extendible minimal (2,1)-polar obstruction H' as an induced subgraph. Moreover, either H' has fewer than six vertices, or it has exactly six vertices and is connected, so it follows from Corollary 25 that $H' \in \{K_1 + 2K_2, \overline{3K_2}, \overline{K_2 + C_4}\}$. If

 $H' \cong \overline{3K_2}$, then H is the graph F_5 in Figure 4. If $H' \cong \overline{K_2 + C_4}$, it is straightforward to verify that H is a (1,2)-polar graph. Otherwise, $H' \cong K_1 + 2K_2$. But H' is contained in a connected component of H on six vertices, which must be isomorphic to $K_1 \oplus (K_1 + 2K_2)$ because H is a P_4 -extendible graph. Then, H is isomorphic to F_3 .

The next technical lemma will be needed to give the complete list of P_4 -extendible minimal 2-polar obstructions with at least eight vertices.

Lemma 29. Let H be a disconnected minimal 2-polar obstruction. If H has a component H' which is not a cograph, then H - H' is a split graph. In consequence, at most one component of H is not a cograph.

Proof: If H - H' is not a split graph it contains $2K_2$, C_4 or C_5 as an induced subgraph, and H would contain F_1 , F_2 or F_{29} as a proper induced subgraph, respectively (see Figures 4 and 9). Now, assume for a contradiction that H has at least two components, H_1 and H_2 , which are not cographs. By the first part of this lemma, $H - H_1$ and $H - H_2$ (and hence H_1) are split graphs, so H is the disjoint union of two split graphs, which implies that it is a (1, 2)-polar graph, contradicting that H is a 2-polar obstruction.



Fig. 9: Family C of P_4 -extendible minimal 2-polar obstructions on 8 vertices.

In the proof of our following lemma, we will implicitly use Lemma 8, and the observation right after it, when analyzing which minimal obstructions appear in the different cases. Hence, it is natural that at most one graph from each figure appears in the proof. For example, since F_{13} appears as an induced subgraph in one of the cases, then none of the graphs F_{14}, \ldots, F_{25} (see Fig. 7) will be explicitly mentioned in the proof.

Lemma 30. The only disconnected P_4 -extendible minimal 2-polar obstructions with at least 8 vertices are the graphs $F_6, F_7, \ldots, F_{25}, F_{27}, F_{28}, \ldots, F_{41}$ depicted in Figures 5 to 7 and 9 to 11.

Proof: Let H be a P_4 -extendible disconnected minimal 2-polar obstruction with at least eight vertices. By the observation after Lemma 8, it is enough to prove that $H \cong F_i$ for some $i \in \{6, \ldots, 41\}$, $i \neq 26$. If H can be transformed by means of partial complementations into a graph with four connected components, we have by Corollary 27 that H is one of F_{13}, \ldots, F_{25} .

Now, assume that H can be transformed by partial complementations into a graph H' with three components, but it cannot be transformed into a graph with four connected components. Notice that at least one component of H' is a cograph, otherwise $3P_4$ is an induced subgraph of H', but F_1 is a proper induced subgraph of $3P_4$, contradicting that H is a minimal 2-polar obstruction. Having a cograph component, H'can be transformed by a finite sequence of partial complementations into a graph H'' with three connected components where at least one of them, B_3 , is a trivial component. Moreover, since H'' is also a minimal



Fig. 10: Family D of P_4 -extendible minimal 2-polar obstructions on 8 vertices.



Fig. 11: Family E of P_4 -extendible minimal 2-polar obstructions on 8 vertices.

2-polar obstruction, $H'' - B_3$ is 2-polar but it is neither a (2, 1)- nor a (1, 2)-polar graph. Therefore, a component B_2 of $H'' - B_3$, is a complete graph while its other component, B_1 , is a (2, 1)-polar graph that is neither a split nor a complete bipartite graph. Without loss of generality we can assume that B_1, B_2 and B_3 are the components of H itself. Denote by m the order of B_2 .

Suppose first that $m \ge 2$. Since B_1 is not a split graph, then it contains some of $2K_2, C_5$ or C_4 as an induced subgraph. If $2K_2 \le B_1$, then H properly contains a copy of F_1 , while if $C_5 \le B_1$, then H must be isomorphic to F_{30} . Otherwise, B_1 contains a copy C of C_4 . Observe that if B_1 contains $K_1 + C_4$ as an induced subgraph, then H properly contains a copy of F_{13} , which is impossible. Hence, any vertex in B_1

not in C is adjacent to some vertex of C. Let u be a vertex in B_1 not in C. If u is adjacent to exactly one vertex of C, then $H \cong F_{32}$; if u is adjacent to two adjacent vertices of C, then $H \cong F_{37}$; if u is adjacent to exactly three vertices of C, $H \cong F_7$; and if u is adjacent to all vertices of C, then H properly contains a copy of F_4 . Thus, if H is none of the graphs mentioned before, any vertex u in B_1 not in C is adjacent to two antipodal vertices in C. In addition, two vertices adjacent to the same pair of antipodal vertices cannot be adjacent to each other, otherwise H contains F_7 as a proper induced subgraph. Furthermore, any two vertices adjacent to distinct pairs of antipodal vertices in C must be adjacent to each other, or H would contain F_{32} as a proper induced subgraph. It is easy to observe that under such restrictions B_1 is a complete bipartite graph, which is impossible.

Now let us consider the case m = 1. We have that B_1 is a connected P_4 -extendible graph with at least six vertices, so B_1 is either an X-spider or the join of two smaller P_4 -extendible graphs. Suppose first that B_1 is an X-spider and let R be its head. If R contains $2K_2, C_4$ or C_5 as an induced subgraph, then H properly contains F_3 , F_4 or F_{28} , respectively, but this is impossible. Then, R is a split graph, which implies that $X \notin \{P_4, F, \overline{F}\}$, or H would be a split graph. We can assume that $X = \overline{P}$. If R contains an induced P_3 , then H properly contains an induced copy of F_{26} , so R must be a cluster. Hence, R is a split graph which is a cluster, so $R = K_a + bK_1$ for some nonnegative integers a and b. Observe that $a \ge 2$ and $b \ge 1$, otherwise H is a 2-polar graph or it contains F_9 as a proper induced subgraph. Then, R contains an induced copy of $\overline{P_3}$, but this implies that H has a proper induced copy of F_3 . Hence, B_1 is not an X-spider, so B_1 is the join of two smaller P_4 -extendible graphs, T_1 and T_2 , and hence $H = T_1 \oplus T_2 + B_2 + B_3$. If the complement of T_i is disconnected for some $i \in \{1, 2\}$, then $\overline{B_1} + \overline{B_2 + B_3}$ has four connected components, a contradiction. Then each T_i has a connected complement, so it is isomorphic to K_1 or it contains P_4 as an induced subgraph. Evidently, at least one of T_1 and T_2 is a nontrivial graph. First assume, without loss of generality, that T_1 is an isolated vertex, then $\overline{B_1} + \overline{B_2 + B_3}$ has three connected components, one of them isomorphic to K_2 , and other isomorphic to K_1 , so we are in the case m = 2. Otherwise, each of T_1 and T_2 contain an induced copy of P_4 , so $\overline{B_1} + \overline{B_2 + B_3}$ contains F_1 as a proper induced subgraph, which is impossible.

Finally, assume that H cannot be transformed by partial complementations into a graph with at least three connected components. Thus, H has exactly two connected components, and the complement of any of them is also a connected graph. Then, by Lemma 29, H is the disjoint union of K_1 and an X-spider, but exactly as in the case m = 1, it can be proved that this is impossible for a P_4 -extendible minimal 2-polar obstruction.

We summarize the results of this section in the following theorem.

Theorem 31. There are exactly 82 P_4 -extendible minimal 2-polar obstructions, they are the graphs F_1, \ldots, F_{41} and their complements.

4 Largest polar subgraphs

In this section, we give algorithms to find maximum order induced subgraphs with some given properties (related to polarity) in P_4 -sparse and P_4 -extendible graphs using their tree representations. Ekim, Mahadev and de Werra [6] previously obtained similar results for cographs using the cotree. Given a graph G, we denote by MC(G), MI(G), and MS(G) a maximum subset of V_G inducing a complete graph, an empty graph, and a split graph, respectively. We use MB(G) and McB(G) to denote a maximum subset of V_G inducing a bipartite and a co-bipartite graph, respectively. We also use MUC(G) and MJI(G)

to denote maximum subsets of V_G inducing a cluster and a complete multipartite graph, respectively; $\mathsf{MM}(G)$, $\mathsf{McM}(G)$, and $\mathsf{MP}(G)$ stand for maximum subsets of V_G inducing a monopolar, a co-monopolar and a polar subgraph of G, while $\mathsf{MU}(G)$ and $\mathsf{McU}(G)$ are used for denoting maximum subsets of V_G inducing a unipolar or a co-unipolar graph, respectively. To simplify the notation, when we are working with preset subgraphs G_i of G, we write MC_i instead of $\mathsf{MC}(G_i)$ and, if there is no possibility of confusion, we write MC instead of $\mathsf{MC}(G)$; we use an analogous notation for all other maximal subgraphs. Given a family \mathcal{F} of subsets of V_G , a witness of $M = \max_{F \in \mathcal{F}}\{|F|\}$ in \mathcal{F} is an element F' of \mathcal{F} such that |F'| = M.

The following proposition provides recursive characterizations for the aforementioned maximum subgraphs in a disconnected graph.

Proposition 32. Let $G = G_0 + G_1$ be a graph, and let W be a subset of V_G . The following statements hold true.

- 1. *W* is a maximum clique of *G* if and only if *W* is a witness of $\max\{|\mathsf{MC}_0|, |\mathsf{MC}_1|\}$.
- 2. *W* is a maximum independent set of *G* if and only if *W* is a witness of $\max\{|\mathsf{MI}_0 \cup \mathsf{MI}_1|\}$.
- 3. W induces a maximum bipartite subgraph of G if and only if W is a witness of $\max\{|\mathsf{MB}_0 \cup \mathsf{MB}_1|\}$.
- 4. W induces a maximum co-bipartite subgraph of G if and only if W is a witness of

 $\max\{|\mathsf{McB}_0|, |\mathsf{McB}_1|, |\mathsf{MC}_0 \cup \mathsf{MC}_1|\}.$

- 5. W induces a maximum split subgraph of G if and only if W is a witness of $\max_{i \in \{0,1\}} \{|\mathsf{MI}_i \cup \mathsf{MS}_{1-i}|\}$.
- 6. W induces a maximum cluster in G if and only if W is a witness of $\max\{|\mathsf{MUC}_0 \cup \mathsf{MUC}_1|\}$.
- 7. W induces a maximum complete multipartite subgraph of G if and only if W is a witness of

 $\max\{|\mathsf{MI}|, |\mathsf{MJI}_0|, |\mathsf{MJI}_1|\}.$

- 8. W induces a maximum monopolar subgraph of G if and only if W is a witness of $\max\{|\mathsf{MM}_0 \cup \mathsf{MM}_1|\}$.
- 9. W induces a maximum co-monopolar subgraph of G if and only if W is a witness of

$$\max_{i \in \{0,1\}} \{ |\mathsf{MS}_i \cup \mathsf{MI}_{1-i}|, |\mathsf{McM}_i|, |\mathsf{MC}_i \cup \mathsf{MJI}_{1-i}| \}.$$

10. W induces a maximum polar subgraph of G if and only if W is a witness of

 $\max\{|\mathsf{MM}|, |\mathsf{MP}_0 \cup \mathsf{MUC}_1|, |\mathsf{MP}_1 \cup \mathsf{MUC}_0|\}.$

11. W induces a maximum unipolar subgraph of G if and only if W is a witness of

 $\max_{i \in \{0,1\}} \{ |\mathsf{MU}_i \cup \mathsf{MUC}_{1-i}|, |\mathsf{MU}_{1-i} \cup \mathsf{MUC}_i| \}.$

12. W induces a maximum co-unipolar subgraph of G if and only if W is a witness of

 $\max\{|\mathsf{MB}|, |\mathsf{MI}_0 \cup \mathsf{McU}_1|, |\mathsf{MI}_1 \cup \mathsf{McU}_0|\}.$

Proof: In this paper, we present several propositions whose proofs are based on similar ideas to those used in the current one. Nonetheless, for the sake of completeness, we include the entire proof for all such propositions. In this particular proof, we consider that property 9 is a good example of the general arguments used for proving the entire statement.

- 1. Let W be a maximum clique of G. Evidently, for some $i \in \{0, 1\}$, $W \cap V_{G_i} = \emptyset$ and $W \cap V_{G_{1-i}}$ is a clique of G_{1-i} . It follows that W is a maximum clique for either G_0 or G_1 such that $|W| = \max\{|\mathsf{MC}_0|, |\mathsf{MC}_1|\}$.
- 2. Let W be a maximum independent set of G. Clearly, $W \cap V_{G_i}$ is an independent set of G_i for each $i \in \{0, 1\}$. It follows that W is the union of a maximum independent set of G_0 with a maximum independent set of G_1 .
- 3. Let W be a set inducing a maximum bipartite subgraph of G. For each $i \in \{0, 1\}$, $G[W \cap V_{G_i}]$ is a bipartite graph, and the disjoint union of two bipartite graphs clearly is a bipartite graph, so the result follows.
- 4. Let W be a set inducing a maximum co-bipartite subgraph of G, and let (A, B) be a partition of W into two cliques. Clearly, each of A and B is completely contained in one of V_{G1} or V_{G2}. If both A and B are contained in V_{Gi} for some i ∈ {0, 1}, then W induces a maximum co-bipartite subgraph of G_i. Otherwise, A ⊆ V_{Gi} and B ⊆ V_{G1-i} for some i ∈ {0, 1}, so G[A] is a maximum clique in G_i and G[B] is a maximum clique in G_{1-i}. The result easily follows from here.
- 5. Let W be a set inducing a maximum split subgraph of G, and let (A, B) be a split partition of G[W]. Since B is a clique, B is contained in either V_{G_0} or V_{G_1} . Hence, for some $i \in \{0, 1\}$, $W \cap V_{G_i}$ induces a split graph while $W \cap V_{G_{1-i}}$ is an independent set. It follows that $W = V_i \cup V_{1-i}$, where V_i is a subset of V_{G_i} inducing a maximum split graph, V_{1-i} is a maximum independent subset of $V_{G_{1-i}}$, and $|W| = \max_{i \in \{0,1\}} \{|\mathsf{MI}_i \cup \mathsf{MS}_{1-i}|\}$.
- 6. Let W be a set inducing a maximum cluster of G. Clearly, for each $i \in \{0, 1\}$, $W \cap V_{G_i}$ induces a cluster. It follows that W is the union of a set inducing a maximum cluster of G_0 with a set inducing a maximum cluster of G_1 .
- 7. Let W be a set inducing a maximum complete multipartite subgraph of G. If W is an independent set, it is evidently a maximum independent set of G. Otherwise, G[W] is a connected graph, so W is completely contained in V_{G_i} for some $i \in \{0, 1\}$, and therefore, W induces a maximum complete multipartite subgraph of G_i . In any case we have that $|W| = \max\{|\mathsf{MI}|, |\mathsf{MJI}_0|, |\mathsf{MJI}_1|\}$.
- 8. Let W be a set inducing a maximum monopolar subgraph of G. Evidently, for any $i \in \{0, 1\}$, $W \cap V_{G_i}$ induces a monopolar graph, so we have that W is the union of a set inducing a maximum monopolar subgraph of G_0 with a set inducing a maximum monopolar subgraph of G_1 .

- 9. Let W be a set inducing a maximum co-monopolar subgraph of G, and let (A, B) be a partition of W such that A induces a complete multipartite graph and B is a clique. Since B is a clique, it is completely contained in either V_{G_0} or V_{G_1} . Now, if A is an independent set, then $W = V_i \cup V_{1-i}$ for some $i \in \{0, 1\}$, where V_i induces a maximum split subgraph of G_i and V_{1-i} induces a maximum independent set of G_{1-i} . Otherwise, if A is not an independent set, it induces a connected graph and is contained in either V_{G_0} or V_{G_1} ; hence, either W induces a maximum co-monopolar subgraph of G_i for some $i \in \{0, 1\}$, or there exists $i \in \{0, 1\}$ such that W is the union of a maximum clique in G_i and a set inducing a maximum complete multipartite subgraph of G_{1-i} .
- 10. Let W be a set inducing a maximum polar subgraph of G, and let (A, B) be a polar partition of G[W]. If A is an independent set, then $W \cap V_{G_i}$ induces a monopolar subgraph of G_i for each $i \in \{0, 1\}$, so W induces a maximum monopolar subgraph of G. Otherwise, if A is not an independent set, G[A] is connected and A is completely contained in V_{G_i} for some $i \in \{0, 1\}$; hence, W is the union of a set inducing a maximum polar subgraph of G_i with a set inducing a maximum cluster of G_{1-i} .
- 11. Let W be a set inducing a maximum unipolar subgraph of G, and let (A, B) be a unipolar partition of G[W]. Since A is a clique, it is completely contained in V_{G_i} for some $i \in \{0, 1\}$. Thus, $W \cap V_{G_{1-i}}$ induces a cluster and $W \cap V_{G_i}$ induces a unipolar graph, so W is the union of a set inducing a maximum unipolar subgraph of G_i with a set inducing a maximum cluster in G_{1-i} .
- 12. Let W be a set inducing a maximum co-unipolar subgraph of G, and let (A, B) be a unipolar partition of G[W]. Since G[B] is a complete multipartite graph, if B ∩ V_{G1} ≠ Ø and B ∩ V_{G2} ≠ Ø, B is an independent set, so W induces a bipartite graph. Otherwise, B ∩ V_{Gi} = Ø for some i ∈ {0,1}, and we have that W ∩ V_{Gi} is an independent set and W ∩ V_{G1-i} induces a co-unipolar graph. The result follows easily from here.

Since $G \oplus H = \overline{G + H}$ for any pair of graphs G and H, the following statement is an immediate consequence of the previous proposition, so we omit the proof. Notice that, by Theorem 1, Propositions 32 and 33 can be used together in a mutual recursive algorithm to determine the maximum subgraphs listed in them for any cograph.

Proposition 33. Let $G = G_0 \oplus G_1$ be a graph, and let W be a subset of V_G . The following statements hold true.

- 1. *W* is a maximum clique of *G* if and only if *W* is a witness of $\max\{|\mathsf{MC}_0 \cup \mathsf{MC}_1|\}$.
- 2. *W* is a maximum independent set of *G* if and only if *W* is a witness of $\max\{|\mathsf{MI}_0|, |\mathsf{MI}_1|\}$.
- 3. W induces a maximum bipartite subgraph of G if and only if W is a witness of $\max\{|MB_0|, |MB_1|, |MI_0 \cup MI_1|\}$.
- 4. W induces a maximum co-bipartite subgraph of G if and only if W is a witness of $\max\{|McB_0 \cup McB_1|\}$.
- 5. W induces a maximum split subgraph of G if and only if, W is a witness of $\max_{i \in \{0,1\}} \{|\mathsf{MC}_i \cup \mathsf{MS}_{1-i}|\}$.

- 6. W induces a maximum cluster in G if and only if W is a witness of $\max\{|\mathsf{MC}_0|, |\mathsf{MUC}_0|, |\mathsf{MUC}_1|\}$.
- 7. W induces a maximum complete multipartite graph of G if and only if W is a witness of

 $\max\{|\mathsf{MUI}_0\cup\mathsf{MUI}_1|\}.$

8. W induces a maximum monopolar subgraph of G if and only if W is a witness of

 $\max_{i \in \{0,1\}} \{|\mathsf{MS}_i \cup \mathsf{MC}_{1-i}|, |\mathsf{MM}_i|, |\mathsf{MI}_i \cup \mathsf{MUC}_{1-i}|\}.$

- W induces a maximum co-monopolar subgraph of G if and only if W is a witness of max{|McM₀∪ McM₁|}.
- 10. W induces a maximum polar subgraph of G if and only if W is a witness of

 $\max\{|\mathsf{McM}|, |\mathsf{MP}_0 \cup \mathsf{MJI}_1|, |\mathsf{MP}_1 \cup \mathsf{MJI}_0|\}.$

11. W induces a maximum unipolar subgraph of G if and only if W is a witness of

 $\max\{|\mathsf{McB}|, |\mathsf{MU}_1 \cup \mathsf{MC}_0|, |\mathsf{MU}_0 \cup \mathsf{MC}_1|\}.$

12. W induces a maximum co-unipolar subgraph of G if and only if W is a witness of

 $\max\{|\mathsf{McU}_0 \cup \mathsf{MJI}_1|, |\mathsf{McU}_1 \cup \mathsf{MJI}_0|\}.$

In the next sections, we characterize maximum subgraphs related to polarity properties in both P_4 -sparse and P_4 -extendible graphs, and we use such characterizations to give linear time algorithms to find the largest subgraphs with such properties in a given graph of the mentioned graph families.

4.1 Largest polar subgraph in P_4 -sparse graphs

We start by introducing a tree representation for P_4 -sparse graphs which is the base for our algorithms. Let $G_1 = (V_1, \emptyset)$ and $G_2 = (V_2, E_2)$ be disjoint graphs such that $V_2 = K \cup R \cup \{s_0\}$, where K is a clique completely adjacent to R, $|K| = |V_1| + 1 \ge 2$ and either $N_{G_2}(s_0) = \{k_0\}$ or $N_{G_2}(s_0) = K \setminus \{k_0\}$ for some vertex k_0 in K. Let f be a bijection from V_1 to $K \setminus \{k_0\}$. We define $G_1 \rtimes G_2$ as the graph G with vertex set $V_1 \cup V_2$ such that $G[V_1] \cong G_1$, $G[V_2] \cong G_2$ and, for each $s \in V_1$, either $N_G(s) = \{f(s)\}$, provided $N_{G_2}(s_0) = \{k_0\}$, or $N_G(s) = K \setminus \{f(s)\}$ otherwise.

Proposition 34. [14] If G is a graph, then G is a spider if and only if there exist graphs G_1 and G_2 such that $G = G_1[*]G_2$.

By Theorem 2, for any nontrivial P_4 -sparse graph G, either G is disconnected, or \overline{G} is disconnected, or G is an spider. Hence, for each P_4 -sparse graph G, a labeled tree T with G as its root and some subgraphs of G as each node can be constructed in the following way. Let H be a node of T. If H is a trivial graph, it is an unlabeled node in T with no children. If H is a disconnected graph, it is labeled as a 0-node and its children are its connected components. If \overline{H} is disconnected, H is labeled as a 1-node and its children are the connected components of \overline{H} . Finally, if H is a spider, let say $H = H_1 * H_2$, H is

labeled as a 2-node and its children are H_1 and H_2 . The labeled tree constructed in this way is called the *ps-tree* of G. The ps-tree of a P_4 -sparse graph was introduced by Jamison and Olariu in [15], where they proved that such representation can be computed in linear time. It follows from the results in [15] that the ps-tree of any P_4 -sparse graph or order n has O(n) nodes. Particularly, it implies that we can compute the lists of children for each node of a ps-tree T in linear time and provide each node with such list preserving the linear space representation for T. Additionally, having the lists of children dor each node of a ps-tree, we can compute in O(n) time the number of unlabeled children that each node has. This will be helpful later. In what follow, we assume that if T is the ps-tree of G, and x is a node of T, then c_1x, c_2x, \ldots denote the children of x. We will use G_x to represent the subgraph of G induced by the leaf descendants of x in T.

The following proposition implies that, given a ps-tree, we can decide in linear time whether the graphs associated to its nodes labeled 2 are thin spiders or thick spiders.

Proposition 35. Let $G = G_1[*]G_2$ be a spider, and let T be its ps-tree. Let w be the only child of G with label 1 in T. If w has two or more unlabeled children, then G is a thick spider. Otherwise, G is a thin spider.

Proof: Let v be the only leg of G in G_2 . Observe that a vertex of G_2 is a universal vertex if and only if it is adjacent to v. Additionally, a vertex of G_2 is universal if and only if it is an unlabeled child of w. Hence, if w has two or more unlabeled children, the degree of v in G is at least two, so G is a thick spider. Otherwise, if w has precisely one unlabeled child, $d_G(v) = 1$ so that G is a thin spider.

Some of the algorithms we give in this section require us to be able to recognize the spider partition of any spider from its associated ps-tree. Nevertheless, this is not always possible, for instance, if we consider any thin spider whose head complement is disconnected, there will be vertices for which it is impossible to decide from the associated ps-tree if they belong to the body or the head of the spider (see Figure 12).



Fig. 12: The ps-tree associated to the thin spider with 2 legs whose head is isomorphic to P_3 . The solid vertices are indistinguishable, but one of them belong to the body of the spider, and the other one belongs to its head.

However, it is clear that, given a ps-tree T, there is a unique P_4 -sparse graph (up to isomorphism) associated with T, and it results that if we fix a spider partition for any node labeled 2 in T, the graph is completely determined. Next, we explain how to fix the spider partition for such nodes, and how to save this data maintaining the linear space needed for storing T.

Let $G = G_1(*)G_2$ be a thin spider, and let T be its associated ps-tree. Let V_1, V_2, K, R , and s_0 be like in the definition of $G_1(*)G_2$, and assume that $N_{G_2}(s_0) = \{k_0\}$. Clearly, the root r of T is labeled 2, and it has precisely two children in T, namely a child v labeled 1 such that $G_v \cong G_2$, and a child u, which is unlabeled if $|V_1| = 1$, or it is labeled 0 otherwise; we call v the 1-child of r. In addition, since G is a thin spider, G_2 can be obtained from $G[R \cup (K \setminus \{k_0\})]$ by adding first an isolated vertex s_0 and then a universal vertex k_0 . Thus, v has precisely two children, namely an unlabeled child (k_0) and a 0-labeled child w, which we will call the 0-child of v. Finally, if |K| > 2 or $R \neq \emptyset$, w has exactly two children, one unlabeled (s_0) and one child x labeled 1 ($G[R \cup (K \setminus \{k_0\})]$) called the 1-child of w. Otherwise, if |K| = 2 and $R = \emptyset$ (in which case $G \cong P_4$), w has exactly two unlabeled children, namely s_0 and the only vertex x in the singleton $K \setminus \{k_0\}$ (see Figure 13).



Fig. 13: General structure of the ps-tree of a thin spider.

As we mentioned before, if |K| = 2 and $R = \emptyset$, then w has precisely two children, s_0 and x, both of them unlabeled. Notice that in G, precisely one child of w is adjacent to the 0-child of r, but we are not able to distinguish from the ps-tree which child of w is such vertex, so we must choose arbitrarily some of them to fix a spider partition (which will completely determine a graph G' isomorphic to G, but possibly different from it, whose ps-tree is T and has the fixed spider partition). Now, if R induces either a disconnected graph or a spider, then x has precisely |K| - 1 unlabeled children, all of them elements of K. Nevertheless, if the complement of R is disconnected, then there are potentially more than |K| - 1unlabeled children of x, and they will be indistinguishable, so we must choose arbitrarily |K| - 1 of them to fix a spider partition.

Now, let $G = G_1[*]G_2$ be a thick spider which is not a thin spider, and let T be its associated ps-tree. Let V_1, V_2, K, R , and s_0 be like in the definition of $G_1[*]G_2$, and assume that $N_{G_2}(s_0) = K \setminus \{k_0\}$. As before, the root r of T is labeled 2, and it has a child v labeled 1, and a child u labeled 0. Since G is a thick spider, G_v is the join of $G[K] - k_0$ with the disjoint union of the graph obtained from $G[R \cup \{k_0\}]$ by adding an isolated vertex s_0 . Thus, v has precisely |K| children, |K| - 1 unlabeled children and a 0-labeled child w. Finally, since $|K| \ge 3$ (because G is not a thin spider), w has exactly two children, one unlabeled (s_0) and one child x labeled 1 ($G[R \cup \{k_0\}]$). Similarly to the case of thin spiders, if R induces either a disconnected graph or a spider, then x has precisely one unlabeled child, k_0 . Nevertheless, if the complement of R is disconnected, then there are potentially more than one unlabeled children of x, and they will be indistinguishable, so we must chose arbitrarily one unlabeled child to fix a spider partition.

As we have seen, to fix the spider partition of a node labeled 2 it is enough to select some unlabeled descendants of such node which will completely determine the body of the associated spider, as well as the entire spider partition. Moreover, we can simply mark the selected vertices for the body of any node labeled 2 and, since these marked vertices are considered only for the spider partition of their great great

grandfather (or great grandfather) in the ps-tree, we can save and process the vertices of the bodies of each node labeled 2 in O(n) space and time, in such a way that any time we need a spider partition of such nodes we use the same fixed partition. It is worth noticing that we could simultaneously mark the vertices of the spider bodies while constructing the ps-tree of a P_4 -sparse graph, avoiding the extra processing time and ensuring that we can recover with precision the original graph from the ps-tree.

The following proposition is to thin spiders as Proposition 32 is to disconnected graphs. In it, we characterize maximum subgraphs of thin spiders with some properties related to polarity.

Proposition 36. Let G = (S, K, R) be a thin spider and let $f : S \to K$ be the bijection such that $N(s) = \{f(s)\}$ for each $s \in S$. Let H be the subgraph of G induced by R. The following statements hold for any subset W of V_G .

- 1. W is a maximum clique of G if and only if W is a witness of $\max_{s \in S} \{|\{s, f(s)\}|, |K \cup \mathsf{MC}(H)|\}$.
- 2. W is a maximum independent set in G if and only if W is a witness of $\max_{s \in S} \{|\{f(s)\} \cup (S \setminus \{s\})|, |S \cup \mathsf{MI}(H)|\}$.
- 3. W induces a maximum bipartite subgraph of G if and only if W is a witness of G

$$\max_{k_1,k_2 \in K} \{ |S \cup \{k_1,k_2\}|, |\mathsf{MI}(H) \cup S \cup \{k_1\}|, |\mathsf{MB}(H) \cup S| \}$$

4. W induces a maximum co-bipartite subgraph of G if and only if W is a witness of

$$\max_{s_1,s_2 \in S} \{ |\{s_1, s_2, f(s_1), f(s_2)\}|, |\mathsf{MC}(H) \cup K \cup \{s_1\}|, |\mathsf{McB}(H) \cup K| \}.$$

- 5. W induces a maximum split subgraph of G if and only if W is a witness of $\max\{|S \cup K \cup \mathsf{MS}(H)|\}$.
- 6. W induces a maximum cluster in G if and only if W is a witness of

$$\max_{\substack{k \in K \\ W' \in \mathcal{X}}} \{ |S \cup \{k\}|, |S \cup \mathsf{MUC}(H)|, |\mathsf{MC}(H) \cup W'| \},$$

where \mathcal{X} is the family of all |S|-subsets W' of $S \cup K$ such that $\{s, f(s)\} \not\subseteq W'$ for any $s \in S$.

7. W induces a maximum complete multipartite subgraph of G if and only W is a witness of

$$\max_{s_1,s_2 \in S} \{|\{s_1, f(s_1), f(s_2)\}|, |\{f(s_1)\} \cup (S \setminus \{s_1\})|, |\{s_1, f(s_1)\} \cup \mathsf{MI}(H)|$$

$$S \cup \mathsf{MI}(H)|, |K \cup \mathsf{MJI}(H)|\}.$$

8. W induces a maximum monopolar subgraph of G if and only W is a witness of

$$\max_{k \in K} \{ |S \cup K \cup \mathsf{MS}(H)|, |S \cup \{k\} \cup \mathsf{MUC}(H)|, |S \cup \mathsf{MM}(H)| \}$$

9. W induces a maximum co-monopolar subgraph of G if and only W is a witness of

$$\max_{s \in S} \{ |S \cup K \cup \mathsf{MS}(H)|, |K \cup \{s\} \cup \mathsf{MJI}(H)|, |K \cup \mathsf{McM}(H)| \}$$

- 10. W induces a maximum polar subgraph of G if and only if W is a witness of $\max\{|S \cup K \cup \mathsf{MP}(H)|\}$.
- 11. W induces a maximum unipolar subgraph of G if and only if W is a witness of $\max\{|S \cup K \cup MU(H)|\}$.
- 12. W induces a maximum co-unipolar subgraph of G if and only if W is a witness of $\max\{|S \cup K \cup McU(H)|\}$.

Proof: This proof is similar in flavor to the proof of Proposition 32. Moreover, we consider that property 7 is a good example of the general arguments used for proving the entire statement.

- 1. Let W be a maximum clique of G. If $s \in W \cap S$, then $W \cap S = \{s\}, W \cap K \subseteq \{f(s)\}$, and $W \cap R = \emptyset$, so in this case $W = \{s, f(s)\}$. Otherwise, $W \cap S = \emptyset$, and since the union of any clique of H with K is a clique, we have that W is the union of K with a maximum clique of H.
- 2. Let W be a maximum stable set in G. If $f(s) \in W \cap K$ for some $s \in S$, then $W \cap K = \{f(s)\}$, $s \notin W \cap S$, and $W \cap R = \emptyset$, so in this case $W = \{f(s)\} \cup (S \setminus \{s\})$. Otherwise, $W \cap K = \emptyset$, and since the union of any independent set in H with S is an independent set, we have that W is the union of S with a maximum independent set of H.
- 3. Let W be a set inducing a maximum bipartite subgraph of G. If $W \cap R$ is a nonempty independent set, then $|W \cap K| \leq 1$. Furthermore, since the union of an independent subset of R with $S \cup \{k\}$ induces a bipartite graph for any $k \in K$, in this case we have that W is the union of $S \cup \{k\}$ with a maximum independent set of H. If $W \cap R$ induces a nonempty bipartite graph, then $W \cap K = \emptyset$ and W clearly is the union of S with a maximum subset of R inducing a bipartite graph. Otherwise, $W \cap R = \emptyset$. Since W induces a bipartite graph and K is a clique, we have that $|W \cap K| \leq 2$. Moreover, since the union of S with any 2-subset of K induces a bipartite graph, in this case W is the union of S with a 2-subset of K.
- 4. Let W be a set inducing a maximum co-bipartite subgraph of G. If W ∩ R is a nonempty clique, |W ∩ S| ≤ 1. Furthermore, since the union of a clique in H with K ∪ {s} induces co-bipartite graph for any s ∈ S, in this case we have that W is the union of K ∪ {s} with a maximum clique of H. If W ∩ R induces a co-bipartite graph which is not a clique, then W ∩ S = Ø and W clearly is the union of K with the vertex set of a maximum co-bipartite subgraph of H. Else, W ∩ R = Ø. Since W induces a co-bipartite graph and S is an independent set, we have that |W ∩ S| ≤ 2. If W ∩ S = {s₁, s₂}, then W ∩ K ⊆ {f(s₁), f(s₂)} and it easily follows that W = {s₁, s₂, f(s₁), f(s₂)}. Notice that W ∩ S ≠ Ø, because otherwise W ⊆ K, but K ∪ {s} induces a co-bipartite graph for any s ∈ S, contradicting the election of W. Thus, W ∩ S = {s₁}. In this case R = Ø or, for any r ∈ R, W ∪ {r} would be a subset of V_G inducing a co-bipartite graph and R = Ø, it follows that W is the union of W. Since K ∪ {s₁} induces a co-bipartite graph and K = Ø, it follows that W is the union of W. Since K ∪ {s₁} induces a co-bipartite graph and K = Ø, it follows that W is the union of a maximum clique of R (which is the empty set) with K ∪ {s₁}.
- 5. For any subset W' of R inducing a graph with split partition (A, B), the graph G[S ∪ K ∪ W'] has (A ∪ S, K ∪ B) as a split partition. Thus, if W is a set inducing a maximum split subgraph of G, W ∩ R is a maximum split subgraph of H, W \ R = S ∪ K, and the result follows.

- 6. Let W be a set inducing a maximum cluster of G. First, assume that W ∩ R = Ø. Since S ∪ {k} induces a cluster of G for any k ∈ K, we have that |W| ≥ |S| + 1, so {s, f(s)} ⊆ W for some s ∈ S. Moreover, since clusters are P₃-free graphs, if {s₁, f(s₁)} ⊆ W, W ∩ K = {f(s₁)}. Thus, in this case W = S ∪ {k} for some k ∈ K. Otherwise, if W ∩ R ≠ Ø, W ∩ R induces a cluster and {s, f(s)} ⊈ W for every s ∈ S, so |W \ R| ≤ |S|. It follows that, if W ∩ R is a clique, then W \ R is an |S|-subset of K ∪ S such that {s, f(s)} ⊈ W \ R for any s ∈ S, and W ∩ R is a maximum clique of H. Otherwise, if W ∩ R has at least two connected components, then W ∩ K = Ø, W \ R = S, and W ∩ R induces a maximum cluster in H.
- 7. Let W be a set inducing a maximum complete multipartite subgraph of G. Notice that, for any subset R' of R inducing a complete multipartite graph, $G[K \cup R']$ is a complete multipartite graph. In consequence, if $W \cap S = \emptyset$, then W is the union of K with a maximum subset of R inducing a complete multipartite graph. Also observe that, since complete multipartite graphs are $\overline{P_3}$ -free graphs, either $W \cap S = \emptyset$ or $W \cap R$ is an independent set.

If $|W \cap K| \ge 3$, then $W \cap S = \emptyset$, so we are done. Now, suppose that $W \cap K = \{f(s_1), f(s_2)\}$ for some $s_1, s_2 \in S$. Observe that in this case $W \cap S$ must be contained in either $\{s_1\}$ or $\{s_2\}$. In addition, some of $W \cap S$ or $W \cap R$ must be an empty set. As in the former case, if $W \cap S = \emptyset$, Wis the union of K with a maximum subset of R inducing a complete multipartite graph. Otherwise, if $W \cap R = \emptyset$, thus $W = \{s_1, f(s_1), f(s_2)\}$ for some $s_1, s_2 \in S$.

Now, suppose that $W \cap K = \{f(s_1)\}$ for some $s_1 \in S$. Notice that either $s_1 \notin W$ or $W \cap S \subseteq \{s_1\}$. Also, $W \cap S \neq \emptyset$, otherwise K would be a subset of W, but $|K| \ge 2$ and we are assuming $|W \cap K| = 1$. Thus, if $W \cap S \subseteq \{s_1\}$, then $W \cap S = \{s_1\}$ and $W \cap R$ is a maximum independent subset of R. Else, if $W \cap S \not\subseteq \{s_1\}$, then $s_1 \notin W$ and there is a vertex $s_2 \in W \cap (S \setminus \{s_1\})$. Hence, $W \cap R = \emptyset$ and $W \cap S = S \setminus \{s_1\}$.

Finally, if $W \cap K = \emptyset$, then $W \cap S \neq \emptyset$, and W is the union of S with a maximum independent subset of R.

8. Let W be a set inducing a maximum monopolar subgraph of G, and let $W' = W \cap R$. If W' induces a graph with split partition (A, B), then $G[S \cup K \cup W']$ is a graph with monopolar partition $(A \cup S, B \cup K)$. Thus, if W' induces a split graph, W is the union of $S \cup K$ with a maximum subset of R inducing a split graph.

Otherwise, if W' induces a cluster which is not a split graph, then W' has a subset inducing a $2K_2$; from here, since $K_2 \oplus 2K_2$ is not a monopolar graph, we have that $|W \cap K| \le 1$, and it follows that $W = W' \cup S \cup \{k\}$ for some $k \in K$.

Finally, if W' induces a monopolar graph which is neither a cluster or a split graph, then any monopolar partition (A, B) of G[W'] is such that $A \neq \emptyset$ and B has at least one pair of nonadjacent vertices; it follows that $W \cap K = \emptyset$, so W is the union of S with a maximum monopolar subgraph of H.

9. Let W be a set inducing a maximum co-monopolar subgraph of G, and let $W' = W \cap R$. If W' induces a graph with split partition (A, B), then $G[S \cup K \cup W']$ is a graph with co-monopolar partition $(B \cup K, A \cup S)$. Thus, if W' induces a split graph, W is the union of $S \cup K$ with a maximum subset of R inducing a split graph.

Otherwise, if W' induces a complete multipartite graph which is not a split graph, then W' has a subset inducing a C_4 ; from here, since $2K_1 + C_4$ is not a co-monopolar graph, we have that $|W \cap S| \leq 1$, and it follows that $W = W' \cup K \cup \{s\}$ for some $s \in S$.

Finally, if W' induces a co-monopolar graph which is neither a complete multipartite graph or a split graph, then any monopolar partition (A, B) of $\overline{G[W']}$ is such that $A \neq \emptyset$ and B has at least one pair adjacent vertices; it follows that $W \cap S = \emptyset$, so W is the union of K with a maximum co-monopolar subgraph of H.

- 10. Let W be a set inducing a maximum polar subgraph of G. Notice that the union of $S \cup K$ with any subset of R inducing a graph with polar partition (A, B), is a graph with polar partition $(A \cup S, B \cup A)$ K). Hence, W is the union of $S \cup K$ with a maximum polar subgraph of H.
- 11. For any subset R' of R inducing a graph with unipolar partition (A, B), the graph $G[S \cup K \cup R']$ has unipolar partition $(A \cup K, B \cup S)$. Thus, if W is a set inducing a maximum unipolar subgraph of $G, W = S \cup K \cup R'$, for some subset R' of R inducing a maximum unipolar graph.
- 12. For any subset R' of R inducing a graph with co-unipolar partition (A, B), the graph $G[S \cup K \cup R']$ has co-unipolar partition $(A \cup S, B \cup K)$. Thus, if W is a set inducing a maximum co-unipolar subgraph of $G, W = S \cup K \cup R'$, for some subset R' of R inducing a maximum co-unipolar graph.

In the following propositions we strongly use the fact that a thin spider is the complement of a thick spider and vice versa. Notice that by a simple complementary argument, analogous results can be given for computing $MI(G_x)$, $McB(G_x)$, $MJI(G_x)$, $McM(G_x)$, and $McU(G_x)$.

Proposition 37. Let G be a P_4 -sparse graph, and let T be its ps-tree. For any node x of T the following assertions hold true.

1. $MC(G_x)$ can be found in linear time.	5. $MM(G_x)$ can be found in linear time.
2. $MB(G_x)$ can be found in linear time.	6 $MD(C)$ can be found in linear time
<i>3.</i> $MS(G_x)$ can be found in linear time.	0. Wr (G_x) can be jound in linear time.
4. $MUC(G_x)$ can be found in linear time.	7. $MU(G_x)$ can be found in linear time.

Proof: The proofs of all items are similar, and we suggest to read 4 as a reference of the general arguments used in the demonstration. We include the proof of all items for the sake of completeness.

1. The assertion is trivially satisfied if x is a leaf of T. If x has type 0, we have by part 1 from Proposition 32 that $MC(G_x)$ is a set realizing $\max_i \{MC(G_{c_ix})\}$. If x has type 1, we have by part 1 from Proposition 33 that $MC(G_x) = \bigcup_i MC(G_{c_ix})$. Finally, let us assume that x has type 2, and let (S, K, R) be the spider partition of G_x . If G_x is a thin spider, we have from item 1 of Proposition 36 that $MC(G_x)$ is a witness of $\max_{s \in S} \{|\{s, f(s)\}|, |K \cup MC(G[R])|\}$, where f(s) is the only neighbor of s in K for each $s \in S$. Otherwise, if G_x is a thick spider, we have from item 2 of Proposition 36 that $\mathsf{MC}(G_x)$ is a witness of $\max_{s \in S} \{ |\{s\} \cup (K \setminus \{f(s)\})|, |K \cup \mathsf{MC}(G[R])| \}$, where, for each $s \in S$, f(s) is the only vertex in K which is not a neighbor of s. The result follows since G_x has O(n) descendants.

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- 2. The assertion is trivially satisfied if x is a leaf of T. If x has type 0, we have by part 3 from Proposition 32 that $\mathsf{MB}(G_x) = \bigcup_i \mathsf{MB}(G_{c_ix})$. If x has type 1, we have by part 3 from Proposition 33 that $\mathsf{MB}(G_x)$ is a set realizing $\max_{i,j} \{\mathsf{MB}(G_{c_ix}), \mathsf{MI}(G_{c_ix}) \cup \mathsf{MI}(G_{c_jx})\}$. Finally, let us assume that x has type 2, and let (S, K, R) be the spider partition of G_x . If G_x is a thin spider, we have from item 3 of Proposition 36 that $\mathsf{MB}(G_x)$ is a witness of $\max_{k_1,k_2 \in K} \{|S \cup \{k_1,k_2\}|, |\mathsf{MI}(H) \cup S \cup \{k_1\}|, |\mathsf{MB}(G[R]) \cup S|\}$. Otherwise, if G_x is a thick spider, we have from item 4 of Proposition 36 that $\mathsf{MB}(G_x)$ is a witness of $\max_{s_1,s_2 \in S} \{|\{f(s_1), f(s_2), s_1, s_2\}|, |\mathsf{MI}(G[R]) \cup S \cup \{f(s_1)\}|, |\mathsf{MB}(G[R]) \cup S|\}$, where f is the bijection from S to K such that $N(s) = K \setminus \{f(s)\}$ for each $s \in S$. The result follows since G_x has O(n) descendants.
- 3. The assertion is trivially satisfied if x is a leaf of T. If x has type 0, we have by part 5 from Proposition 32 that MS(G_x) is a set realizing max_i{MS(G_{cix}) ∪ ⋃_{j≠i} MI(G_{cjx})}. If x has type 1, we have by part 5 from Proposition 33 that MS(G_x) is a set realizing max_i{MS(G_{cix}) ∪ ⋃_{j≠i} MC(G_{cjx})}. If x has type 2, we have from item 5 of Proposition 36 that MS(G_x) is the union of a maximum subset of R inducing a split graph with S ∪ K. The result follows since G_x has O(n) descendants.
- 4. The assertion is trivially satisfied if x is a leaf of T. If x has type 0, we have by part 6 of Proposition 32 that $MUC(G_x)$ is a set realizing $\bigcup_i MUC(G_{c_ix})$. If x has type 1, we have by part 6 of Proposition 33 that $MUC(G_x)$ is a set realizing $\max_i \{MC(G_x), MUC(G_{c_ix})\}$. Finally, let us assume that x has type 2, and let (S, K, R) be the spider partition of G_x . If G_x is a thin spider, we have from item 6 of Proposition 36 that $MUC(G_x)$ is a witness of

$$\max_{\substack{k \in K \\ X \in \mathcal{X}}} \{ |S \cup \{k\}|, |S \cup \mathsf{MUC}(G[R])|, |\mathsf{MC}(G[R]) \cup X| \},$$

where \mathcal{X} is the family of all |S|-subsets X of $S \cup K$ such that $\{s, f(s)\} \not\subseteq X$ for any $s \in S$, being f as usual. Notice that, when computing the maximum described above, we do not need to check each member of \mathcal{X} , because all of them have the same number of vertices. In addition, if such maximum is attained by $|\mathsf{MC}(G[R]) \cup X|$ for an $X \in \mathcal{X}$, then any $X \in \mathcal{X}$ can be used to construct a maximum cluster of G_x , particularly, $|\mathsf{MC}(G[R]) \cup K|$ is a maximum cluster of G_x .

Finally, if G_x is a thick spider, we have from item 7 of Proposition 36 that $MUC(G_x)$ is a witness of

$$\max_{s_1, s_2 \in S} \{ |\{f(s_1), s_1, s_2\}|, |\{s_1\} \cup (K \setminus \{f(s_1)\})|, |\{s_1, f(s_1)\} \cup \mathsf{MC}(G[R])|, |\{s_1, f(s_1)\} \cup \mathsf{MC}(G$$

$$|K \cup \mathsf{MC}(G[R])|, |S \cup \mathsf{MUC}(G[R])|\},\$$

where f is the bijection from S to K such that $N(s) = K \setminus \{f(s)\}$ for each $s \in S$. The result follows since G_x has O(n) descendants.

5. The assertion is trivially satisfied if x is a leaf of T. If x has type 0, we have by part 8 of Proposition 32 that $\mathsf{MM}(G_x)$ is a set realizing $\bigcup_i \mathsf{MM}(G_{c_ix})$. If x has type 1, we have by part 8 from Proposition 33 that $\mathsf{MM}(G_x)$ is a set realizing $\max_{i,j} \{\mathsf{MM}(G_{c_ix}), \mathsf{MS}(G_{c_ix}) \cup \bigcup_{j \neq i} \mathsf{MC}(G_{c_jx}), \mathsf{MI}(G_{c_ix}) \cup \bigcup_{i \neq i} \mathsf{MUC}(G_{c_ix})\}$. Finally, let us assume that x has type 2, and let (S, K, R) be the

spider partition of G_x . No matter if G_x is a thin or a thick spider, we have from items 8 and 9 of Proposition 36 that $MM(G_x)$ is a witness of

 $\max_{k \in K} \{ |S \cup K \cup \mathsf{MS}(G[R])|, |S \cup \{k\} \cup \mathsf{MUC}(G[R])|, |S \cup \mathsf{MM}(G[R])| \}.$

The result follows since G_x has O(n) descendants.

- 6. The assertion is trivially satisfied if x is a leaf of T. If x has type 0, we have by part 10 from Proposition 32 that $\mathsf{MP}(G_x)$ is a set realizing $\max_i \{\mathsf{MM}(G_x), \mathsf{MP}(G_{c_ix}) \cup \bigcup_{j \neq i} \mathsf{MUC}(G_{c_jx})\}$. If x has type 1, we have by part 10 from Proposition 33 that $\mathsf{MP}(G_x)$ is a set realizing $\max_i \{\mathsf{MCM}(G_x), \mathsf{MP}(G_{c_ix}) \cup \bigcup_{j \neq i} \mathsf{MJI}(G_{c_jx})\}$. Finally, let us assume that x has type 2, and let (S, K, R) be the spider partition of G_x . No matter if G_x is a thin or a thick spider, we have from item 10 of Proposition 36 that $\mathsf{MP}(G_x)$ is the union of $S \cup K$ with a maximum subset of R inducing a polar graph. The result follows since G_x has O(n) descendants.
- 7. The assertion is trivially satisfied if x is a leaf of T. If x has type 0, we have by part 11 from Proposition 32 that MU(G_x) is a set realizing max_i{MU(G_{cix}) ∪ ⋃_{j≠i} MUC(G_{cjx})}. If x has type 1, we have by part 11 from Proposition 33 that MU(G_x) is a set realizing max_i{McB(G_x), MU(G_{cix}) ∪ ⋃_{j≠i} MC(G_{cjx})}. Finally, let us assume that x has type 2, and let (S, K, R) be the spider partition of G_x. No matter if G_x is a thin or a thick spider, we have from items 11 and 12 of Proposition 36 that MU(G_x) is the union of S ∪ K with a maximum subset of R inducing a unipolar graph. The result follows since G_x has O(n) descendants.

We obtain the main result of this section as a direct consequence of the proposition above.

Theorem 38. For any P_4 -sparse graph G, maximum order subgraphs of G with the properties of being monopolar, unipolar, or polar, can be found in linear time. In consequence, the problems of deciding whether a P_4 -sparse graph is either a monopolar graph, a unipolar graph, or a polar graph are linear-time solvable.

Proof: From Proposition 37, $MM(G_x)$, $MU(G_x)$ and $MP(G_x)$ can be found in linear time for any node x of the ps-tree associated to a P_4 -sparse graph. Particularly, it can be done for the root of the ps-tree, so the result follows.

4.2 Largest polar subgraph in P_4 -extendible graphs

Based on Theorem 4, it is possible to represent each P_4 -extendible graph G by means of a labeled tree T with root G, which can be constructed in the following way. Let H be a node of T. If H is a trivial graph, it is an unlabeled node of T with no children. If H is a disconnected graph, it is labeled 0 and its children are its connected components. If \overline{H} is disconnected, then H is labeled 1 and its children are the components of \overline{H} . If H is an extension graph, it is a node labeled 2 with as many children as the order of H which has additional information encoding the graph induced by its children. Finally, if H is an X-spider with nonempty head whose spider partition is (S, K, R), H is a node labeled 3 and has exactly two children: its left child, $H[S \cup K]$, and its right child, H[R]. We will call the tree constructed in this way the *parse tree* of G. Hochstättler and Schindler [12] showed that the problems of recognizing

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 P_4 -extendible graphs and computing the parse tree of a P_4 -extendible graph can be solved in linear time.⁽ⁱ⁾ From the results by Hochstättler and Schindler in [12], it follows that the parse tree of a P_4 -extendible graph of order n has O(n) nodes. It implies that it takes linear time to compute the lists of children for all nodes of the parse tree. Since such lists can be considered additional information for each node, this preserves the condition that the parse tree uses only linear space

It is really easy to provide characterizations for maximal substructures associated to polarity in extension graphs but, for the sake of brevity, we only notice that, since there is a finite number of extension graphs, it is possible to compute the aforementioned maximal substructures in constant time.

We continue with propositions characterizing maximal substructures associated to polarity in both P-spiders and F-spiders. As the reader can notice, the proofs are very similar in nature to those of Proposition 36. Notice that P_4 -spiders are special cases of thin (and thick) spiders, so a Lemma analogous to Lemmas 39 and 40, for P_4 -spiders, is simply a particular case of Proposition 36.

Lemma 39. Let G = (S, K, R) be a \overline{P} -spider, where $S = \{a, a', d\}$, $K = \{b, c\}$ and $\{a, a', b\}$ induces C_3 . Let W be a subset of V_G , and let H = G[R]. The following statements hold.

- 1. W is a maximum clique of G if and only it is a witness of $\max\{|\{a, a', b\}|, |\mathsf{MC}(H) \cup K|\}$.
- 2. W is a maximum independent set of G if and only if it is a witness of

 $\max\{|\{a,c\}|, |\{a,d\}|, |\{a',c\}|, |\{a',d\}|, |\{b,d\}|, |\mathsf{MI}(H) \cup \{a,d\}|, |\mathsf{MI}(H) \cup \{a',d\}|\}.$

3. W induces a maximum bipartite graph if and only if W is a witness of

 $\max\{|(S \cup K) \setminus \{a\}|, |(S \cup K) \setminus \{a'\}|, |\mathsf{MI}(H) \cup S \cup (K \setminus \{b\})|, |\mathsf{MB}(H) \cup S|\}.$

4. W induces a maximum co-bipartite graph if and only if W is a witness of

 $\max\{|S \cup K|, |\mathsf{MC}(H) \cup S \cup (K \setminus \{d\})|, |\mathsf{McB}(H) \cup \{c, b\}|\}.$

5. W induces a maximum split graph in G if and only it is a witness of

$$\begin{split} \max\{|\{a,a',b,d\}|,|\{a,a',b,c\}|,|\{a',b,c,d\}|,|\{a,b,c,d\}|,|\mathsf{MI}(H)\cup\{a,a',b,d\}|,\\ |\mathsf{MS}(H)\cup\{a',b,c,d\}|,|\mathsf{MS}(H)\cup\{a,b,c,d\}|\}. \end{split}$$

6. W induces a maximum cluster in G if and only if it is a witness of

 $\max\{|\{a, a', b, d\}|, |\{a, a', c, d\}|, |\mathsf{MC}(H) \cup \{a, a', c\}|, |\mathsf{MC}(H) \cup S|, |\mathsf{MUC}(H) \cup S|\}.$

7. W induces a maximum complete multipartite graph in G if and only if it is a witness of

 $\begin{aligned} \max\{|\{a,a',b\}|,|\{a,b,c\}|,|\{a',b,c\}|,|\{b,c,d\}|,|\mathsf{MI}(H)\cup\{a,b\}|,|\mathsf{MI}(H)\cup\{a',b\}|,\\ |\mathsf{MI}(H)\cup\{c,d\}|,|\mathsf{MI}(H)\cup\{a,d\}|,|\mathsf{MI}(H)\cup\{a',d\}|,|\mathsf{MJI}(H)\cup K|\}. \end{aligned}$

⁽i) Actually, the parse tree defined in [12] is slightly different than the one we introduce, due to the fact that they assume by convention that the father of a node labeled 2 is always a node labeled 3, that the root is always a node labeled 1, and that nodes labeled 1 and 3 may have only one child. Nevertheless, with some minor changes, the algorithm in [12] can be adapted to construct our version of the parse tree.

8. W induces a maximum monopolar graph in G if and only if it is a witness of

 $\max\{|\mathsf{MC}(H) \cup S \cup K|, |\mathsf{MUC}(H) \cup \{a, a', c, d\}|, |\mathsf{MUC}(H) \cup \{a, a', b, d\}|, \\ |\mathsf{MS}(H) \cup \{a, b, c, d\}|, |\mathsf{MS}(H) \cup \{a', b, c, d\}|, |\mathsf{MS}(H) \cup \{a, a', c, d\}|, |\mathsf{MM}(H) \cup S|\}.$

9. W induces a maximum co-monopolar graph in G if and only if it is a witness of

$$\begin{split} \max\{|\mathsf{MI}(H)\cup S\cup K|, |\mathsf{MS}(H)\cup \{a,b,c,d\}|, |\mathsf{MS}(H)\cup \{a',b,c,d\}|, \\ |\mathsf{MJI}(H)\cup \{a,a',b,c\}|, |\mathsf{McM}(H)\cup K|\}. \end{split}$$

- 10. W induces a maximum polar graph in G if and only if W is the union of a maximum subset of R inducing a polar graph with $S \cup K$.
- 11. W induces a maximum unipolar graph in G if and only if W is the union of a maximum subset of R inducing a unipolar graph with $S \cup K$.
- 12. W induces a maximum co-unipolar graph in G if and only if and only if W is a witness of

 $\max\{|\mathsf{MI}(H) \cup S \cup K|, |\mathsf{MB}(H) \cup S \cup \{b\}|, |\mathsf{McU}(H) \cup K \cup \{a, d\}|, |\mathsf{McU}(H) \cup K \cup \{a', d\}|\}.$

Proof: Since the nature of the proofs is very similar, we consider that reading 8 is enough to get an idea of the general arguments used throughout the entire proof. Nonetheless, for the sake of completeness, we include the entire argument.

- 1. Let W be a maximum clique of G. If $W \cap R = \emptyset$, then $W = \{a, a', b\}$. Otherwise, if $W \cap R \neq \emptyset$, then $W \cap S = \emptyset$ and W is the union of K with a maximum clique in H.
- 2. Let W be a maximum independent set of G. If $W \cap R = \emptyset$, then W is a maximum independent subset of $S \cup K$, i.e., $W \in \{\{a, c\}, \{a, d\}, \{a', c\}, \{a', d\}, \{b, d\}\}$. Otherwise, if $W \cap R \neq \emptyset$, then $W \cap K = \emptyset$, and W is the union of a maximum independent set in H with a maximum independent subset of S.
- 3. Let W be a set inducing a maximum bipartite subgraph of G. Notice that, since {a, a', b} induces a triangle, |W ∩ {a, a', b}| ≤ 2. It follows from the previous observation that, if W ∩ R = Ø, W is some of (S∪K) \{a}, (S∪K) \{a'}, or (S∪K) \{b}. Else, if W ∩ R is a nonempty independent set, |W ∩ K| ≤ 1. Moreover, it is a simple observation that the union of any independent subset of R with S ∪ (K \ {b}) induces a bipartite graph, but the union of an independent subset of R with any other 4-subset of S ∪ K does not induce a bipartite graph. Thus, when W ∩ R is a nonempty independent set, W is the union of a maximum independent subset of R with S ∪ (K \ {b}). Finally, if W ∩ R induces a nonempty bipartite graph, then W ∩ K = Ø and we trivially have that W \ R = S.
- 4. Let W be a set inducing a maximum co-bipartite subgraph of G. Since P admits a partition in two cliques, if W ∩ R = Ø, W = S ∪ K. Else, if W ∩ R induces a nonempty clique, neither {a, d} or {a', d} is a subset of W. Moreover, the union of a clique contained in R with (S ∪ K) \ {d}

induces a co-bipartite graph, and the union of a nonempty subset of R with any other 4-subset of $S \cup K$ does not induce a co-bipartite graph. Thus, if $W \cap R$ induces a nonempty clique, W is the union of a maximum clique in H with $(S \cup K) \setminus \{d\}$. Otherwise, $W \cap R$ is a co-bipartite graph which is not a clique, and then $W \cap S = \emptyset$, so clearly $W \setminus R = K$; the result follows.

- 5. Let W be a set inducing a split subgraph of G. If $W \cap R = \emptyset$, W is a maximum subset of $S \cup K$ inducing a split graph, so W is one of $\{a, a', b, d\}$, $\{a, a', b, c\}$, $\{a', b, c, d\}$, or $\{a, b, c, d\}$. Now, assume that $W \cap R \neq \emptyset$. Notice that in this case $\{a, a', c\} \not\subseteq W$, otherwise $\{a, a', c, r\}$ would induce $2K_2$ for any $r \in W \cap R$. Thus, if $W \cap R$ is an independent set, $W \setminus R$ is any of $\{a, b, c, d\}$, $\{a', b, c, d\}$, or $\{a, a', b, d\}$. Else, if $W \cap R$ induces a split graph which is not empty, $\{a, a'\}$ could not be a subset of W, because $\{a, a', r, r'\}$ would induce $2K_2$ for any adjacent vertices $r, r' \in W \cap R$. Thus, if $W \cap R$ is not an independent set, $W \setminus R$ must be one of $\{a, b, c, d\}$, or $\{a', b, c, d\}$, and the result follows.
- 6. Let W be a set inducing a maximum cluster of G. If W ∩ R = Ø, W is a maximum subset of S ∪ K inducing a cluster, i.e., W ∈ {{a, a', b, d}, {a, a', c, d}}. Now, assume that W ∩ R ≠ Ø. If W ∩ R is a clique, then W cannot have simultaneously c and d, or b and any of a or a'. Thus, in this case W is the union of a maximum subset of R inducing a clique with one of {a, a', c} or {a, a', d}. Otherwise, if W ∩ R induces a cluster which is not a complete graph, then W ∩ K = Ø, and W is the union of S with a maximum subset of R inducing a cluster.
- 7. Let W be a set inducing a maximum complete multipartite subgraph of G. If W ∩ R = Ø, W is a maximum subset of S ∪ K inducing a complete multipartite graph, i.e., W is one of {a, a', b}, {a, b, c}, {a', b, c}, or {b, c, d}. Now, assume that W ∩ R ≠ Ø. Notice that in this case, W ∩ S is completely adjacent to W ∩ K. In addition, W cannot have both, a and a'. It follows that, if W ∩ R is an independent set, then W \ R is one of {a, b}, {a', b}, {c, d}, {a, d}, {a', d}, or K. Otherwise, if W ∩ R induces a maximum complete multipartite graph of R which is not empty, W ∩ S = Ø and W \ R = K, so the result follows.
- Let W be a set inducing a maximum monopolar subgraph of G. If W∩R is a clique, then {a, a', c}∪ (W ∩ R) induces a cluster and, since {b, d} is an independent set, we have that W \ R = S ∪ K, so W is the union of a maximum clique of H with S ∪ K.

When $W \cap R$ induces a noncomplete graph which is simultaneously a split graph and a cluster, since $W \cap R$ is not a clique, $\{a, a', b, c\}$ could not be a subset of W or, for any nonadjacent vertices $r, r' \in W \cap R$, $\{a, a', b, c, r, r'\}$ would induce $K_1 \oplus (K_2 + P_3)$, which is not a monopolar graph. Moreover, some simple verifications show that $W \setminus R$ is any of $\{a, b, c, d\}$, $\{a', b, c, d\}$, $\{a, a', c, d\}$, or $\{a, a', b, d\}$.

Else, if $W \cap R$ induces a cluster which is not a split graph, then it has a subset U inducing $2K_2$, so $\{b, c\} \not\subseteq W$, or $G[\{b, c\} \cup U] \cong K_2 \oplus 2K_2$, which is not a monopolar graph. From here, it is easy to verify that $W \setminus R$ is any of $\{a, a', b, d\}$, or $\{a, a', c, d\}$.

Now, assume that $W \cap R$ induces a split graph which is not a cluster. Since $K_1 \oplus (K_2 + P_3)$ is not a monopolar graph and $W \cap R$ has a subset W' inducing P_3 , we have that $\{a, a', b\}$ is not a subset of W. From here, we can easily check that $W \setminus R$ is any of $\{a, b, c, d\}$, $\{a', b, c, d\}$, or $\{a, a', c, d\}$.

Finally, suppose that $W \cap R$ induces a monopolar graph which is neither a cluster or a split graph. Suppose that there exists $k \in K \cap W$, and let (A, B) be a monopolar partition of G[W]. If $k \in A$, then $W \cap R \subseteq B$, implying that $W \cap R$ induce a cluster, which is not the case. Then, it must be that $k \in B$, but then $W \cap R \cap B$ would be a clique and, since $(W \cap R) \setminus B \subseteq A$, we have that $W \cap R$ would induce a split graph, but we are assuming it does not. Therefore, $K \cap W = \emptyset$, and it follows that W is the union of S with a maximum subset of R inducing a monopolar graph.

9. Let W be a set inducing a maximum co-monopolar subgraph of G. If W ∩ R is an independent set, then ({a, a', b}, {c, d} ∪ (W ∩ R)) is a co-monopolar partition of G[(W ∩ R) ∪ S ∪ K]. Hence, if W ∩ R is an independent set, W \ R = S ∪ K.

Notice that, if $W \cap R$ is not an independent set, then $S \not\subseteq W$, otherwise W would have a subset inducing $K_1 + 2K_2$, which is not a co-monopolar graph. In addition, if $W \cap R$ induces a graph with split partition (A, B) and $W \setminus R$ is any of $\{a, b, c, d\}$, or $\{a', b, c, d\}$, then W induces a graph with co-monopolar partition $(A \cup \{a, d\}, B \cup \{b, c\})$ or $(A \cup \{a', d\}, B \cup \{b, c\})$. Also, if $W \cap R$ induces a complete multipartite graph and $W \setminus R = \{a, a', b, c\}$, then G[W] has the co-monopolar partition $(\{a, a'\}, (W \cap R) \cup \{b, c\})$.

If $W \cap R$ induces a split graph which is not a complete multipartite graph, then $\{a, a'\} \not\subseteq W$ or, for any subset $\{r_1, r_2, r_3\}$ of W inducing $\overline{P_3}$, $\{a, a', r_1, r_2, r_3\}$ would induce $K_1 + 2K_2$, which is not a co-monopolar graph. In addition, since $W \cap R$ is a split graph, $\{a, b, c, d\} \cup (W \cap R)$ and $\{a', b, c, d\} \cup (W \cap R)$ induce split graphs, and hence co-monopolar graphs, so in this case W is the union of a maximum subset of R inducing a split graph with one of $\{a, b, c, d\}$ or $\{a', b, c, d\}$.

Else, if $W \cap R$ induces a complete multipartite graph which is not a split graph, then W has a subset W' inducing C_4 . Therefore, neither $\{a, d\} \subseteq W$ or $\{a', d\} \subseteq W$, otherwise W would have a subset inducing $C_4 + 2K_1$, which is not a co-monopolar graph. Moreover, the union of any subset of R inducing a complete multipartite graph with $\{a, a', b, c\}$ induces a co-monopolar graph, so in this case W is precisely the union of a maximal subset of R inducing a complete multipartite graph with $\{a, a', b, c\}$.

Finally, assume that $W \cap R$ induces a co-monopolar graph which is neither a split graph or a complete multipartite graph. Suppose for a contradiction that there exist a vertex $s \in S \cap W$, and let (A, B) be a co-monopolar partition of G[W]. If $s \in A$, then $W \cap R \subseteq B$, which is impossible since $G[W \cap R]$ is not a complete multipartite graph. Then, $s \in B$, but in such a case $B \cap W \cap R$ is an independent set, and $(W \cap R) \setminus B \subseteq A$, implying that $W \cap R$ induces a split graph, contradicting our initial assumption. Hence $S \cap W = \emptyset$. Additionally, for any subset W' of R inducing a co-monopolar graph, $W' \cup K$ is also a co-monopolar graph, so in this case W is the union of K with a maximum subset of R inducing a co-monopolar graph.

- 10. Let W be a set inducing a maximum polar subgraph of G. If (A, B) is a polar partition of $G[W \cap R]$, then $(A \cup K, B \cup S)$ is a polar partition of G[W].
- 11. Let W be a set inducing a maximum unipolar subgraph of G. If (A, B) is a unipolar partition of $G[W \cap R]$, then $(A \cup K, B \cup S)$ is a polar partition of G[W].
- 12. Let W be a set inducing a maximum co-unipolar subgraph of G. Notice that, for any independent subset R' of R, $(\{a, d\} \cup R', \{a'\} \cup K)$ is a co-unipolar partition of $G[S \cup K \cup R']$. Therefore,

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if $W \cap R$ is an independent subset of R, we have that W is the union of a maximum independent subset of R with $S \cup K$.

Observe that $K_2 + K_3$ is not a co-unipolar graph. Hence, if $W \cap R$ is not an independent set, either $\{a, a'\} \not\subseteq W$ or $c \notin W$. It easily follows from the previous observation that, if $W \cap R$ induces a nonempty graph with bipartition (A, B), then W is the union of a maximum subset of R inducing a bipartite graph with some of $K \cup \{a, d\}, K \cup \{a', d\}$, or $S \cup \{b\}$.

Now, assume that $W \cap R$ induces co-unipolar graph which is a nonempty bipartite graph. We claim that, in such case, $\{a, a'\} \not\subseteq W$, and we prove it by means of contradiction. Suppose that $a, a' \in W$, and let (A, B) be a co-unipolar partition of G[W]. Since $G[W \cap R]$ is not an empty graph, $W \cap B \neq \emptyset$, and thus, either $a \in A$ and $a' \in B$, or vice versa. However, due to $B \cap \{a, a'\} \neq \emptyset$ we have that $W \cap R \cap B$ is an independent set, but then $W \cap R$ induces a bipartite graph, reaching a contradiction. From here, it is easy to conclude that, in this case, W is the union of a maximum subset of R inducing a co-unipolar graph with some of $K \cup \{a, d\}$ or $K \cup \{a', d\}$.

Lemma 40. Let G = (S, K, R) be an *F*-spider, where $S = \{a, a', d\}$, $K = \{b, c\}$ and $\{a, a', b\}$ induces P_3 . Let *W* be a subset of V_G , and let H = G[R]. The following statements hold true.

1. W is a maximum clique of G if and only if W is a witness of

$$\max\{|\{a,b\}|, |\{a',b\}|, |\{b,c\}|, |\{c,d\}|, |\mathsf{MC}(H) \cup K|\}.$$

2. W is a maximum independent set of G if and only if W is a witness of

 $\max\{|\{a, a', c\}|, |\mathsf{MI}(H) \cup S|\}.$

3. W is a set inducing a maximum bipartite subgraph of G if and only if W is a witness of

 $\max\{|S \cup K|, |\mathsf{MI}(H) \cup S \cup (K \setminus \{b\})|, |\mathsf{MI}(H) \cup S \cup (K \setminus \{c\})|, |\mathsf{MB}(H) \cup S|\}.$

4. W is a set inducing a maximum co-bipartite subgraph of G if and only if W is a witness of

$$\begin{split} \max\{|(S \cup K) \setminus \{a\}|, |(S \cup K) \setminus \{a'\}|, |\mathsf{MC}(H) \cup K \cup \{a\}|, |\mathsf{MC}(H) \cup K \cup \{a'\}|, \\ |\mathsf{MC}(H) \cup K \cup \{d\}|, |\mathsf{McB}(H) \cup K|\}. \end{split}$$

- 5. W induces a maximum split graph in G if and only if W is the union of a maximum subset of R inducing a split graph with $S \cup K$.
- 6. W induces a maximum cluster of G if and only if W is a witness of

$$\max\{|\{a, a', c, d\}|, |\mathsf{MC}(H) \cup \{a, a', c\}|, |\mathsf{MUC}(H) \cup S|\}.$$

7. W induces a maximum complete multipartite graph in G if and only if W is a witness of

 $\max\{|\{a, a', b, c\}|, |\mathsf{MI}(H) \cup S|, |\mathsf{MI}(H) \cup \{a, a', b\}|, |\mathsf{MIJ}(H) \cup K|\}.$

8. W induces a maximum monopolar graph in G if and only if W is a witness of

 $\max\{|\mathsf{MS}(H) \cup S \cup K|, |\mathsf{MUC}(H) \cup \{a, a', b, d\}|, |\mathsf{MUC}(H) \cup \{a, a', c, d\}|, |\mathsf{MM}(H) \cup S|\}.$

9. W induces a maximum co-monopolar graph in G if and only if W is a witness of

 $\max\{|\mathsf{MS}(H) \cup S \cup K|, |\mathsf{MJI}(H) \cup \{a, b, c\}|, |\mathsf{MJI}(H) \cup \{a', b,$

 $|\mathsf{MJI}(H) \cup \{b, c, d\}|, |\mathsf{McM}(H) \cup K|\}.$

- 10. W induces a maximum polar graph in G if and only if W is the union of a maximum subset of R inducing a polar graph with $S \cup K$.
- 11. W induces a maximum unipolar graph in G if and only if W is the union of a maximum subset of R inducing a unipolar graph with $S \cup K$.
- 12. W induces a maximum co-unipolar graph in G if and only if W is the union of a maximum subset of R inducing a co-unipolar graph with $S \cup K$.

Proof: Again, due to the similarities in the proofs, we consider that reading 4 is enough to have a general idea of the arguments used throughout the whole proof.

- Let W be a maximum clique of G. If R = Ø, W clearly is one of {a, b}, {a', b}, {b, c}, or {c, d}. Otherwise, if R ≠ Ø, R' ∪ K is a clique, for any clique R' contained in R, so in this case W ∩ R is a nonempty clique. It follows that W ∩ S = Ø and W is the union of K a maximum clique contained in R.
- Let W be a maximum independent set of G. If R = Ø, W evidently is one of {a, a', c} or S. Otherwise, if R ≠ Ø, R' ∪ S is an independent set, for any independent subset R' of R. Thus, if R ≠ Ø, W ∩ R is a nonempty independent subset of R, so W ∩ K = Ø. Hence, in this case W is the union of S with a maximum independent subset of R.
- 3. Let W be a set inducing a maximum bipartite subgraph of G. If $W \cap R = \emptyset$, then clearly $W = S \cup K$. Else, if $W \cap R$ is a nonempty independent set, then $|W \cap K| \le 1$. In addition, for any independent subset R' of R, both $R' \cup S \cup \{b\}$ and $R' \cup S \cup \{c\}$ induce bipartite graphs, so in this case W is the union of a maximum independent set of R with either $S \cup \{b\}$ or $S \cup \{c\}$. Otherwise, $W \cap R$ induces a nonempty bipartite graph and $W \cap K = \emptyset$, where it easily follows that W is the union of S with a maximum bipartite subgraph of H.
- 4. Let W be a set inducing a maximum co-bipartite subgraph of G. It is an easy observation that the only subsets of S∪K inducing a maximum co-bipartite graph are (S∪K) \{a} and (S∪K) \{a'}; hence, if W ∩ R = Ø, W must be one of these sets. Notice that if W ∩ R ≠ Ø then |W ∩ S| ≤ 1. From here, it is easy to observe that if W ∩ R is a nonempty clique, then W \ R is one of K ∪ {a}, K ∪ {a'}, or K ∪ {a}, so in this case W is the union of one of these sets with a maximum clique of H. Finally, if W ∩ R induces a co-bipartite graph which is not a clique, then W ∩ S = Ø and W clearly is the union of K with a maximum set inducing a co-bipartite subgraph of H.

- 5. Let W be a set inducing a maximum split subgraph of G. Just notice that, for any subset R' of R inducing a graph with split partition (A, B), $(A \cup S, B \cup K)$ is a split partition of $G[S \cup K \cup R']$.
- 6. Let W be a set inducing a maximum cluster of G. If R = Ø, then W = {a, a', c, d}. Otherwise, the union of S with any subset of R inducing a cluster is also a cluster. Thus, we may assume that |W \ R| ≥ 3. Moreover, if W ≠ {a, a', c, d}, then W ∩ R ≠ Ø and then, none of {a, b}, {a', b}, or {c, d}, is a subset of W, or W would have a subset inducing P₃. From here, it is a easy to conclude that, if W ∩ R is a clique, then W \ R ∈ {S, {a, a', c}}, while, if W ∩ R induces a cluster which is not complete graph, then W \ R = S.
- 7. Let W be a set inducing a maximum complete multipartite subgraph of G. If W ∩ R = Ø, then W is a maximum subset of S ∪ K inducing a complete multipartite graph, so W = {a, a', b, c}. Otherwise, W ∩ R ≠ Ø, and since G[W] is P₃-free, none of {c, a}, {c, a'}, or {b, d}, could be a subset of W. It follows that, in this case, |W \ R| ≤ 3. Notice that the union S with any independent subset of R is an independent set, so it induces a complete multipartite graph. Hence, if W ∩ R is an independent set, |W \ R| = 3 and a simple verification yields that W \ R can be any of S or {a, a', b}. Finally, if W ∩ R induces a complete multipartite graph which is not an empty graph, then W ∩ S = Ø, and W \ R = K.
- 8. Let W be a set inducing a maximum monopolar subgraph of G. If W ∩ R induces a graph with split partition (A, B), then (A ∪ S, B ∪ K) is a split partition of G[S ∪ K ∪ (W ∩ R)]. Else, if W ∩ R induces a cluster which is not a split graph, then W ∩ R has a subset inducing 2K₂, so K ⊈ W, because K₂ ⊕ 2K₂ is not a monopolar graph. In addition, it is easy to corroborate that for any subset R' of R inducing a cluster, R ∪ {a, a', b, d} and R' ∪ {a, a', c, d} induce monopolar graphs. Finally, assume that W ∩ R induces a monopolar graph which is neither a split graph or a cluster. Suppose for a contradiction that there exists a vertex k ∈ K ∩ W, and let (A, B) be a monopolar partition of G[W]. If k ∈ A, then W ∩ R ⊆ B, so W ∩ R induces a cluster, but we are assuming this is not the case. Thus, k ∈ B, but then, B ∩ W ∩ R is a clique, and (W ∩ R) \ B ⊆ A, so W ∩ R induces a split graph, which is impossible. Therefore, K ∩ W = Ø. Moreover, if R' is a subset of R inducing a graph with monopolar partition (A, B), then (A ∪ S, B) is a monopolar partition of G[R' ∪ S], where the result follows.
- 9. Let W be a set inducing a maximum co-monopolar subgraph of G. If a subset R' of R induces a graph with split partition (A, B), then (B∪K, A∪S) is a co-monopolar partition of G[R'∪S∪K]. Thus, if W ∩ R induces a split graph, then W \ R = S ∪ K.

Now, if $W \cap R$ induces a complete multipartite graph which is not a split graph, there exists a subset W' of $W \cap R$ inducing a 4-cycle. Hence, since $C_4 + 2K_1$ is not a co-monopolar graph, $|W \cap S| \leq 1$. Moreover, for any subset R' of R inducing a complete multipartite graph and any $s \in S$, $(\{s\}, R' \cup K)$ is a co-monopolar partition of $G[R' \cup K \cup \{s\}]$. Thus, if $W \cap R$ induces a complete multipartite graph which is not a split graph, then $W \setminus R$ is one of $\{a, b, c\}, \{a', b, c\}, \{b, c, d\}$.

Finally, assume that $W \cap R$ induces a co-monopolar graph which is neither a complete multipartite graph or a split graph. Suppose for a contradiction that there exists a vertex $s \in S \cap W$, and let (A, B) be a co-monopolar partition of G[W]. If $s \in A$, then $W \cap R \cap A = \emptyset$, so $W \cap R$ must induce a complete multipartite graph, which is not the case. Thus, $s \in B$, so $B \cap W \cap R$ is an

independent set, because complete multipartite graphs are $\overline{P_3}$ -free graphs. But then, $W \cap R$ induces a split graph, which is impossible. Therefore $W \cap S = \emptyset$. In addition, if R' is any subset of Rinducing a graph with co-monopolar partition (A, B), then $(A \cup K, B)$ is a co-monopolar partition of $G[R' \cup K]$. Hence, if $W \cap R$ induces a co-monopolar graph which is neither a split graph or a complete multipartite graph, then $W \setminus R = K$.

- 10. Let W be a set inducing a maximum polar subgraph of G. The result follows since, for any subset R' of R inducing a graph with polar partition (A, B), $(A \cup K, B \cup S)$ is a polar partition of $G[S \cup K \cup R']$.
- 11. Let W be a set inducing a maximum unipolar subgraph of G. It is enough to notice that, for any subset R' of R inducing a graph with unipolar partition (A, B), $(A \cup K, B \cup S)$ is a unipolar partition of $G[S \cup K \cup R']$.
- 12. Let W be a set inducing a maximum co-unipolar subgraph of G. The result follows since, for any subset R' of R inducing a graph with co-unipolar partition (A, B), we have that $(A \cup S, B \cup K)$ is a co-unipolar partition of $G[S \cup K \cup R']$.

For the proof of the next proposition we strongly use, without explicit mention, that the complements of *P*-spiders and the complements of *F*-spiders are, respectively, \overline{P} -spiders and \overline{F} -spiders. Notice that by a simple complementary argument, analogous results can be given for computing $MI(G_x)$, $McB(G_x)$, $MJI(G_x)$, $McM(G_x)$, and $McU(G_x)$.

Proposition 41. Let G be a P_4 -extendible graph, and let T be its associated parse tree. For any node x of T the followings assertions are satisfied.

1.	$MC(G_x)$ can be computed in linear time.	5.	$MM(G_x)$ can be computed in linear time.
2.	$MB(G_x)$ can be computed in linear time.	6	MD(C) age to computed in linear time
3.	$MS(G_x)$ can be computed in linear time.	0.	$\operatorname{MF}(G_x)$ can be computed in linear time.
4.	$MUC(G_x)$ can be computed in linear time.	7.	$MU(G_x)$ can be computed in linear time.

Proof: The assertions trivially hold whenever x is a leaf of T. Also, if x is a node labeled 0 or 1, the proof follows exactly as in Proposition 37. Thus, we will assume for the rest of the proof that x has label either 2 or 3. Even in these cases the proof is similar in flavor to Proposition 37, but we use Lemmas 39 and 40 besides Proposition 36. Hence, we only write the proof for item 6.

If x is a node labeled 2, it is not hard to verify that $MP(G_x) = G_x$. Otherwise, x is a node labeled 3, so G_x is an X-spider. By Proposition 36 and Lemmas 39 and 40, if G_x is a graph with X-spider partition (S, K, R), then $MP(G_x)$ is the union of $S \cup K$ with a maximum subset of R inducing a polar graph. The result follows since G_x has O(n) descendants.

The main results of this section are summarized in the next theorem, which is a direct consequence of the proposition above.

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Theorem 42. For any P_4 -extendible graph G, maximum order subgraphs of G with the properties of being monopolar, unipolar, or polar, can be found in linear time. In consequence, the problems of deciding whether a P_4 -extendible graph is either a monopolar graph, a unipolar graph, or a polar graph are linear-time solvable.

Proof: From Proposition 41, $MM(G_x)$, $MU(G_x)$ and $MP(G_x)$ can be found in linear time for any node x of the parse tree associated to a P_4 -extendible graph. Particularly, it can be done for the root of the parse tree, so the result follows.

5 Conclusions

This work must be considered a sequel and a complement of [3], where, among other things, some properties related to polarity on P_4 -sparse and P_4 -extendible graphs were characterized by finite families of forbidden induced subgraphs. Specifically, the families of minimal (s, 1)-polar obstructions for any nonnegative integer s, as well as the families of minimal monopolar, unipolar, and polar obstructions, when restricted to the mentioned graph classes, were exhibited in the aforementioned paper. It is worth noticing that, from such characterizations, it directly follows that there exist brute force algorithms of polynomialtime complexity for deciding whether a P_4 -sparse or a P_4 -extendible graph is monopolar, unipolar, or polar.

The results in this work are divided in two parts. First, we adapt the techniques used in [11] to generalize the characterization of cograph minimal 2-polar obstructions given in that paper, by explicitly exhibiting complete lists of minimal 2-polar obstructions when restricted to either P_4 -sparse or P_4 -extendible graphs. The following proposition summarize our main result on this topic.

Theorem 43. Let \mathcal{G} be any subfamily of either P_4 -sparse or P_4 -extendible graphs which is both, hereditary and closed under complements. Let \mathcal{F} be the family of graphs depicted in Figure 14. A graph G in \mathcal{G} is a minimal 2-polar obstruction if and only G can be obtained from some graph in $\mathcal{G} \cap \mathcal{F}$ by a finite sequence of partial complementations.



Fig. 14: Some minimal 2-polar obstructions.

For the second part, based on unique tree representations for P_4 -sparse and P_4 -reducible graphs, we present linear time algorithms for finding largest subgraphs with properties related to polarity in any graph of such families (see Theorems 38 and 42). These results generalize the one given by Ekim, Mahadev and de Werra [6] for finding the largest polar subgraph in cographs based on their cotree.

Our algorithms can be easily adapted to give back yes-certificates, so we wonder whether it can be adapted, preserving its time-complexity, to also return no-certificates.

Problem 1. Can we adapt our algorithms to make them linear-time certifying algorithms?

We also think it is possible to use an approach similar to the one used for proving Theorems 38 and 42, to extend such results to wider classes of graphs having a simple enough tree representation. Specifically, we pose the next problem.

Problem 2. Can we give a linear time algorithm to find maximum monopolar, maximum unipolar, and maximum polar subgraphs on P_4 -tidy or extended P_4 -laden graphs?

In the context of matrix partitions, it was shown by Feder, Hell and Xie in [8] that, for any pair of fixed nonnegative integers, s and k, there is only a finite number of minimal (s, k)-polar obstructions, so that theoretically there is a polynomial-time brute force algorithm to decide whether a given graph is an (s, k)-polar graph. Moreover, Feder, Hell, Klein and Motwani present in [7] an explicit polynomial-time algorithm for solving the problem of deciding whether an input graph admits a fixed sparse-dense partition. Particularly, since both, complete *s*-partite graphs and *k*-clusters can be recognized in quadratic time, we have that (s, k)-polar graphs can be recognized in $O(|V|^{4+2\max\{s,k\}})$ -time. The aforementioned results make us wonder if it is possible to improve the time complexity of such algorithms by restricting the input graph to some of the graph classes with relatively few induced paths on four vertices.

Problem 3. Given arbitrary fixed nonnegative integers s and k, can we a give linear-time algorithm for finding a maximum order (s, k)-polar subgraph of a cograph G?

We also propose to solve the next natural problem which is closely related to the previous question.

Problem 4. Give an efficient algorithm for computing the minimum value of z = s + k such that an input cograph G is an (s, k)-polar graph.

Finally, we think that an approach similar to the one used here can be helpful to find the complete family of minimal 2-polar obstructions for general graphs, so we pose such problem as a future line of work.

References

- [1] J. A. Bondy and U.S.R. Murty, Graph Theory, Springer, Berlin, 2008.
- [2] A. Bretscher, D. Corneil, M. Habib and C. Paul, A simple linear time LexBFS cograph recognition algorithms, SIAM Journal on Discrete Mathematics 22(4) (2008) 1277–1296.
- [3] F. E. Contreras-Mendoza and C. Hernández-Cruz, Minimal obstructions for polarity, monopolarity, unipolarity and (s, 1)-polarity in generalizations of cographs, arXiv e-prints, 2022, arXiv: 2203.04953.
- [4] D.G. Corneil, H. Lerchs and L. Stewart Burlingham, Complement reducible graphs, *Discrete Applied Mathematics* 3 (3) (1981) 163–174.

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- [5] D.G. Corneil, Y. Perl and L.K. Stewart Burlingham, A linear recognition algorithm for cographs, *SIAM Journal on Computing* 14(4) (1985), 926–934.
- [6] T.Ekim, N. V. R. Mahadev and D. de Werra, Polar cographs, *Discrete Applied Mathematics* 156 (2008) 1652–1660.
- [7] T. Feder, P. Hell, S. Klein and R. Motwani, List partitions, *SIAM Journal of Discrete Mathematics* 16(3) (2003), 449-478.
- [8] T. Feder, P. Hell and W. Xie, Partitions with finitely many minimal obstructions, *Electronic Notes in Discrete Mathematics* 28 (2007) 371–378.
- [9] S. Foldes and P. L. Hammer, Split graphs, *in: Proc. 8th Southeastern Conf. on Combinatorics, Graph Theory, and Computing*, 1977, 311–315.
- [10] C. Hannebauer, *Matrix colorings of P*₄-sparse graphs, Master's thesis (2010), FernUniversität in Hagen.
- [11] P. Hell, C. Hernández-Cruz and C. Linhares-Sales, Minimal obstructions to 2-polar cographs, Discrete Applied Mathematics 261 (2019) 219–228.
- [12] W. Hochstättler and H. Schindler, *Recognizing P*₄-extendible graphs in linear time, Technical Report No. 95.188, Universität zu Köln (1995).
- [13] B. Jamison and S. Olariu, On a unique tree representation for P₄-extendible graphs, *Discrete Applied Mathematics* 34 (1991) 151–164.
- [14] B. Jamison and S. Olariu, A tree representation for P₄-sparse graphs, Discrete Applied Mathematics 35 (1992) 115–129.
- [15] B. Jamison and S. Olariu, Recognizing P₄-sparse graphs in linear time, SIAM Journal on Computing 21 (2) (1992) 381–406.