Discrete Mathematics and Theoretical Computer Science ....
.... **vol. 26:3 #16 (2024)**

On the mod k chromatic index of graphs

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revisions 7th Mar. 2024, 22nd Oct. 2024; accepted 24th Oct. 2024.

For a graph G and an integer $k \geq 2$, a χ'_k-coloring of G is an edge coloring of G such that the subgraph induced by the edges of each color has all degrees congruent to $1 \pmod{k}$, and $\chi'_k(G)$ is the minimum number of colors in a χ'_k-coloring of G. In ["The mod k chromatic index of graphs is $O(k)$", J. Graph Theory. 2023; 102: 197-200], Botler, Colucci and Kohayakawa proved that $\chi'_k(G) \leq 198k - 101$ for every graph G. In this paper, we show that $\chi'_k(G) \leq 177k - 93$.

Keywords: edge coloring, modulo, orientation, maximum average degree

1 Introduction

All graphs considered here are simple. Let $G = (V, E)$ be a graph, and $v(G) := |V(G)|$ and $e(G) := |E(G)|$. If $X \subseteq V(G)$, then $G[X]$ is the subgraph of G induced by X. For an integer $k \geq 2$, a χ'_k-coloring of G is a coloring of the edges of G such that the subgraph induced by the edges of each color has all degrees congruent to $1 \pmod{k}$, and the $\bmod\ k$ chromatic index of graph G, denoted by $\chi'_k(G)$, is the minimum number of colors in a χ'_k-coloring of G. Pyber (1992) proved that $\chi'_2(G) \leq 4$ for every graph G and asked whether $\chi'_k(G)$ is bounded by some function of k only. Scott (1997) proved that $\chi'_k(G) \leq 5k^2 \log k$ for any graph G, and in turn asked if $\chi'_k(G)$ is in fact bounded by a linear function of k. Botler et al. (2023) answers Scott's question affirmatively by proving the following theorem.

Theorem 1.1 (Botler et al. (2023)) *For every graph G we have $\chi'_k(G) \leq 198k - 101$.*

Also in Botler et al. (2023), Botler, Colucci, and Kohayakawa proposed the following conjecture:

Conjecture 1.2 (Botler et al. (2023)) *There is a constant C s.t. $\chi'_k(G) \leq k + C$ for every graph G.*

In this paper, we improve the upper bound of the mod k chromatic index of graphs by proving the following theorem.

Theorem 1.3 *For every graph G we have $\chi'_k(G) \leq 177k - 93$.*

In the proof of Theorem 1.1, Botler et al. (2023) applies the following two lemmas.

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ISSN 1365–8050

Lemma 1.4 (Mader (1972)) *If $k \geq 1$, G is a graph with $e(G) \geq 2kv(G)$, then G contains a k-connected subgraph.*

Lemma 1.5 (Thomassen (2014)) *If $k \geq 1$ and G is a $(12k-7)$-edge-connected graph with an even number of vertices, then G has a spanning subgraph in which each vertex has degree congruent to $k \pmod{2k}$.*

A graph G is *k-divisible* if k divides the degree of each vertex of the graph G. By applying Lemma 1.4 and Lemma 1.5, Botler et al. (2023) proved the following lemma.

Lemma 1.6 (Botler et al. (2023)) *If graph G does not contain a nonempty k-divisible subgraph, then $e(G) < 2(12k-6)v(G)$.*

For a graph G, let $N_G(v)$ denote the neighbors of v, $E_G(v)$ denote the edges that are incident to v, and let $d_G(v)$ be the degree of v, i.e., $d_G(v) = |E_G(v)|$. Let $\vec{G} = (V, \vec{E})$ be an orientation of G, for $v \in V$, let $N^+_{\vec{G}}(x)$ denote the out-neighbor(s) of x, i.e., $N^+_{\vec{G}}(x) = \{y : x \to y\}$, let $d^+_{\vec{G}}(x)$ be the out-degree of x, i.e., $d^+_{\vec{G}}(x) = |N^+_{\vec{G}}(x)|$; if y is an out-neighbor of x, then we say edge $\overrightarrow{xy}$ an out-edge of x. Let $N^-_{\vec{G}}(x)$ denote the in-neighbor(s) of x, i.e., $N^-_{\vec{G}}(x) = \{y : x \leftarrow y\}$, let $d^-_{\vec{G}}(x)$ be the in-degree of x, i.e., $d^-_{\vec{G}}(x) = |N^-_{\vec{G}}(x)|$; if y is an in-neighbor of x, then we say edge $\overleftarrow{xy}$ an in-edge of x. Let $\Delta^+\left(\vec{G}\right) = \max_{v \in V} d^+_{\vec{G}}(v)$, $\Delta^-\left(\vec{G}\right) = \max_{v \in V} d^-_{\vec{G}}(v)$. We drop the subscripts G or $\vec{G}$ in the above notations when G or $\vec{G}$ is clear from the context.

The *maximum average degree* of a graph G, denoted by $\text{mad}(G)$, is defined as

$$\text{mad}(G) = \max_{H \subseteq G} \frac{2e(H)}{v(H)},$$

which places a bound on the average vertex degree in all subgraphs. It has already attracted a lot of attention and has a lot of applications. The following theorem is well-known (cf. Hakimi (1965), Theorem 4), we use it in our proof of Theorem 1.3.

Theorem 1.7 *Let G be a graph. Then G has an orientation $\vec{G}$ such that $\Delta^+(\vec{G}) \leq d$ if and only if $\text{mad}(G) \leq 2d$.*

2 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. The proof uses the following lemma, which was available in Yang (2009). For the completeness of this paper, we present its short proof here.

Lemma 2.1 *((Yang, 2009, Lemma 1.7), adapted) Let $d \geq 0$ be an integer. If an oriented graph $\vec{G}$ has $\Delta^+(\vec{G}) \leq d$, then there exists a linear order σ of $V(\vec{G})$, such that for any vertex $u \in V(\vec{G})$, the number of vertices that are the in-neighbors of u, and precede u in σ is at most d.*

Proof: We recursively construct a linear ordering $\sigma = v_1 v_2 \ldots v_n$ of $V = V(\vec{G})$ as follows. Suppose that we have constructed the final sequence $v_{i+1} \ldots v_n$ of L. (If $i = n$ then this sequence is empty.) Let $M = \{v_{i+1}, \ldots, v_n\}$ be the set of vertices that have already been ordered and $U = V - M$ be the set of vertices that have not yet been ordered. Let $\vec{G_U} \subseteq \vec{G}$ be the subgraph of $\vec{G}$ induced by U. If we have not yet finished constructing σ, we choose $v_i \in U$ so that $d^-_{\vec{G_U}}(v_i)$ is minimal in $\vec{G_U}$. Since

$\sum_{v\in U} d^-_{\vec{G_U}}(v) = \sum_{v\in U} d^+_{\vec{G_U}}(v)$, and $\Delta^+\left(\vec{G_U}\right) \le \Delta^+\left(\vec{G}\right) \le d$, we have $d^-_{\vec{G_U}}(v_i) \le d$. This proves the lemma. □

Combining Lemma 2.1 and the techniques used in Botler et al. (2023), we prove the following lemma, which is the key to the proof of Theorem 1.3.

Lemma 2.2 *Let $d \ge 1$ be an integer. If a graph G has an orientation $\vec{G}$ such that $\Delta^+(\vec{G}) \le d$, then $\chi'_k(G) \le 7d + 2k - 3$.*

Proof: If a graph G has an orientation $\vec{G}$ such that $\Delta^+(\vec{G}) \le d$, by applying Lemma 2.1, we can suppose linear ordering $\sigma := v_1v_2\ldots v_n$ of $V(G)$ satisfying that for any vertex $u \in V(\vec{G})$, the number of vertices that are the in-neighbors of u, and precede u in σ is at most d.

Following the above linear ordering σ, we give a χ'_k-coloring of G by coloring the edges incident with v_i for each $v_i \in \{v_1, \ldots, v_{n-1}\}$ in turn. At step v_i, we name *this procedure as processing vertex v_i, which means that we color all the edges incident with v_i that are not colored yet at this time*. After we have finished processing vertex v_i, we shall maintain that we have a χ'_k-coloring of the graph spanned by the edges incident with $v_1, \ldots, v_i$, we call this a *good partial χ'_k-coloring after step v_i*.

To define the coloring method, we partition all the colors into two sets C_1 and C_2 such that $|C_1| = 3d-1$ and $|C_2| = 4d + 2k - 2$, note that $|C_1| + |C_2| = 7d + 2k - 3$. For each $1 \le i \le n-1$, we use the colors in C_1 to color the uncolored out-edges of v_i and use the colors in C_2 to color the uncolored in-edges of v_i. Equivalently, for any directed edge uv ($u \to v$ in $\vec{G}$), if u is processed before v, then edge uv is colored with a color in C_1; if v is processed before u, then edge uv is colored with a color in C_2.

By induction on i, we give a good partial χ'_k-coloring of G after processing v_i. For the induction hypothesis, suppose when we begin to process vertex v_i, all the edges that are incident with a vertex v that precedes v_i in σ have already been colored.

For each $1 \le i \le n-1$, when v_i is processed, let $U^+(v_i)$ denote the unprocessed out-neighbor(s) of v_i, i.e., $U^+(v_i) = N^+_{\vec{G}}(v_i) \cap \{v_{i+1}, \ldots, v_n\}$; let $U^-(v_i)$ denote the unprocessed in-neighbor(s) of v_i, i.e., $U^-(v_i) = N^-_{\vec{G}}(v_i) \cap \{v_{i+1}, \ldots, v_n\}$. When we process v_i, we color the uncolored out-edges $\{v_iv_j : v_j \in U^+(v_i)\}$ such that all out-edges of v_i are colored with distinct colors; to color the uncolored in-edges $\{v_jv_i : v_j \in U^-(v_i)\}$, we use colors in C_2 that are different than having been used in the out-edges of v_i (refer $X(v_i)$ in the following paragraph); and we do the coloring in this order.

For the induction step, suppose we process vertex v_i, where $i \in \{1, 2, \ldots, n-1\}$. Suppose $N^+_{\vec{G}}(v_i) \cap \{v_1, \ldots, v_{i-1}\} = \{x_1, \ldots, x_\ell\} = X(v_i)$. Since $\Delta^+(\vec{G}) \le d$, $|X(v_i)| \le d$. Suppose $N^-_{\vec{G}}(v_i) \cap \{v_1, \ldots, v_{i-1}\} = \{y_1, \ldots, y_r\} = Y(v_i)$, then by Lemma 2.1, $|Y(v_i)| \le d$. For the induction hypothesis, we suppose each edge y_jv_i, where $y_j \in Y(v_i)$, is colored with a color in C_1; and each edge v_ix_j, where $x_j \in X(v_i)$, is colored with a color in C_2. Now we process vertex v_i, i.e., color the remaining uncolored edges incident with v_i, we do this in two steps.

In the first step, we use the colors in C_1 to color the uncolored out-edges $\{v_iv_j : v_j \in U^+(v_i)\}$, such that all the out-edges of v_i have distinct colors. We show that we can do this for any edge v_iv_j with $v_j \in U^+(v_i)$. For vertex v_j, since $v_j \in U^+(v_i)$, v_j has not been processed yet. Therefore, by Lemma 2.1 and the induction hypothesis, the in-edges of v_j that have been colored with colors in C_1 is at most $d-1$ (note that in the counting we removed the in-edge v_iv_j of v_j). For vertex v_i, when we begin to process vertex v_i, by the induction hypothesis, the edges incident with v_i and colored with colors in C_1 are y_jv_i, where $y_j \in Y(v_i)$. During processing vertex v_i, for the edge v_iv_j with $v_j \in U^+(v_i)$, at most $|U^+(v_i)|-1$ edges incident with v_i are colored with colors in C_1. Note that,

$$|C_1| - |Y(v_i)| - (|U^+(v_i)| - 1) - (d-1) \geq (3d-1) - d - (d-1) - (d-1) \geq 1.$$

This proves that there is a color left in C_1 for v_iv_j with $v_j \in U^+(v_i)$.

In the second step, we color the uncolored in-edges $R(v_i) = \{v_jv_i : v_j \in U^-(v_i)\}$ of v_i. Note that v_i has at most $|X(v_i)| \leq d$ processed out-neighbors before v_i is processed. Observe that for any edge $x_ix_{\ell'}$ with $x_{\ell'} \in X(v_i)$, $x_ix_{\ell'}$ is an in-edge of $x_{\ell'}$. By the induction hypothesis, $x_ix_{\ell'}$ is colored when $x_{\ell'}$ is processed, and is colored with a color in C_2. After removing the colors that used by edges $x_ix_{\ell'}$ with $x_{\ell'} \in X(v_i)$, there are at least $|C_2| - |X(v_i)| \geq 4d+2k-2-d = 3d+2k-2$ colors left in C_2 that can be used to color edges in $R(v_i)$.

We partition these left colors in C_2 arbitrarily into sets $A(v_i)$ and $B(v_i)$ so that $|A(v_i)| = d+k$ and $|B(v_i)| \geq 2d+k-2$. For each $v_j \in U^-(v_i)$ is an unprocessed in-neighbor of v_i, we say that a color c is $forbidden$ at v_j if there is out-edge of v_j is colored with c, and we call the colors in $A(v_i)$ that are not forbidden at v_j $available$ at v_j. Note that at most $d-1$ out-edges of v_j are colored (removing the out-edge v_jv_i of v_j in the counting). This implies that at least $k+1$ colors in $A(v_i)$ are available at v_j.

Let $R^*(v_i)$ be the maximal subset of $R(v_i)$ that can be colored with colors in $A(v_i)$ in a way such that:

(a) each in-edge $v_jv_i \in R^*(v_i)$ of v_i is colored with a color available at v_j;

(b) the number of edges in $R^*(v_i)$ colored with any color is congruent to $1 \pmod{k}$.

Let $\bar{R}(v_i) = R(v_i) \setminus R^*(v_i)$ be the set of the remaining edges in $R(v_i)$. We claim that $|\bar{R}(v_i)| < |A(v_i)|$.

Assume otherwise that $|\bar{R}(v_i)| \geq |A(v_i)|$, and suppose $A(v_i) = \{a_i : 1 \leq i \leq d+k\}$, $\bar{R}(v_i) = \{e_j = w_jv_i : w_j \in U^+(v_i),\ 1 \leq j \leq t,\ \text{and}\ t \geq d+k\}$. We define an auxiliary bipartite graph T with vertices bipartition $A(v_i)$ and $\bar{R}(v_i)$, edges $E(T) = \{a_ie_j : \text{where } e_j = w_jv_i, a_i \text{ is available at } w_j\}$.

Since, for each $w_j \in U^+(v_i)$, there are at least $k+1$ colors in $A(v_i)$ available at w_j, we have $d_T(e_j) \geq k+1$ for every $e_j \in \bar{R}(v_i)$. Therefore,

$$\sum_{a_i \in A(v_i)} d_T(a_i) = |E(T)| = \sum_{e_j \in \bar{R}(v_i)} d_T(e_j) \geq (k+1)t \geq (k+1)(d+k).$$

Since $|A(v_i)| = d+k$, we concluded that there exists a color a_i in $A(v_i)$, $d_T(a_i) \geq k+1$, which means that color a_i is available on at least $k+1$ edges in $\bar{R}(v_i)$.

If some edge in $R^*(v_i)$ is already colored with a_i, then we color k edges in $\bar{R}(v_i)$ with color a_i. If no edge in $R^*(v_i)$ is colored with a_i, then we color $k+1$ edges in $\bar{R}(v_i)$ with color a_i. Both of these cases contradict with the maximality of $R^*(v_i)$. This proves that $|\bar{R}(v_i)| < |A(v_i)| = d+k$.

Finally we show that we can color all the edges in $\bar{R}(v_i)$ with distinct colors in $B(v_i)$. For this, it suffices to note that, for each $w_jv_i \in \bar{R}(v_i)$, there are at most $d-1+|\bar{R}(v_i)|-1 \leq 2d+k-3 < |B(v_i)|$ colors of $B(v_i)$ that are either forbidden at w_j, or were used on previous edges of $\bar{R}(v_i)$. □

We prove our main result by using Lemma 1.6, Theorem 1.7, Lemma 2.1, and Lemma 2.2. The proof is similar to Theorem 5 in Botler et al. (2023), the differences are applications of Theorem 1.7 and Lemma 2.1 here, and Lemma 2.2 is stronger than the corresponding one in Botler et al. (2023).

Theorem 1.3. *For every graph G we have $\chi'_k(G) \leq 177k - 93$.*

Proof: Let H be a maximal subgraph of G such that $d_H(v) \equiv 1 \pmod{k}$ for every $v \in V(H)$, and let $G' = G \backslash E(H)$. Then $V(G) \setminus V(H)$ is independent. Since otherwise, there exists an edge e with both ends in $V(G) \setminus V(H)$; then $H' = H + e$ would be a graph for which $d_{H'}(v) \equiv 1 \pmod{k}$; but this contradicts the maximality of H.

Similarly, by the maximality of H, $G'[V(H)]$ has no nonempty k-divisible subgraph. By Lemma 1.6, for every nonempty $J \subseteq G'[V(H)]$, we have $e(J) < 2(12k-6)v(J)$. Thus,

$$\mathrm{mad}(G'[V(H)]) = \max_{J \subseteq G'[V(H)]} \frac{2e(J)}{v(J)} < \max_{J \subseteq G'[V(H)]} \frac{2(24k-12)v(J)}{v(J)} = 2(24k-12).$$

By Theorem 1.7, $G'[V(H)]$ has an orientation $\overrightarrow{G'[V(H)]}$ such that $\Delta^+(\overrightarrow{G'[V(H)]}) \leq 24k - 12$.

For every vertex $u \in V(H)$, by the maximality of H, u has at most $k-1$ neighbors in $V(G) \setminus V(H)$. For every edge $e = uv$ in G' with $u \in V(H)$ and $v \in V(G) \setminus V(H)$, we orient e from u to v.

Thus there exists an orientation $\vec{G'}$ of G', such that $\Delta^+(\vec{G'}) \leq 25k - 13$. By Lemma 2.2, there exists a χ'_k-coloring of G' using at most $177k - 94$ colors. Then color all $E(H)$ with a new color, this proves the theorem. □

Remark. In the above proof of Theorem 1.3, for all the edges $e = uv$ in G' with $u \in V(H)$ and $v \in V(G) \setminus V(H)$, we can orient e from u to v. Then define a linear ordering σ' beginning with vertices in $V(G) \backslash V(H)$, and concatenating a linear ordering of vertices in $G'[V(H)]$ that has been proved existing by Lemma 2.1, but use $\Delta^+(\overrightarrow{G'[V(H)]}) \leq 24k - 12$ here (instead of using of $\Delta^+(\vec{G'}) \leq 25k - 13$ as the proof of Theorem 1.3). By using this σ', and the above orientation, following the methodology of Lemma 2.2, we can first color all the edges incident with $V(G) \setminus V(H)$, and then the edges in $G'[V(H)]$, by processing the vertices one by one following linear ordering σ'. This coloring process can be used to prove that $\chi'_k(G) \leq 171k - 87$. The proof for this comes from tweaking the proofs of Lemma 2.2. As the authors in Botler et al. (2023) have mentioned, we think we would be far from the truth still (refer Conjecture 1.2), we skip the details of this small improvement here for the readability of this paper.

References

F. Botler, L. Colucci, and Y. Kohayakawa. The mod k chromatic index of graphs is $O(k)$. *J. Graph Theory*, 102(1):197–200, 2023. ISSN 0364-9024,1097-0118. URL `https://doi.org/10.1002/jgt.22866`.

S. L. Hakimi. On the degrees of the vertices of a directed graph. *J. Franklin Inst.*, 279:290–308, 1965. ISSN 0016-0032. URL `https://doi.org/10.1016/0016-0032(65)90340-6`.

W. Mader. Existenz n-fach zusammenhängender Teilgraphen in Graphen genügend grosser Kantendichte. *Abh. Math. Sem. Univ. Hamburg*, 37:86–97, 1972. ISSN 0025-5858,1865-8784. URL `https://doi.org/10.1007/BF02993903`.

L. Pyber. Covering the edges of a graph by . . .. In *Sets, graphs and numbers (Budapest, 1991)*, volume 60 of *Colloq. Math. Soc. János Bolyai*, pages 583–610. North-Holland, Amsterdam, 1992. ISBN 0-444-98681-2.

A. D. Scott. On graph decompositions modulo k. *Discrete Math.*, 175(1-3):289–291, 1997. ISSN 0012-365X,1872-681X. URL `https://doi.org/10.1016/S0012-365X(96)00109-4`.

C. Thomassen. Graph factors modulo k. *J. Combin. Theory Ser. B*, 106:174–177, 2014. ISSN 0095-8956,1096-0902. URL `https://doi.org/10.1016/j.jctb.2014.01.002`.

D. Yang. Generalization of transitive fraternal augmentations for directed graphs and its applications. *Discrete Math.*, 309(13):4614–4623, 2009. ISSN 0012-365X,1872-681X. URL `https://doi.org/10.1016/j.disc.2009.02.028`.