

On the Advice Complexity of Online Matching on the Line*

Béla Csaba

Judit Nagy-György†

University of Szeged, Szeged, Hungary

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We consider the matching problem on the line with advice complexity. We give a 1-competitive online algorithm with advice complexity $n - 1$, and show that there is no 1-competitive online algorithm reading less than $n - 1$ bits of advice. Moreover, for each $0 < k < n$ we present a $c(n/k)$ -competitive online algorithm with advice complexity $O(k(\log N + \log n))$ where n is the number of servers, N is the distance of the minimal and maximal servers, and $c(n)$ is the complexity of the best online algorithm without advice.

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1 Introduction

In the problem of online minimum matching (also known as online weighted matching), defined by Kalyanasundaram and Pruhs (1993) and Khuller et al. (1994), n points of a metric space s_1, \dots, s_n , called *servers*, are given. The *requests* r_1, \dots, r_n are the elements of the metric space as well, arriving one by one according to their indices, and upon arrival each one has to be matched to an unmatched server at a cost equal to their distance. The goal is to minimize the total cost. We say that n is the size of the input.

In this paper, the metric space is the real line. We can assume without loss of the generality that $s_1 \leq \dots \leq s_n$. Observe that each matching corresponds to a permutation.

Competitive analysis is commonly used to evaluate the performance of online algorithms. Here we consider a minimization problem. Algorithm A is c -competitive if the cost of A is at most c times larger than the optimal cost (plus some constant). For randomized algorithms, the expected value of the cost is compared to the optimum. The competitive ratio of algorithm A is the minimal c for which A is c -competitive. Let us mention that this notion is also called *weak competitive ratio* by some researchers, since it includes an additive constant.

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There are no tight bounds for the competitive ratio on the line. Antoniadis et al. (2019) have developed an $O(n^{\log_2(3+\epsilon)-1}/\epsilon)$ -competitive deterministic algorithm. An $O(\log n)$ -competitive randomized algorithm is presented by Gupta and Lewi (2012). Peserico and Scquizzato (2023) have given a lower bound $\Omega(\sqrt{\log n})$ for the competitive ratio of randomized algorithms.

The term advice complexity for online algorithms was introduced by Dobrev et al. (2009). In this paper we work in the *tape model* introduced by Böckenhauer et al. (2009). In this model, the online algorithm may read an infinite advice tape written by the oracle, and the advice complexity is the number of bits read. The question is the following: *How many bits of advice are necessary and sufficient to achieve a competitive ratio c ?* This includes determining the number of bits an optimal algorithm needs (in case $c = 1$). Advice complexity has been investigated for many online problems, see the survey paper Boyar et al. (2017) for more information.

We note that the results of Mikkelsen (2016) and Peserico and Scquizzato (2023) imply that no online algorithm with sublinear advice complexity can be $O(1)$ -competitive.

In the next section we prove that the advice complexity of the best 1-competitive matching algorithm is $n - 1$. In Section 3 we present a family of online matching algorithms. The algorithm, indexed by a positive integer k , reads $O(k(\log N + \log n))$ bits of advice and has competitive ratio $c(n/k)$ if it uses a $c(n)$ -competitive matching algorithm as a subroutine. This shows how increasing the number of advice bits help to decrease the competitive ratio.

2 The advice complexity of 1-competitive online algorithms

In this section we give matching upper and lower bounds for the advice complexity of 1-competitive online algorithms. Let us begin with two folklore results on the structure of any optimal matching on the line. For completeness we present the proofs.

Proposition 1. *Consider an optimal matching corresponding to permutation π (i.e. r_i is matched to $s_{\pi(i)}$).*

- *If $r_i \leq s_{\pi(j)} < s_{\pi(i)}$ for some i, j then $r_j \leq s_{\pi(j)}$.*
- *If $r_i \geq s_{\pi(j)} > s_{\pi(i)}$ for some i, j then $r_j \geq s_{\pi(j)}$.*

Proof: Consider the case $r_i \leq s_{\pi(j)} < s_{\pi(i)}$ (the other case is similar). Suppose, to the contrary, that $r_j > s_{\pi(j)}$. If $r_j \leq s_{\pi(i)}$ then

$$\begin{aligned} \text{dist}(r_i, s_{\pi(j)}) + \text{dist}(r_j, s_{\pi(i)}) &= s_{\pi(j)} - r_i + s_{\pi(i)} - r_j \\ &< s_{\pi(i)} - r_i + r_j - s_{\pi(j)} \\ &= \text{dist}(r_i, s_{\pi(i)}) + \text{dist}(r_j, s_{\pi(j)}) \end{aligned}$$

by assumption $r_j > s_{\pi(j)}$ but this is a contradiction. If $r_j > s_{\pi(i)}$ then

$$\begin{aligned} \text{dist}(r_i, s_{\pi(j)}) + \text{dist}(r_j, s_{\pi(i)}) &= s_{\pi(j)} - r_i + r_j - s_{\pi(i)} \\ &< s_{\pi(i)} - r_i + r_j - s_{\pi(j)} \\ &= \text{dist}(r_i, s_{\pi(i)}) + \text{dist}(r_j, s_{\pi(j)}) \end{aligned}$$

by assumption $s_{\pi(i)} > s_{\pi(j)}$ but this is a contradiction. □

Proposition 2. Consider an optimal matching corresponding to permutation π . Suppose that

$$\max\{r_i, r_j\} \leq \min\{s_{\pi(i)}, s_{\pi(j)}\} \text{ or } \min\{r_i, r_j\} \geq \max\{s_{\pi(i)}, s_{\pi(j)}\}.$$

Then the matching corresponding to π' where $\pi'(i) = \pi(j)$, $\pi'(j) = \pi(i)$ and $\pi'(\ell) = \pi(\ell)$ if $\ell \neq i, j$ is also optimal.

Proof: Suppose that $\max\{r_i, r_j\} \leq \min\{s_{\pi(i)}, s_{\pi(j)}\}$ (the other case is similar).

$$\begin{aligned} \text{dist}(r_i, s_{\pi(i)}) + \text{dist}(r_j, s_{\pi(j)}) &= s_{\pi(i)} - r_i + s_{\pi(j)} - r_j \\ &= s_{\pi'(j)} - r_j + s_{\pi'(i)} - r_i \\ &= \text{dist}(r_j, s_{\pi'(j)}) + \text{dist}(r_i, s_{\pi'(i)}), \end{aligned}$$

therefore the sums of the distances of the matched points in the two matchings are equal. \square

Consider an optimal matching corresponding to permutation π . Let

$$L_\pi = \{r_i : s_{\pi(i)} \leq r_i, 1 \leq i \leq n\},$$

the set of the requests matched to their left, and

$$R_\pi = \{r_i : s_{\pi(i)} > r_i, 1 \leq i \leq n\},$$

the set of the requests matched to their right. Algorithm LR serves r_i in the following way:

- if there is an unmatched server s_j equal to r_i , then match them;
- if all unmatched servers are greater than r_i , then match r_i to the least unmatched server;
- if all unmatched servers are less than r_i , then match r_i to the largest unmatched server;
- otherwise read a bit of advice:
 - if it is 0 (it means that $r_i \in L_\pi$), then match r_i to the greatest unmatched server less than r_i ;
 - if it is 1 (it means that $r_i \in R_\pi$) then match r_i to the least unmatched server greater than r_i .

Theorem 3. Algorithm LR reads at most $n - 1$ bits of advice, and gives an optimal matching.

Proof: Note that algorithm LR reads at most one advice bit per request, and it does not read any for serving the last request.

We prove optimality by induction on n . The case $n = 1$ is trivial. Suppose that $n > 1$ and the statement holds for all $i < n$. Consider an optimal matching corresponding to permutation π . If $r_1 \in L_\pi$ and $s_{\pi(1)} = \max\{s_j : s_j \leq r_1, 1 \leq j \leq n\}$, then the statement is true by the induction hypothesis. Similarly, if $r_1 \in R_\pi$ and $s_{\pi(1)} = \min\{s_j : s_j > r_1, 1 \leq j \leq n\}$, then we are ready.

Suppose now, that $r_1 \in L_\pi$ and $s_{\pi(1)} < \max\{s_j : s_j \leq r_1, 1 \leq j \leq n\} = s_{\pi(i)}$ for some $i > 1$. Proposition 1 implies that $r_i \in L_\pi$ too. Therefore, by Proposition 2, there is an optimal matching corresponding to permutation π' such that $s_{\pi'(1)} = \max\{s_j : s_j \leq r_1, 1 \leq j \leq n\}$, and we can apply the argument above. Similarly, if $r_1 \in R_\pi$, then there is an optimal matching corresponding to permutation π' such that $s_{\pi'(1)} = \min\{s_j : s_j > r_1, 1 \leq j \leq n\}$, therefore the statement holds. \square

The following theorem shows that there is no better optimal algorithm.

Theorem 4. *There is no online algorithm for the matching problem on the line which provides an optimal matching while reading less than $n - 1$ bits of advice.*

Proof: Consider an optimal algorithm A. For each $n \in \mathbb{Z}^+$ we construct a set \mathcal{I}_n of inputs, each of size n , in a recursive way. An input consists of a set of servers and a sequence of requests. The set of servers is the same for every input in \mathcal{I}_n : $s_i = i$ for each $i \in \{1, \dots, n\}$. We only need to determine the sequences of requests in \mathcal{I}_n . The first input \mathcal{I}_1 has one element consisting of one request: $r_1 = 1$.

Suppose, that $n > 1$. The request sequence ρ_0 , consisting of the requests $r_i = n - 2^{-i}$, $i = 1, \dots, n$ plays a special role in \mathcal{I}_n . It belongs to \mathcal{I}_n , and every other request sequence of \mathcal{I}_n can be obtained from prefixes of ρ_0 by extending it to having length n with request sequences from \mathcal{I}_k , $1 \leq k < n$.

More formally, repeat the following for every $1 \leq k < n$. Take the prefix of ρ_0 of length $n - k$, denote it by ρ , its elements are $r_i = n - 2^{-i}$, $i = 1, \dots, n - k$. Then for every $\rho' \in \mathcal{I}_k$ we form a new element $\rho\rho'$ of \mathcal{I}_n . Hence, every prefix ρ of ρ_0 with length $n - k$ is extended into $|\mathcal{I}_k|$ different request sequences of \mathcal{I}_n . It is easy to see, that s_n is matched to r_{n-k} in all optimal serving of the request sequence $\rho\rho'$. Note also, that $s_n = n$ is matched to r_n in every optimal serving of ρ_0 .

Using induction, one can easily show that

$$|\mathcal{I}_n| = 1 + \sum_{i=1}^{n-1} |\mathcal{I}_i| = 2^{n-1}.$$

We prove that A needs different advice words for any two elements of \mathcal{I}_n , and none of these advice words can be a prefix of the other. We proceed by induction on n . The statement is trivial for $n = 1$. Suppose that $n > 1$, and the statement holds for all $k < n$. Let ρ consisting of r_1, \dots, r_n and ρ' consisting of r'_1, \dots, r'_n two different elements of \mathcal{I}_n . We have three cases.

If $\rho = \rho_0$ and ρ' is constructed for an element of \mathcal{I}_k for some $k < n$, then the first $n - k$ requests are identical, i.e. $r_1 = r'_1, \dots, r_{n-k} = r'_{n-k}$, moreover, in any optimal serving of ρ , s_n is matched to r_n , and in any optimal serving of ρ' , s_n is matched to r'_{n-k} . Therefore A cannot distinguish between the two inputs knowing only the first $n - k$ requests, but the $(n - k)$ th requests require different treatment in the two inputs, therefore the statement holds.

If ρ is constructed for an element of \mathcal{I}_k and ρ' is constructed for an element of \mathcal{I}_l for some $k < l < n$, then the first $n - l$ requests are identical, i.e. $r_1 = r'_1, \dots, r_{n-l} = r'_{n-l}$. Moreover in any optimal serving of ρ , s_n is matched to r_{n-k} and any optimal serving of ρ' , s_n is matched to r'_{n-l} . Therefore A cannot distinguish between the two inputs knowing only the first $n - l$ requests, but the $(n - l)$ th requests require different treatment in the two inputs, therefore the statement holds.

If ρ and ρ' are constructed for two elements of \mathcal{I}_k for some $k < n$, then the first $n - k$ requests are identical, i.e. $r_1 = r'_1, \dots, r_{n-k} = r'_{n-k}$, moreover $(n - n')$ th request is matched to s_n and the first $n - k - 1$ requests are matched to s_{k+1}, \dots, s_{n-1} in both cases, so the statement holds by the induction hypothesis. \square

3 Algorithms DIVIDE_k and RESCALE

In this section we present an algorithm for each $k > 0$ integer. We will see that there is a trade-off between the amount of advice and the competitive ratio: larger k means a better competitive ratio but also more advice.

First we assume that $s_1 = 1$, $s_n = N - 1$, $N \in \mathbb{Z}^+$ and $r_i \in \mathbb{Z}$ for each $i = 1, \dots, n$. Algorithm DIVIDE_k determines k blocks on the line, each containing about n/k servers, identifies requests whose pairs are outside, and works as a shell algorithm using algorithm LR to handle these requests, and a $c(n)$ -competitive algorithm A inside the blocks as subroutines.

Consider an optimal matching corresponding to permutation π for which $\pi(i) < \pi(j)$ if $r_i < r_j$. Note that it is well-known that such an optimal matching exists (it follows by Proposition 1 and Proposition 2 as well).

A detailed description of Algorithm DIVIDE_k is as follows.

Pre-processing: Let $\ell \equiv n \pmod k$ (so $0 \leq \ell < k$), and consider the following partitioning of the servers:

$$S_i = \{s_{(i-1)\lceil \frac{n}{k} \rceil + 1}, \dots, s_{i\lceil \frac{n}{k} \rceil}\} \quad \text{for every } 1 \leq i \leq \ell,$$

$$S_{\ell+j} = \{s_{\ell\lceil \frac{n}{k} \rceil + (j-1)\lfloor \frac{n}{k} \rfloor + 1}, s_{\ell\lceil \frac{n}{k} \rceil + j\lfloor \frac{n}{k} \rfloor}\} \quad \text{for every } 1 \leq j \leq k - \ell - 1.$$

Let

$$p_i = \frac{1}{2} (\max\{s : s \in S_i\} + \min\{s : s \in S_{i+1}\}), \quad \text{for every } i = 1, \dots, k - 1.$$

Determining the blocks:

$$B_1 = (-\infty, p_1], \quad B_i = (p_{i-1}, p_i], \quad i = 2, \dots, k - 1, \quad B_k = (p_{k-1}, \infty).$$

Note that if $p_{i-1} = p_i$ then $B_i = \emptyset$.

Partitioning the points:

- For each $i = 2, \dots, k$ the algorithm reads $\lceil \log_2 N \rceil$ bits of advice, *i.e.*, a word $q_{i,L}$, which is the minimum of N and the position of the rightmost request of block B_i , such that this request has to be served by a server in S_j for some $j < i$. If there is no such a request in block B_i , then word $q_{i,L}$ is 0. If $q_{i,L} = N$ then set $q_{i,L} := \infty$. Let

$$L = \bigcup_{i=2}^k (p_{i-1}, q_{i,L}],$$

where for $a > b$ we let $(a, b] = \emptyset$.

- For each $i = 1, \dots, k - 1$ the algorithm reads $\lceil \log_2 N \rceil$ bits of advice, *i.e.*, a word $q_{i,R}$, which is the maximum of 0 and the position of the leftmost request of block B_i , such that this request has to be served by a server in S_j for some $j > i$. If there is no such a request in block B_i , then word $q_{i,R}$ is the all-1 word of length $\lceil \log_2 N \rceil$. If $q_{i,R} = 0$, then set $q_{i,R} := -\infty$. Let

$$R = \bigcup_{i=1}^{k-1} [q_{i,R}, p_i], \quad \text{and } B = \mathbb{R} \setminus \{L \cup R\}.$$

Observe, that if a request r belongs to B , then its pair is in the block of r . Note that there may be requests equal to some $q_{i,L}$ or $q_{i,R}$ with pairs in S_i .

- (First marking procedure) Set $M_R = \emptyset$. For each $i = 1, \dots, k - 1$ if $q_{i,R} > 0$, then the algorithm reads 2 times $\lfloor \log_2 n \rfloor$ bits of advice, *i.e.*, the number $d_{i,R}$ of requests equal to $q_{i,R}$ with pairs in S_i and the number $m = m_{i,R}$ of requests in $R \cap B_i$ minus $d_{i,R}$, *i.e.*, the number of requests in B_i matched outside to the right in the matching corresponding to π . If $m > 0$, let s_j be the server in $\bigcup_{j>i} S_j \setminus M_R$ with minimal index. Set $M_R := M_R \cup \{s_j, \dots, s_{j+m-1}\}$.
- (Second marking procedure) Set $M_L = \emptyset$. For each $i = 0, \dots, k - 2$ if $q_{k-i,L} > 0$, then the algorithm reads 2 times $\lfloor \log_2 n \rfloor$ bits of advice, *i.e.*, the number $d_{k-i,L}$ of requests equal to $q_{k-i,L}$ with pairs in S_{k-i} and the number $m = m_{k-i,L}$ of requests in $L \cap B_{k-i}$ minus $d_{k-i,L}$, *i.e.*, the number of requests in B_{k-i} matched outside to the left in the matching corresponding to π . If $m > 0$, let s_j be the server in $\bigcup_{j<k-i} S_j \setminus M_L$ with maximal index. Set $M_L := M_L \cup \{s_j, \dots, s_{j-m+1}\}$.

Let $M = M_R \cup M_L$ be the set of the marked servers and U the set of the unmarked servers.

Servicing requests. Set an empty auxiliary advice tape for algorithm LR.

Handling request r_i :

- If $r_i \in B \cap B_j$ for some $1 \leq j \leq k$, then $U := U \cup \{r_i\}$ and use algorithm A on inputs in $U \cap (S_j \cup B_j)$ to serve r_i .
- If $r_i = q_{j,R}$ for some $1 \leq j \leq k - 1$ and the number of unmarked requests in U equal to $q_{i,R}$ is less than $d_{i,R}$, then $U := U \cup \{r_i\}$, and use algorithm A on inputs in $U \cap (S_j \cup B_j)$ to serve r_i .
- If $r_i = q_{j,L}$ for some $2 \leq j \leq k$ and the number of unmarked requests in U equal to $q_{i,L}$ is less than $d_{i,L}$, then $U := U \cup \{r_i\}$, and use algorithm A on inputs in $U \cap (S_j \cup B_j)$ to serve r_i .
- If $r_i \in R \cap B_j$, $r_i > q_{j,R}$ for some $1 \leq j \leq k - 1$ and the number of marked requests in B_j is less than $m_{j,R}$, then mark r_i , let $M := M \cup \{r_i\}$, write bit 1 at the end of the bit sequence on the auxiliary tape, and use algorithm LR on inputs in M to serve r_i . If algorithm A did not read any bits of advice to serve request r_i , then delete the last bit of the content of the auxiliary tape.
- If $r_i = q_{j,R}$ for some $1 \leq j \leq k - 1$ and the number of unmarked requests equal to $q_{i,R}$ is $d_{i,R}$, then mark r_i , let $M := M \cup \{r_i\}$, write bit 1 at the end of the bit sequence on the auxiliary tape and use algorithm LR on inputs in M to serve r_i . If algorithm A did not read any bits of advice to serve request r_i , then delete the the last bit of the content of the auxiliary tape.
- If $r_i \in L \cap B_j$, $r_i < q_{j,L}$ for some $2 \leq j \leq k$ and the number of marked requests in B_j is less than $m_{j,L}$, then mark r_i , let $M := M \cup \{r_i\}$, write bit 0 at the end of the bit sequence on the auxiliary tape, and use algorithm LR on inputs in M to serve r_i . If algorithm A did not read any bits of advice to serve request r_i , then delete the the last bit of the content of the auxiliary tape.
- If $r_i = q_{j,L}$ for some $2 \leq j \leq k$ and the number of unmarked requests equal to $q_{i,L}$ is $d_{i,L}$, then mark r_i , let $M := M \cup \{r_i\}$, write bit 0 at the end of the bit sequence on the auxiliary tape, and use algorithm LR on inputs in M to serve r_i . If algorithm A did not read any bits of advice to serve request r_i , then delete the the last bit of the content of the auxiliary tape.

Theorem 5. *Algorithm DIVIDE_k is $c(n/k)$ -competitive and reads $O(k(\log N + \log n))$ bits of advice.*

Proof: The second part of the statement follows immediately from the definition of algorithm DIVIDE_k .

We can assume without loss of generality that whenever $r_i = r_{i'}$ and r_i is matched to s_j and $r_{i'}$ is matched to $s_{j'}$ by DIVIDE_k where $j < j'$, then $\pi(i) < \pi(i')$. We will prove that if $r_j \in B_i \cap U$ then $s_{\pi(j)} \in S_i \cap U$, moreover, if $r_j \in M$, then $s_{\pi(j)} \in M$.

At first we need to see that the algorithm terminates. The number of servers and the number of requests are equal by definition of the model. The first and second marking procedure do not get stuck by definition of the matching corresponding to π and the marking procedures. Moreover $M_L \cap M_R = \emptyset$ by definition of π . Indeed, suppose, to the contrary that s_i is marked in the first and the second marking procedure as well. There are requests $r_j \leq s_i$ and $r_\ell > s_i$ such that $\pi(j) \geq i$ and $\pi(\ell) \leq i$ by definition of π and the pigeonhole principle, therefore $\pi(j) > \pi(\ell)$, but this is a contradiction.

The algorithm uses LR to match requests in M to servers in M , and it uses algorithm A to match unmarked requests in B_i to unmarked servers in S_i for each $i \in \{1, \dots, k\}$. By definition of algorithm DIVIDE_k , the number of the marked servers and the number of the marked requests are equal, and there is always an unread bit on the auxiliary advice tape whenever it is necessary.

Moreover, the number of unmarked requests in B_i and the number of unmarked servers in S_i are equal, and the matching corresponding to π match them to each other for all $i \in \{1, \dots, k\}$ (this implies that a request in M is matched to a server in M in the optimal matching). We will prove this by induction on $m = \sum_{i=1}^{k-1} (m_{i,R} + m_{i+1,L})$. It is easy to see that the statement holds for $m = 0$.

Now suppose that $m > 0$, and the statement holds for all smaller cases. Let i be the smallest index with $m_{i,R} > 0$ and j the index of the first server in block B_{i+1} . Then there is a request $r_\ell \in B_i \cap R$ matched to s_j in the optimal matching, *i.e.*, $j = \pi(\ell)$, otherwise by the minimality of i there is a request $r_{\ell'} > p_i(\geq r_\ell)$ such that $j = \pi(\ell')$ and $j' > j$ such that for the first request $r_\ell \in B_i \cap R$ and $j' = \pi(\ell)$, but this is a contradiction by Proposition 1. Deleting r_ℓ and s_j from the model we are ready by the induction hypothesis.

If $\sum_{i=1}^{k-1} m_{i,R} = 0$, then let i be the largest index with $m_{i,L} > 0$ and j the index of the last server in block B_{i-1} . Then there is a request $r_\ell \in B_i \cap L$ matched to s_j in the optimal matching, *i.e.*, $j = \pi(\ell)$, otherwise, by the maximality of i , there is a request $r_{\ell'} \leq p_i(< r_\ell)$ such that $j = \pi(\ell')$ and $j' < j$ such that for the first request $r_\ell \in B_i \cap L$ and $j' = \pi(\ell)$, but it is a contradiction by Proposition 1. Again, deleting r_ℓ and s_j from the model we are ready by the induction hypothesis.

Since algorithm LR is optimal on M and algorithm A is $c(n/k)$ -competitive on the blocks, we conclude that algorithm DIVIDE_k is $c(n/k)$ -competitive. \square

Corollary 6. *If $N = n^b$ for some constant b , then Algorithm DIVIDE_k is $c(n/k)$ -competitive and reads $O(k \log n)$ bits of advice.*

Now we assume that the positions of requests and servers may be arbitrary real numbers. Algorithm RESCALE works in the following way:

Fix $k \in \mathbb{Z}^+$. Set $s'_i = n^3(s_i - s_1) + 1$, $r'_i = \lfloor n^3(r_i - s_1) \rfloor + 1$ and $N = \lceil s'_n - s'_1 \rceil - 2$. Execute algorithm DIVIDE_k on this modified input, and match r_i to s_j when DIVIDE_k matches r'_i to s'_j .

Theorem 7. *Algorithm RESCALE is $c(n/k)$ -competitive, and it reads $O(k(\log(s_n - s_1) + \log n))$ bits of advice.*

Proof: The number of bits read is $O(k(\log N + \log n)) = O(k(\log(s_n - s_1) + \log n))$ by the choice of N .

For bounding the optimal costs, observe that

$$\text{OPT}(I') \geq n^3 \cdot \text{OPT}(I) - 1/n^2,$$

where I is the original, I' is the modified input. Moreover

$$\text{RESCALE}(I') \leq n^3 \cdot \text{DIVIDE}_k(I) + 1/n^2,$$

therefore, by Theorem 5 we have

$$\text{RESCALE}(I') \leq n^3 \cdot c(n/k) \cdot \text{OPT}_k(I) + \frac{1}{n^2} \leq c(n/k) \cdot \text{OPT}_k(I') + \frac{c(n/k) + 1}{n^2},$$

where $\frac{c(n/k)}{n^2} = o(1)$, since we can assume that $c(n) = o(n^2)$. \square

Using the algorithm of Antoniadis et al. (2019) as algorithm A we get $O((n/k)^{\log_2(3+\epsilon)-1}/\epsilon)$ -competitive algorithms.

4 Conclusions and further questions

Note that A may be randomized. In this case we use the expected value of the cost of A instead of the cost of A in the definition of the competitive ratio. Applying the $O(\log n)$ -competitive randomized algorithm of Gupta and Lewi (2012) as algorithm A we get $O(\log(n/k))$ -competitive randomized algorithms in the previous section.

There is no known lower bound on the advice complexity of non-optimal online algorithms other than the result of Mikkelsen (2016), *e.g.* for constant competitiveness greater than 1, and it is not known whether a linear number of bits of advice is sufficient for non-constant competitiveness.

The following research question arises: Can one modify the methods of the present paper for other metric spaces, *e.g.* tree metric spaces?

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