

Uniquely hamiltonian graphs for many sets of degrees

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revisions 27th Feb. 2024, 6th Nov. 2024; accepted 7th Sep. 2024.

We give constructive proofs for the existence of uniquely hamiltonian graphs for various sets of degrees. We give constructions for all sets with minimum 2 (a trivial case added for completeness), all sets with minimum 3 that contain an even number (for sets without an even number it is known that no uniquely hamiltonian graphs exist), and all sets with minimum 4, except $\{4\}$, $\{4, 5\}$, and $\{4, 6\}$. For minimum degree 3 and 4, the constructions also give 3-connected graphs.

We also introduce the concept of seeds, which makes the above results possible and might be useful in the study of Sheehan's conjecture. Furthermore, we prove that 3-connected uniquely hamiltonian 4-regular graphs exist if and only if 2-connected uniquely hamiltonian 4-regular graphs exist.

Keywords: graph, hamiltonian cycle, degree

1 Introduction

The most important problem for hamiltonian cycles is of course which properties guarantee the existence of a hamiltonian cycle, but as soon as the existence of a hamiltonian cycle is known, the question arises how many hamiltonian cycles exist. In [4], recent results and an overview of older results on graphs with few hamiltonian cycles are given. The extremal case is when a graph contains a single hamiltonian cycle, that is: it is *uniquely hamiltonian*. A crucial role for the existence of a uniquely hamiltonian graph is played by the combination of vertex degrees present in the graph. Already in 1946 Tutte reported a result by Smith that uniquely hamiltonian cubic graphs don't exist [9]. A long standing conjecture by Sheehan [7] states that this should in fact be the case for all d -regular graphs with $d > 2$. The result by Smith was later improved by Thomason [8] showing that uniquely hamiltonian graphs where all vertices have odd degree don't exist. In [5] it is shown that no d -regular uniquely hamiltonian graphs exist if $d \geq 23$. So while there are e.g. neither uniquely hamiltonian graphs with all degrees 3 nor with all degrees 24, a special case of what we will prove will be that there are uniquely hamiltonian graphs if both these vertex degrees are allowed. For even d with $4 \leq d \leq 22$ it is not known whether d -regular uniquely hamiltonian graphs exist. In [3] Fleischner shows that there are uniquely hamiltonian graphs with minimum degree 4. He constructs graphs with vertices of degree 4 and 14 and graphs where *the maximum degree can grow even larger* – without specifying which degrees can occur. We will use an improved version of his method to prove that for all sets M with minimum 4, except maybe for $\{4\}$, $\{4, 5\}$, and $\{4, 6\}$, uniquely hamiltonian graphs exist, so that the set of vertex degrees is exactly M . Furthermore we characterize sets of degrees with minimum 2 or 3 for which uniquely hamiltonian graphs exist completely.

The term *graph* always refers to a simple undirected graph, that is: without multiple edges and without loops. If multiple edges are allowed, we use the term *multigraph*. Loops are never allowed, as they are trivial in the context of uniquely hamiltonian graphs.

We define the degree set $M_{deg}(G)$ of a graph (or multigraph) G with vertex set V as $M_{deg}(G) = \{\deg(v) \mid v \in V\}$.

For a set $M = \{d_0, d_1, d_2, \dots, d_k\}$ with $d_0 < d_1 < \dots < d_k$, we say that a 2-connected (if $2 \in M$), resp. 3-connected (otherwise) uniquely hamiltonian graph G realizes M if $M_{deg}(G) = M$. If such a G exists, we define M to be *uhc-realizable*.

Next to the question whether a set M is uhc-realizable, it is also interesting which role is played by the larger degrees. Our emphasis is on the smallest degree d_0 and we want to know whether the number of times that the degrees d_1, \dots, d_k occur can be bounded by a constant even for very large graphs, so that the average degree can be arbitrarily close to the smallest degree. On the other hand it might also be interesting to know, whether the larger degrees can occur an unbounded number of times and maybe also occur for at least a fixed fraction of the vertices also in arbitrarily large graphs. The average degree would in that case be bounded from below by the minimum degree times a constant factor $c > 1$. The strongest requirement is, if both can occur and even in combination depending on the d_i . We formalize that by the following definition:

For a set $M = \{d_0, d_1, d_2, \dots, d_k\}$ with $k > 0$, $d_0 < d_1 < \dots < d_k$ we say that M is *strongly uhc-realizable*, if for each partition D_1, D_2 of $\{d_1, \dots, d_k\}$ (with one of D_1, D_2 possibly empty) there are constants $c_1 \in \mathbb{N}$, $c_2 \in \mathbb{R}$, $c_2 > 0$, and an infinite sequence of graphs $G_i = (V_i, E_i)$ realizing M , so that for all $d \in D_1$ each G_i has at most c_1 vertices of degree d , and for each $d' \in D_2$ each G_i has at least $c_2|V_i|$ vertices of degree d' .

2 Minimum degree 2 or 3

We will start with an easy remark that is mainly contained for completeness:

Remark 2.1 Any finite set $M = \{d_0 = 2, d_1, d_2, \dots, d_k\} \subset \mathbb{N}$ with $2 < d_1 < d_2 < \dots < d_k$ is uhc-realizable and if $k > 0$, it is also strongly uhc-realizable.

Proof:

We will first prove that M is uhc-realizable. $|M| = 1$ is trivial. If $|M| = 2$, one can take K_{d_1+1} and subdivide the edges of a hamiltonian cycle. If $|M| > 2$ one can e.g. take complete graphs $K_{d_1+1}, \dots, K_{d_k+1}$, remove an edge $e_{d_i} \in K_{d_i+1}$ for $1 \leq i < k$, an edge $e'_{d_i} \in K_{d_i+1}$ for $2 \leq i \leq k$ with $e_{d_i} \cap e'_{d_i} = \emptyset$ for $2 \leq i < k$, and then connect the endpoints of e_{d_i} and $e'_{d_{i+1}}$ for $1 \leq i < k$. The result is obviously hamiltonian and 2-connected and after subdividing the edges of a hamiltonian cycle, one has a uniquely hamiltonian graph with exactly the vertex degrees in M .

To show that M is strongly uhc-realizable, assume a partition D_1, D_2 to be given. If $D_2 = \emptyset$ one can subdivide edges on the hamiltonian cycle arbitrarily often to obtain the sequence of graphs. If $D_2 \neq \emptyset$ one can use the above construction for multisets M'_j containing the same elements as M , but numbers in D_1 exactly once and numbers in D_2 exactly j times. □

3 Minimum degree 3 and 4

The following construction is a slight modification of a construction by H. Fleischner [3].

Let $P = (V, E)$ be a graph and $s, t, v \in V$ be vertices. If there is a unique hamiltonian path from s to t in the graph $P_{-v} = P[V \setminus \{v\}]$ induced by $V \setminus \{v\}$, we call $\mathcal{P} = (P, s, t, v)$ a *weak H-plugin* or just an H-plugin. If in addition there is no hamiltonian path from s to t in P (so also containing v), we call $\mathcal{P} = (P, s, t, v)$ a *strong H-plugin*.

In cases where s, t , and v are clear from the context, we will also refer to the graph P alone as an H-plugin.

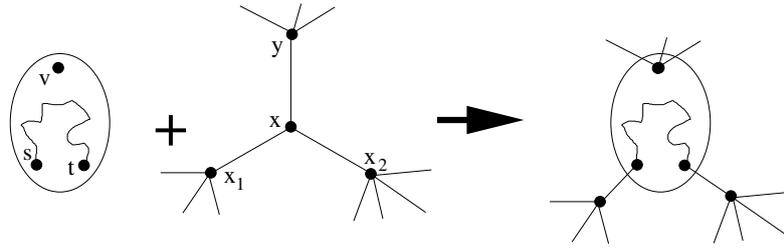


Fig. 1: The splicing operation.

For an H-plugin (P, s, t, v) and a graph G with a vertex x of degree 3 with neighbours y, x_1, x_2 , we define the P -splice of G at $\{x, y\}$, denoted as $O(x, y, \mathcal{P})$ as the graph obtained by removing x , connecting x_1 with the vertex s in a copy P' of P , x_2 with the vertex t in P' and identifying the vertex v in P' with y . This operation is sketched in Figure 1. We will also refer to it shortly as *splicing the edge* $\{x, y\}$. The notation $O(x, y, \mathcal{P})$ does not take into account which of the vertices is x_1 and which is x_2 , so in general $O(x, y, \mathcal{P})$ is one of the two possibilities. Elementary arguments show that if P – or at least P together with a new vertex connected to s, t , and v – as well as G are 3-connected, then $O(x, y, \mathcal{P})$ is 3-connected.

The following lemma and corollary are stronger versions of Lemmas 1,2, and 3 in [3].

Lemma 3.1 (parts already in [3]) *Let $G = (V, E)$ be a graph with a unique hamiltonian cycle C_H , $x \in V$ of degree 3 with neighbour y , so that the edge $\{x, y\}$ is not on C_H . Let $\mathcal{P} = (P, s, t, v)$ be an H-plugin.*

If at least one of the following three conditions is fulfilled, then $O(x, y, \mathcal{P})$ has a unique hamiltonian cycle $C_{H,O}$. Except for the edges incident with x , all edges of C_H are also contained in $C_{H,O}$.

- (i) $G[V \setminus \{y\}]$ is not hamiltonian.
- (ii) $\{x, y\}$ lies in a triangle.
- (iii) \mathcal{P} is a strong H-plugin.

Condition (iii) also explains the name *strong* H-plugin: while in general the splicing of edges that are not on the unique hamiltonian cycle only guarantees a unique hamiltonian cycle in the result if the edges satisfy some extra condition, this extra condition is not necessary if \mathcal{P} is strong.

Proof: As s and t have only one edge to the outside of (the copy of) P in $O(x, y, \mathcal{P})$, none of them can be incident only with edges of a hamiltonian cycle $C_{H,O}$ of $O(x, y, \mathcal{P})$ that lie outside P . To this end there are in principle three ways how $C_{H,O}$ could pass through P :

- a.) by a hamiltonian path of P_{-v} from s to t while the vertex $v = y$ is incident to two edges of $C_{H,O}$ not in P ,
- b.) by a hamiltonian path of P from v to s or to t ,
- c.) by a hamiltonian path of P from s to t .

In all three cases (i), (ii), and (iii) of the lemma, we can get a hamiltonian cycle of $O(x, y, \mathcal{P})$ passing P like described in a.) if we replace the part x_1, x, x_2 in C_H by x_1, s, \dots, t, x_2 with the middle part the unique hamiltonian path from s to t in P_{-v} . So there is always a hamiltonian cycle for case a.), but that cycle is unique due to the two paths in P_{-v} and outside P_{-v} being unique.

Assume now that $O(x, y, \mathcal{P})$ has a hamiltonian cycle passing P as in case b.) and assume w.l.o.g. that the endpoint is s . Replacing the part $y = v, \dots, s, x_1$ by y, x, x_1 , we get a hamiltonian cycle of G

containing $\{x, y\}$, which does not exist, as C_H is unique. So a hamiltonian cycle falling into case b.) does not exist.

It remains to be shown that also case c.) can not occur under the additional prerequisites.

(i)

Assume that $O(x, y, \mathcal{P})$ has a hamiltonian cycle passing P as in case c.). Replacing the part x_1, s, \dots, t, x_2 (now also containing $v = y$) by x_1, x, x_2 , we get a cycle in G missing only y – that is: a hamiltonian cycle of $G[V \setminus \{y\}]$, which does by assumption not exist.

(ii) This is a special case of (i). Assume that $G[V \setminus \{y\}]$ contains a hamiltonian cycle C'_H . Then C'_H passes x by x_1, x, x_2 , but replacing this part by x_1, y, x, x_2 or x_1, x, y, x_2 – depending on whether the triangle is x_1, x, y or x_2, x, y – we get a hamiltonian cycle of G containing e , which does not exist, as C_H is unique.

(iii) In this case the prerequisites are exactly that a path as in c.) does not exist. □

Corollary 3.2 (parts already in [3]) *Let $G = (V, E)$ be a graph with a unique hamiltonian path P_H from $s \in V$ to $t \in V$. Assume $x \in V$, $x \notin \{s, t\}$ is of degree 3 with neighbour y , so that the edge $\{x, y\}$ is not on P_H . Let $\mathcal{P} = (P, s', t', v)$ be an H -plugin.*

If at least one of the following four conditions is fulfilled, then $O(x, y, \mathcal{P})$ has a unique hamiltonian path $P_{H,O}$ from s to t . Except for the edges incident with x , all edges of P_H are also contained in $P_{H,O}$.

(i) $y \notin \{s, t\}$, and $G[V \setminus \{y\}]$ has no hamiltonian path from s to t .

(ii) $\{x, y\}$ lies in a triangle.

(iii) \mathcal{P} is a strong H -plugin.

(iv) $y \in \{s, t\}$.

Proof:

Adding a new vertex to G and connecting it with s and t , the resulting graph G' has a unique hamiltonian cycle if and only if G has a unique hamiltonian path from s to t . Applying Lemma 3.1 to G' we get the results. Case (iv) follows by case (i) of Lemma 3.1. □

We can now prove the main theorem for minimum degree 3:

Theorem 3.3 *A finite set $M = \{d_0 = 3, d_1, d_2, \dots, d_k\}$ with $3 < d_1 < d_2 < \dots < d_k$ of natural numbers is uhc-realizable if and only if M contains an even number. In that case it is also strongly uhc-realizable.*

Proof:

The fact that there is no uniquely hamiltonian graph G with $M_{deg}(G) = M$ if M contains no even number, is a well known result of Thomason [8] – no matter what the condition on connectivity is. To show that M is uhc-realizable if M contains an even number, we will explicitly construct a 3-connected uniquely hamiltonian graph G with $M_{deg}(G) = M$ in that case.

Figure 2 shows one of the five smallest uniquely hamiltonian graphs G with $M_{deg}(G) = \{3, 4\}$ (see [4]). By using the strong plugin given in Figure 3 to an edge not on the hamiltonian cycle and incident to a vertex of degree 4, we can increase the degree of that vertex by 2. Doing that recursively, we can increase the degree of that vertex to any even degree. Applying the plugin to an edge incident with two vertices of degree 3, we can increase the degree of one of them to 5 and recursively to any odd degree. As the number of vertices of degree 3 can be increased by replacing a vertex by a triangle – and

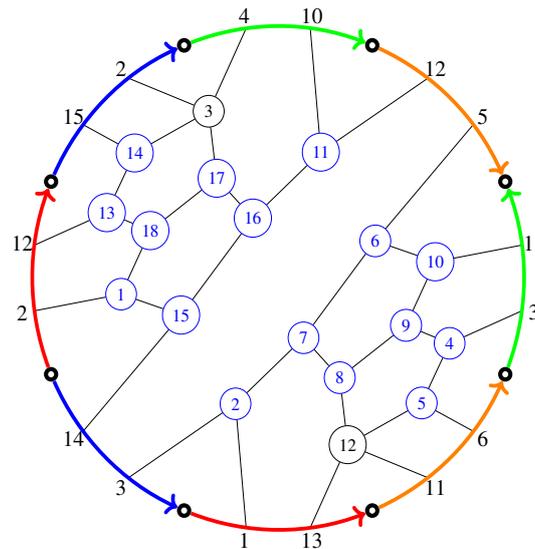


Fig. 2: The graph $U_{3,4}$ drawn as a minimum genus embedding. Sides with the same colour have to be identified. This is one of the five smallest uniquely hamiltonian graphs with only degrees 3 and 4 as given in [4]. The unique hamiltonian cycle is $1, 2, \dots, 18$. The vertices 3 and 12 are the only vertices of degree 4.

keeping the graph uniquely hamiltonian – we can conclude that there are infinitely many (3-connected) uniquely hamiltonian graphs G_M for any degree set $M = \{3, d_1, d_2, \dots, d_k\}$ containing one or two even degrees. If we take two graphs realizing degree sets M, M' , remove one vertex of degree 3 in each of them and connect the neighbours in a way that the parts of the unique hamiltonian cycles are connected to each other, we get a graph $G_{M \cup M'}$ realizing the degree set $M \cup M'$. This way we get that for each $M = \{3, d_1, d_2, \dots, d_k\}$ with at least one even element there are infinitely many uniquely hamiltonian graphs G_M realizing it.

Assume now that for a degree set $M = \{3, d_1, d_2, \dots, d_k\}$ containing an even degree a partition D_1, D_2 of $\{d_1, d_2, \dots, d_k\}$ is given. There is a uniquely hamiltonian graph G_M realizing M . If D_2 is empty, we can recursively replace vertices of degree 3 by triangles to get an infinite sequence of uniquely hamiltonian graphs realizing M and having the same number of vertices of degree $d \in D_1$. If D_2 contains an even degree, we can make arbitrarily many copies of a graph realizing $D_2 \cup \{3\}$ and recursively combine them in the way described above with G_M . The result has a constant number of vertices with degree in D_1 and at least a constant fraction of vertices with degree in D_2 . If finally D_2 does not contain a vertex of even degree, we can recursively replace vertices of degree 3 in G_M by triangles, so that for each $k \in \mathbb{N}$ and each $d \in D_2$ we can use the plugin to make k vertices with degree d . As all graphs constructed in this proof are 3-connected, this final construction proves that M is strongly uhc-realizable. \square

The repeated application of $P_{3,+2}$ does not give smallest possible graphs with this degree sequence – in fact not even smallest graphs constructed by using plugins. There is e.g. a plugin on 15 vertices increasing the degree of the identified vertex by 4 and increasing the number of vertices by 13 instead of 16 when applying $P_{3,+2}$ twice.

For minimum degree 4, it is unfortunately not so easy to give a strong plugin, but we have to construct it, starting from weak plugins.

We do not only want to splice one edge in a graph G , but each edge in some set of edges. This is in

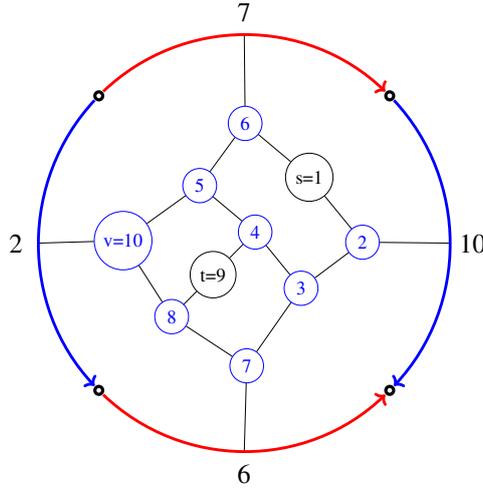


Fig. 3: The Petersen graph with one edge removed gives a strong plugin $P_{3,+2}$ for $s = 1$, $t = 9$ and $v = 10$. It can be easily checked by hand that $1, 2, \dots, 9$ is the unique hamiltonian path from s to t if v is removed and of course there is no hamiltonian path from s to t without removing v as it would imply a hamiltonian cycle in the Petersen graph. When used as a plugin, the degrees in the copy of $P_{3,+2}$ are 3 and $d + 2$ if the vertex identified with v has degree d .

general not possible, if the edges only satisfy condition (i) of Lemma 3.1 or Corollary 3.2 for G : if z is a vertex, so that $G[V \setminus \{z\}]$ has no hamiltonian cycle or hamiltonian path between two vertices a, b , it is possible that after splicing an edge $\{x, y\}$ not even close to z , the result $O(x, y, \mathcal{P})$ has a hamiltonian path or cycle in the graph with z removed. If on the other hand we have a set E_O of candidate edges $\{x_1, y_1\}, \dots, \{x_k, y_k\}$ to be spliced with different x_i in different triangles, or the y_i are one of the starting points s, t of the unique hamiltonian path, these properties are preserved after splicing an edge in E_O . This implies that in that situation we can apply the splicing operation also with a weak H-plugin to all edges simultaneously or in any order and still draw the conclusions of Lemma 3.1 or Corollary 3.2.

Let $G = (V, E)$ be a graph with $s, t \in V$, $v \notin V$ and a unique hamiltonian path from s to t . For a set $V' \subseteq V$ we define $W_{V'}(G)$ as the graph obtained from G by adding the vertex v and connecting it to all vertices in V' – or formally: $W_{V'}(G) = (V_W, E_W)$ with $V_W = V \cup \{v\}$, $E_W = E \cup \{\{v, w\} | w \in V'\}$. For a set $\{4, d_1, \dots, d_k\}$ with $4 < d_1 < d_2 < \dots < d_k$ we call $G = (V, E)$ a $\{4, d_1, \dots, d_k\}$ -seed, if there is a set $V' \subseteq V$, so that if $W_{V'}(G)$ is used for splicing an edge with both endpoints of degree 3 in a 3-connected graph, the result is 3-connected and the set of degrees that occur in the copy of $W_{V'}(G)$ is exactly $\{4, d_1, \dots, d_k\}$.

Remark 3.4 Let $G = (V, E)$ be a graph with a unique hamiltonian path from $s \in V$ to $t \in V$, $V' \subseteq V$ and $v \notin V$. Then we have:

- (i) $W_{V'}(G)$ is an H-plugin.
- (ii) If $V' = \{x, y\}$ and x, y are the endpoints of an edge not on the unique hamiltonian path from s to t , then $W_{\{x, y\}}(G)$ is a strong H-plugin.

This remark follows immediately from the definitions of H-plugin and strong H-plugin and the fact that a hamiltonian path from s to t containing v would imply a hamiltonian path in G containing the edge $\{x, y\}$.

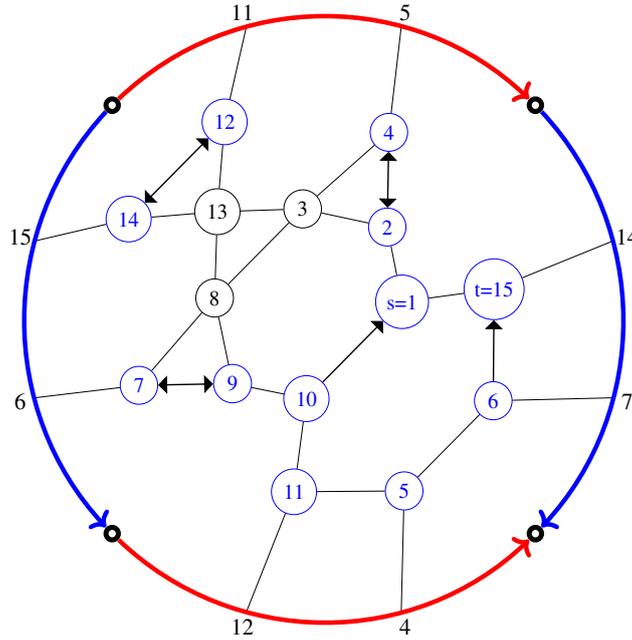


Fig. 4: The graph P^- from [3], which has two hamiltonian cycles: $1, 2, 3, \dots, 15$ and $1, 2, 3, 4, 5, 11, 12, 13, 14, 15, 6, 7, 8, 9, 10$. As only one of them contains the edge $\{1, 15\}$ it has a unique hamiltonian path $1, 2, \dots, 15$ from $s = 1$ to $t = 15$. Edges with both endpoints of degree 3 to which the splicing operation with a weak H-plugin can be applied while the uniqueness of the hamiltonian path is preserved, are drawn as arrows pointing at the vertices which can or must be chosen as y .

If $G = (V, E)$ is an M -seed for some set M and $V' \subseteq V$ is a set proving this, then the plugin $W_{V'}(G)$ is also called an M -plugin.

We will use seeds to construct weak H-plugins, use those to construct strong H-plugins, and the strong H-plugins to construct uniquely hamiltonian graphs with certain sets of degrees.

We will first use the splicing operation to show how weak H-plugins imply the existence of certain strong H-plugins:

Lemma 3.5 *If for a set $M = \{4, d_1, \dots, d_k\}$ with $4 < d_1 < \dots < d_k$ there is an M -seed S , then there is a strong H-plugin \mathcal{P}_M^{str} , so that when \mathcal{P}_M^{str} is used for splicing an edge with both endpoints of degree 3, the set of vertex degrees of the vertices in the copy of \mathcal{P}_M^{str} is exactly M .*

If there are infinitely many M -seeds, each with for $1 \leq i \leq k$ exactly C_i vertices with degree d_i when used for splicing an edge with both endpoints of degree 3, then there are infinitely many strong M -plugins \mathcal{P}_M^{str} , each with $5C_i$ vertices with degree d_i after splicing.

Proof:

Let \mathcal{P}_M be the (weak) M -plugin constructed from S as described in Remark 3.4 and assume that for $1 \leq i \leq k$ exactly C_i vertices in \mathcal{P}_M have degree d_i when it is used for splicing an edge.

Figure 4 shows the graph P^- with a unique hamiltonian path $1, 2, \dots, 15$ from $s = 1$ to $t = 15$ (given in [3]). Edges with both endpoints of degree 3 to which the splicing operation with a weak H-plugin can be applied in a way that there is still a unique hamiltonian path between s and t are drawn as arrows pointing at the vertices which can or must be chosen as the vertex y in the operation. If we splice these edges with \mathcal{P}_M , we get a graph with $5C_1, \dots, 5C_k$ vertices with degrees d_1, \dots, d_k , 2 vertices (the vertices 5 and

11) with degree 3, and all other vertices with degree 4. Due to Corollary 3.2, this graph still has a unique hamiltonian path from s to t not containing the edge $\{5, 11\}$. If we remove the edge between s and t , add a new vertex v , and connect it to the vertices 5 and 11, due to Remark 3.4 we get a strong H-plugin \mathcal{P}_d^{str} . Each of the vertices s and t now has a degree $d' - 1$ with $d' \in M$, so when applied in a splicing operation the degree is again d' . Before splicing, v has degree 2, so splicing an edge with both endpoints of degree 3 it gets degree 4. All other vertices have a degree in M . If we apply the \mathcal{P}_d^{str} -splice to an edge with both endpoints of degree 3, one of them is deleted and the other one is identified with v and gets degree 4. If there are infinitely many H-plugins \mathcal{P}_M , each with C_1, \dots, C_k vertices with degrees d_1, \dots, d_k , we get infinitely many strong H-plugins \mathcal{P}_M^{str} with $5C_1, \dots, 5C_k$ vertices with degrees $4 < d_1 < \dots < d_k$. \square

Lemma 3.6 *For each $k \in \mathbb{N}$ there are 3-connected uniquely hamiltonian graphs $G_k = (V_k, E_k)$ with $M_{deg}(G_k) = \{3, 4\}$, so that the edges not on the hamiltonian cycle form a 2-regular subgraph containing all vertices of degree 4 together with a matching of size at least k containing all vertices of degree 3.*

Proof:

We can apply a well known technique from [6] to obtain a uniquely hamiltonian graph from a graph with two hamiltonian cycles that contains a cubic vertex that is passed by the two hamiltonian cycles in different ways. We take two copies of P^- and in each of them an arbitrary cubic vertex that is traversed by the two hamiltonian cycles in two different ways. Say these vertices are v and v' , that the neighbours are a, b, c , resp. a', b', c' and that the hamiltonian cycles pass v as a, v, b and a, v, c (and accordingly for v'). Removing v and v' and adding the edges $\{a, c'\}, \{b, b'\}, \{c, a'\}$, only one hamiltonian cycle remains – using the paths a to c in one copy and c' to a' in the other.

As in both hamiltonian cycles the vertices of degree 4 are traversed in a way so that the edges not on the hamiltonian cycle and incident with the 4-valent vertices form a triangle, the result will in each case have a unique hamiltonian cycle with two triangles of edges not on the hamiltonian cycle containing all 6 vertices of degree 4. As each cubic vertex has exactly one edge not on the hamiltonian cycle, these edges form the required matching. Starting from this graph, we can replace vertices of degree 3 by triangles to increase the number of cubic vertices and therefore also the size of the matching until we have a matching of size at least k . \square

We get the following theorem as an immediate consequence:

Theorem 3.7 *Let $M \subset \{4, 5, 6, \dots\}$ with $4 \in M$ be a set, so that there are sets M_1, M_2, \dots, M_k with $\bigcup_{i=1}^k M_i = M$ and for $1 \leq i \leq k$ there is an M_i -seed S_i . Then M is uhc-realizable.*

If $|M| > 1$, $|M_i| = 2$ for $1 \leq i \leq k$, and for each i there are infinitely many M_i -seeds with the same number of vertices with degree different from 4, then M is also strongly uhc-realizable.

Proof:

Given the set M , we can take any uniquely hamiltonian graph $G_{k'}$ from Lemma 3.6 with $k' \geq k$, $k' > 0$ and splice the edges of the matching using each of the strong H-plugins $\mathcal{P}_{M_1}^{str}, \dots, \mathcal{P}_{M_k}^{str}$ at least once. This removes all vertices of degree 3 or increases their degree to 4. Furthermore outside the H-plugins only degree 4 occurs and in the H-plugins exactly all vertex degrees in M occur, while the graph has still one unique hamiltonian cycle.

To show that M is strongly uhc-realizable for $k \geq 1$, assume a partition D_1, D_2 of $M \setminus \{4\}$ to be given. If $D_2 = \emptyset$, to construct the sequence of graphs we can use increasingly large strong H-plugins – keeping the numbers of vertices of degree d constant for $d \in \{d_1, \dots, d_k\}$. If $D_2 \neq \emptyset$, we can use graphs $G_{k'}$ for increasingly large k' and use the same arbitrarily large number of copies of strong H-plugins \mathcal{P}_d^{str} for each $d \in D_2$. \square

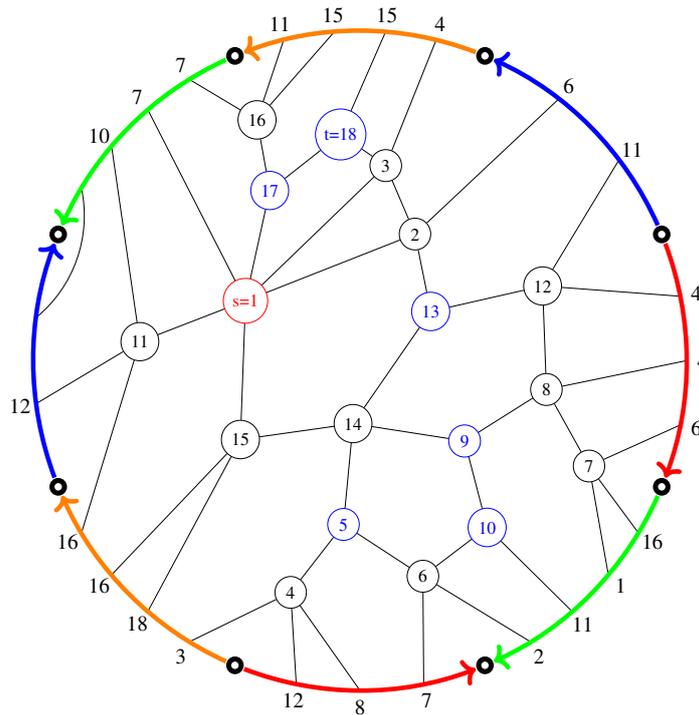


Fig. 5: A $\{4, 7\}$ -seed G with the unique hamiltonian path $1, 2, 3, \dots, 18$ from $s = 1$ to $t = 18$. The set V' that fulfills the requirements of the definition is the set containing all vertices of degree 3, except for vertex $t = 18$, so $V' = \{5, 9, 10, 13, 17\}$. The uniqueness of the hamiltonian path as well as the fact that $G -$ so also the result of $W_{V'}(G)$ used for splicing in a 3-connected graph – is 3-connected, have been checked by computer, but as the graph is relatively small, these properties can – though tedious – still be checked by hand.

Remark 3.8 For each $k \geq 7$ there are $\{4, k\}$ -seeds.

For each $k \geq 8$ there are infinitely many $\{4, k\}$ -seeds, so that the corresponding plugins after being used for splicing an edge with both endpoints of degree 3 contain two vertices of degree k and all other vertices have degree 4.

Proof: We start from the $\{4, 7\}$ -seed S_7 in Figure 5. We use the triangle $1, 2, 3$ for constructing $\{4, k\}$ -seeds for $k \geq 8$ as in Figure 6: new vertices t_1, \dots, t_{k-7} are inserted, the edge $\{2, 3\}$ is replaced by the path $2, t_1, \dots, t_{k-7}, 3$, and edges $\{1, t_1\}, \dots, \{1, t_k\}$ are added. Each hamiltonian path from $s = 1$ to $t = 18$ that is not $1, 2, t_1, \dots, t_{k-7}, 3, \dots, 18$ could be transformed to a hamiltonian path contradicting the uniqueness of the hamiltonian path in S_7 . Also the connectivity requirements can be easily checked.

For $k \geq 8$ there is a vertex t_{k-7} and the number of vertices of degree 4 can be increased by steps of 1 always producing new $\{4, k\}$ -seeds for the same k . This procedure is described in Figure 7. Any hamiltonian path from 1 to 18 traversing the vertices in a different way than given in Figure 7 would imply a second hamiltonian path from 1 to 18 in S_7

□

The construction of the $\{4, k\}$ -seeds is exclusively to show that such seeds do exist and by no means meant to construct minimal ones. For $k > 7$ smaller $\{4, k\}$ -seeds are known – e.g. a $\{4, 10\}$ -seed with 10 vertices. This $\{4, 10\}$ -seed has only vertices of degree 2, 3, and 4 and the hamiltonian path goes from

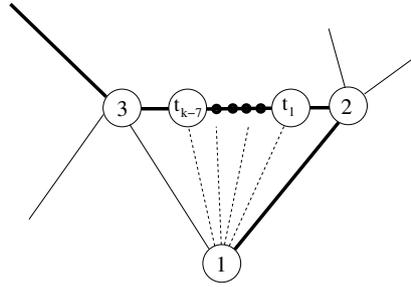


Fig. 6: Constructing $\{4, k\}$ -seeds for $k > 8$.

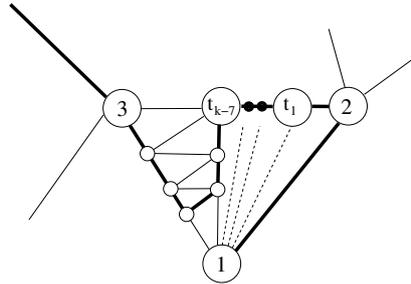


Fig. 7: Extending a $\{4, k\}$ -seed for $k \geq 8$ constructed from the $\{4, 7\}$ -seed in Figure 5 by increasing the number of vertices of degree 4. The number of vertices of degree 4 can be increased by steps of 1 vertex.

a vertex of degree 2 to a vertex of degree 3 – see Figure 11.

Unfortunately in spite of an extensive computer search, no $\{4\}$ -, $\{4, 5\}$ -, or $\{4, 6\}$ -seeds were found. See Section 4 for details.

Remark 3.9 For each $k \geq 7$ there are $\{4, 6, k\}$ -seeds.

For each $k \geq 8$ there are infinitely many $\{4, 6, k\}$ -seeds, so that the corresponding plugins after being used for splicing an edge with both endpoints of degree 3 contain one vertex of degree 6, 2 vertices of degree k , and all other vertices have degree 4.

Proof:

In Figure 8 a $\{4, 6, 7\}$ -seed is given that contains a triangle 1, 2, 3 and the unique hamiltonian cycle from 1 to 16 contains the edges $\{1, 2\}$ and $\{2, 3\}$. Except for vertex 1 none of the vertices has degree 7 after splicing an edge, so the seed can be extended in the same way as in the proof of Remark 3.8 to seeds for larger k and for $k \geq 8$ also to the infinite sequence. □

Remark 3.10 For each $k \geq 6$ there are $\{4, 5, k\}$ -seeds.

For each $k \geq 7$ there are infinitely many $\{4, 5, k\}$ -seeds, so that the corresponding plugins after being used for splicing an edge with both endpoints of degree 3 contain one vertex of degree 5, 2 vertices of degree k , and all other vertices have degree 4.

Proof:

In Figure 9 a $\{4, 5, 6\}$ -seed is given that contains a triangle 1, 2, 3 and the unique hamiltonian cycle from 1 to 16 contains the edges $\{1, 2\}$ and $\{2, 3\}$. Except for vertex 1 none of the vertices has degree 6 after splicing an edge, so the seed can be extended in the same way as in the proof of Remark 3.8 to seeds for larger k and for $k \geq 7$ also to the infinite sequence.

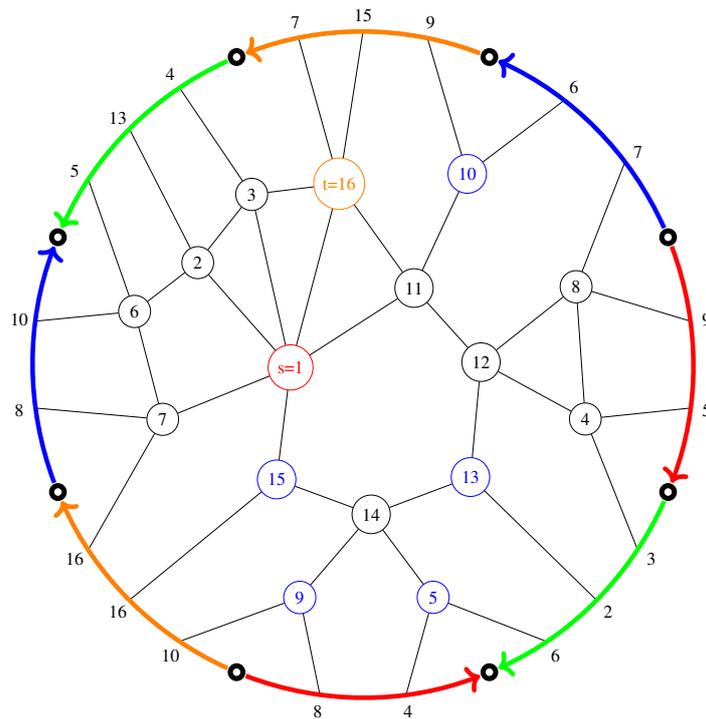


Fig. 8: A $\{4, 6, 7\}$ -seed G with the unique hamiltonian path $1, 2, 3, \dots, 16$ from $s = 1$ to $t = 16$. The set V' that fulfills the requirements of the definition is the set containing all vertices of degree 3.

□

Theorem 3.7 and Remarks 3.8, 3.9, and 3.10 now immediately imply the main result for minimum degree 4:

Theorem 3.11 • Except for maybe $\{4\}$, $\{4, 5\}$, and $\{4, 6\}$, any set $M = \{4, d_1, d_2, \dots, d_k\}$ with $4 \leq d_1 < d_2 < \dots < d_k$ is *uhc-realizable*.

• Any set $M = \{4, d_1, d_2, \dots, d_k\}$ with $8 \leq d_1 < d_2 < \dots < d_k$ and $k \geq 1$ is *strongly uhc-realizable*.

Due to Theorem 3.7 the existence of a 4-seed implies the existence of a 3-connected uniquely hamiltonian 4-regular graph, but in fact also the other direction is correct:

Corollary 3.12 *There is a 3-connected uniquely hamiltonian 4-regular graph, if and only if there is a $\{4\}$ -seed. In that case there are infinitely many 3-connected uniquely hamiltonian 4-regular graphs and every set M of natural numbers $d \geq 2$ with $4 \in M$ and $|M| \geq 2$ is strongly uhc-realizable.*

Proof: From a 3-connected uniquely hamiltonian 4-regular graph G we can get a $\{4\}$ -seed with $\deg(s) = 3$ and $\deg(t) = 2$ by choosing a vertex of G as s , subdivide an edge $\{s, x\}$ on the hamiltonian cycle incident with s with a new vertex t , and remove an edge $\{s, y\}$ that is not on the hamiltonian cycle. The set $\{y, t\}$ shows that it is a $\{4\}$ -seed. The 3-connectivity after using plugins constructed from it follows by standard arguments from Menger's theorem. A way to construct seeds with more vertices of degree

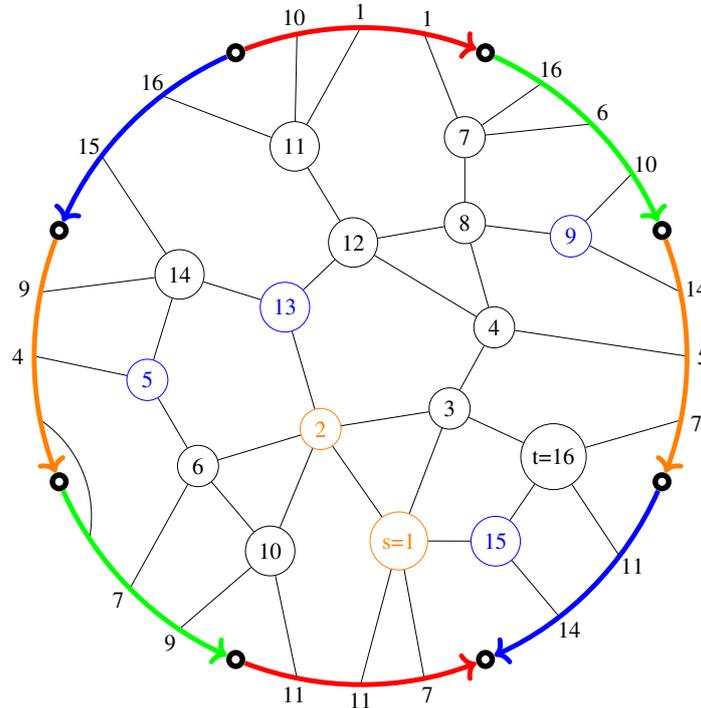


Fig. 9: A $\{4, 5, 6\}$ -seed G with the unique hamiltonian path $1, 2, 3, \dots, 16$ from $s = 1$ to $t = 16$. The set V' that fulfills the requirements of the definition is the set containing all vertices of degree 3.

4 and $\{4, k\}$ -seeds for $k > 4$ is given in Figure 10. The rest of the statement is a direct consequence of Remark 2.1, Theorem 3.3, and Theorem 3.7. \square

Furthermore, for 4-regular graphs, the existence of a 2-connected uniquely hamiltonian graph also implies the existence of a 3-connected uniquely hamiltonian graph:

Lemma 3.13 *There is a 3-connected uniquely hamiltonian 4-regular graph, if and only if there is a 2-connected uniquely hamiltonian 4-regular graph.*

Proof:

As 3-connected graphs are also 2-connected, the only thing to prove is that the existence of a uniquely hamiltonian 4-regular graph with a 2-cut implies the existence of a 3-connected uniquely hamiltonian 4-regular graph.

Let $G = (V, E)$ be a uniquely hamiltonian 4-regular graph with a 2-cut and $\{s, t\}$ be vertices of a 2-cut, so that one of the components of $G[V \setminus \{s, t\}]$ – say C_0 – has minimum size. Let $G_0 = G[C_0 \cup \{s, t\}]$. Then there is a unique hamiltonian path in G_0 from s to t and due to the minimality of C_0 the vertices s and t have degree at least 2 in G_0 . If one has degree 2, they are non-adjacent. As the number of vertices with odd degree must be even and as they both have neighbours in more than one component, they both have degree 2 or both have degree 3. In case of degree 2 we can add the edge $\{s, t\}$, so that in each case we have a graph, which we will call again G_0 with a unique hamiltonian path P_H from s to t , where s and t are of degree 3 and all other vertices of degree 4. Let now G_0^v be G_0 with an edge $e \neq \{s, t\}$ that is not part of P_H subdivided with a new vertex v . By construction G_0^v does not have a hamiltonian path from s to t , but a unique hamiltonian path in $(G_0^v)_{-v} = G_0$. So G_0^v is a strong H-plugin that when applied to two connected copies of P^- like in Lemma 3.6 gives a 4-regular uniquely hamiltonian graph.

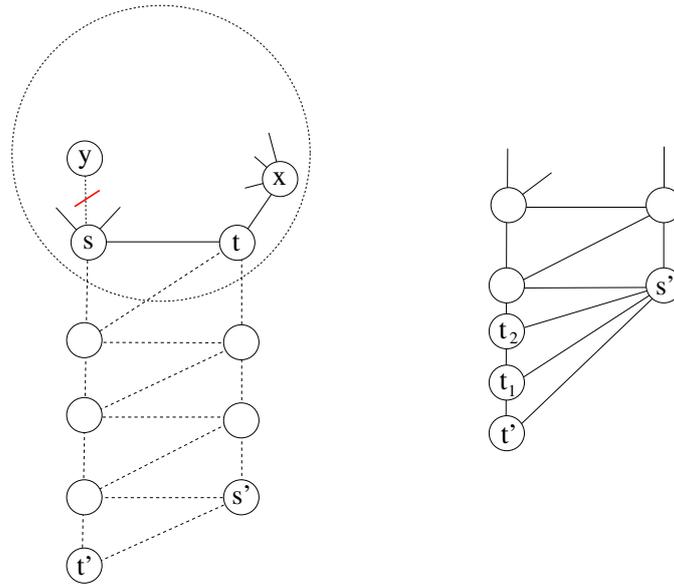


Fig. 10: Extending a $\{4\}$ -seed with more vertices of degree 4 and making $\{4, k\}$ -seeds from it, depicted for the example $k = 6$, where the set V' from the definition of seeds would be $\{y, t', t_1, t_2\}$.

It remains to be shown that for a 3-connected graph G' and suitable $x, y \in G'$ the graph $O(x, y, G_0^v)$ is 3-connected. It is sufficient to show that the graph G_1 obtained from G_0^v by adding a new vertex v' and connecting it to s, t , and v is 3-connected.

Assume to the contrary that G_1 has a 2-cut K . Note that $K \neq \{s, t\}$ as C_0 is a component and v and through v also v' are connected to it. If s and t are in different components of $G_1 \setminus K$, then the common neighbour v' must be in K . So $K \setminus \{v'\}$ is a 1-cut of G_0^v . If $K = \{v, v'\}$, choose w as a neighbour of v different from s, t , otherwise let w be the vertex in $K \setminus \{v'\}$. Then w is a cutvertex of G_0^v and also of G_0 . Together with s or t it forms a 2-cut contradicting the minimality of C_0 .

If s and t are in the same component of $G_1 \setminus K$ or one is in K , there is a vertex $x \notin \{v, v'\}$ in a component not containing s or t . But then K – possibly after replacing v or v' in K by a neighbour – again contradicts the minimality of C_0 , so G_1 does not have a 2-cut. □

In [2] Fleischner proved that there are 4-regular uniquely hamiltonian multigraphs and in fact $2k$ -regular uniquely hamiltonian multigraphs with arbitrarily high degree. Another direct consequence of Lemma 3.6 is the following simple generalisation:

Corollary 3.14 *For a set $M = \{d_1, \dots, d_k\}$ with $2 \leq d_1 < d_2 < \dots < d_k$ of natural numbers there is a uniquely hamiltonian multigraph G with $M_{deg}(G) = M$ if and only if M contains an even number. In that case there are infinitely many 3-connected uniquely hamiltonian multigraphs G with $M_{deg}(G) = M$.*

Proof:

In [8] it is shown that uniquely hamiltonian multigraphs do not exist if all degrees are odd, so we only have to prove that they do exist if an even degree is contained.

For $2 \in M$ this is even proven for simple graphs in Remark 2.1, so assume that all elements of M are at least 3. Taking graphs $G_{k'}$ with $k' \geq k$ from Lemma 3.6 with the matching and 2-factor with the described properties, we can multiply the edges of the 2-factor containing the 4-regular vertices until the vertices all have an even degree contained in M . For each remaining degree d_i , we can now choose an

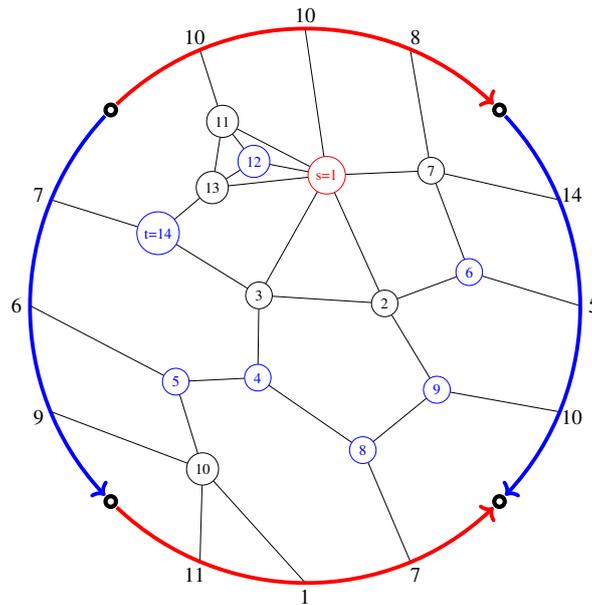


Fig. 12: A graph with a unique hamiltonian path from $s = 1$ to $t = 14$. The set $V' = \{4, 5, 6, 8, 9, 12\}$ shows that it is a $\{4, 8\}$ -seed and the set $V' = \{1, 4, 5, 6, 8, 9, 12\}$ shows that it is a $\{4, 9\}$ -seed. There are no smaller $\{4, 8\}$ - or $\{4, 9\}$ -seeds.

To be exact: for $k \in \{6, 7\}$ we did a complete search only up to 15 vertices. No $\{4, 6\}$ - or $\{4, 7\}$ -seeds exist for these vertex numbers. For larger vertex numbers we focused on seeds without interior vertices with large degree. The smallest such 7-seeds have 18 vertices – an example is given in Figure 5. For 6-seeds the existence of seeds without internal vertices of large degree was only checked up to 17 vertices. No such 6-seeds were found. For 18, 19 and 20 vertices we restricted the search to 3 cases: for no internal vertices with degree larger than 4 the cases $\{\deg(s), \deg(t)\} = \{5, 5\}$ and $\{\deg(s), \deg(t)\} = \{3, 5\}$ were checked. For one internal vertex with degree 5, the case $\{\deg(s), \deg(t)\} = \{3, 3\}$ with 5 vertices of degree 3, one (internal) vertex of degree 5 and the rest of degree 4 was checked. No such seeds were found and the total CPU time needed was about 18 years on an AMD EPYC 7532.

Even for carefully designed and implemented algorithms independent tests are necessary. As runs without any output are not very good tests for the programs, the two programs were also compared when generating 10-seeds and 12-seeds with $\{\deg(s), \deg(t)\} = \{2, 3\}$ and no internal vertices with degree larger than 4. The output of the two programs was compared for their number and for isomorphism up to 20 vertices. For 10-seeds there were in total 4.689 non-isomorphic seeds and for 12-seeds there were in total 1.414.640 non-isomorphic seeds. In addition k -seeds with $\{\deg(s), \deg(t)\} = \{3, k - 1\}$, no internal vertices of degree larger than 4, and $6 \leq k \leq 10$ on up to 16 vertices (in total 4.907 seeds) were compared. For seeds, *isomorphism* means that the two endpoints of the hamiltonian path are marked vertices and are distinguished from the other vertices, so some seeds that are non-isomorphic as seeds can be isomorphic as graphs. There was complete agreement. The program used for the larger runs can be obtained from the authors.

5 Final remarks

In this article we are interested only in uniquely hamiltonian graphs. Nevertheless the method of splicing can also be useful when constructing graphs with few hamiltonian cycles. We will only give a short sketch

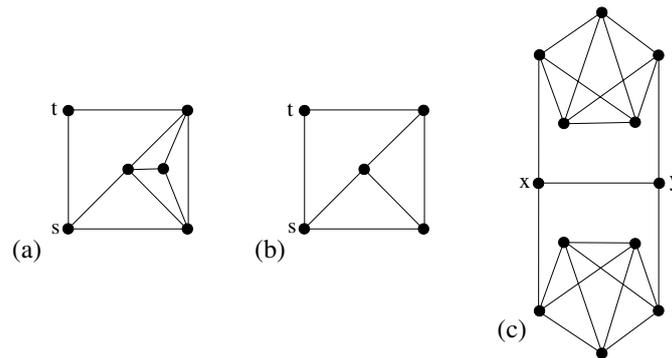


Fig. 13: The splicing operation for more than one hamiltonian cycle with a *generalized 4-seed* and a *generalized 6-seed*.

of the possibilities. We will not formally state results, as we do not give formal proofs. The following statements should be considered as preliminary as long as no proofs are given somewhere.

If we allow n_s hamiltonian paths from s to t in a seed and n_G hamiltonian cycles in a graph G – none of them containing the edge e of G – then with the otherwise same prerequisites of Lemma 3.1, the proof can be repeated, this time showing that the result after splicing has $n_s \cdot n_G$ hamiltonian cycles. This implies that for any set M of natural numbers with minimum 4 there is a constant C and an infinite series of graphs with degree set M and at most C hamiltonian cycles. In fact there is also one constant working as an upper bound for all sets M . The constants we get from our proof that used P^- are nevertheless very large and far worse for the 4-regular case than in [10]. For better constants one has to search for starting graphs that need fewer splicing operations, but can have more than one hamiltonian cycle. An example is the construction in [10] proving that there are infinitely many (2-connected) 4-regular graphs with 144 hamiltonian cycles. It was found and proven in a completely different way, but can be interpreted making use of splicing:

The graph in Figure 13(c) has 36 hamiltonian cycles – none of them containing $\{x, y\}$. Furthermore removing y , the graph is non-hamiltonian. The *generalized 4-seed* (that is: allowing more than one hamiltonian path from s to t) in Figure 13(a) has 4 hamiltonian paths from s to t , so with plugins obtained from it and its extensions, the results of splicing $\{x, y\}$ have 144 hamiltonian cycles. The generalized seed in Figure 13(b) has 2 hamiltonian paths from s to t and would give one vertex of degree 6, so splicing $\{x, y\}$ would give 72 hamiltonian cycles for the degree set $M = \{4, 6\}$ and replacing a vertex of degree 3 by a triangle also for $M = \{4, 8\}$.

All graphs explicitly given in the previous sections can be inspected at and downloaded from the database *House of Graphs* [1]. They can be found by searching for the keyword `UHG_degree_sequence`.

All properties about small graphs stated here have been checked by computer, but can – though sometimes tedious – be confirmed by hand.

6 Acknowledgement

A large part of the computational resources (Stevin Supercomputer Infrastructure) and services used in this work were provided by the VSC (Flemish Supercomputer Center), funded by Ghent University, FWO and the Flemish Government – department EWI.

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