

A study of k -dipath colourings of oriented graphs

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We examine t -colourings of oriented graphs in which, for a fixed integer $k \geq 1$, vertices joined by a directed path of length at most k must be assigned different colours. A homomorphism model that extends the ideas of Sherk for the case $k = 2$ is described. Dichotomy theorems for the complexity of the problem of deciding, for fixed k and t , whether there exists such a t -colouring are proved.

Keywords: Directed Graph, Oriented Graph, Graph Colouring, Graph Homomorphism

1 Introduction

Recall that an *oriented graph* is a digraph obtained from a simple, undirected graph by giving each edge one of its two possible orientations. Recall, also, that if G and H are oriented graphs, then a *homomorphism of G to H* is a function ϕ from the vertices of G to the vertices of H such that $\phi(x)\phi(y) \in E(H)$ whenever $xy \in E(G)$. If G and H are oriented graphs such that there is a homomorphism ϕ of G to H , then we write $\phi : G \rightarrow H$, or $G \rightarrow H$ if the name of the function ϕ is not important.

Let k and t be positive integers, and let G be an oriented graph. Chen and Wang (2006) defined a *k -dipath t -colouring* of G to be an assignment of t colours to the vertices of G so that any two vertices joined by a directed path of length at most k are assigned different colours. A 1-dipath t -colouring of an oriented graph G is a t -colouring of the underlying undirected graph of G . See Figure 1 for an example of a 3-dipath 4-colouring of an oriented graph. The *k -dipath chromatic number of G* , denoted by $\chi_{k\text{-dip}}(G)$, is the smallest positive integer t such that there exists a k -dipath t -colouring of G . Chen and Wang (2006) showed that any orientation of a Halin graph has 2-dipath chromatic number at most 7, and there are infinitely many such graphs G with $\chi_{2\text{-dip}}(G) = 7$.

For a positive integer t , an *oriented t -colouring* of an oriented graph G is a homomorphism of G to some oriented graph on t vertices. Oriented colourings were first introduced by Courcelle (1994), and have been a topic of considerable interest in the literature since then; see the recent survey by Sopena (2015), and also work by Borodin et al. (1999), Dolama and Sopena (2006), Sopena and Wu (2013) for

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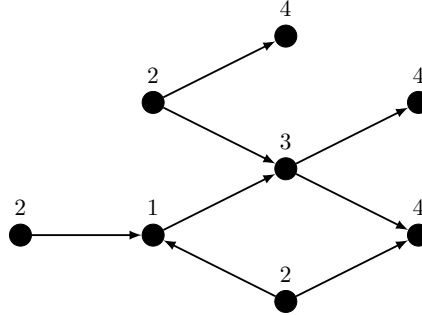


Fig. 1: A 3-dipath 4-colouring.

related topics. The 2-dipath chromatic number is of interest, in part, because it gives a lower bound for the *oriented chromatic number* $\chi_o(G)$ – the smallest positive integer t such that G admits an oriented t -colouring. Since oriented graphs have no directed cycles of length two, the definition implies that any two vertices of G joined by a directed path of length at most two are assigned different colours (*i.e.*, they have different images) in an oriented colouring of G . It follows from the definition that any oriented colouring of G is a 2-dipath colouring of G ; hence $\chi_{2\text{-dip}}(G) \leq \chi_o(G)$.

In her Master’s thesis Young (2009) gives a *homomorphism model* for 2-dipath t -colouring. For each positive integer t , she describes an oriented graph \mathcal{G}_t with the property that an oriented graph G has a 2-dipath t -colouring if and only if there is a homomorphism of G to \mathcal{G}_t . As is common with such theorems, it is possible to use the homomorphism to \mathcal{G}_t to find a 2-dipath t -colouring of G : the colour assigned to a vertex is determined by its image (but is not equal to it). The existence of this model implies an upper bound for the oriented chromatic number as a function of the 2-dipath chromatic number. It also leads to a proof that deciding whether a given oriented graph has a 2-dipath t -colouring is Polynomial if the fixed integer $t \leq 2$, and NP-complete if $t \geq 3$.

A natural question is whether Sherk’s results can be generalized to k -dipath t -colouring. We seek a model similar to hers, where the homomorphism to the target oriented graph can be used to find a k -dipath t -colouring of the given oriented graph. In particular, vertices of the given oriented graph G with the same image should be assigned the same colour. We suggest that such model will likely exist only for colouring oriented graphs with no directed cycles of length k or less. Consider the case of 3-dipath t -colouring, where $t \geq 3$. Suppose there exists a digraph $H_{3,t}$ with the property that an oriented graph G has a 3-dipath t -colouring if and only if there is a homomorphism of G to $H_{3,t}$. The digraph $H_{3,t}$ has no loops, otherwise all vertices of G can be assigned the same colour.

By definition, the directed 3-cycle has a 3-dipath 3-colouring: assign each vertex a different colour. Thus, there is a homomorphism of the directed 3-cycle to $H_{3,t}$. Consequently, $H_{3,t}$ has a directed 3-cycle. But, now there is a homomorphism of a directed path of length three to $H_{3,t}$ in which the two end vertices have the same image. Since the ends of a directed path of length 3 must be assigned different

colours in a 3-dipath t -colouring, this model will not have the desired property. Similar considerations apply to k -dipath t -colouring for all pairs of positive integers k and t with $t \geq k$. Hence, a homomorphism model of the type we seek will not exist if the oriented graphs being coloured can have directed cycles of length k or less. Finally, we note that Sherk's homomorphism model for 2-dipath t -colouring is for oriented graphs. These have no directed cycles of length two or less.

The main result of this paper is the construction of a homomorphism model for k -dipath t -colouring of oriented graphs with no directed cycles of length k or less. That is, for all positive integers k and t we describe an oriented graph $\mathcal{G}_{k,t}$ with the property that an oriented graph G , with no directed cycle of length at most k , has a k -dipath t -colouring if and only if G admits a homomorphism to $\mathcal{G}_{k,t}$ and, further, the homomorphism can be used to find a k -dipath t -colouring of G .

After presenting this result in Section 3, in Section 4 we determine the complexity of deciding the existence of a k -dipath t -colouring for all pairs of fixed positive integers k and t . When instances are restricted to oriented graphs with no directed cycles of length k or less, it is shown that that this problem is NP-complete whenever $t > k \geq 3$, and Polynomial if $t = k$ or $k \leq 2$. When there are no restrictions, it is shown that this problem is NP-complete whenever $k \geq 3$ and $t \geq 3$, and Polynomial whenever $k \leq 2$ or $t \leq 2$.

2 Preliminaries

In this section we review relevant definitions, the homomorphism model for 2-dipath colourings, and make small improvements to the known upper bounds on the oriented chromatic number of oriented graphs with 2-dipath chromatic number 3 or 4. We also observe some straightforward extensions to k -dipath colouring of known results for 2-dipath colouring.

Let G be an oriented graph, and let $x, y \in V(G)$. A *directed walk* is a sequence of vertices $W = v_0, v_1, v_2, \dots, v_{\ell-1}, v_\ell$, such that $v_i v_{i+1} \in E(G)$ for $i = 0, 1, \dots, \ell - 1$. The integer ℓ is the *length* of W . If $v_0 = x$ and $v_\ell = y$, then W is a *directed walk from x to y* . Note that the vertices belonging to W need not be different. If no two vertices of W are the same, then W is a *directed path*. If all vertices of W are different except v_0 and v_ℓ , then W is a *directed cycle*.

If there is a directed walk from x to y , then the vertex y is said to be *reachable* from x . The *distance* from x to y is defined to be the smallest length of a directed walk from x to y , or infinity if no such walk exists. The *weak distance between x and y* , denoted $d_{weak}(x, y)$, is defined to be the minimum of the distance from x to y and the distance from y to x .

This parameter is ∞ if neither of x and y is reachable from the other.

The *weak diameter* of an oriented graph G is the maximum of the weak distance between any two distinct vertices of G . A directed graph is called *weakly connected* if its weak diameter is finite.

Let G be a directed graph, and $x \in V(G)$. The *out-neighbourhood* of x is $N^+(x) = \{y : xy \in E(G)\}$, and the *in-neighbourhood* of x is $N^-(x) = \{y : yx \in E(G)\}$. The vertex x is a *source* if $N^-(x) = \emptyset$, and is a *sink* if $N^+(x) = \emptyset$. A *universal source* is a source such that $N^+(x) = V(G) - \{x\}$, and a *universal sink* is a sink such that $N^-(x) = V(G) - \{x\}$. More generally, the ℓ -*out-neighbourhood* of x is the set of all vertices reachable from x by a directed walk of length at most ℓ , and the ℓ -*in-neighbourhood* of x is the set of all vertices which can reach x by a directed walk of length at most ℓ .

The *directed girth* of a directed graph H is defined to be the minimum length of a directed cycle in H , or infinity if H has no directed cycle. Our homomorphism model for k -dipath colouring applies only to the family of oriented graphs with directed girth at least $k + 1$.

Let \mathcal{F} be a family of oriented graphs. The *oriented chromatic number* of \mathcal{F} , denoted by $\chi_o(\mathcal{F})$, is the least integer t so that $\chi_o(F) \leq t$ for all $F \in \mathcal{F}$. We say that an oriented graph H is a *universal target* for \mathcal{F} if $F \rightarrow H$ for all $F \in \mathcal{F}$. If H is a universal target for \mathcal{F} , then $\chi_o(\mathcal{F}) \leq |V(H)|$. For example, using the quadratic residue tournament on seven vertices as a universal target for the family \mathcal{O} of orientations of outerplanar graphs, Sopena (1997) shows $\chi_o(\mathcal{O}) \leq 7$.

Let G be an oriented graph with directed girth at least $k+1$. We say G is a *k-dipath clique* if $\chi_{k\text{-dip}}(G) = |V(G)|$. The terminology arises by analogy with undirected graphs, where a clique is a graph for which the chromatic number equals the number of vertices.

Let G be an oriented graph. Define G^k to be the simple graph formed from G as follows:

- $V(G^k) = V(G)$, and
- $E(G^k) = \{uv \mid 0 < d_{\text{weak}}(u, v) \leq k\}$.

It follows from the definitions that there is an equivalence between 2-dipath colourings of G and proper vertex colourings of the simple graph G^2 (see MacGillivray and Sherk (2014)). This equivalence extends to k -dipath colourings of G and proper vertex colourings of G^k .

Observation 1 *If G is an oriented graph with directed girth at least $k + 1$, then there is a one-to-one correspondence between k -dipath colourings of G and proper colourings of G^k .*

Using this observation we generalize a result of Bensmail et al. (2017) for 2-dipath cliques. This result will be used in Section 4.

Proposition 2 (Bensmail et al. (2017)) *An oriented graph is a 2-dipath clique if and only if it has weak diameter at most 2.*

Proposition 3 *Let $k \geq 2$ be an integer. An oriented graph is a k -dipath clique if and only if it has weak diameter at most k .*

Proof: Let G be an oriented graph with directed girth at least $k + 1$. We observe that G^k is a complete graph if and only if for each pair of non-adjacent vertices, say u and v , there is a directed path of length at most k , in some direction, between u and v . Equivalently, G has weak diameter at most k . \square

We now review the homomorphism model for 2-dipath colouring given by MacGillivray and Sherk (2014). Let t be a positive integer. The oriented graph \mathcal{G}_t is defined as follows. Its vertices are $(t + 1)$ -tuples in which the position among $1, 2, \dots, t$ indicated by the 0-th entry is filled with the place-holder “.”, and the remaining positions among $1, 2, \dots, t$ are filled with a 0 or a 1.

$$V(\mathcal{G}_t) = \{(u_0; u_1, u_2, \dots, u_t) : u_0 \in \{1, 2, \dots, t\}, u_{u_0} = \cdot, u_i \in \{0, 1\} \text{ for } 1 \leq i \leq t \text{ and } i \neq u_0\}$$

$$E(\mathcal{G}_t) = \{(u_0; u_1, u_2, \dots, u_t)(x_0; x_1, x_2, \dots, x_t) : u_{x_0} = 1, x_{u_0} = 0\}.$$

It is then proved that an oriented graph G admits a homomorphism to \mathcal{G}_t if and only if G has 2-dipath chromatic number at most t . That is, \mathcal{G}_t is a universal target for the family of oriented graphs with 2-dipath chromatic number at most t . Further, suppose $\phi : G \rightarrow \mathcal{G}_t$. For each vertex x , assigning colour u_0 to x if and only if the mapping $\phi, \phi(x) = (u_0; u_1, u_2, \dots, u_t)$, is a 2-dipath t -colouring of G . Since homomorphisms compose, the oriented chromatic number of \mathcal{G}_t is an upper bound for the oriented chromatic number of G . MacGillivray et al. (2010) show that $\chi_o(\mathcal{G}_t) \leq 2^t - 1$, which gives the following.

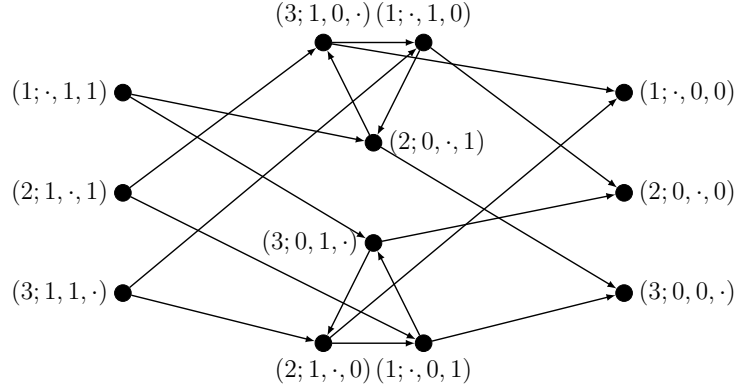


Fig. 2: The universal target for the family of oriented graphs with $\chi_{2\text{-dip}} \leq 3$.

Theorem 4 (MacGillivray et al. (2010)) *If G is an oriented graph, then*

$$\chi_{2\text{-dip}}(G) \leq \chi_o(G) \leq 2^{\chi_{2\text{-dip}}(G)} - 1.$$

The topic of universal targets for k -dipath colourings is considered in Section 3. Here we offer improvements for the cases $t = 3, 4$ of Theorem 4.

Proposition 5 *Let $t \geq 2$ be an integer and let G be an oriented graph with $\chi_{2\text{-dip}} \leq t$.*

- *If $t \leq 3$, then $\chi_o(G) \leq 5$.*
- *If $t \leq 4$, then $\chi_o(G) \leq 12$.*

Proof:

Figure 2 shows \mathcal{G}_3 , except for arcs between the source vertices on the left and the sink vertices on the right.

By inspection, \mathcal{G}_3 admits a homomorphism to the tournament of order 5 formed from a copy of a directed 3-cycle together with a universal source vertex and universal sink vertex. This proves the first statement.

Let H be the oriented graph obtained from \mathcal{G}_4 by deleting all sources and all sinks. Figure 3 gives a mapping of H to the oriented graph of order 10 shown. Thus, this oriented graph, together with a universal source and universal sink vertex, is a homomorphic image of \mathcal{G}_4 . This proves the second statement. \square

For $k \geq 2$, a k -dipath colouring of an oriented graph G is a 2-dipath colouring of G . Thus, Theorem 4 implies the following result for the k -dipath chromatic number.

Corollary 6 *If G is an oriented graph with directed girth at least $k + 1$, then*

$$\chi_o(G) \leq 2^{\chi_{k\text{-dip}}(G)} - 1.$$

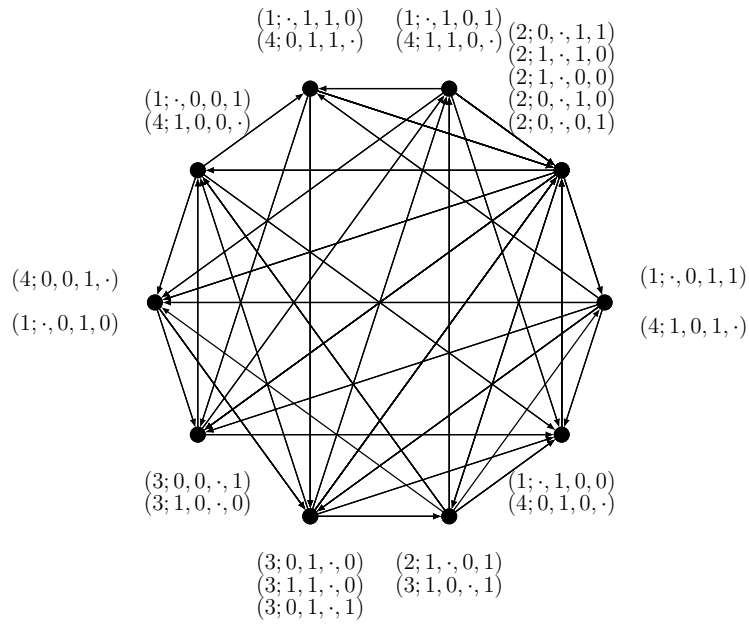


Fig. 3: A homomorphic image of the modified universal target for the family of oriented graphs with $\chi_{2\text{-dip}} \leq 4$.

Note, however, that by Theorem 17, $\chi_{k\text{-dip}}$ can always be replaced by $\chi_{2\text{-dip}}$. By contrast, the lower bound for $\chi_o(G)$ given in Theorem 4 does not hold if $\chi_{2\text{-dip}}$ is replaced by $\chi_{k\text{-dip}}$. The oriented chromatic number of a directed path on 4 vertices is 3, but its 4-dipath chromatic number is 4.

3 Homomorphisms and k -dipath Colouring

We begin our study of homomorphisms and k -dipath colourings by considering oriented graphs without directed cycles. The transitive tournament on t vertices is denoted by T_t .

Theorem 7 (Nešetřil and Pultr (1978)) *If G is an acyclic oriented graph, then $G \rightarrow T_t$ if and only if there is no homomorphism of a directed path on $t + 1$ vertices to G .*

A different way to phrase the condition in the theorem is that if $G \rightarrow T_t$, then G has no directed cycles, and no directed path of length t or more.

Corollary 8 *If G is an acyclic oriented graph and the longest directed path in G has t vertices, then $\chi_{k\text{-dip}}(G) = t$ for all $k \geq t > 1$.*

Proof: Suppose $k \geq t$.

Observe that since G has a path on t vertices, we have $\chi_{k\text{-dip}}(G) \geq t$. By Theorem 7, there exists a homomorphism $\phi : G \rightarrow T_t$. Since T_t has no directed cycle, any two vertices joined by a directed path must have different images. If the vertices of T_t are regarded as the colours $1, 2, \dots, t$, then ϕ is a k -dipath colouring of G . Thus $\chi_{k\text{-dip}}(G) \leq t$. \square

Since any oriented graph which admits a homomorphism to T_t is acyclic, it follows that T_t is a universal target for k -dipath t -colouring of acyclic oriented graphs.

Corollary 9 *Let k and t be positive integers. An acyclic oriented graph G has $\chi_{k\text{-dip}}(G) \leq t$ if and only if G admits a homomorphism to T_t .*

Corollary 10 *Let k and t be positive integers. An acyclic oriented graph G has $\chi_{k\text{-dip}}(G) \leq t$ if and only if G has no directed path on at least $t + 1$ vertices.*

Though not a direct analogue, Corollary 8 has a similar flavour to the early results on graph colourings of Vitaver (1962), Hasse (1965), Gallai (1967), and Roy (1967).

Observe that for $k \geq t$, the k -dipath chromatic number of T_t equals t . Thus, Corollary 8 states that the k -dipath chromatic number of T_t is an upper bound on the k -dipath chromatic number of any oriented graph which admits a homomorphism to T_t . The same statement holds if T_t is replaced by any oriented graph with large enough directed girth.

Theorem 11 *Let G and H be oriented graphs such that H has directed girth at least $k + 1$. If $G \rightarrow H$, then $\chi_{k\text{-dip}}(G) \leq \chi_{k\text{-dip}}(H)$.*

Proof: Suppose $\phi : G \rightarrow H$. Let c be a k -dipath t -colouring of H . Let $c' : V(G) \rightarrow \{1, 2, 3, \dots, t\}$ be defined by $c'(v) = c(\phi(v))$, for all $v \in V(G)$. We claim that this is a k -dipath t -colouring of G .

Let $u, v \in V(G)$. Suppose that there is a directed path from u to v of length $\ell \leq k$. Then there is a directed walk W of length k in H from $\phi(u)$ to $\phi(v)$. Since the directed girth of H is $k + 1$ or more, W

is a directed path. Thus, $c(\phi(u)) \neq c(\phi(v))$. Therefore $c'(u) \neq c'(v)$, and c' is a k -dipath colouring of G . \square

We now describe a homomorphism model for k -dipath t -colouring of oriented graphs with directed girth at least $k + 1$. That is, for all positive integers k and t , we will describe an oriented graph $\mathcal{G}_{k,t}$ with the property that an oriented graph G , with no directed cycle of length at most k , has a k -dipath t -colouring if and only if G admits a homomorphism to $\mathcal{G}_{k,t}$ and, further, the homomorphism can be used to find a k -dipath t -colouring of G . The graph $\mathcal{G}_{k,t}$ is a universal target for k -dipath t -colouring. Once this model is in place, Theorem 11 could be viewed as a direct consequence of the fact that a composition of homomorphisms is a homomorphism; it arises in the proof for the model, however.

We begin by defining a special set of matrices. Let $k \geq 2$ be an integer. Suppose that an oriented graph G , with directed girth at least $k + 1$, admits a k -dipath t -colouring, c . For $x \in V(G)$, given $c(x)$, we want to encode information about the colours of vertices in the k -in-neighbourhood of x and k -out-neighbourhood of x .

We define the k -dipath colouring matrix of x with respect to c to be the $(2k - 1) \times t$ zero-one matrix $A_{x,c}(G)$ with rows indexed by $-(k - 1), -(k - 2), \dots, -1, 0, 1, \dots, (k - 2), (k - 1)$, columns indexed by $1, 2, 3, \dots, t$, and (i, j) -entry equal to 1 if and only if there exists a vertex $y \in V(G)$ such that $c(y) = j$, and

- if $i \in \{-(k - 1), -(k - 2), \dots, -1\}$, then there is a directed path from y to x of length i ;
- if $i \in \{1, 2, \dots, (k - 1)\}$, then there is a directed path from x to y of length i ; and
- if $i = 0$, then $x = y$.

When the graph G is clear from the context, or unimportant in the discussion, the k -dipath colouring matrix of x with respect to c is denoted by $A_{x,c}$.

We illustrate the definition with an example. Consider the colouring, c , given in Figure 1. Let x be the unique vertex such that $c(x) = 3$. The 3-dipath colouring matrix of x is given by

	1	2	3	4
-2	0	1	0	0
-1	1	1	0	0
0	0	0	1	0
1	0	0	0	1
2	0	0	0	0

For example, the value, 1, in the entry $(-1, 2)$ arises because there is a vertex w such that $c(w) = 2$ and there is a directed path of length 1 from w to x .

Let $\mathcal{A}_{k,t}$ denote the set of all possible k -dipath colouring matrices over all k -dipath t -coloured oriented graphs with directed girth at least $k + 1$. We note that $\mathcal{A}_{k,t}$ is necessarily finite, as each member is a $(2k - 1) \times t$ matrix with entries from $\{0, 1\}$.

Since vertices at weak distance at most k must be assigned different colours, each element of $\mathcal{A}_{k,t}$ is a matrix that satisfies several conditions. Suppose A is the k -dipath colouring matrix of x with respect to c . If the colour of x is j , then no vertex at weak distance at most k from x can be assigned colour j . Thus, the entry of A in column j and row 0 equals 1, and all other entries in column j are zero. If there is a directed path of length $i < k$ from y to x , then no vertex w for which there is a directed path of length

$k - i$ from x to w can have the same colour as y . If the colour of y is j , then the entry of A in column j and row $-i$ equals 1, and the entries of A in column j and rows $0, 1, \dots, k - i$ are all equal to zero. Note that no assertion can be made about the entries of A in column j and rows $-(i - 1), -(i - 2), \dots, -1$ because there is no guarantee that a vertex v for which there is a directed path of length at most $i - 1$ to x lies on the directed path from y to x . Similar considerations apply to vertices w for which there is a directed path of length $i < k$ from x to w . These conditions are formalized succinctly as follows.

Observation 12 Let $A = [a_{ij}] \in \mathcal{A}_{k,t}$. Suppose $a_{ij} = 1$. Then,

- if $i = 0$, then $a_{\ell j} = 0$ for all $\ell \neq i$;
- if $i < 0$, then $a_{\ell j} = 0$, $0 \leq \ell \leq k - i$; and
- if $i > 0$, then $a_{-\ell j} = 0$, $0 \leq \ell \leq k - i$.

We now construct an oriented graph, $\mathcal{G}_{k,t}$, which is a universal target for the family of k -dipath t -colourable oriented graphs of directed girth at least $k + 1$. The vertex set $V(\mathcal{G}_{k,t}) = \mathcal{A}_{k,t}$. We describe the edge set informally at first, and then formally. Suppose the matrices $A = A_{x,c}$ and $B = B_{y,c'}$ are vertices of $\mathcal{G}_{k,t}$. Let $c(x) = m_1$ and $c'(y) = m_2$. Then $AB \in E(G)$ if

1. A encodes the fact that x has an out-neighbour of colour m_2 ;
2. B encodes the fact that y has an in-neighbour of colour m_1 ;
3. if A encodes the fact that a vertex of colour m_3 is joined to x by a directed path of length $i < k$, then B must encode the fact that a vertex of colour m_3 is joined to y by a directed path of length $i + 1$;
4. if B encodes the fact that y is joined to a vertex of colour m_3 by a directed path of length $i < k$, then A must encode the fact that x is joined to a vertex of colour m_3 by a directed path of length $i + 1$.

We now formally define the edge set of $\mathcal{G}_{k,t}$. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be vertices of $\mathcal{G}_{k,t}$ (i.e., matrices in $\mathcal{A}_{k,t}$). Then $AB \in E(\mathcal{G}_{k,t})$ if, whenever $a_{0m_1} = b_{0m_2} = 1$, the following conditions all hold:

1. $a_{1m_2} = 1$;
2. $b_{-1m_1} = 1$;
3. if $0 < i < k - 1$ and $a_{-im_3} = 1$, then $b_{-(i-1)m_3} = 1$; and
4. if $0 < i < k - 1$ and $b_{im_3} = 1$, then $b_{(i+1)m_3} = 1$.

We now establish some properties of $\mathcal{G}_{k,t}$.

Lemma 13 The digraph $\mathcal{G}_{k,t}$ has directed girth at least $k + 1$. In particular, it is an oriented graph.

Proof: Let $A_1, A_2, \dots, A_\ell, A_1$ be a directed cycle in $\mathcal{G}_{k,t}$. Suppose $\ell \leq k$, and A_1 has a 1 in entry $(0, m_1)$. This implies A_2 has a 1 in entry $(-1, m_1)$ and a 1 in entry $((\ell - 1), m_1)$, contrary to Observation 12. \square

Lemma 14 For integers $t \geq 2$ and $k \geq 2$ the k -dipath chromatic number of $\mathcal{G}_{k,t}$ is at most t .

Proof: Consider the colouring, c , given by $c(A) = m_1$, where m_1 is the unique column of the matrix of A for which the entry $(0, m_1)$ of A is a 1. We claim that c is a k -dipath colouring of $\mathcal{G}_{k,t}$. Let $A_1 A_2, \dots, A_\ell$ be a directed path of length $\ell \leq k$ in $\mathcal{G}_{k,t}$. If there exists a pair of indices i and j such that $1 \leq i < j \leq k$ and $c(A_i) = c(A_j)$, then A_{i+1} has a 1 in entry $(-1, c(A_i))$ and a 1 in entry $((j - (i + 1)), c(A_i))$, contrary to Observation 12. \square

Theorem 15 *Let $t \geq 2$ and $k \geq 2$ be integers. If G is an oriented graph with directed girth at least $k + 1$, then $\chi_{k\text{-dip}}(G) \leq t$ if and only if $G \rightarrow \mathcal{G}_{k,t}$.*

Proof: Let G be an oriented graph with directed girth at least $k + 1$. If $G \rightarrow \mathcal{G}_{k,t}$, then by Lemmas 13 and 14, and by Theorem 11, $\chi_{k\text{-dip}}(G) \leq t$.

Suppose $\chi_{k\text{-dip}}(G) \leq t$. Let c be a k -dipath colouring of G using t colours. Consider the mapping $\phi : V(G) \rightarrow V(\mathcal{G}_{k,t})$, where for all $v \in V(G)$, $\phi(v) = A_v$, the k -dipath colouring matrix of v with respect to c . Let uv be an arc of G . We claim $A_u A_v$ is an arc of $\mathcal{G}_{k,t}$. Suppose A_u has a 1 in entry $(0, m_1)$, and A_v has a 1 in entry $(0, m_2)$ (i.e., $c(u) = m_1$ and $c(v) = m_2$). To show that A_u and A_v are adjacent in $\mathcal{G}_{k,t}$ we must check that the four conditions in the definition are satisfied:

1. Since $c(v) = m_2$, A_u has a 1 in entry $(1, m_2)$.
2. Since $c(u) = m_1$, A_v has a 1 in entry $(-1, m_1)$.
3. Suppose there exists $i > 0$ such that A_u has a 1 in entry $(-i, m_3)$. Since c is a k -dipath colouring the entries of A_u in column m_3 and rows $0, 1, \dots, k - i$ are all equal to 0.
4. Suppose there exists $i > 0$ such that A_v has a 1 in entry (i, m_3) . Since c is a k -dipath colouring, the entries of A_v in column m_3 and rows $-(k - i), -(k - i + 1), \dots, -1, 0$ are all equal to zero.

This proves the claim. Therefore $\phi : G \rightarrow \mathcal{G}_{k,t}$ is a homomorphism. \square

Corollary 16 *For integers $t \geq 2$ and $k \geq 2$, $\chi_{k\text{-dip}}(\mathcal{G}_{k,t}) = t$.*

Proof: We show $\chi_{k\text{-dip}}(\mathcal{G}_{k,t}) \geq t$. The result then follows from Lemma 14. Consider the transitive tournament T_t . Assigning each vertex a different colour is a k -dipath t -colouring of T_t . Hence, by Theorem 15, there is a homomorphism $T_t \rightarrow \mathcal{G}_{k,t}$.

Since T_t does not have a k -dipath m -colouring for $m < t$, by Theorem 15, there is no homomorphism $T_t \rightarrow \mathcal{G}_{k,m}$. Since a composition of homomorphisms is a homomorphism, it follows that $\mathcal{G}_{k,m} \not\rightarrow \mathcal{G}_{k,t}$ when $m < t$. Therefore, $\chi_{k\text{-dip}}(\mathcal{G}_{k,t}) > t - 1$. The result now follows. \square

Our final result of this section shows that the homomorphism model captures three facts. First, if $t < k$, then no oriented graph with directed girth at least $k + 1$ and a k -dipath t -colouring can have a directed path of length greater than t . Any such digraph is t -dipath t -colourable (as the proof shows). Second, if $t \geq k$, then any k -dipath t -colouring of an oriented graph is also a k -dipath $(t + 1)$ -colouring, and there exist oriented graphs with k -dipath chromatic number $t + 1$. Third, every $(k + 1)$ -dipath t -colouring of an oriented graph is a k -dipath t -colouring.

Theorem 17 *Let $k \geq 2$ and $t \geq 2$ be integers. Then*

1. if $t \leq k$, then $\mathcal{G}_{k,t} \rightarrow \mathcal{G}_{t,t}$;
2. $\mathcal{G}_{k,t} \rightarrow \mathcal{G}_{k,(t+1)}$ and $\mathcal{G}_{k,(t+1)} \not\rightarrow \mathcal{G}_{k,t}$;
3. if $t > k$, then $\mathcal{G}_{(k+1),t} \rightarrow \mathcal{G}_{k,t}$ and $\mathcal{G}_{k,t} \not\rightarrow \mathcal{G}_{(k+1),t}$.

Proof: We first prove (1). Suppose $t \leq k$. Then no oriented graph with directed girth at least $k + 1$ and a k -dipath t -colouring has a directed path of length greater than t . By Theorem 7 there is a homomorphism of G to the transitive tournament T_t . In particular, $\mathcal{G}_{k,t} \rightarrow T_t$. But, by Theorem 15, $T_t \rightarrow \mathcal{G}_{t,t}$. Therefore, $\mathcal{G}_{k,t} \rightarrow \mathcal{G}_{t,t}$.

We now prove (2). It is clear that the subgraph of $\mathcal{G}_{k,(t+1)}$ induced by the vertices (colouring matrices) in which every entry in column $t + 1$ is zero (i.e., k -dipath $(t + 1)$ -colourings in which colour $t + 1$ is never used) is isomorphic to $\mathcal{G}_{k,t}$. Thus, $\mathcal{G}_{k,t} \rightarrow \mathcal{G}_{k,(t+1)}$. On the other hand, the transitive tournament on $t + 1$ vertices has a homomorphism to $\mathcal{G}_{k,(t+1)}$ but not to $\mathcal{G}_{k,t}$. Therefore, $\mathcal{G}_{k,(t+1)} \not\rightarrow \mathcal{G}_{k,t}$.

Finally, we prove (3). Suppose $t > k$. Since a $(k + 1)$ -dipath t -colouring is a k -dipath t -colouring, it follows from Theorem 15 that $\mathcal{G}_{(k+1),t} \rightarrow \mathcal{G}_{k,t}$. To see the second statement, note that a directed cycle of length $k + 1$ has a homomorphism to $\mathcal{G}_{k,t}$ but, by Lemma 13, no homomorphism to $\mathcal{G}_{(k+1),t}$. Therefore, $\mathcal{G}_{k,t} \not\rightarrow \mathcal{G}_{(k+1),t}$. \square

Directed graphs, G and H are called *homomorphically equivalent* if there are homomorphisms $G \rightarrow H$ and $H \rightarrow G$. If G and H are homomorphically equivalent, then a directed graph admits a homomorphism to G if and only if it admits a homomorphism to H . A directed graph is a *core* if it is not homomorphically equivalent to any proper subgraphs. Every directed graph G has a minimal subgraph H which is a core, and to which G is homomorphically equivalent; H is unique up to isomorphism, and is called *the core of G* (see Fellner (1982) and Welzl (1982)). We note that the core, H , of G is an induced subgraph of G because, by minimality, any homomorphism $G \rightarrow H$ must map H onto itself.

Corollary 18 *Let t and k be positive integers. If $t \leq k$, then the core of $\mathcal{G}_{k,t}$ is T_t .*

Proof: Note that T_t is a core. The existence of the required homomorphism is noted in the proof of statement (1) in Theorem 17. \square

4 Complexity of k -dipath Colourings

In this section we consider the following decision problem.

k -DIPATH t -COLOURING

Instance: an oriented graph, G .

Question: does G have a k -dipath t -colouring?

The dichotomy theorem stated below covers the cases where $k = 2$. We shall find a similar result for all remaining cases.

Theorem 19 (MacGillivray and Sherk (2014); Young (2009)) *Let $t \geq 1$ be a fixed integer. If $t \leq 2$, then 2-DIPATH t -COLOURING is Polynomial. If $t \geq 3$, then 2-DIPATH t -COLOURING is NP-complete.*

Given a simple graph G , we construct an oriented graph, $H_{k,t}$ ($t > k \geq 3$), such that $\chi_{k\text{-dip}}(H_{k,t}) \leq t$ if and only if $\chi(G) \leq t$. Let G be a simple graph, and let \tilde{G} be an arbitrary acyclic orientation of G . Corresponding to each vertex $v \in V(\tilde{G})$ the oriented graph $H_{k,t}$ contains:

1. vertices v_{in} and v_{out} ;
2. a transitive tournament on $t - k + 1$ vertices with source vertex s_v and sink vertex t_v ;
3. vertices $v'_1, v'_2, \dots, v'_{k-2}$;
4. the directed path $t_v, v'_1, v'_2, \dots, v'_{k-2}, v_{\text{in}}$, which has length $k - 1$; and
5. an arc $v_{\text{out}}s_v$.

For each arc $uv \in E(\tilde{G})$, the oriented graph $H_{k,t}$ is augmented by adding the arc $u_{\text{out}}v_{\text{in}}$.

This completes the construction of $H_{k,t}$. It can clearly be carried out in polynomial time. We note that since \tilde{G} has no directed cycles, by construction the same is true of $H_{k,t}$. Hence $H_{k,t}$ (formally) has directed girth at least $k + 1$. We also note that each vertex v_{out} has in-degree 0 and each vertex v_{in} has out-degree 0.

Observation 20 $\chi_{k\text{-dip}}(H_{k,t}) \geq t$.

Proof: For any vertex $v \in V(G)$, observe that the subgraph of $H_{k,t}$ induced by the $t - k + 1$ vertices of the transitive tournament corresponding to v , together with the vertices $v'_1, v'_2, \dots, v'_{k-2}, v_{\text{in}}$ is a k -dipath clique on t vertices. \square

Observation 21 Let $t > k \geq 3$. If $H_{k,t}$ has a k -dipath t -colouring, then for every $v \in V(G)$ and every k -dipath t -colouring, c , of $H_{k,t}$, $c(v_{\text{out}}) = c(v_{\text{in}})$.

Proof: For any $v \in V(G)$, observe that the subgraph of $H_{k,t}$ induced by the vertex v_{out} , the $t - k + 1$ vertices of the transitive tournament corresponding to v and the vertices $v'_1, v'_2, \dots, v'_{k-2}$ is a k -dipath clique on t vertices. Since the subgraph of $H_{k,t}$ induced by the $t - k + 1$ vertices of the transitive tournament corresponding to v , together with the vertices $v'_1, v'_2, \dots, v'_{k-2}$ and v_{in} is also k -dipath clique on t vertices, and c is a k -dipath t -colouring, it follows that $c(v_{\text{out}}) = c(v_{\text{in}})$. \square

Lemma 22 If G is a simple graph and $H_{k,t}$ is constructed from G as above, then, for all $t > k \geq 3$, the graph G is t -colourable if and only if $H_{k,t}$ has a k -dipath t -colouring.

Proof: Suppose $H_{k,t}$ has a k -dipath t -colouring, c . Let $\phi : V(G) \rightarrow \{1, 2, 3, \dots, t\}$ be defined by $\phi(v) = c(v_{\text{in}})$. We claim that ϕ is a proper colouring of the graph G . Suppose $ab \in E(\tilde{G})$. Then, $a_{\text{out}}b_{\text{in}} \in E(H_{k,t})$, and so $c(a_{\text{out}}) \neq c(b_{\text{in}})$. By Observation 21, $c(b_{\text{out}}) = c(b_{\text{in}})$. Therefore, $c(a_{\text{in}}) \neq c(b_{\text{in}})$, and $\phi(a) \neq \phi(b)$. This proves the claim. Hence G is t -colourable.

Suppose now that G has a t -colouring, ϕ . We construct a k -dipath t -colouring, c , of $H_{k,t}$. For each vertex $v \in V(G)$, set $c(v_{\text{out}}) = c(v_{\text{in}}) = \phi(v)$. We claim that c can be completed to a k -dipath t -colouring of $H_{k,t}$.

Since, for every $v \in V(G)$, if $\phi(v_{\text{out}}) = i$, then vertices of the transitive tournament corresponding to v together with the vertices $v'_1, v'_2, \dots, v'_{k-2}$ can be assigned colours from the set $\{1, 2, 3, \dots, t\} \setminus \{i\}$ so

that the resulting colouring has the property that no two vertices at weak distance at most k are assigned the same colour.

This proves the claim. \square

Theorem 23 *Let t and $k \geq 3$ be fixed positive integers. When restricted to instances of directed girth at least $k + 1$, the decision problem k -DIPATH t -COLOURING is NP-complete if $t > k$, and Polynomial if $t \leq k$.*

Proof: The problem is clearly in NP. If $t > k$ then NP-completeness follows from Lemma 22.

Suppose $t \leq k$. By Corollary 18, an oriented graph G with directed girth at least $k + 1$ has a k -dipath t -colouring if and only if it admits a homomorphism to the transitive tournament T_t . Homomorphism to T_t can be checked in polynomial time as shown by Bang-Jensen et al. (1988). The result now follows. \square

We now show that if the girth restriction is removed, then the dichotomy changes.

Theorem 24 *Let k and t be positive integers. If $t \leq 2$, then k -DIPATH t -COLOURING is Polynomial. If $t \geq 3$, then k -DIPATH t -COLOURING is NP-complete.*

Proof: Suppose first that $t \leq 2$. If $k = 1$, then the condition that vertices joined by a directed path of length at most k must be assigned different colours is the same as the condition that adjacent vertices must be assigned different colours. Hence, a digraph G has a k -dipath t -colouring if and only if the underlying graph of G has a t -colouring. The latter problem is Polynomial. If $k = 2$, k -dipath t -colouring is Polynomial by Theorem 19. For $k \geq 3$, if G has a directed path of length at least 3 or a directed 3-cycle (which is easy to check), then, since $t \leq 2$, it has no k -dipath t -colouring. Otherwise, (G has no directed path of length greater than 2), a k -dipath t -colouring is a 2-dipath t -colouring, which is Polynomial.

Now suppose $t \geq 3$. If $k = 2$, then k -dipath t -colouring is NP-complete by Theorem 19. Hence, assume $k \geq 3$. If $t > k$, the result follows from Theorem 23. Hence we may also assume $t \leq k$. Similarly as above, if $t < k$ then no oriented graph with a directed path of length greater than t (which is easy to check), can have a k -dipath t -colouring. Otherwise (G has no directed path of length greater than t), a k -dipath t -colouring is a t -dipath t -colouring. Thus, we may further assume that $t = k \geq 3$.

We have previously noted that the problem belongs to NP. The transformation is from t -colouring. Suppose an instance of t -colouring, a simple, undirected graph G is given. We will replace each edge of G by the oriented graph which we now construct.

Let C be the directed cycle of length t with vertex sequence $v_0, v_1, \dots, v_{t-1}, v_0$. Add four new vertices x_0, x_1, y_0, y_1 , and arcs to make the directed paths x_0, x_1, v_1 and y_0, y_1, v_2 . This completes the construction of F . Observe that x_0 is joined to every vertex of C except v_0 by a directed path of length at most t . Every vertex of C is assigned a different colour in a k -dipath t -colouring of F . Hence, the vertices x_0 and v_0 must be assigned the same colour. Similarly, the vertices y_0 and v_1 must be assigned the same colour. In particular, x_0 and y_0 must be assigned different colours. Furthermore, any assignment of two different colours to x_0 and y_0 can be extended to a k -dipath t -colouring of F .

Construct an oriented graph G' from G as follows. Replace each edge xy of G with a copy F_{xy} of F by identifying x_0 with x and y_0 with y . The construction can clearly be carried out in polynomial time. Observe that each vertex in $V(G) \cap V(G')$ has in-degree 0.

We claim that G is t -colourable if and only if G' has a k -dipath t -colouring. Suppose that a t -colouring of G is given. For each copy of F in G' , this assignment gives different colours to x_0 and y_0 . By our

earlier observation, this assignment can be extended to a k -dipath t -colouring of F . Since each vertex in $V(G) \cap V(G')$ has in-degree 0, there is no directed path of positive length joining vertices in different copies of F , this results in a k -dipath t -colouring of G' . Now suppose a k -dipath t -colouring of G' is given. By our observation on colourings of F , any two vertices which are adjacent in G are assigned different colours. Hence restricting this colouring to $V(G)$ gives a t -colouring of G . This completes the proof. \square

The problem of deciding whether the k -th power of a graph G is t -colourable is known to be NP-complete (see Lin and Skiena (1995) and McCormick (1983)). The results above imply that, if $t > k$, the problem of deciding whether the underlying graph of the k -th power of an oriented graph G (i.e., two vertices are adjacent if and only if they are at weak distance at most k) is t -colourable is NP-complete, even when restricted to powers of oriented graphs with directed girth at least $k + 1$, and also that, if $t = k \geq 3$, the problem of deciding whether the underlying graph of the k -th power of an oriented graph G is t -colourable is NP-complete.

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