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A polynomial bound on the number of minimal separators and potential maximal cliques in P_6-free graphs of bounded clique number

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In this note we show a polynomial bound on the number of minimal separators and potential maximal cliques in P_6-free graphs of bounded clique number.

Keywords: potential maximal cliques, minimal separators, tame class, P_6-free graphs

1 Introduction

Let G be a graph. For a set $X \subseteq V(G)$, we say that a connected component C of $G - X$ is a *full component* of X if $N_G(C) = X$, that is, every vertex of X has a neighbor in C. A set $X \subseteq V(G)$ is a *minimal separator* if it has at least two full components.

Intuitively, the space of all minimal separators of a graph G reflects the space of all possible separations that can be used to solve some computational problem on G via dynamic programming. A related notion of *potential maximal clique* (not defined formally in this work) corresponds to all reasonable choices of a bag in a tree decomposition of G.

Bouchitté and Todinca (2001) (with some results generalized by Fomin et al. (2015)) showed that indeed these notions are useful to solve a wide family of graph problems.

Theorem 1 (Bouchitté and Todinca (2001, 2002))**.** *If G is an n-vertex graph with a minimal separators and b potential maximal cliques, then $b \leq n(a^2 + a + 1)$ and $a \leq nb$. Furthermore, given a graph G, one can in time polynomial in the input and compute the list of all its minimal separators and potential maximal cliques.*

Theorem 2 (Bouchitté and Todinca (2001); Fomin et al. (2015), informal statement)**.** *A wide family of graph problems, including* Maximum Weight Independent Set *and* Feedback Vertex Set*, can be solved in time polynomial in the size of the input graph and the number of its potential maximal cliques.*

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As a result, if for a graph class $\mathcal{G}$ one can prove a (purely graph-theoretical) polynomial bound on the number of minimal separators or potential maximal cliques in $\mathcal{G}$, then one can immediately obtain polynomial-time algorithms for a wide family of problems on $\mathcal{G}$. We call such a graph class *tame*. Recently, a methodological study of which graph classes are tame was initiated, see, e.g., the work of Abrishami et al. (2022); Gajarský et al. (2022); Gartland and Lokshtanov (2023a,b); Milanič and Pivač (2021).

We say that a graph G is *H-free* for a graph H if no induced subgraph of G is isomorphic to H. The *clique number* of a graph G is the maximum cardinality of a set pairwise adjacent vertices of G. For an integer t, by P_t we denote the path on t vertices (and $t-1$ edges). The main result of this note is a short proof that P_6-free graphs of bounded clique number have a polynomial number of minimal separators and potential maximal cliques.

Theorem 3. *Let G be an n-vertex P_6-free graph of clique number k. Then, G contains at most $(2n)^{k+1}$ minimal separators and at most $2^{2k+2}n^{2k+3}$ potential maximal cliques.*

We remark that some additional condition to just P_6-freeness is needed for the conclusion of Theorem 3, as even the class of P_5-free graphs is not tame. This can be witnessed by *prisms*: An n-prism consists of two n-vertex cliques with a matching in between; it is P_5-free but admits $2^n - 2$ minimal separators. Furthermore, the P_6 cannot be replaced with P_7: Chudnovsky et al. (2023) provide a construction of P_7-free graphs G_n that have clique number 2, $|V(G_n)| = 6n + 2$, and at least 3^n minimal separators.

Our motivation for proving Theorem 3 is two-fold. First, the proof is very simple, much simpler than the arguments of Grzesik et al. (2022) giving polynomial-time algorithms for MAXIMUM WEIGHT INDEPENDENT SET and related problems in P_6-free graphs without any assumption on the clique number. Second, it gives examples of graph classes that are tame, but in which such problems as 5-COLORING or ODD CYCLE TRANSVERSAL are NP-hard: Huang (2016) proved that 5-COLORING is NP-hard in P_6-free graphs (and trivial in graphs of clique number larger than 5) and Chudnovsky et al. (2021); Dabrowski et al. (2020) proved that ODD CYCLE TRANSVERSAL is NP-hard in P_6-free graphs of clique number at most 3. This answers negatively a question of (Milanič and Pivač, 2021, Open problem 3).

2 Proof of Theorem 3

If G is edgeless, then the statement is immediate (there are no minimal separators and n potential maximal cliques), so we assume $E(G) \neq \emptyset$ and thus $n, k \geq 2$.

Assuming $n, k \geq 2$, we will prove that G contains at most $(2n)^k \cdot (2n-1)$ minimal separators. This directly implies the bound for minimal separators of Theorem 3 and also implies the bound on the number of potential maximal cliques in G via Theorem 1.

We need *modules* and some basic facts about them. Let G be a graph. A *module* is a nonempty set $M \subseteq V(G)$ such that every $u \in V(G) \setminus M$ is adjacent to either all vertices of M or to no vertex of M. A module M is:

- *strong* if for every other module M' either $M \subseteq M'$, $M' \subseteq M$, or $M \cap M' = \emptyset$;
- *proper* if it is different than $V(G)$;
- *maximal* if it is proper and strong and no other proper strong module contains it;
- connected if $G[M]$ is connected.

We need the following properties of modules, which can be distilled from Lemma 2 and Theorem 2 and the discussion around them in the work of Habib and Paul (2010).

Lemma 4. *In a graph on at least two vertices, the maximal modules form a partition of the vertex set.*

Lemma 5. *An n-vertex graph G contains at most $2n-1$ strong modules. Furthermore, for every module M in G, there exists a unique strong module M' that contains M and is inclusion-wise minimal with this property, and one of the following holds:*

- *$M = M'$ and M is a strong module in G;*
- *$G[M']$ is disconnected and M is a union of at least two connected components of $G[M']$;*
- *the complement of $G[M']$ is disconnected and M is a union of at least two connected components of the complement of $G[M']$.*

In the context of graphs of bounded clique number, we observe the following immediate corollary.

Corollary 6. *An n-vertex graph of clique number k contains at most $2^k(2n-1)$ connected modules.*

Proof: Fix a connected module M and let M' be as in Lemma 5. Then, Lemma 5 implies that M is a union of some connected components of the complement of $G[M']$ (possibly all of them for the first option of Lemma 5). There are at most $(2n-1)$ choices for M' and at most 2^k choices which components of the complement of $G[M']$ form M, as the clique number of G is k. □

We will also need the following lemma of Grzesik et al. (2022).

Lemma 7 (cf. Lemma 4.2 of Grzesik et al. (2022))**.** *Let G be a graph, $X \subseteq V(G)$, and let A be a full component of X with $|A| > 1$. Let $p, q \in A$ be any two vertices in different maximal modules of $G[A]$ (which exist by Lemma 4) that are adjacent (i.e., $pq \in E(G)$). Then, for every $x \in X$ either:*

- *There exists an induced P_4 in G with one endpoint in x and the remaining three vertices in A.*
- *The vertex x is adjacent to p or to q (or to both).*
- *The complement of $G[A]$ is disconnected and $N(x) \cap A$ consists of some connected components of the complement of $G[A]$.*

We remark that if the complement of $G[A]$ is disconnected, then the connected components of the complement of $G[A]$ are exactly the maximal proper strong modules of $G[A]$.

We deduce the following.

Lemma 8. *Let G be a P_6-free graph of clique number $k \geq 2$, let X be a minimal separator in G, and let A and B be two distinct full sides of X. Then there exists a set $Q \subseteq A$ of size at most k such that every vertex of $X \setminus N(Q)$ is complete to B.*

Proof: If $|A| = 1$, take $Q = A$, so $X \setminus N(Q) = \emptyset$ and we are done. Assume then $|A| > 1$.

If the complement of $G[A]$ is disconnected, take Q to be any set consisting of one vertex from each connected component of the complement of $G[A]$. Otherwise, take Q of size 2, consisting of any two vertices of two different maximal modules of $G[A]$ that are adjacent (such two modules exist as $|A| > 1$ and $G[A]$ is connected). Since the clique number of G is k, we have $|Q| \leq \max(k, 2) = k$. By Lemma 7, for every $x \in X \setminus N(Q)$ there exists a P_4 with one endpoint in x and the remaining three vertices in A; denote it P^x. If x is not complete to B, then, as $G[B]$ is connected and B is a full component of X, there are $y, z \in B$ with $xy, yz \in E(G)$ but $xz \notin E(G)$. Then, P^x, prolonged with y and z is an induced P_6 in G, a contradiction. $\square$

By Lemma 8, we infer that B is a connected module of $G - N(Q)$. As $|Q| \leq k$, there are at most n^k choices of Q. By Corollary 6, for fixed Q, there are at most $2^k(2n-1)$ choices of B. As $X = N_G(B)$, there are at most $(2n)^k \cdot (2n-1)$ minimal separators in G, as promised.

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References

T. Abrishami, M. Chudnovsky, C. Dibek, S. Thomassé, N. Trotignon, and K. Vušković. Graphs with polynomially many minimal separators. *J. Comb. Theory, Ser. B*, 152:248–280, 2022. doi: 10.1016/j.jctb.2021.10.003. URL `https://doi.org/10.1016/j.jctb.2021.10.003`.

V. Bouchitté and I. Todinca. Treewidth and minimum fill-in: Grouping the minimal separators. *SIAM J. Comput.*, 31(1):212–232, 2001. doi: 10.1137/S0097539799359683. URL `https://doi.org/10.1137/S0097539799359683`.

V. Bouchitté and I. Todinca. Listing all potential maximal cliques of a graph. *Theor. Comput. Sci.*, 276 (1-2):17–32, 2002. doi: 10.1016/S0304-3975(01)00007-X. URL `https://doi.org/10.1016/S0304-3975(01)00007-X`.

M. Chudnovsky, J. King, M. Pilipczuk, P. Rzążewski, and S. Spirkl. Finding large H-colorable subgraphs in hereditary graph classes. *SIAM J. Discret. Math.*, 35(4):2357–2386, 2021. doi: 10.1137/20M1367660. URL `https://doi.org/10.1137/20M1367660`.

M. Chudnovsky, R. McCarty, M. Pilipczuk, M. Pilipczuk, and P. Rzążewski. Sparse induced subgraphs in P_6-free graphs. *CoRR*, abs/2307.07330, 2023. doi: 10.48550/arXiv.2307.07330. URL `https://doi.org/10.48550/arXiv.2307.07330`.

K. K. Dabrowski, C. Feghali, M. Johnson, G. Paesani, D. Paulusma, and P. Rzążewski. On cycle transversals and their connected variants in the absence of a small linear forest. *Algorithmica*, 82 (10):2841–2866, 2020. doi: 10.1007/s00453-020-00706-6. URL `https://doi.org/10.1007/s00453-020-00706-6`.

F. V. Fomin, I. Todinca, and Y. Villanger. Large induced subgraphs via triangulations and CMSO. *SIAM J. Comput.*, 44(1):54–87, 2015. doi: 10.1137/140964801. URL `https://doi.org/10.1137/140964801`.

J. Gajarský, L. Jaffke, P. T. Lima, J. Novotná, M. Pilipczuk, P. Rzążewski, and U. S. Souza. Taming graphs with no large creatures and skinny ladders. In S. Chechik, G. Navarro, E. Rotenberg, and G. Herman, editors, *30th Annual European Symposium on Algorithms, ESA 2022, September 5-9, 2022, Berlin/Potsdam, Germany*, volume 244 of *LIPIcs*, pages 58:1–58:8. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022. doi: 10.4230/LIPIcs.ESA.2022.58. URL `https://doi.org/10.4230/LIPIcs.ESA.2022.58`.

P. Gartland and D. Lokshtanov. Graph classes with few minimal separators. I. Finite forbidden induced subgraphs. In N. Bansal and V. Nagarajan, editors, *Proceedings of the 2023 ACM-SIAM Symposium on Discrete Algorithms, SODA 2023, Florence, Italy, January 22-25, 2023*, pages 3063–3097. SIAM, 2023a. doi: 10.1137/1.9781611977554.ch119. URL `https://doi.org/10.1137/1.9781611977554.ch119`.

P. Gartland and D. Lokshtanov. Graph classes with few minimal separators. II. A dichotomy. In N. Bansal and V. Nagarajan, editors, *Proceedings of the 2023 ACM-SIAM Symposium on Discrete Algorithms, SODA 2023, Florence, Italy, January 22-25, 2023*, pages 3098–3178. SIAM, 2023b. doi: 10.1137/1.9781611977554.ch120. URL `https://doi.org/10.1137/1.9781611977554.ch120`.

A. Grzesik, T. Klimošová, M. Pilipczuk, and M. Pilipczuk. Polynomial-time algorithm for maximum weight independent set on P_6-free graphs. *ACM Trans. Algorithms*, 18(1):4:1–4:57, 2022. doi: 10.1145/3414473. URL `https://doi.org/10.1145/3414473`.

M. Habib and C. Paul. A survey of the algorithmic aspects of modular decomposition. *Comput. Sci. Rev.*, 4(1):41–59, 2010. doi: 10.1016/J.COSREV.2010.01.001. URL `https://doi.org/10.1016/j.cosrev.2010.01.001`.

S. Huang. Improved complexity results on k-coloring P_t-free graphs. *Eur. J. Comb.*, 51:336–346, 2016. doi: 10.1016/j.ejc.2015.06.005. URL `https://doi.org/10.1016/j.ejc.2015.06.005`.

M. Milanič and N. Pivač. Polynomially bounding the number of minimal separators in graphs: Reductions, sufficient conditions, and a dichotomy theorem. *Electron. J. Comb.*, 28(1):1, 2021. doi: 10.37236/9428. URL `https://doi.org/10.37236/9428`.