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Planar cycle-extendable graphs

Three of the authors dedicate this work to their coauthor, the late Ajit A Diwan

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For most problems pertaining to perfect matchings, one may restrict attention to *matching covered graphs* — that is, connected nontrivial graphs with the property that each edge belongs to some perfect matching. There is extensive literature on these graphs that are also known as 1-extendable graphs (since each edge extends to a perfect matching) including an ear decomposition theorem due to Lovász and Plummer.

A cycle C of a graph G is conformal if G - V(C) has a perfect matching; such cycles play an important role in the study of perfect matchings, especially when investigating the Pfaffian orientation problem. A matching covered graph G is cycle-extendable if — for each even cycle C — the cycle C is conformal, or equivalently, each perfect matching of C extends to a perfect matching of G, or equivalently, C is the symmetric difference of two perfect matchings of G, or equivalently, C extends to an ear decomposition of G. In the literature, these are also known as cycle-nice or as 1-cycle resonant graphs.

Zhang, Wang, Yuan, Ng and Cheng, 2022, provided a characterization of claw-free cycle-extendable graphs. Guo and Zhang, 2004, and independently Zhang and Li, 2012, provided characterizations of bipartite planar cycle-extendable graphs. In this paper, we establish a characterization of all planar cycle-extendable graphs — in terms of K_2 and four infinite families.

Keywords: matchings, perfect matchings, matching covered graphs, cycle-extendability

1 Cycle-extendability

All graphs considered in this paper are loopless. However, we allow multiple/parallel edges. For graph-theoretic notation and terminology, we follow Bondy and Murty (2008), whereas for terminology pertaining to matching theory, we follow Lucchesi and Murty (2024).

A graph is said to be *matchable* if it has a perfect matching. Most problems pertaining to the study of perfect matchings may be reduced to *matching covered graphs* — that is, nontrivial connected graphs

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with the property that each edge lies in some perfect matching. There is extensive literature on matching covered graphs. In Lovász and Plummer (1986), they are referred to as 1-extendable graphs, since each edge extends to a perfect matching. There is also literature on k-extendable graphs — matching covered graphs with the additional property that each matching of cardinality k extends to a perfect matching; see Plummer (1991). In a similar spirit, a matching covered graph G is cycle-extendable if, for each even cycle C, either perfect matching of C extends to a perfect matching of G. This leads us to the following decision problem that is not known to be in \mathbf{NP} .

Decision Problem 1.1 Given a matching covered graph G, decide whether G is cycle-extendable.

Before stating our contributions and related prior work on the above problem, let us take a closer look at conformality and cycle-extendability. A cycle C of a matchable graph G is a *conformal cycle* if the graph G - V(C) is matchable. Thus, a matching covered graph is cycle-extendable if and only if each even cycle is conformal. From this viewpoint, it is easy to see that Decision Problem 1.1 belongs to **co-NP**.

In Figure 1a, the 6-cycle denoted by the dashed line is not conformal, and using the symmetries of the Cube graph, the reader may verify that there are precisely four such 6-cycles, whereas every other cycle is conformal. In particular, the Cube graph is not cycle-extendable. Figures 1b and 1c depict examples of bipartite and nonbipartite cycle-extendable graphs, respectively.

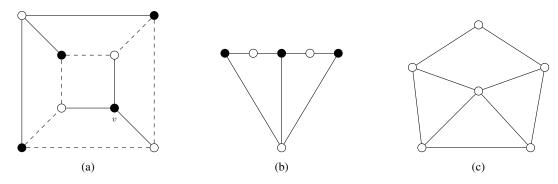


Fig. 1: (a) the Cube graph is not cycle-extendable; (b) a bipartite cycle-extendable graph; (c) a nonbipartite cycle-extendable graph W_5^-

Observe that, in a matchable graph, a cycle is conformal if and only if it may be expressed as the symmetric difference of two perfect matchings. It is for this reason that they are also known as *alternating cycles*; see Carvalho and Little (2008). It is easily observed that a connected matchable graph, distinct from K_2 , is matching covered if and only if each edge belongs to a conformal cycle. Little (1975) proved the stronger statement that, in a matching covered graph, any two edges belong to a common conformal cycle. There are several other terms used in the literature for conformal cycles: *well-fitted* cycles in McCuaig (2004); *nice* cycles in Lovász (1987); and *central* cycles in Robertson et al. (1999).

Zhang, Wang, Yuan, Ng and Cheng Zhang et al. (2022) provided a characterization of claw-free cycle-extendable (aka cycle-nice) matching covered graphs; their work implies that Decision Problem 1.1 belongs to **NP** as well as **P** for claw-free graphs.

Guo and Zhang (2004), and independently Zhang and Li (2012), provided characterizations of bipartite planar cycle-extendable (aka 1-cycle resonant) graphs. The second author discovered a couple of these

characterizations independently, and they appear in their MSc thesis Pause (2022); they also made partial progress towards characterizing planar cycle-extendable graphs.

In this paper, we characterize all planar cycle-extendable graphs and our result implies that Decision Problem 1.1 belongs to **NP** as well as **P** for planar graphs. The results reported here comprise a proper subset of those that appear in the MS thesis Dalwadi (2025) of the first author.

Organization of this paper

In Section 1.1, we discuss the ear decomposition theory for matching covered graphs that provides an alternative definition of cycle-extendable graphs that motivates their investigation. In Section 1.2, we discuss series and parallel reductions that help us in restricting ourselves to irreducible graphs — that is, simple graphs whose vertices of degree two comprise a stable set.

In Section 2, we describe the tight cut decomposition theory for matching covered graphs and its applications towards characterizing cycle-extendable graphs. In particular, in Section 2.2, we use a result of Carvalho and Little (2008) to deduce that every planar cycle-extendable irreducible graph, except K_2 , either has a vertex of degree two or otherwise is a "brick" — a special class of 3-connected nonbipartite matching covered graphs.

In Section 3, we first discuss a necessary condition ($K_{2,3}$ -freeness) for a planar matching covered graph to be cycle-extendable, and then a necessary condition for a 3-connected graph to be $K_{2,3}$ -free. In Section 4, we use results of Section 3 and the well-known brick generation theorem of Norine and Thomas (2007) to characterize planar cycle-extendable bricks.

In Section 5, we describe four infinite families of nonbipartite planar cycle-extendable irreducible graphs using a special class of bipartite cycle-extendable graphs called "half biwheels". Finally, in Section 5.7, we prove that every planar cycle-extendable irreducible graph is either K_2 or is a member of one of these four families; this proves a conjecture of the third author, and places Decision Problem 1.1 in **NP** as well as in **P** for planar graphs.

1.1 Ear decompositions of matching covered graphs

Observe that matching covered graphs, except K_2 , are 2-connected. The ear decomposition theory of 2-connected graphs, due to Whitney (1933), is well-known. We now discuss a refinement of this theory, due to Hetyei, that is applicable to the subclass of bipartite matching covered graphs; see Lovász and Plummer (1986).

Given a graph G and a (proper) subgraph H, an ear (aka a single ear) of H in G is an odd path P whose ends are in V(H) but is otherwise disjoint with H. For a bipartite graph G, a sequence of subgraphs (G_0, G_1, \ldots, G_r) is called a bipartite ear decomposition of G if (i) G_0 is an (even) cycle, (ii) $G_r = G$, and (iii) $G_{i+1} = G_i \cup P_i$ where P_i is an ear of G_i (in G) for each $i \in \{0, 1, \ldots, r-1\}$. Figure 2 shows a bipartite ear decomposition of a bipartite matching covered graph where each ear is denoted by thick lines.

A subgraph H of a graph G is *conformal* if the graph G - V(H) is matchable. Given any bipartite ear decomposition of a bipartite graph, it is easily observed that each subgraph in the sequence is conformal. Furthermore, the following theorem due to Hetyei, implies that each subgraph is also matching covered and establishes the converse as well.

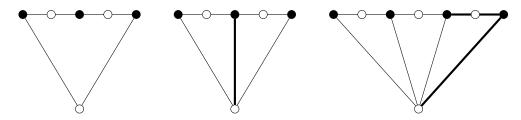


Fig. 2: a bipartite ear decomposition of a bipartite matching covered graph

Theorem 1.2 [BIPARTITE EAR DECOMPOSITION THEOREM]

A bipartite graph G, distinct from K_2 , is matching covered if and only if each of its conformal cycles extends to a bipartite ear decomposition of G.

For the Cube graph, depicted in Figure 1a, the reader may observe that each conformal cycle extends to a bipartite ear decomposition; furthermore, the non-conformal 6-cycle (shown using dashed lines) does not extend to a bipartite ear decomposition.

Theorem 1.2 implies that any bipartite matching covered graph (distinct from K_2) can be constructed in a straightforward manner, from any conformal cycle, by adding a single ear at a time so that each subgraph is also matching covered graph. Unfortunately, in the case of nonbipartite matching covered graphs, one can not restrict to the addition of single ears. For instance, in the case of K_4 , we must start from C_4 ; now observe that we must add the remaining two edges simultaneously in order to get a bigger matching covered subgraph (that is, K_4 itself). Lovász and Plummer (1986) proved the surprising result that every matching covered graph may be constructed, from any conformal cycle, by adding either a single ear or a "double ear" at a time so that each subgraph is also matching covered. We state this more formally below.

Given a graph G and (proper) subgraph H, a double ear of H in G is a pair of vertex-disjoint single ears of H (in G). For a matching covered graph G, a sequence of matching covered subgraphs (G_0,G_1,\ldots,G_r) is called an ear decomposition of G if (i) G_0 is an even cycle, (ii) $G_r=G$, and (iii) $G_{i+1}=G_i\cup R_i$ where R_i is either a single or a double ear of G_i (in G) for each $i\in\{0,1,\ldots,r-1\}$. As in the case of bipartite ear decompositions, it is easy to see that each subgraph in an ear decomposition is also conformal. We are now ready to state the aforementioned theorem of Lovász and Plummer.

Theorem 1.3 [EAR DECOMPOSITION THEOREM]

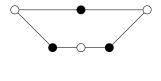
Each conformal cycle of a matching covered graph G extends to an ear decomposition of G.

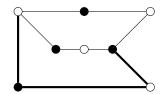
Figure 3 shows an ear decomposition of the nonbipartite graph R_8^- that will play an important role in our work, and is obtained from the bicorn R_8 , shown in Figure 6a, by deleting an edge; each ear addition is denoted by a thick line.

The statement of Theorem 1.3 begs the question: for which matching covered graphs, is it possible to extend each even cycle to an ear decomposition? Clearly, this is possible only for those matching covered graphs that are cycle-extendable. This provides an alternative motivation for characterizing cycle-extendable graphs.

Proposition 1.4 [AN ALTERNATIVE VIEWPOINT OF CYCLE-EXTENDABILITY]

A matching covered graph G is cycle-extendable if and only if each of its even cycles extends to an ear decomposition of G.





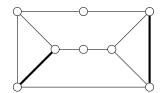


Fig. 3: an ear decomposition of R_8^-

We conclude this section by describing a special class of nonbipartite matching covered graphs that play a crucial role in the theory of matching covered graphs, as well as in our work. Before doing so, we remark that, in an ear decomposition of a matching covered graph, a double ear is added only when neither of the constituent single ears can be added in order to obtain a matching covered graph.

Let G be a nonbipartite matching covered graph and $(G_0, G_1, G_2, \ldots, G_r = G)$ be an ear decomposition. Note that G_0 is an even cycle. Let G_k denote the first nonbipartite subgraph in this sequence. It follows from our above remark that G_k is also the first subgraph in the sequence that is obtained by adding a double ear, say R_{k-1} , to the previous (bipartite) subgraph G_{k-1} . We say that G_k is a near-bipartite graph. Below, we provide an alternative definition that is independent of the ear decomposition.

For a matching covered graph G, an odd path P is a *removable single ear* of G if each internal vertex of P (if any) has degree two in G and the graph G-P is also matching covered. (Here, by G-P, we mean the graph obtained from G by deleting all of the edges and internal vertices of P.) Likewise, a pair of vertex-disjoint odd paths $R:=\{P_1,P_2\}$ is a *removable double ear* of G if each internal vertex of P_1 as well as P_2 has degree two in G and the graph $G-P_1-P_2$ is matching covered. A *near-bipartite graph* is a matching covered graph, say G, that has a removable double ear $R:=\{P_1,P_2\}$ such that $G-R:=G-P_1-P_2$ is bipartite. In this sense, near-bipartite graphs comprise a subset of nonbipartite matching covered graphs whose members are closest to being bipartite.

In the next section, we discuss a reduction of the Decision Problem 1.1 to a more restricted class of matching covered graphs.

1.2 Irreducible graphs

Given a graph G and a parallel edge e, we say that G - e is obtained from G by an application of *parallel reduction*. Observe that G is matching covered if and only if G - e is matching covered; furthermore, G is cycle-extendable if and only if G - e is cycle-extendable.

Let G be a graph that has a path P := wxyz of length three, each of whose internal vertices, x and y, is of degree two in G. Let J denote the graph obtained from G by replacing the path P with a single edge e joining w and z. That is, J := G - P + e. We say that J is obtained from G by an application of series reduction. The reader may observe that G is matching covered if and only if G is matching covered except when G is G is cycle-extendable if and only if G is cycle-extendable.

A graph G is *irreducible* if it is simple and its degree two vertices comprise a stable set. The above observations prove the following.

Proposition 1.5 Let H be an irreducible (matching covered) graph that is obtained from a matching covered graph G by repeated applications of series and parallel reductions. Then G is cycle-extendable if and only if H is cycle-extendable.

Thus, in order to settle the complexity status of Decision Problem 1.1, it suffices to focus on the following decision problem.

Decision Problem 1.6 Given an irreducible matching covered graph G, decide whether G is cycle-extendable or not.

In the next section, we discuss a necessary matching-theoretic condition for a graph to be cycle-extendable. In order to do so, we shall require some concepts from the theory of matching covered graphs.

2 Applications of the tight cut decomposition theory

For a nonempty proper subset X of the vertices of a graph G, we denote by $\partial(X)$ the cut associated with X, that is, the set of all edges that have one end in X and the other end in $\overline{X} := V(G) - X$. We refer to X and \overline{X} as the *shores* of $\partial(X)$. For a vertex v, we simplify the notation $\partial(\{v\})$ to $\partial(v)$, and we refer to such a cut as *trivial*.

For a cut $\partial(X)$ of a graph G, we denote by $G/(X \to x)$, or simply by G/X, the graph obtained from G by shrinking the shore X to a single vertex x called the *contraction vertex*. The graph G/\overline{X} is defined analogously. The graphs G/\overline{X} and G/X are called the $\partial(X)$ -contractions of G. Observe that each edge in either $\partial(X)$ -contraction corresponds to an edge in G. Conversely, each edge of G that is not in $\partial(X)$ corresponds to an edge in precisely one of the two $\partial(X)$ -contractions, whereas each edge of G that is in $\partial(X)$ corresponds to an edge in both $\partial(X)$ -contractions; it is customary and convenient to use the same label for all these edges. For the graph shown in Figure 4, the blue line indicates a cut; one of the contractions is K_4 , whereas the other contraction is obtained from $K_{3,3}$ by replacing any vertex by a triangle.

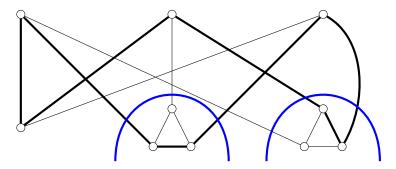


Fig. 4: a matching covered graph and its nontrivial tight cuts

2.1 Tight cut decomposition, bricks and braces

Let G be a matching covered graph. A cut $\partial(X)$ is a *tight cut* if $|M \cap \partial(X)| = 1$ for every perfect matching M. In Figure 4, the blue lines indicate nontrivial tight cuts. It is easy to see that if $\partial(X)$ is a nontrivial tight cut then each $\partial(X)$ -contraction is a matching covered graph that has fewer vertices than G. If either of the $\partial(X)$ -contractions has a nontrivial tight cut, then that graph can be further decomposed

into smaller matching covered graphs. We may repeat this procedure until we obtain a list of matching covered graphs — each of which is free of nontrivial tight cuts. This process is known as the *tight cut decomposition procedure*.

A matching covered graph free of nontrivial tight cuts is called a *brace* if it is bipartite; otherwise, it is called a *brick*. Thus, an application of the tight cut decomposition procedure to a matching covered graph results in a list of bricks and braces. For the graph shown in Figure 4, an application of the tight cut decomposition procedure yields two copies of K_4 and one copy of $K_{3,3}$. We now make a simple observation pertaining to bricks and braces.

Let $S \coloneqq \{u,v\}$ denote a 2-vertex-cut in a matching covered graph G on six or more vertices. Let J denote a component of G-S. If J is a nontrivial odd component then $\partial(V(J))$ is a nontrivial tight cut; whereas, if J is an even component then $\partial(V(J) \cup \{u\})$ is a nontrivial tight cut. This immediately proves the following well-known fact.

Proposition 2.1 Every simple brick/brace, except K_2 and C_4 , is 3-connected.

A matching covered graph may admit several applications of the tight cut decomposition procedure. However, Lovász (1987) proved the following remarkable result.

Theorem 2.2 [UNIQUE TIGHT CUT DECOMPOSITION THEOREM]

Any two applications of the tight cut decomposition procedure to a matching covered graph yield the same list of bricks and braces (up to multiplicities of edges).

The following characterization of planar cycle-extendable braces is an immediate consequence of Proposition 2.1 and a result of (Klavžar and Salem, 2012, Corollary 2.3).

Corollary 2.3 The only simple planar cycle-extendable braces are K_2 and C_4 .

We now switch our attention to proving the following lemma that relates cycle-extendability with tight cuts.

Lemma 2.4 Let G be a matching covered graph and let $\partial(X)$ denote a nontrivial tight cut. If G is cycle-extendable then both $\partial(X)$ -contractions of G are also cycle-extendable.

Proof: We let $G_1 := G/(X \to x)$ and $G_2 := G/(\overline{X} \to \overline{x})$. Assume that G is cycle-extendable. It suffices to prove that G_1 is cycle-extendable. To this end, let Q_1 denote an even cycle of G_1 .

If $x \notin V(Q_1)$ then Q_1 is a cycle of G. We let M denote a perfect matching of $G - V(Q_1)$. Observe that, since M extends to a perfect matching of G, and since $\partial(X)$ is tight, $|M \cap \partial(X)| = 1$. Consequently, $M_1 := M \cap E(G_1)$ is a perfect matching of $G_1 - V(Q_1)$.

Now suppose that $x \in V(Q_1)$, and let e and f denote the two edges of Q_1 incident with x. Let Q_2 denote an even cycle of G_2 containing e and f. (To see why such a cycle exists, consider the symmetric difference of two perfect matchings of G_2 : one containing e and another containing f.) Observe that $Q := Q_1 \cup Q_2$ is an even cycle in G, and let M denote a perfect matching of G - V(Q). Note that $M \cap \partial(X) = \emptyset$ and that $M_1 := M \cap E(G_1)$ is a perfect matching of $G_1 - V(Q_1)$.

In both cases, we have shown that Q_1 is a conformal cycle of G_1 , whence G_1 is cycle-extendable. This completes the proof of Lemma 2.4.

It is worth noting that the converse of the above lemma does not hold in general. For instance, the graph shown in Figure 4 is not cycle-extendable since the cycle shown using a thick line is not conformal; however, its bricks and braces are cycle-extendable. Lemma 2.4 immediately yields the following.

Corollary 2.5 *Each brick and brace of a cycle-extendable graph is also cycle-extendable.*

Lovázs's theorem (2.2) leads to certain graph invariants that play a crucial role in the theory of matching covered graphs. In the following two sections, we discuss a couple of these invariants and related concepts that are relevant to our work.

2.2 Near-bricks versus near-bipartite graphs

It follows from the Unique Tight Cut Decomposition Theorem (2.2) that the number of bricks of a matching covered graph G (obtained by any application of the tight cut decomposition procedure) is an invariant; we denote this by b(G). For instance, b(G) = 2 for the graph G shown in Figure 4.

It is worth noting that b(G) = 0 if and only if G is bipartite. We say that G is a *near-brick* if b(G) = 1. In particular, every brick is a near-brick. The following is an easy consequence of Theorem 2.2 that we will find useful in the next section.

Corollary 2.6 For a near-brick, given any nontrivial tight cut $\partial(X)$, one of the $\partial(X)$ -contractions is bipartite and the other one is a near-brick.

We now observe an easy refinement of the above corollary. Let G be a near-brick that is not a brick and let $\partial(X)$ denote a nontrivial tight cut. By Corollary 2.6, one of the $\partial(X)$ -contractions, say $G_1 \coloneqq G/\overline{X}$, is bipartite. Let Y be a minimal (not necessarily proper) subset of X such that $|Y| \ge 3$ and $D \coloneqq \partial_{G_1}(Y)$ is a tight cut of G_1 . By choice of Y, the graph G_1/\overline{Y} is a brace, where $\overline{Y} \coloneqq V(G_1) - Y$. Observe that D is a nontrivial tight cut of G and that one of the D-contractions of G is isomorphic to G_1/\overline{Y} . This proves the following well-known fact.

Corollary 2.7 In any near-brick that is not a brick, there exists a nontrivial tight cut $\partial(X)$ such that one of the $\partial(X)$ -contractions is a brace and the other one is a near-brick.

Recall from Section 1.1 that a near-bipartite graph is a matching covered graph, say G, that has a removable double ear $R := \{P_1, P_2\}$ such that $G - R := G - P_1 - P_2$ is bipartite. The following result of Carvalho et al. (2002) implies that every near-bipartite graph is a near-brick.

Theorem 2.8 Let G be a matching covered graph, and let R be a removable double ear. Then b(G-R) = b(G) - 1.

The following result is an immediate consequence of Theorem 2.8, and it implies that every near-brick that has a removable double ear is near-bipartite. For an example, see the graph R_8^- shown in Figure 3.

Corollary 2.9 Let G be a near-brick, and let R be a removable double ear. Then G - R is bipartite and matching covered.

We remark that a near-brick need not be near-bipartite. For instance, wheels (defined in Section 4.1), of order six or more, are bricks that are not near-bipartite. This is easily seen using the next proposition — that follows from the facts: (i) the number of odd faces of a plane graph G, denoted by $f_{odd}(G)$, is even, and (ii) deleting any edge may reduce $f_{odd}(G)$ by at most two.

Proposition 2.10 Every near-bipartite plane graph G satisfies $f_{odd}(G) \in \{2,4\}$.

Lastly, we use p(G) to denote the number of Petersen bricks of a matching covered graph G (obtained by any application of the tight cut decomposition procedure), where *Petersen brick* refers to the Petersen graph up to multiple edges. In the next section, we apply a result of Carvalho and Little (2008) to deduce

that every cycle-extendable graph is either bipartite or otherwise a near-brick whose unique brick is not a Petersen brick.

2.3 Cycle-extendable implies bipartite or near-brick

The aforementioned result of Carvalho and Little pertains to the cycle space $\mathcal{C}(G)$ and its various subspaces that are well-studied vector spaces; see Bondy and Murty (2008). We let $\mathcal{C}^e(G)$ denote the *even space* of a graph G — that is, the subspace of the cycle space $\mathcal{C}(G)$ spanned by the even cycles. Likewise, $\mathcal{A}(G)$ denotes the *alternating space* — that is, the subspace of $\mathcal{C}^e(G)$ spanned by the conformal cycles.

For 2-connected graphs, it is well-known that the $C^e(G)$ is a proper subspace of C(G) if and only if G is nonbipartite. In the same spirit, the following result of Carvalho and Little (2008) characterizes the matching covered graphs for which the alternating space A(G) is a proper subspace of the even space $C^e(G)$ — in terms of the invariants b and b.

Theorem 2.11 For a matching covered graph G, the following are equivalent:

- (i) $\mathcal{A}(G)$ is a proper subspace of $\mathcal{C}^e(G)$,
- (ii) there exists an even cycle $C \in C^e(G) A(G)$, and
- (iii) b(G) + p(G) > 1.

Observe that, for a matching covered graph G, the inequality b(G) + p(G) > 1 holds if and only if G is nonbipartite, and either (i) G is not a near-brick or otherwise (ii) G is a near-brick whose unique brick is the Petersen brick. The following is an immediate consequence of the above theorem that we alluded to earlier.

Corollary 2.12 [CYCLE-EXTENDABLE IMPLIES BIPARTITE OR NEAR-BRICK]

Every cycle-extendable graph is either bipartite or otherwise a near-brick whose unique brick is not the Petersen brick.

Proof: We shall prove the contrapositive. Let G be a matching covered graph such that b(G) + p(G) > 1. By Theorem 2.11, there exists an even cycle $C \in C^e(G) - \mathcal{A}(G)$. Clearly, C is not a conformal cycle; whence G is not cycle-extendable.

We now deduce another consequence of the above corollary and some of the earlier results; in the case of bipartite graphs, we shall find a result of Chartrand, Kaugars and Lick useful.

Corollary 2.13 [MINIMUM DEGREE THREE OR MORE IMPLIES BRICK]

Every simple planar cycle-extendable graph, with minimum degree three or more, is a brick.

Proof: Let G be a simple planar cycle-extendable graph with $\delta(G) \geq 3$. First, suppose that G is bipartite. Consider a planar embedding of G. Since G is 2-connected and $\delta(G) \geq 3$, it follows from a result of Chartrand et al. (1972) that there exists a vertex v so that G - v is a 2-connected (plane) graph. We invoke Whitney's well-known result Bondy and Murty (2008), and we let G denote the facial (even) cycle that bounds the face containing the point corresponding to the deleted vertex G. By planarity, all neighbors of G lie on G. Thus, G is not conformal and G is not cycle-extendable; contradiction.

Hence, G is nonbipartite. By Corollary 2.12, G is a near-brick. Suppose to the contrary that G is not a brick. Using Lemma 2.7, there exists a nontrivial tight cut $\partial(X)$ so that one of the $\partial(X)$ -contractions,

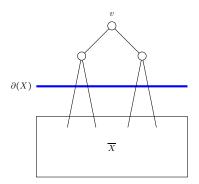


Fig. 5: illustration for the proof of Theorem 2.13

say $G_1 \coloneqq G/\overline{X}$, is a brace. We invoke Corollaries 2.5 and 2.3 to infer that G_1 is isomorphic to G_2 (up to multiple edges). Since G_2 is simple, the multiple edges of G_2 (if any) are incident with the contraction vertex. Let G_2 that is not incident with any edge in G_2 (if any) see Figure 5. Observe that G_2 (if any) = 2 and that G_2 (if any) = 2. This contradicts our hypothesis that G_2 is isomorphic to G_2 (if any) are incident with the contraction vertex.

Inspired by the proof of the above corollary in the bipartite case, we introduce the following definition that we shall find useful later. For a vertex v of a graph G, a cycle C (in G - v) is said to be v-isolating if v is an isolated vertex in the graph G - V(C).

In order to characterize planar cycle-extendable graphs, it follows from Proposition 1.5 and Corollary 2.13 that it suffices to characterize (i) planar cycle-extendable bricks, and (ii) planar cycle-extendable irreducible graphs that have a vertex of degree two. In the case of planar bricks, we shall solve a more general problem that we discuss in the next section.

3 $K_{2,3}$ -freeness

To bisubdivide an edge means to subdivide it by inserting an even number of subdivision vertices. A graph H is a bisubdivision of a graph G if H may be obtained from G by selecting any (possibly empty) subset $F \subseteq E(G)$ and bisubdividing each edge in F, or equivalently, if G may be obtained from H by a sequence of series reductions.

Given a fixed graph J, a graph G is said to be J-free if G does not contain any subgraph H that is a bisubdivision of J; otherwise, we say that G is J-based. The bicorn R_8 , shown in Figure 6a, is $K_{2,3}$ -based; the thick lines indicate a bisubdivision of $K_{2,3}$. The pentagonal prism minus a specific edge, shown in Figure 6b, is $K_{2,3}$ -free; the reader may verify this using Lemma 3.1 and Proposition 4.3.

3.1 Planarity: cycle-extendable implies $K_{2,3}$ -free

We now discuss a necessary condition for a planar graph to be cycle-extendable.

Lemma 3.1 [CYCLE-EXTENDABILITY IMPLIES $K_{2,3}$ -FREENESS IN PLANAR GRAPHS] *If a planar matching covered graph is cycle-extendable then it is* $K_{2,3}$ -free.

Proof: We prove the contrapositive. Let G denote any planar embedding of a planar matching covered graph that is $K_{2,3}$ -based, and let H denote a subgraph that is a bisubdivision of $K_{2,3}$. Observe that H

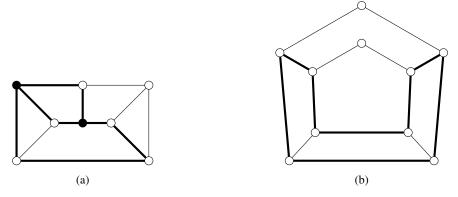


Fig. 6: (a) the bicorn R_8 ; (b) the pentagonal prism minus one edge

comprises two cubic vertices, say u and v, and three internally-disjoint even uv-paths, say P_1 , P_2 and P_3 . See Figure 7. Any two of these paths, say P_i and P_j (where i < j), comprise a facial cycle of H, say Q_{ij} . We will argue that at least one of these three cycles is not conformal.

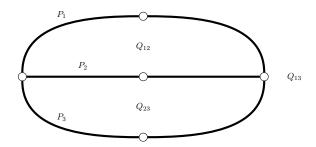


Fig. 7: an illustration for the proof of Lemma 3.1

As per the Jordan Curve Theorem, each cycle Q_{ij} partitions the rest of the plane into two regions: (i) the interior denoted by $int(Q_{ij})$ and (ii) the exterior denoted by $ext(Q_{ij})$. We adjust notation so that $int(Q_{ij})$ refers to the region that does not meet the path P_k (where k is distinct from i and j) as shown in Figure 7.

Now, assume without loss of generality that each of Q_{12} and Q_{23} is conformal. Since Q_{12} is conformal, $\operatorname{int}(Q_{12})$ contains an even number of vertices. Likewise, $\operatorname{int}(Q_{23})$ contains an even number of vertices. Observe that $\operatorname{ext}(Q_{13})$ contains precisely all of the vertices that lie in $\operatorname{int}(Q_{12}) \cup \operatorname{int}(Q_{23})$, plus the internal vertices of the path P_2 . In particular, $\operatorname{ext}(Q_{13})$ contains an odd number of vertices; ergo, Q_{13} is not conformal. Thus G is not cycle-extendable. This completes the proof of Lemma 3.1.

The above lemma was proved by the second author and appeares in his MSc thesis Pause (2022). However, we learnt later this was already observed by Zhang and Li (2012).

It is worth noting that the converse of the above lemma does not hold in general. Figure 6b shows an example of a (nonbipartite) planar matching covered graph that is $K_{2,3}$ -free but not cycle-extendable; the even cycle depicted by a thick line is not conformal. However, the converse indeed holds in the case of

bipartite graphs; this was proved by Zhang and Li (2012), and independently by the second author Pause (2022). This inspires the following decision problem.

Decision Problem 3.2 Given a matching covered graph G, decide whether G is $K_{2,3}$ -free.

Thus, in the context of planar nonbipartite matching covered graphs, cycle-extendable graphs comprise a proper subset of $K_{2,3}$ -free graphs. We are currently working on characterizing $K_{2,3}$ -free matching covered graphs, and have managed to solve the problem in the case of 3-connected graphs.

In light of Lemma 3.1, in order to conclude that a planar graph is not cycle-extendable, it suffices to show the existence of a bisubdivision of $K_{2,3}$. In the following section, we discuss a sufficient condition for the existence of a bisubdivision of $K_{2,3}$ that is applicable to 3-connected nonbipartite graphs. Since bricks are 3-connected (see Proposition 2.1) and nonbipartite, this will help us in characterizing planar cycle-extendable bricks. In fact, we will obtain a characterization of planar $K_{2,3}$ -free bricks.

3.2 3-connectedness: mixed bicycle implies $K_{2,3}$ -based

A pair of vertex-disjoint simple cycles, say (Q_o, Q_e) , is said to be a *mixed bicycle* if Q_o is odd and Q_e is even. (Here, simple just means that the length of Q_e is four or more.) The following is our promised sufficient condition for 3-connected (nonbipartite) graphs.

Lemma 3.3 [MIXED BICYCLE IMPLIES $K_{2,3}$ -BASED IN 3-CONNECTED GRAPHS] If a 3-connected graph has a mixed bicycle then it is $K_{2,3}$ -based.

Proof: Let (Q_o,Q_e) denote a mixed bicycle in a 3-connected graph G. By Menger's Theorem, there exist three vertex-disjoint paths — say P_i (where $i \in \{1,2,3\}$) — each of which has one end, say u_i , in Q_o and the other end, say v_i , in Q_e . Let Q_1,Q_2 and Q_3 denote the three edge-disjoint paths of Q_e that have both ends in $\{v_1,v_2,v_3\}$. In particular, $Q_e=Q_1\cup Q_2\cup Q_3$.

Since Q_e is even, at least one of Q_1, Q_2 and Q_3 is even. Adjust notation so that Q_1 is even, and let v_1 and v_2 denote the ends of Q_1 . Note that Q_1 and $Q_2 \cup Q_3$ are edge-disjoint even v_1v_2 -paths. It remains to find one more even v_1v_2 -path that is edge-disjoint with $Q_1 \cup (Q_2 \cup Q_3)$. Such a path is easily obtained by combining $P_1 \cup P_2$ with the u_1u_2 -path of Q_o of appropriate parity. This completes the proof.

We remark that the converse of the above lemma does not hold in general. Clearly, 3-connected bipartite matching covered graphs are counterexamples; however, one can also construct nonbipartite counterexamples such as certain subgraphs of K_5 and K_6 . We conclude this section with the following consequence of Lemmas 3.3 and 3.1.

Corollary 3.4 If a 3-connected planar matching covered graph has a mixed bicycle then it is not cycle-extendable.

4 Planar $K_{2,3}$ -free bricks

Our characterization of planar $K_{2,3}$ -free bricks relies on two main ingredients, one of which is Lemma 3.3. The other one is the brick generation theorem due to Norine and Thomas (2007), which states that all simple bricks may be constructed from five infinite families and the Petersen graph by means of four operations. We refer to these families as *Norine-Thomas families*; furthermore, we refer to their members, as well as to the Petersen graph, as *Norine-Thomas bricks*.

In the next section, we discuss the planar Norine-Thomas families, and we use Lemma 3.3 to classify the $K_{2,3}$ -free ones. This classification will serve as the base case in our inductive proof of the characterization of planar $K_{2,3}$ -free bricks.

4.1 Planar Norine-Thomas bricks

We begin this section by describing two of the five Norine-Thomas families — odd wheels and odd prisms — whose members happen to be planar as well as cycle-extendable.

A graph obtained from an odd cycle $Q := u_0 u_1 \dots u_{2k}$, by adding a new vertex h and edges hu_i for each $i \in \{0, \dots, 2k\}$, is called an *odd wheel*, or simply a *wheel*. The vertex h is called its *hub* and the cycle Q is called its *rim*. Let G be a wheel and let C denote an even cycle. Clearly, $h \in V(C)$ and G - V(C) is an odd path, whence C is conformal. This proves the following.

Proposition 4.1 Wheels are cycle-extendable.

A graph obtained from two vertex-disjoint copies of an odd cycle, say $Q := w_0 w_1 \dots w_{2k}$ and $Q' := z_0 z_1 \dots z_{2k}$, by adding edges $w_i z_i$ for each $i \in \{0, 1, \dots, 2k\}$, is called an *odd prism*, or simply a *prism*. Observe that prisms are *cubic* — that is, the degree of each vertex is precisely three.

In order to prove that prisms are cycle-extendable, we will first observe that they are near-bipartite and we shall locate all of their removable double ears; the reader may recall definitions from Section 2.2. Before that, we state a simplification of the notion of removable double ears that is applicable to irreducible graphs.

For a matching covered graph G, an edge e is removable if G-e is matching covered, and a pair of distinct edges $R \coloneqq \{\alpha, \beta\}$ is a removable doubleton if neither α nor β is removable but G-R is matching covered. For instance, the graph R_8^- , shown in Figure 3, has a removable doubleton depicted by thick lines. Note that, if G is irreducible, each removable double ear is a removable doubleton and vice-versa. Furthermore, if G is near-bipartite then, by Corollary 2.9, the graph G-R is bipartite for each removable doubleton $R \coloneqq \{\alpha, \beta\}$; also, if A and B denote the color classes of G-R then (up to relabeling) the edge α has both ends in A whereas β has both ends in B. Using these observations, the following is easily proved; we shall it find very useful in future sections, and we shall invoke it implicitly.

Lemma 4.2 For a near-bipartite graph G, if R is any removable doubleton and C is any even cycle, then $|C \cap R| \in \{0,2\}$.

Let G be a prism with vertex-disjoint odd cycles $Q := w_0w_1 \dots w_{2k}$ and $Q' := z_0z_1 \dots z_{2k}$ as in the definition stated earlier. Observe that $\{w_iw_{i+1}, z_iz_{i+1}\}$ is a removable doubleton of G for each $i \in \{0,1,\dots,2k\}$, where arithmetic is modulo 2k. In particular, each vertex is incident with two distinct removable doubletons. Using this observation and Lemma 4.2, and the fact that G is cubic, we infer that an even cycle C of G contains w_i if and only if it contains z_i ; consequently, $M := \{w_iz_i|i\in\{0,1,\dots,2k\} \text{ and } \{w_i,z_i\}\cap V(C)=\varnothing\}$ is a perfect matching of G-V(C); thus, C is conformal. This proves the following.

Proposition 4.3 Prisms are cycle-extendable.

There are two more Norine-Thomas families each of whose member is planar; these are "staircases" and "truncated biwheels". The smallest member of each of these families is the triangular prism $\overline{C_6}$. We refer the reader to Kothari and Murty (2016) for their descriptions — using which they may easily verify the following.

Proposition 4.4 Every staircase as well as truncated biwheel, except $\overline{C_6}$, has a mixed bicycle; consequently, each of them is $K_{2,3}$ -based.

To summarize, we have proved the following.

Corollary 4.5 [PLANAR $K_{2,3}$ -FREE NORINE-THOMAS BRICKS]

Wheels and prisms are the only planar Norine-Thomas bricks that are $K_{2,3}$ -free; in fact, they are also cycle-extendable.

In the next section, we describe the Norine-Thomas brick generation theorem.

4.2 Norine-Thomas brick generation theorem

We first state the induction viewpoint of the Norine-Thomas result using the terminology of Carvalho et al. (2015).



Fig. 8: an illustration of the bicontraction operation

Let G be a matching covered graph, and let v_0 be a vertex of degree two that has two distinct neighbors v_1 and v_2 . The *bicontraction* of v_0 is the operation of contracting the two edges v_0v_1 and v_0v_2 , and we use G/v_0 to denote the resulting graph; see Figure 8. Note that $\partial(X)$ is a tight cut, where $X := \{v_0, v_1, v_2\}$, and G/v_0 is simply the $\partial(X)$ -contraction G/X, as defined in Section 2.1; consequently, G/v_0 is matching covered. It is worth noting that even if G is simple, G/v_0 need not be simple. The following is easily proved and we shall use its contrapositive in our proof of the characterization of planar $K_{2,3}$ -free bricks.

Lemma 4.6 Let G be a matching covered graph and let v_0 denote a vertex of degree two that has two distinct neighbors. For any graph K with maximum degree $\Delta(K) \leq 3$, if G/v_0 is K-based then G is also K-based.

Now, let simple G be a brick and e denote a removable edge as defined in the preceding section. Since each vertex of G has three or more distinct neighbors, the matching covered graph G-e is irreducible and it has zero, one or two vertices of degree two. The matching covered graph J obtained from G-e by bicontracting each of its vertices of degree two (if any) is referred to as the *retract of* G-e. Note that its minimum degree $\delta(J) \geq 3$; however, J need not be a brick. For instance, in the Petersen graph \mathbb{P} , shown in Figure 9a, each edge is removable; however, the reader may verify that b(J) = 2, where J is retract of $\mathbb{P} - e$ for any edge e.

In light of the above discussion, a removable edge e of a brick G is a thin edge if the retract of G - e is also a brick. For instance, the bicorn R_8 , shown in Figure 9b, has a unique removable edge e that also

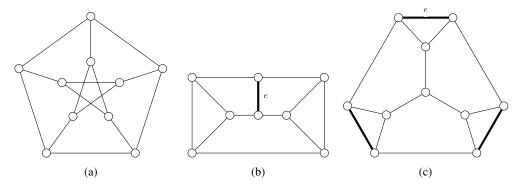


Fig. 9: (a) the Petersen graph \mathbb{P} ; (b) the bicorn R_8 ; (c) the tricorn R_{10} .

happens to be thin — since the retract of $R_8 - e$ is the brick K_4 with multiple edges. Carvalho et al. (2006) proved that every brick, except K_4 , triangular prism $\overline{C_6}$ and the Petersen graph, has a thin edge.

A thin edge e of a simple brick G is a *strictly thin edge* if the retract of G-e is also simple. For instance, the tricorn R_{10} , shown in Figure 9c, has precisely three strictly thin edges; for each such edge, say e, the retract of $R_{10} - e$ is precisely the wheel W_5 . On the other hand, the bicorn R_8 has no strictly thin edges. We are now ready to state the generation theorem for simple bricks due to Norine and Thomas (2007).

Theorem 4.7 [STRICTLY THIN EDGE THEOREM]

Every simple brick, except for the Norine-Thomas bricks, has a strictly thin edge.

In order to prove our characterization of planar $K_{2,3}$ -free bricks, we require the generation viewpoint of the above theorem which states that every simple brick may be generated from a Norine-Thomas brick by a sequence of four "expansion operations" as described by Carvalho et al. (2006).

Let H be a graph that has a vertex of degree at least four, say v. Consider any graph G that is obtained from H by replacing the vertex v by two new vertices v_1 and v_2 , distributing the edges of $\partial_H(v)$ amongst v_1 and v_2 so that each of them receives at least two, and then adding a new vertex v_0 as well as edges v_0v_1 and v_0v_2 . We say that G is obtained from H by bisplitting the vertex v. Observe that $H = G/v_0$; see Figure 8. It is easily seen that G is matching covered if and only if H is matching covered.

Each of the aforementioned four "expansion operations" consists of bisplitting zero, one or two vertices of a simple brick J and then adding a suitable edge e in order to obtain a larger simple brick G. Conversely, e is a strictly thin edge of G and J is the retract of G-e. The interested reader may refer to Carvalho et al. (2006) for the exact descriptions of these expansion operations. However, we shall only use the bisplitting operation in our proof of the characterization of planar $K_{2,3}$ -free bricks; in particular, we will find the following observation useful.

Corollary 4.8 Let G denote a graph obtained from a graph H by bisplitting a vertex of degree four or more. If H contains a mixed bicycle then so does G.

The above can also be deduced from Lemma 4.6 by noting that a mixed bicycle is simply a bisubdivision of the disjoint union of C_3 and C_4 , and that bicontraction may be viewed as the "inverse" of bisplitting.

4.3 Characterization of planar $K_{2,3}$ -free bricks

We are now ready to present our characterization of planar $K_{2,3}$ -free bricks.

Theorem 4.9 [PLANAR $K_{2,3}$ -FREE BRICKS]

A simple planar brick is $K_{2,3}$ -free if and only if it is either a wheel or a prism.

Proof: In light of Corollary 4.5, we only need to prove the forward direction. To this end, let G denote a simple planar brick that is $K_{2,3}$ -free. We proceed by induction on the number of edges.

If G is a Norine-Thomas brick then we are done by Corollary 4.5. Otherwise, by Theorem 4.7, G has a strictly thin edge, say e, and we let J denote the retract of G - e. Since G is $K_{2,3}$ -free, so is G - e; by Lemma 4.6, so is J. Ergo, J is a simple planar brick that is $K_{2,3}$ -free; by the induction hypothesis, J is either a wheel or a prism. In particular, G is obtained from J by zero, one or two bisplitting operations followed by adding an edge. We shall consider two cases depending on the number of bisplitting operations — either zero or at least one.

First, suppose that G is obtained from J by adding an edge; that is, G = J + e. Since J is either a wheel or a prism, the reader may easily verify that if $|V(J)| \ge 8$ then G contains a mixed bicycle; by Lemma 3.3, G is $K_{2,3}$ -based; contradiction. On the other hand, if |V(J)| = 6, then G is obtained from W_5 or $\overline{C_6}$ by adding the edge e; in each case, one may observe that $K_{2,3}$ is a subgraph of G; contradiction.

Now suppose that G is obtained from J by one or two bisplitting operations followed by adding an edge. In particular, J has a vertex of degree four or more. Ergo, J is an odd wheel of order six or more. Furthermore, the first bisplitting operation must bisplit the hub h of J, the reader may easily verify that the resulting graph has a mixed bicycle. By Lemma 4.8, G contains a mixed bicycle. By Lemma 3.3, G is $K_{2,3}$ -based; contradiction. This completes the proof of Theorem 4.9.

The above theorem, along with Corollary 4.5, yields the following.

Corollary 4.10 [PLANAR CYCLE-EXTENDABLE BRICKS]

A simple planar brick is cycle-extendable if and only if it is either a wheel or a prism.

We now proceed towards our final goal of characterizing planar cycle-extendable irreducible graphs.

5 Planar cycle-extendable irreducible graphs

In light of Proposition 1.5, Corollary 2.13 and Theorem 4.9, it remains to characterize those planar cycle-extendable irreducible graphs that have a vertex of degree two; let G be such a graph and let x_0 denote any vertex of degree two. In our inductive proof of the Main Theorem (5.20), we shall consider the smaller graph $J' := G/x_0$. Observe that J' is a planar matching covered graph. By Lemma 2.4, J' is also cycle-extendable. However, J' need not be irreducible; in fact, J' need not be simple. We now introduce the notion of an "osculating bicycle" that will help us in deducing that J' is either simple, or otherwise has exactly two multiple edges.

5.1 Osculating bicycle

A pair of cycles (Q, Q') in a graph G is said to be an *osculating bicycle* if (i) they intersect in precisely one vertex, and (ii) they have the same parities. Furthermore, if each of them is an odd cycle then we refer to (Q, Q') as an *odd* osculating bicycle. Likewise, if each of them is an even cycle then (Q, Q') is an *even* osculating bicycle.

We recall the following definition from Section 2.3. For a vertex v of a graph G, a cycle C (in G-v) is v-isolating if v is an isolated vertex in the graph G-V(C). The next lemma provides an easy sufficient condition to deduce that a graph (obtained via bisplitting a vertex) is not cycle-extendable. We shall find this lemma immensely useful in our proof of the Main Theorem (5.20).

Lemma 5.1 [OSCULATING BICYCLE LEMMA]

Let G be a graph and x_0 denote a vertex of degree two that has two distinct neighbors. If $J' := G/x_0$ has an osculating bicycle (Q,Q') such that neither Q nor Q' is a cycle in G, then $Q \cup Q'$ is an x_0 -isolating even cycle in G.

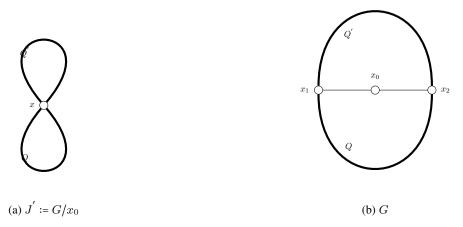


Fig. 10: an illustration for the Osculating Bicycle Lemma and its proof

Proof: We let x_1 and x_2 denote the two distinct neighbors of x_0 in G, and let x denote the contraction vertex in $J' := G/x_0$. Suppose that (Q, Q') is an osculating bicycle in $J' := G/x_0$ such that neither Q nor Q' is a cycle in G. Since Q is not a cycle in G, the vertex x belongs to Q, and one of the two edges of $Q \cap \partial_{J'}(x)$ is incident with x_1 in G and the other edge is incident with x_2 in G; see Figure 10. An analogous statement holds for Q'. Since (Q, Q') is an osculating bicycle in J', we infer that Q and Q' meet precisely at the vertex x in J'. This fact, combined with the fact that their parities are the same, implies that $Q \cup Q'$ is an x_0 -isolating even cycle in G, as shown in Figure 10b.

For distinct vertices u and v of a graph G, we let $\mu_G(u,v)$ denote the number of edges joining u and v. We shall find this notation useful in the proof of the following corollary of the above lemma.

Corollary 5.2 Let G be a simple cycle-extendable graph, x_0 be a vertex of degree two and let $J' := G/x_0$. Then either J' is simple or otherwise J' has precisely two multiple edges. Furthermore, if G is irreducible then the underlying simple graph J of J' is also irreducible.

Proof: Observe that, since G is simple, all multiple edges in $J^{'}$ (if any) are incident with the contraction vertex that we denote by x.

To prove the first part, suppose to the contrary that $J^{'}$ has more than two multiple edges; we consider two cases. First suppose that there exists a vertex $u \in V(J^{'}) - x$ such that $\mu_{J^{'}}(x,u) \geq 3$. Since $J^{'} = G/x_0$, it follows from the pigeonhole principle that G is not simple; contradiction. Otherwise, there exist distinct $u_1, u_2 \in V(J^{'}) - x$ such that $\mu_{J^{'}}(x,u_1) = \mu_{J^{'}}(x,u_2) = 2$. Let e and e' denote distinct edges joining u_1 and x; likewise, let f and f' denote distinct edges joining u_2 and x. Observe that (Q := ee', Q' := ff') is an osculating bicycle in $J^{'}$ and neither Q nor $Q^{'}$ is a cycle in the simple graph G. By the Osculating Bicycle Lemma (5.1), $Q \cup Q'$ is an x_0 -isolating 4-cycle in G; this contradicts the hypothesis that G is cycle-extendable. This proves the first part.

Now suppose that G is irreducible and let x_1 and x_2 denote the neighbors of x_0 ; thus, $d_G(x_1) \geq 3$ and $d_G(x_2) \geq 3$. Consequently, $d_{J'}(x) \geq 4$ and degree two vertices of J' comprise a stable set. If J' = J then we are done. Otherwise, by the first part, there exists a unique vertex $u \in V(J') - x$ such that $\mu_{J'}(x,u) = 2$; let e and e' denote the two edges joining x and u. Adjust notation so that J = J' - e'. Since G is simple, adjust notation so that $e \in \partial_G(x_1)$ and $e' \in \partial_G(x_2)$. Suppose to the contrary that, unlike J', the vertices of degree two of J do not comprise a stable set. Since $d_J(x) \geq 3$, we infer that $d_J(u) = 2$ and that the neighbour of u, distinct from x, also has degree two in J. We choose $f \in \partial_G(x_1) - e - x_1x_0$ and $f' \in \partial_G(x_2) - e' - x_2x_0$. Since $f, f' \in \partial_J(x)$, we let Q denote a conformal cycle containing f and f' in f. Since f and f is a cycle in f in f and neither f nor f is a cycle in f. By the Osculating Bicycle Lemma (5.1), f is an f is an f-isolating cycle in f. This contradicts the hypothesis that f is cycle-extendable, and proves the second part.

We now proceed to describe an important class of bipartite cycle-extendable graphs that will play an important role in our description as well as appreciation of planar cycle-extendable irreducible graphs.

5.2 Half biwheels

A half biwheel is any graph H obtained from an even path P[A, B] with ends, say u and v in A, by introducing a new vertex, say h, and joining h with each vertex in A. Figure 2 shows half biwheels of orders six and eight. The ends of P are called the *corners*, and h is called the *hub*, of H. Observe that any half biwheel is a cycle-extendable bipartite matching covered graph. Also, we allow P to be K_1 ; in this case, H is the smallest half biwheel K_2 , either vertex may be regarded as the hub h and the other one as the (only) corner u = v. Note that C_4 is the second smallest half biwheel; any vertex may be regarded as the hub h and its two distinct neighbors as the corners u and v. In all other cases, the hub and (distinct) corners are uniquely determined. The reader may easily verify the following.

Proposition 5.3 Every half biwheel H is bipartite and cycle-extendable. Furthermore, each path of H, starting at the hub and ending at a corner, is conformal.

In each of the following four sections, we describe a family of nonbipartite planar cycle-extendable irreducible graphs $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2$ and \mathcal{G}_3 . Each member of these families is constructed from $k \geq 1$ half biwheel(s) — by perhaps identifying all of the hubs (only in the case of \mathcal{G}_1) — and introducing additional vertices (only in the case of \mathcal{G}_3) as well as edges. For the sake of brevity, we shall find it convenient to use arithmetic modulo k. The reader may verify that these families are pairwise disjoint.

Additionally, the reader may verify that each member of these families is a subdivision of a 3-connected planar graph. Hence, by Whitney's Theorem (Bondy and Murty, 2008, Theorem 10.28), it admits a unique planar embedding. We will provide planar embeddings of certain small members of these families. Some

of our propositions shall refer to the cyclic ordering of edges (incident at a common vertex) induced by the planar embedding.

For each family, we will state propositions pertaining to the existence of specific even cycles and osculating bicycles. We shall find these very useful in our proof of the Main Theorem in order to invoke the Osculating Bicycle Lemma (5.1) and thus significantly reduce the amount of case analysis required. In particular, propositions pertaining to even cycles shall be useful in the case where x is a cubic vertex of J, as per the notation in the proof of Lemma 5.1 and Figure 10, whereas propositions pertaining to osculating bicycles shall be invoked in the case where x has degree four or more in J.

5.3 Generalized prisms \mathcal{G}_0

In this section, we introduce our first family of graphs, generalized prisms, denoted by \mathcal{G}_0 . Each member of \mathcal{G}_0 , say J, is constructed from the union of k disjoint half biwheels, say $H_i[A_i, B_i]$ for $i \in \{0, 1, \dots, k-1\}$, where k is odd and at least three, each labeled as follows — if the hub h_i belongs to A_i then label h_i as $w_i = x_i$ and the corners as y_i and z_i , and if the hub h_i belongs to B_i then label h_i as $y_i = z_i$ and the corners as w_i and x_i — by adding the following edges: $\alpha_i \coloneqq x_i w_{i+1}$ and $\beta_i \coloneqq y_i z_{i+1}$ for each $0 \le i \le k-1$. Figure 11 depicts an example where the half biwheels H_i are shown in blue.

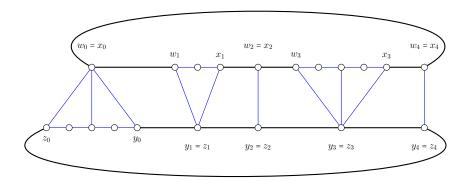


Fig. 11: a generalized prism

Each member J of \mathcal{G}_0 is a near-bipartite planar (matching covered) graph. Furthermore, $R_i \coloneqq \{\alpha_i, \beta_i\}$ denotes a removable doubleton for each $0 \le i \le k-1$, and these are the only removable doubletons. The components of $J - R_0 - R_1 - \ldots - R_{k-1}$ are precisely the half biwheels H_0, H_1, \ldots, H_k . Observe that if each half biwheel H_i is isomorphic to K_2 then J is simply a prism. In this sense, the family \mathcal{G}_0 indeed generalizes the prisms. In order to prove that members of \mathcal{G}_0 are cycle-extendable, we shall find the following lemma useful; the reader may prove it easily.

Lemma 5.4 Let J be a graph, let $(V_0, V_1, \ldots, V_{k-1})$ denote a partition of V(J), let H_i denote the induced subgraph $J[V_i]$ for each $0 \le i \le k-1$, and let L denote a subgraph of J. If $L_i := L \cap H_i$ is a conformal subgraph of H_i for each $0 \le i \le k-1$, then L is a conformal subgraph of J.

We are now ready to prove that generalized prisms are cycle-extendable.

Proposition 5.5 *Each member of* G_0 *is cycle-extendable.*

Proof: Let $J \in \mathcal{G}_0$ and let $R_0, R_1, \ldots, R_{k-1}$ denote its removable doubletons. Let $H_0, H_1, \ldots, H_{k-1}$ denote the (half biwheel) components of $J - R_0 - R_1 - \ldots - R_{k-1}$; adjust notation so that $\partial(V(H_i)) = R_{i-1} \cup R_i$ and let $V_i := V(H_i)$ for each $0 \le i \le k-1$.

Let C denote an even cycle of J. Let us pick $i \in \{0,1,\ldots,k-1\}$, and make some observations. By Lemma 4.2, $|C \cap R_i| \in \{0,2\}$. Since $\partial(V(H_i)) = R_{i-1} \cup R_i$, we infer that $C \cap \partial(V(H_i)) \in \{\varnothing, R_{i-1}, R_i, R_{i-1} \cup R_i\}$. Let $C_i \coloneqq C \cap H_i$. Note that $C \cap \partial(V(H_i)) = \varnothing$ if and only if either $C_i = C$ or C_i is the null subgraph of H_i . Secondly, $C \cap \partial(V(H_i)) \in \{R_{i-1}, R_i\}$ if and only if C_i is a path of the half biwheel H_i that starts at the hub and ends at a corner. Lastly, $C \cap \partial(V(H_i)) = R_{i-1} \cup R_i$ if and only if C_i is a spanning subgraph of H_i . By Proposition 5.3, C_i is a conformal subgraph of H_i . It follows immediately from Lemma 5.4, with C playing the role of C_i , that C_i is a conformal cycle of C_i . Thus C_i is cycle-extendable.

The following property of generalized prisms, pertaining to the existence of even cycles containing a specified cubic vertex but not containing a specified neighbor, is easily verified.

Proposition 5.6 [EVEN CYCLES IN GENERALIZED PRISMS]

Let $J \in \mathcal{G}_0$ and let $R_0, R_1, \ldots, R_{k-1}$ denote its removable doubletons. Let x denote a cubic vertex and let H denote the (half biwheel) component of $J - R_0 - R_1 - \ldots - R_{k-1}$ that contains x. Let w denote any neighbor of x in J so that (i) either $w \notin V(H)$ or (ii) $w \in V(H)$ and $d_J(w) = 2$. Then J - w has an even cycle that contains x.

Proof: We adopt notation from the definition of generalized prisms, and adjust notation so that $H = H_1$. We leave it as an exercise for the reader to locate the desired even cycle in the induced subgraph $J[V(H_0) \cup V(H_1) \cup V(H_2)]$ — by considering various cases depending on whether H_1 is K_2 or not, and depending on the specific choices of x and w.

For any member $J \in \mathcal{G}_0$, it is easy to see that the planar embedding of J has precisely two odd faces whose boundaries are vertex-disjoint and comprise a spanning subgraph of J. We use D and $D^{'}$ to denote these facial cycles. By adjusting notation, one of these cycles contains each α_i whereas the other one contains each β_i , where $\{R_i := \{\alpha_i, \beta_i\}, 0 \le i \le k-1\}$ is the set of all removable doubletons. Using these observation, we now proceed to discuss the existence of specific osculating bicycles in generalized prisms.

Proposition 5.7 [OSCULATING BICYCLES IN GENERALIZED PRISMS]

Let $J \in \mathcal{G}_0$ and let x denote a vertex of degree four or more. Let α_0 and α_1 denote removable doubleton edges in $\partial(x)$ and let $f, f' \in \partial(x) - \{\alpha_0, \alpha_1\}$ so that $\alpha_0, \alpha_1, f', f$ appear in this cyclic order in the planar embedding of J. Then there exist:

- (i) an odd osculating bicycle $(Q,Q^{'})$ such that $\alpha_{0},\alpha_{1}\in E(Q)$ and $f,f^{'}\in E(Q^{'})$, and
- (ii) an even osculating bicycle (C, C') such that $\alpha_0, f \in E(C)$ and $\alpha_1, f' \in E(C')$.

Proof: We adopt notation from the definition of generalized prisms; thus $x = h_1$. Let $f := h_1 t$ and $f' := h_1 t'$. Let D and D' denote the odd facial cycles as discussed earlier, and adjust notation so that $h_1 \in V(D)$.

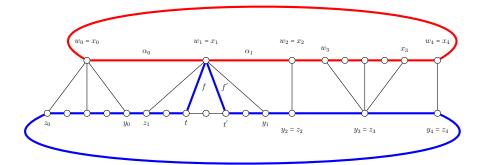


Fig. 12: Illustration for the proof of Proposition 5.7 (i)

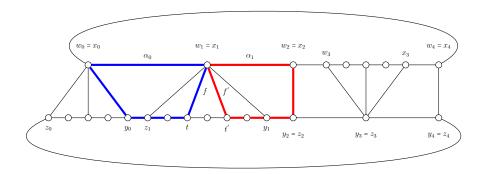


Fig. 13: Illustration for the proof of Proposition 5.7 (ii)

To prove (i), we display an odd osculating bicycle (Q, Q') whose each constituent cycle uses precisely one edge from each of the removable doubletons of J as follows: Q := D and $Q' := D' + f + f' - t(H_1 - h)t'$; see Figure 12.

To prove (ii), we display an even osculating bicycle (C,C') whose each constituent cycle uses either zero or two edges from each of the removable doubletons of J as follows: $C \coloneqq h_1 ft(H_1 - h)z_1\beta_0 y_0 x_0\alpha_0 h_1$ and $C' \coloneqq h_1 f't'(H_1 - h)y_1\beta_1 z_2 w_2\alpha_1 h_1$; see Figure 13.

In each of the following three sections, we will define a family of graphs G_i where $i \in \{1, 2, 3\}$. Each member of G_i will be defined using some number of half biwheels. For each such half biwheel H[A, B], we adopt the convention that h denotes the hub and that u and v denote the corners (that are not necessarily distinct). Also, this convention is extended naturally to accommodate the use of subscripts.

5.4 Generalized wheels \mathcal{G}_1

We describe a family of graphs \mathcal{G}_1 . Each member of \mathcal{G}_1 , say J, is constructed from k disjoint half biwheels, say H_i for $0 \le i \le k-1$, where k is odd and at least three, by identifying all of their hubs into a single vertex h and adding the set of edges $E_3 := \{v_i u_{i+1} : 0 \le i \le k-1\}$. Figure 14 depicts two examples where the half biwheels H_i are shown in blue. We refer to h as a *hub of* J.

Observe that $J - E_3$ is precisely the (bipartite) graph obtained by identifying the hubs of $k = |E_3|$ half biwheels and h is its unique cut vertex. Furthermore, when J is not K_4 , the set E_3 comprises precisely those edges whose both ends are cubic. If each half biwheel H_i (in the above definition) is isomorphic to K_2 , then J is simply a wheel. In this sense, the family \mathcal{G}_1 indeed generalizes wheels. Note that the hub h is the unique vertex of degree four or more except when J is K_4 .

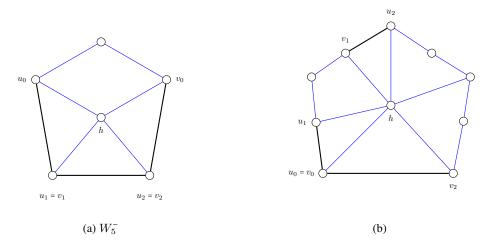


Fig. 14: generalized wheels

Each member J of \mathcal{G}_1 is a planar matching covered graph and $f_{\text{odd}}(J) = k+1$. Consequently, they are not near-bipartite in general; see Proposition 2.10. The graph J is near-bipartite if and only if k=3 and at least one of H_0 , H_1 and H_2 is isomorphic to K_2 . Furthermore, if k=3 and if H_i denotes a half biwheel that is isomorphic to K_2 then $R_i := \{u_i h, v_{i+1} u_{i+2}\}$ is a removable doubleton; it follows that the number of half biwheels (among H_0 , H_1 , and H_2) that are isomorphic to K_2 equals the number of removable doubletons in J.

To see that each member J of \mathcal{G}_1 is cycle-extendable, note that J-h is an odd cycle; consequently, if C is any even cycle, then $h \in V(C)$ and J-V(C) is an odd path.

Proposition 5.8 *Each member of* G_1 *is cycle-extendable.*

We now make an observation pertaining to the existence of even cycles in generalized wheels containing a specified cubic vertex but not containing a specified neighbor.

Proposition 5.9 [EVEN CYCLES IN GENERALIZED WHEELS]

Let h denote the hub of $J \in \mathcal{G}_1 - K_4$ and let E_3 denote the set of edges whose both ends are cubic. Let x

denote a cubic vertex of J and let $w \neq h$ denote a neighbor of x in J. If there is no even cycle in J - w that contains x, then (i) $|E_3| = 3$ and (ii) each of x and w is an isolated vertex in $J - E_3 - h$.

Proof: We adopt notation from the definition of generalized wheels, and adjust notation so that $x \in V(H_1)$ and $w \in V(H_0) \cup V(H_1)$. Assume that there is no even cycle in J - w that contains x.

First, consider the case in which $w \in V(H_1)$. If $x \notin \{u_1, v_1\}$, then the reader may locate an even cycle in $H_1 - w$ that contains x; a contradiction. Otherwise, we may adjust notation so that $x = u_1$. Observe that $Q := hu_1v_0(H_0 - h)u_0v_{k-1}h$ is an even cycle in J - w that contains x; a contradiction.

Next, consider the case in which $w \in V(H_0)$; thus $x = u_1$ and $w = v_0$. If H_1 is not K_2 , then H_1 has an even cycle containing x; this contradicts our assumption. Now suppose that H_1 is K_2 ; in other words, $x = u_1 = v_1$. Observe that $Q := hv_1u_2(H_2 - h)v_2u_3h$ is an even cycle that contains x; furthermore, Q does not contain $w = v_0$ if and only if $u_3 \neq v_0$. Hence, by our assumption, $u_3 = v_0$. It follows from the definition of generalized wheels that conditions (i) and (ii) hold.

Finally, we proceed to discuss the existence of certain osculating bicycles in generalized wheels.

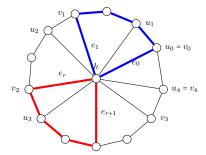


Fig. 15: illustration for the proof of Proposition 5.10 (iii)

Proposition 5.10 [OSCULATING BICYCLES IN GENERALIZED WHEELS]

Let h denote the hub of $J \in \mathcal{G}_1 - K_4$ and let E_3 denote the set of edges whose both ends are cubic. Let H_0, H_1, \ldots, H_k denote the (half biwheel) blocks of $J - E_3$ so that they appear in this cyclic order (around the hub h) in the planar embedding of J, and adjust notation so that $E_3 = \{v_i u_{i+1} : v_i \in V(H_i) \text{ and } u_{i+1} \in V(H_{i+1})\}$. The following statements hold:

- (i) for any r such that $0 \le r < k$, if the four edges hu_0, hv_r, hu_{r+1} and hv_k are pairwise distinct then $(Q := hu_0v_kh, Q' := hv_ru_{r+1}h)$ is an odd osculating bicycle in J.
- (ii) for any $0 \le i < r \le k$, for distinct e and f in $E(H_i) \cap \partial_J(h)$ and distinct e' and f' in $E(H_r) \cap \partial_J(h)$, there exists an even osculating bicycle (Q,Q') in J such that $e, f \in E(Q)$ and $e', f' \in E(Q')$.
- (iii) for any $0 < r \le k$, for any four pairwise distinct edges $e_0, e_1, e_r, e_{r+1} \in \partial_J(h)$ that appear in this cyclic order in the planar embedding of J, such that $e_i \in E(H_i)$, there exists an odd osculating bicycle (Q, Q') in J such that $e_0, e_1 \in E(Q)$ and $e_r, e_{r+1} \in E(Q')$.

Proof: The reader may easily verify statement (i), whereas statement (ii) follows from the fact that any two adjacent edges (in H_i ; likewise, in H_r) belong to an even cycle.

We now display an odd osculating bicycle (Q, Q') for statement (iii). For each $i \in \{0, 1, r, r+1\}$, we let w_i denote the end of e_i that is distinct from h. We define them as follows: $Q := hw_0(H_0 - h)v_0u_1(H_1 - h)w_1h$ and $Q' := hw_r(H_r - h)v_ru_{r+1}(H_{r+1} - h)w_{r+1}h$; see Figure 15. Both cycles use precisely one edge from the set E_3 ; thus, they are odd cycles.

5.5 Double half biwheels \mathcal{G}_2

This section describes another family of graphs denoted as \mathcal{G}_2 . Each member of \mathcal{G}_2 , say J, is constructed from the disjoint union of two half biwheels neither of which is isomorphic to K_2 , say H_0 and H_1 , by adding the following four edges: $\alpha_0 := h_1 h_0$, $\beta_0 := v_1 v_0$, $\alpha_1 := u_0 h_1$, and $\beta_1 := h_0 u_1$. Figure 16 depicts an example where the half biwheels H_0 and H_1 are shown in blue. (The reader may verify that if any of H_0 and H_1 is allowed to be isomorphic to K_2 , then J is a generalized wheel.)

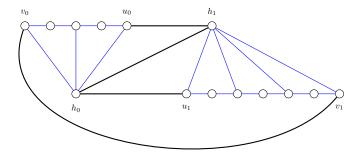


Fig. 16: a double half biwheel

The reader may observe that, in the above definition, the half biwheels H_0 and H_1 are interchangeable and that h_0 and h_1 are the only vertices of degree four or more. Each member of \mathcal{G}_2 , say J is a near-bipartite planar (matching covered) graph. Furthermore, $R_0 = \{\alpha_0, \beta_0\}$ and $R_1 = \{\alpha_1, \beta_1\}$ are the only removable doubletons, and the components of $J - R_0 - R_1$ are precisely the half biwheels H_0 and H_1 . Using ideas similar to the ones in the proof of Proposition 5.5, the reader may verify the following.

Proposition 5.11 *Each member of* G_2 *is cycle-extendable.*

As usual, we make an observation concerning the existence of even cycles in double half biwheels containing a specified cubic vertex but not containing a specified neighbor. We leave its proof as an exercise for the reader.

Proposition 5.12 [EVEN CYCLES IN DOUBLE HALF BIWHEELS]

Let $J \in \mathcal{G}_2$ and let R_0, R_1 denote the removable doubletons of J. Let x denote a cubic vertex and let H_1 denote the (half biwheel) component of $J - R_0 - R_1$ that contains x. Let w denote a neighbor of x in J so

that (i) either $w \notin V(H_1)$ or (ii) $w \in V(H_1)$ and $d_J(w) = 2$. Then, J - w has an even cycle that contains x.

We now proceed to state two observations pertaining to the existence of osculating bicycles; the first of these deals with vertices of degree four or more.

Proposition 5.13 [OSCULATING BICYCLES IN DOUBLE HALF BIWHEELS - I]

Let $J \in \mathcal{G}_2$ and let x denote a vertex of degree four or more. Let α_0 and α_1 denote removable doubleton edges in $\partial(x)$ and let $f, f' \in \partial(x) - \{\alpha_0, \alpha_1\}$ so that $\alpha_0, \alpha_1, f', f$ appear in this cyclic order in the planar embedding of J. Then there exists an odd osculating bicycle (Q, Q') such that $\alpha_0, f \in E(Q)$ and $\alpha_1, f' \in E(Q')$.

Proof: We adopt notation from the definition of double half biwheels; thus $x = h_1$. Let $f := h_1 w$ and $f' := h_1 w'$. We display an odd osculating bicycle (Q, Q') whose each constituent cycle uses precisely one edge from each of the two removable doubletons of J as follows: $Q := h_1 f w (H_1 - h_1) u_1 \beta_1 h_0 \alpha_0 h_1$ and $Q' := h_1 f' w' (H_1 - h_1) v_1 \beta_0 v_0 (H_0 - h_0) u_0 \alpha_1 h_1$. Figure 17a shows an illustration.

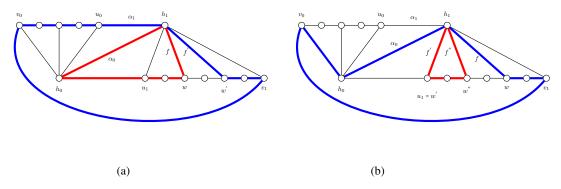


Fig. 17: illustration for the proofs of Propositions 5.13 and 5.14

The next proposition deals with vertices of degree five or more.

Proposition 5.14 [OSCULATING BICYCLES IN DOUBLE HALF BIWHEELS - II]

Let $J \in \mathcal{G}_2$ and let x denote a vertex of degree five or more. Let α_0 and α_1 denote removable doubleton edges in $\partial(x)$ and let $f, f', f'' \in \partial(x) - \{\alpha_0, \alpha_1\}$ so that $\alpha_1, \alpha_0, f', f'', f$ appear in this cyclic order in the planar embedding of J. Then there exists an even osculating bicycle (Q, Q') such that $\alpha_0, f \in E(Q)$ and $f', f'' \in E(Q')$.

Proof: We adopt notation from the definition of double half biwheels; thus $x = h_1$. Let $f := h_1 w$, $f' := h_1 w'$ and $f'' := h_1 w''$. We display an even osculating bicycle (Q, Q') whose each constituent cycle meets each of the two removable doubletons in zero or two edges as follows: $Q := h_1 f w (H_1 - h_1) v_1 \beta_0 v_0 h_0 \alpha_0 h_1$ and $Q' := h_1 f' w' (H_1 - h_1) w'' f'' h_1$. See Figure 17b for an example.

5.6 Hexagon half biwheels \mathcal{G}_3

This section introduces our last family of graphs denoted as \mathcal{G}_3 . A member of \mathcal{G}_3 , say J, is obtained from the disjoint union of a hexagon (that is, a 6-cycle) $H_0 := a_0a_1a_2a_3a_4a_5a_0$ and a half biwheel H_1 by adding the following edges: $\alpha_0 := h_1a_4$, $\beta_0 := v_1a_1$, $\alpha_1 := a_3h_1$ and $\beta_1 := a_0u_1$. Figure 18 depicts two examples where H_0 and H_1 are shown in blue. Observe that if H_1 is isomorphic to K_2 then J is the graph R_8 .

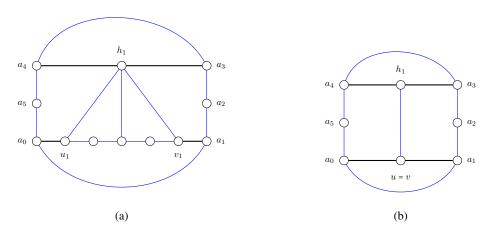


Fig. 18: hexagon half biwheels

The reader may easily verify that each member of \mathcal{G}_3 , say J, is also near-bipartite (matching covered) graph and $R_0 = \{\alpha_0, \beta_0\}$, $R_1 = \{\alpha_1, \beta_1\}$ are the only removable doubletons. Furthermore, the components of $J - R_0 - R_1$ are precisely the 6-cycle H_0 and the half biwheel H_1 . Using ideas similar to the ones in the proof of Proposition 5.5, the reader may verify the following.

Proposition 5.15 *Each member of* G_3 *is cycle-extendable.*

We now proceed to make a couple of observations pertaining to the existence of even cycles containing a specified cubic vertex but not containing a specified neighbor; the first of these considers the case in which the cubic vertex belongs to the hexagon. We leave their proofs as exercises for the reader.

Proposition 5.16 [EVEN CYCLES IN HEXAGON HALF BIWHEELS - I]

Let $J \in \mathcal{G}_3$ and let R_0, R_1 denote the removable doubletons of J. Let H_0 denote the component of $J - R_0 - R_1$ that is isomorphic to a 6-cycle. Let x denote a cubic vertex such that $x \in V(H_0)$, and w denote any neighbor of x. Then J - w has an even cycle that contains x.

The next proposition considers the case in which the cubic vertex belongs to the half biwheel.

Proposition 5.17 [EVEN CYCLES IN HEXAGON HALF BIWHEELS - II]

Let $J \in \mathcal{G}_3$ and let R_0, R_1 denote the removable doubletons of J. Let H_1 denote the component of $J - R_0 - R_1$ that is a half biwheel. Let x denote a cubic vertex such that $x \in V(H_1)$ and w denote a neighbor of x in J so that (i) either $w \notin V(H_1)$ or (ii) $w \in V(H_1)$ and $d_J(w) = 2$. Then J - w has an even cycle that contains x.

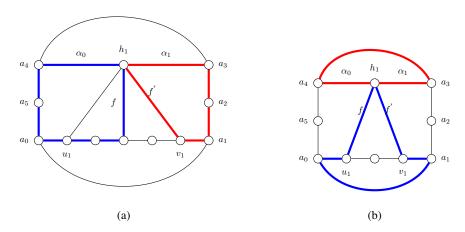


Fig. 19: illustration for the proof of Proposition 5.18

Finally, we discuss the existence of specific osculating bicycles in hexagon half biwheels.

Proposition 5.18 [OSCULATING BICYCLES IN HEXAGON HALF BIWHEELS]

Let $J \in \mathcal{G}_3$ and let x denote a vertex of degree four or more. Let α_0 and α_1 denote removable doubleton edges in $\partial(x)$ and let $f, f' \in \partial(x) - \{\alpha_0, \alpha_1\}$ so that $\alpha_0, \alpha_1, f', f$ appear in this cyclic order in the planar embedding of J. Then there exist:

- (i) an odd osculating bicycle (Q,Q') such that $\alpha_0,\alpha_1 \in E(Q)$ and $f,f' \in E(Q')$, and
- (ii) an odd osculating bicycle (C, C') such that $\alpha_0, f \in E(C)$ and $\alpha_1, f' \in E(C')$.

Proof: We adopt notation from the definition of hexagon half biwheels; thus $x = h_1$. Let $f := h_1 w$ and $f' := h_1 w'$.

To prove (i), we display an odd osculating bicycle (Q,Q'), whose each constituent cycle uses precisely one edge from each of the two removable doubletons of J as follows: $Q := h_1 w (H_1 - h_1) u_1 \beta_1 a_0 a_5 a_4 \alpha_0 h_1$ and $Q' := h_1 w' (H_1 - h_1) v_1 \beta_0 a_1 a_2 a_3 \alpha_1 h_1$. Figure 19a shows an illustration.

Likewise, to prove (ii), we display an odd osculating bicycle (C, C'), whose each constituent cycle uses precisely one edge from each of the two removable doubletons of J as follows: $C := h_1 \alpha_1 a_3 a_4 \alpha_0 h_1$ and $C' := h_1 w (H_1 - h_1) u_1 \beta_1 a_0 a_1 \beta_0 v_1 (H_1 - h_1) w' h_1$. Figure 19b shows an illustration.

5.7 Main Theorem: planar cycle-extendable irreducible graphs

The following observation pertaining to all of the families defined in the previous section may be easily verified by the reader; see (Bondy and Murty, 2008, Theorem 10.7, Corollary 10.8).

Proposition 5.19 Let $J \in \mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ and let x denote a vertex of degree four or more. Then J - x is a 2-connected (planar) graph. Consequently, the neighbors of x lie on a cycle of J - x.

We are now ready to state and prove the Main Theorem.

Theorem 5.20 [Planar Cycle-Extendable Irreducible Graphs]

A planar irreducible matching covered graph G is cycle-extendable if and only if G belongs to $\{K_2\} \cup \mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$.

Proof: Propositions 5.5, 5.8, 5.11 and 5.15 prove the reverse implication. For the forward implication, let G be a planar irreducible matching covered graph that is cycle-extendable. We proceed by induction on the number of edges. Firstly, if $\delta(G) \geq 3$ then, by Corollary 2.13 and Corollary 4.10, G is either a wheel or a prism; thus $G \in \mathcal{G}_0 \cup \mathcal{G}_1$.

Now suppose that G has a vertex of degree two, say x_0 , and let x_1 and x_2 denote its neighbors. Let $J^{'} := G/x_0$ and let x denote its bicontraction vertex. Since G is irreducible, $d_{J^{'}}(x) \geq 4$. Since G is cycle-extendable, by the first part of Corollary 5.2, $J^{'}$ has at most two multiple edges. Clearly, multiple edges (if any) are incident at x. In case $J^{'}$ is not simple we use e and $e^{'}$ to denote its multiple edges. We let J denote the underlying simple graph of $J^{'}$. Furthermore, either $J = J^{'}$ or otherwise we adjust notation so that $J = J^{'} - e^{'}$. Thus, $d_{J}(x) \geq 3$. Since G is irreducible, by the second part of Corollary 5.2, J is also irreducible.

Clearly, J and $J^{'}$ are planar matching covered graphs. By Lemma 2.4, they are also cycle-extendable. Since J is irreducible and $d_{J}(x) \ge 3$, by the induction hypothesis, J belongs to $\mathcal{G}_{0} \cup \mathcal{G}_{1} \cup \mathcal{G}_{2} \cup \mathcal{G}_{3}$.

We will divide the proof into cases depending on the degree of vertex x and whether J belongs to \mathcal{G}_0 , \mathcal{G}_1 , \mathcal{G}_2 or \mathcal{G}_3 . Within each case, we will consider various subcases. In each subcase, we will either display an osculating bicycle (Q,Q') in J' such that neither of its constituent cycles is a cycle in G and invoke Lemma 5.1 to arrive at a contradiction (by inferring that G is not cycle-extendable), or otherwise conclude that G belongs to \mathcal{G}_0 , \mathcal{G}_1 , \mathcal{G}_2 or \mathcal{G}_3 . Depending on the family that J belongs to, we shall adopt the following notation.

Notation 5.21 If $J \in \mathcal{G}_0 \cup \mathcal{G}_2 \cup \mathcal{G}_3$, then we let R_0, R_1, \dots, R_k denote all of the removable doubletons of J, and H_0, H_1, \dots, H_k denote the components of $G - R_0 - R_1 - \dots - R_k$. Furthermore, if $J \in \mathcal{G}_3$, then we let H_0 denote the component that is a 6-cycle.

Notation 5.22 If $J \in \mathcal{G}_1 - K_4$, then we let E_3 denote the set of edges whose both ends are cubic, h denote the unique cut vertex of $J - E_3$, and H_0, H_1, \ldots, H_k denote the (half biwheel) blocks of $J - E_3$ so that they appear in this cyclic order (around the hub h) in the planar embedding of J and adjust notation so that $E_3 = \{v_i u_{i+1} : v_i \in V(H_i) \text{ and } u_{i+1} \in V(H_{i+1})\}$.

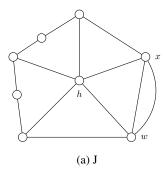
Case 1: $d_J(x) = 3$.

Observe that $J^{'}$ is obtained from J by adding a multiple edge $e^{'}$ incident with x. Let $e=e^{'}=xw$ in $J^{'}$ and let $Q^{'}$ denote 2-cycle $ee^{'}$. Observe that, since $d_{J^{'}}(x)=4$ and since G is simple, there is a unique way to bisplit x and obtain the graph G from $J^{'}$. Consequently, if there exists an even cycle Q in $J^{'}$ such that $Q,Q^{'}$ is an osculating bicycle in $J^{'}$, then neither Q nor $Q^{'}$ is a cycle in G; in this case, by Lemma 5.1, we arrive at a contradiction to the assumption that G is cycle-extendable. Observe that Q exists if and only if J-w has an even cycle that contains x. From the preceding discussion, it suffices to consider only those cases in which J-w has no even cycle that contains x. In the following two paragraphs, we shall heavily exploit this observation.

If $J \in \mathcal{G}_l$ where $l \in \{0, 2, 3\}$, we adopt Notation 5.21. If l = 3, we invoke Proposition 5.16 to infer that $x \notin V(H_0)$. Now, we may adjust notation so that $x \in V(H_1)$. Depending on the value of l, we invoke Proposition 5.6, or 5.12, or 5.17, to infer that $d_J(w) \ge 3$ and that w is a hub of the same half biwheel H_1 .

Since the bisplitting is unique (as noted earlier), the reader may easily verify that G belongs to the same family \mathcal{G}_l .

Now consider the case in which $J \in \mathcal{G}_1$. If $J = K_4$ then the reader may easily verify that $G = W_5^-$. Now suppose that $J \neq K_4$ and adopt Notation 5.22. We invoke Proposition 5.9 to infer that either w = h, or otherwise, $|E_3| = 3$ and $J - E_3 - h$ has precisely two isolated vertices — namely, x and y. If y = h, then using the fact that bisplitting of y is unique, one may easily see that $y \in \mathcal{G}_1$. We consider the remaining case below.



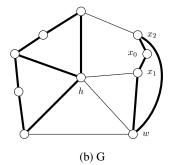


Fig. 20: the case in which $J \in \mathcal{G}_1$ and $G \in \mathcal{G}_2$

Suppose that $|E_3| = 3$ and that $J - E_3 - h$ has precisely two isolated vertices — namely, x and w. Based on the description of \mathcal{G}_1 , this is equivalent to saying that k = 2 and precisely two of the half biwheels H_0 , H_1 and H_2 are isomorphic to K_2 . We adjust notation so that each of H_1 and H_2 is isomorphic to K_2 and that $x \in V(H_1)$ and $w \in V(H_2)$. Since the bisplitting is unique, observe that $H := G[w, x_1, x_0, x_2]$ is isomorphic to C_4 , and that G is a member of G_2 — wherein G_2 wherein G_3 (with hub G_4) are the half biwheels as per the definition of double half biwheels in Section 5.5. See Figure 20.

Case 2: $d_J(x) \ge 4$.

Note that, unlike the previous case, we must consider all possible bisplittings of the vertex x that lead to a planar graph (instead of just one bisplitting). Observe that $\partial_{J'}(x) = \partial_G(\{x_0, x_1, x_2\})$. It follows from Proposition 5.19 that the cyclic order of these edges must be the same in G around the set $\{x_0, x_1, x_2\}$ as the cyclic order in J'; otherwise, it is easy to see that the resulting graph has a subdivision of $K_{3,3}$.

We consider subcases depending on whether J belongs to $\mathcal{G}_0 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ or to \mathcal{G}_1 . Note that an osculating bicycle in J also exists in J. It is for this reason that, when applicable, we simply display an osculating bicycle in J with the desired properties in order to arrive at a contradiction as discussed earlier.

Case 2.1: $J \in \mathcal{G}_0 \cup \mathcal{G}_2 \cup \mathcal{G}_3$.

We adopt Notation 5.21. Since $d_J(x) \ge 4$, we may adjust notation so that x is a hub of H_1 that is not isomorphic to K_2 . Let α_0 and α_1 denote the removable doubleton edges in $\partial_J(x)$. Since G is obtained from J' by bisplitting the vertex x, the edges α_0 and α_1 may or may not be adjacent in G; we consider these cases separately.

First suppose that α_0 and α_1 are adjacent in G and adjust notation so that $\alpha_0, \alpha_1 \in \partial_G(x_1)$. We choose $f, f' \in \partial_G(x_2) - x_2x_0$ so that $\alpha_0, \alpha_1, f', f$ appear in this cyclic order in the planar embedding of J. Depending on whether J belongs to $\mathcal{G}_0, \mathcal{G}_2$ or \mathcal{G}_3 , we invoke Proposition 5.7 (ii), 5.13 or 5.18 (ii), in order to locate the desired osculating bicycle (Q, Q') in J.

Now suppose that α_0 and α_1 are nonadjacent in G and adjust notation so that $\alpha_0 \in \partial(x_1)$ and $\alpha_1 \in \partial(x_2)$. We consider two subcases depending on whether $J \in \mathcal{G}_0 \cup \mathcal{G}_3$ or $J \in \mathcal{G}_2$. If $J \in \mathcal{G}_0 \cup \mathcal{G}_3$, we choose $f \in \partial_G(x_1) - x_1x_0 - \alpha_0$ and $f' \in \partial_G(x_2) - x_2x_0 - \alpha_1$ so that $\alpha_0, \alpha_1, f', f$ appear in this cyclic order in the planar embedding of J. Depending on whether J belongs to \mathcal{G}_0 or \mathcal{G}_3 , we invoke Proposition 5.7 (i) or 5.18 (i), in order to locate the desired osculating bicycle (Q, Q') in J. Henceforth $J \in \mathcal{G}_2$; we consider two subcases depending on whether $d_J(x) = 4$ or $d_J(x) \ge 5$.

Let us first suppose that $d_J(x) = 4$. Equivalently, H_1 is isomorphic to a 4-cycle, say $h_1u_1yv_1h_1$, where $h_1 := x$. Since $\alpha_0 \in \partial_G(x_1)$ and $\alpha_1 \in \partial_G(x_2)$, observe that $x_1u_1yv_1x_2x_0x_1$ is a 6-cycle in G and that $G \in \mathcal{G}_3$ as per the definition of hexagon half biwheels in Section 5.6; see Figure 21.

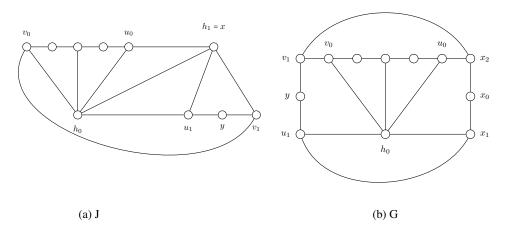


Fig. 21: the case in which $J \in \mathcal{G}_2$ and $G \in \mathcal{G}_3$

Lastly, suppose that $d_J(x) \ge 5$; consequently, at least one of x_1 and x_2 has degree four or more. Adjust notation so that $d_G(x_2) \ge 4$. Now, we may choose $f' \in \partial_G(x_1) - x_1x_0 - \alpha_0$ and distinct $f'', f \in \partial_G(x_2) - x_2x_0 - \alpha_1$ so that $\alpha_1, \alpha_0, f', f'', f$ appear in this cyclic order in the planar embedding of J. We invoke Proposition 5.14 to locate the desired osculating bicycle (Q, Q') in J.

Case 2.2: $J \in \mathcal{G}_1$.

We adopt Notation 5.22. Since $d_J(x) \ge 4$, the vertex x is precisely the hub h of J. Note that, for each $0 \le i \le k$, the edge set of the half biwheel H_i (in J) may or may not form a half biwheel in G. Furthermore, $E(H_i)$ forms a half biwheel in G if and only if all edges in $E(H_i) \cap \partial_J(x)$ are incident with (precisely) one of x_1 and x_2 (in G); in this case, we say that the half biwheel H_i remains intact; otherwise we say that the half biwheel H_i is destroyed. It follows from planarity that the number of half biwheels that are destroyed is either zero, two or one; we shall consider these cases separately in that order.

First we consider the case in which zero half biwheels are destroyed. We may adjust notation so that there exists $0 \le r < k$ such that: (i) for each $0 \le i \le r$, all edges in $E(H_i) \cap \partial_J(x)$ are incident with x_1 (in G), and likewise (ii) for each $r < i \le k$, all edges in $E(H_i) \cap \partial_J(x)$ are incident with x_2 . It follows from the irreducibility of G that the four edges xu_0, xv_r, xu_{r+1} and xv_k are pairwise distinct. We invoke

Proposition 5.10 (i) to locate the desired osculating bicycle (Q, Q') in J.

Next we consider the case in which two half biwheels, say H_i and H_r , are destroyed. Let e denote an edge in $E(H_i) \cap \partial_J(x)$ that is incident with x_1 (in G) and f denote an edge in $E(H_i) \cap \partial_J(x)$ that is incident with x_2 . Likewise, let e' denote an edge in $E(H_r) \cap \partial_J(x)$ that is incident with x_1 and f' denote an edge in $E(H_r) \cap \partial_J(x)$ that is incident with x_2 . We invoke Proposition 5.10 (ii) to locate the desired osculating bicycle (Q,Q') in J.

Lastly we consider the case in which precisely one half biwheel, say H_0 , is destroyed. we may adjust notation so that there exists $0 < r \le k$ such that: (i) for each $0 < i \le r$, all edges in $E(H_i) \cap \partial_J(x)$ are incident with x_1 (in G), and likewise (ii) for each $r < i \le k$, all edges in $E(H_i) \cap \partial_J(x)$ are incident with x_2 . Unless H_0 is isomorphic to C_4 , k = 2, each of H_1 and H_2 is isomorphic to K_2 , and r = 1, the reader may verify that it is possible to choose $e_0, e_1, e_r, e_{r+1} \in \partial_J(h)$ that satisfy the conditions stated in Proposition 5.10 (iii) so that e_0 and e_1 are nonadjacent in G and, likewise, e_r and e_{r+1} are nonadjacent in G; we invoke Proposition 5.10 (iii) to locate the desired osculating bicycle (Q, Q') in J.

Now suppose that H_0 is isomorphic to C_4 , k=2, each of H_1 and H_2 is isomorphic to K_2 , and r=1. Observe that J is the graph W_5^- as per the labels shown in Figure 14a. First suppose that $J \neq J'$; up to symmetry, either $e^{'}=hu_1$ or otherwise $e^{'}=hu_0$. In the former case, since H_0 is destroyed and G is simple, $(Q\coloneqq H_0,Q^{'}\coloneqq ee^{'})$ is the desired even osculating bicycle in J'. In the latter case, it follows from irreducibility and planarity that $(Q\coloneqq hv_0u_2u_1h,Q^{'}\coloneqq ee^{'})$ is the desired even osculating bicycle in J'. Finally, if J=J', then it follows from the facts that H_0 is destroyed and G is irreducible, that G is the graph $R_8^- \in \mathcal{G}_3$ shown in Figure 18b.

This completes our proof of the Main Theorem (5.20).

We conclude our paper with the following corollaries of our Main Theorem (5.20) and Proposition 1.5.

Corollary 5.23 A planar bipartite matching covered graph G is cycle-extendable if and only if any irreducible graph H, obtained from G by repeated applications of series and parallel reductions, is isomorphic to K_2 .

Corollary 5.24 A planar nonbipartite matching covered graph G is cycle-extendable if and only if any irreducible graph H, obtained from G by repeated applications of series and parallel reductions, belongs to $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$.

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