Discrete Mathematics and Theoretical Computer Science ....
.... **vol. 27:1 Permutation Patterns 2024 #13 (2025)**

Pattern avoidance in nonnesting permutations

Sergi Elizalde[1*] Amya Luo[2†]

[1] *Department of Mathematics, Dartmouth College, Hanover, NH, USA*
[2] *School of Mathematics, Georgia Institute of technology, Atlanta, GA, USA*

revisions 30th May 2025, 3rd Sep. 2025; accepted 4th Sep. 2025.

Nonnesting permutations are permutations of the multiset $\{1, 1, 2, 2, \dots, n, n\}$ that avoid subsequences of the form $abba$ for any $a \neq b$. These permutations have recently been studied in connection to noncrossing (also called quasi-Stirling) permutations, which are those that avoid subsequences of the form $abab$, and in turn generalize the well-known Stirling permutations. Inspired by the work by Archer et al. on pattern avoidance in noncrossing permutations, we consider the analogous problem in the nonnesting case. We enumerate nonnesting permutations that avoid each set of two or more patterns of length 3, as well as those that avoid some sets of patterns of length 4. We obtain closed formulas and generating functions, some of which involve unexpected appearances of the Catalan and Fibonacci numbers. Our proofs rely on decompositions, recurrences, and bijections.

Keywords: nonnesting permutation, pattern avoidance, nonnesting matching

1 Introduction

Let $[n] = \{1, 2, \dots, n\}$, and let $\mathcal{S}_n$ be the set of permutations of $[n]$. We denote by $[n] \sqcup [n] = \{1, 1, 2, 2, \dots, n, n\}$ the multiset consisting of two copies of each integer between 1 and n.

Given two words $\pi = \pi_1\pi_2 \dots \pi_m$ and $\sigma = \sigma_1\sigma_2 \dots \sigma_k$ over the positive integers $\mathbb{N}$, we say that π *contains* the pattern σ if there exist indices $1 \leq i_1 < i_2 < \cdots < i_k \leq m$ such that the subsequence $\pi_{i_1}\pi_{i_2} \dots \pi_{i_k}$ is in the same relative order as σ, that is,

- $\pi_{i_r} < \pi_{i_s}$ if and only if $\sigma_r < \sigma_s$, and
- $\pi_{i_r} = \pi_{i_s}$ if and only if $\sigma_r = \sigma_s$,

for all $r, s \in [k]$. This subsequence is called an *occurrence* of σ. If π does not contain σ, we say that π *avoids* the pattern σ.

A *Stirling permutation* is a permutation π of $[n] \sqcup [n]$ that avoids the pattern 212; equivalently, there do not exist indices $1 \leq i_1 < i_2 < i_3 \leq 2n$ such that $\pi_{i_2} < \pi_{i_1} = \pi_{i_3}$. Stirling permutations were introduced by Gessel and Stanley (1978) in connection to certain generating functions for Stirling numbers of the second kind.

*Partially supported by Simons Collaboration Grant #929653
†Partially supported by a Presidential Scholarship from Dartmouth College

ISSN 1365–8050

A *quasi-Stirling permutation* is a permutation π of $[n] \sqcup [n]$ that avoids the patterns 1212 and 2121; equivalently, there do not exist indices $1 \leq i_1 < i_2 < i_3 < i_4 \leq 2n$ such that $\pi_{i_1} = \pi_{i_3}$ and $\pi_{i_2} = \pi_{i_4}$. Quasi-Stirling permutations were introduced by Archer et al. (2019) as a generalization of Stirling permutations that arises in connection with labeled trees, and were further studied by Elizalde (2021).

One can view permutations π of $[n] \sqcup [n]$ as labeled matchings of $[2n]$, by placing an arc with label ℓ between i and j if $\pi_i = \pi_j = \ell$. With this interpretation, a permutation of $[n] \sqcup [n]$ is quasi-Stirling if and only if the corresponding matching is noncrossing, i.e., there are no two arcs (i_1, i_3) and (i_2, i_4) where $i_1 < i_2 < i_3 < i_4$. For this reason, quasi-Stirling permutations are also called *noncrossing permutations* in Elizalde (2024).

With this perspective, it is natural to consider permutations of $[n] \sqcup [n]$ whose corresponding matching is nonnesting, i.e., there are no two arcs (i_1, i_4) and (i_2, i_3) where $i_1 < i_2 < i_3 < i_4$. They can be defined as permutations of $[n] \sqcup [n]$ that avoid the patterns 1221 and 2112. Following Elizalde (2024), we call these *nonnesting permutations*, and we denote by $\mathcal{C}_n$ the set of nonnesting permutations of $[n] \sqcup [n]$. For example, as shown in Figure 1, $1521352434 \in \mathcal{C}_5$, but $13241342 \notin \mathcal{C}_4$, because the subsequence 2442 is in the same relative order as 1221. It is well known (see Stanley (2015)) that both noncrossing and nonnesting matchings of $[2n]$ are counted by the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Since there are $n!$ ways to assign the labels to the arcs, it follows (see Elizalde (2024)) that both the number noncrossing and the number of nonnesting permutations of $[n] \sqcup [n]$ are given by

$$|\mathcal{C}_n| = n!C_n = \frac{(2n)!}{(n+1)!}.$$

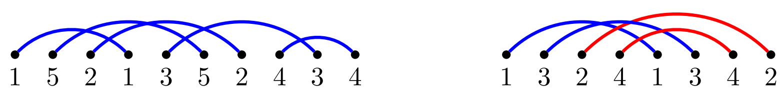

Fig. 1: The permutation 1521352434 is nonnesting, but the permutation 13241342 is not.

In Archer et al. (2019), the authors consider quasi-Stirling permutations that avoid other patterns. Specifically, they enumerate quasi-Stirling (i.e. noncrossing) permutations that avoid any set of at least two elements from $\mathcal{S}_3$. The goal of this paper is to extend the results from Archer et al. (2019) to the nonnesting case, by providing the enumeration of nonnesting permutations that avoid any set of at least two elements from $\mathcal{S}_3$, as well as those that avoid some patterns of length 4.

The known bijections between noncrossing and nonnesting matchings (see e.g. Athanasiadis (1998)), when extended to permutations of $[n]\sqcup[n]$, do not generally behave well with respect to pattern avoidance. In particular, with the exception described in Remark 2.10, there is no straightforward way to translate the results from Archer et al. (2019) to our setting. This is reflected in the fact that the enumeration formulas that we obtain in the nonnesting case are mostly different from those in (Archer et al., 2019, Fig. 6).

Noncrossing and nonnesting permutations also have a very different behavior when enumerated with respect to the number of descents. For noncrossing permutations, this refined enumeration is obtained in Elizalde (2021) by using a recursive decomposition of certain rooted trees to derive an implicit formula for the corresponding bivariate generating function. On the other hand, it is shown in Elizalde (2024) that the distribution of the number of descents on nonnesting permutations has some unexpected properties, such as being symmetric, as well as close connections to standard Young tableaux (see Elizalde (2025)).

As we will discuss in Section 4, analogous properties hold for nonnesting permutations avoiding certain patterns.

Additional motivation for the study of pattern-avoiding nonnesting permutations comes from Bernardi's work on deformations of the braid arrangement (see Bernardi (2018)). He shows that certain configurations, called *annotated* 1*-sketches*, naturally index the regions of the Catalan arrangement, and that certain subsets of them index the regions of other important hyperplane arrangements. It turns out that annotated 1-sketches are precisely nonnesting permutations, and that the relevant subsets can be described as nonnesting permutations avoiding *vincular patterns*, which are patterns where some entries are required to be adjacent in an occurrence. For example, regions of the semiorder arrangement are indexed by permutations $\pi \in \mathcal{C}_n$ with no subsequence $\pi_i\pi_{i+1}\pi_j\pi_{j+1}$ such that $\pi_i = \pi_j < \pi_{i+1} = \pi_{j+1}$ (in vincular pattern notation, we say that π avoids 12-12), regions of the Shi arrangement are indexed by permutations $\pi \in \mathcal{C}_n$ with no subsequence $\pi_i\pi_j\pi_{j+1}\pi_k$ such that $\pi_i = \pi_j < \pi_{j+1} = \pi_k$ (we say that π avoids 1-12-2), and regions of the Linial arrangement are indexed by permutations $\pi \in \mathcal{C}_n$ that avoid both 12-12 and 1-12-2. The numbers of regions of such hyperplane arrangements are well known, so one immediately deduces formulas for the number of nonnesting permutations avoiding these specific vincular patterns. This suggests the problem of enumerating nonnesting permutations that avoid other patterns. In this paper we will tackle this problem for the case of classical patterns, that is, with no adjacency requirements. In general, the number of permutations avoiding a vincular pattern is bounded from below by the number of permutations avoiding the corresponding classical pattern where the adjacency requirements have been removed.

In Section 2, we study nonnesting permutations that avoid sets of patterns of length 3, completing the enumeration for all sets $\Lambda \subseteq \mathcal{S}_3$ consisting of at least 2 patterns, in analogy with the work in Archer et al. (2019) for the noncrossing case. In Section 3, we enumerate nonnesting permutations avoiding several sets of patterns of length 4, which often require more complicated proofs. Our results include some unexpected new interpretations of the Catalan numbers. We conclude with a few conjectures in Section 4.

The proof techniques include bijections, generating functions (both ordinary and exponential), and decompositions of the permutations into smaller pieces obtained by analyzing their structure, which often give rise to recurrences or summation formulas.

Let us finish this section by introducing some notation. Given a set Λ of finite words over $\mathbb{N}$, let $\mathcal{C}_n(\Lambda)$ denote the set of permutations in $\mathcal{C}_n$ that avoid all the patterns in Λ, and let its cardinality be $c_n(\Lambda) = |\mathcal{C}_n(\Lambda)|$. If $\Lambda = \{\sigma, \tau, \dots\}$, we often write $\mathcal{C}_n(\sigma, \tau, \dots)$ instead of $\mathcal{C}_n(\{\sigma, \tau, \dots\})$.

For any word $\alpha = \alpha_1\alpha_2\dots\alpha_k$ over $\mathbb{N}$, define its reversal by $\alpha^r = \alpha_k \dots \alpha_2\alpha_1$. If n is the largest entry of α, define its complement α^c to be the word whose ith entry is $n + 1 - \alpha_i$ for all $i \in [k]$. The composition of these two operations gives the reverse-complement α^{rc}. Clearly, π avoiding σ is equivalent to π^r avoiding σ^r, to π^c avoiding σ^c, and to π^{rc} avoiding σ^{rc}. In particular, the set $\mathcal{C}_n$ is closed under the operations of reversal and complementation.

Denote by $\mathrm{st}(\alpha)$ the *standardization* of α, which is obtained by replacing the copies of the smallest entry with 1, the copies of the second smallest entry with 2, and so on.

Denote by $\mathrm{Set}(\alpha)$ the set of different entries in α, without multiplicities. For example, we have $\mathrm{Set}(113232) = \{1, 2, 3\}$. Given two sets A and B, we write $A < B$ to mean that $a < b$ for every $a \in A$ and $b \in B$. Given two words α and β, we write $\alpha < \beta$ to mean $\mathrm{Set}(\alpha) < \mathrm{Set}(\beta)$. We define other relations $>$, $\leq$ and $\geq$ similarly. Note that $A \leq B$ implies that $|A \cap B| \leq 1$. Note also that these relations are not transitive or antisymmetric, since the empty word, which we denote by ε, trivially satisfies that $\varepsilon \leq \alpha$ and $\varepsilon \geq \alpha$ for any α.

If $\pi \in \mathcal{C}_n$ avoids a pattern σ, then the permutation $\pi' \in \mathcal{C}_{n-1}$ obtained by removing the two copies of n from π also avoids σ. In this case, we will say that π' *generates* π.

2 Patterns of length 3

In this section we consider nonnesting permutations avoiding patterns of length 3. Applying reversal and complementation, the enumeration of $\mathcal{C}_n(\Lambda)$ for all $\Lambda \subseteq \mathcal{S}_3$ with $|\Lambda| \geq 2$ can be reduced to the sets listed in Table 1, which serves as a summary of the results in this section.

Λ	Formula for $c_n(\Lambda)$	OEIS code	Result in the paper
$\{112\}$	C_n	A000108	Theorem 2.1
$\{121\}$	$n!$	A000142	Theorem 2.2
$\{123, 321\}$	0, for $n \geq 5$	N/A	Corollary 2.4
$\{123, 231\}$	$\dfrac{n^2+5n-6}{2}$, for $n \geq 2$	A055999	Theorem 2.5
$\{132, 213\}$	F_n^2	A007598	Theorem 2.6
$\{132, 231\}$	2^n, for $n \geq 2$	A000079	Theorem 2.7
$\{132, 312\}$	$4 \cdot 3^{n-2}$, for $n \geq 2$	A003946	Theorem 2.8
$\{123, 213\}$			Theorem 2.9
$\{123, 132, 213\}$	OGF: $\dfrac{1-x}{1-2x-2x^2+2x^3}$	A052528	Theorem 2.11
$\{123, 213, 312\}$	$n+2$, for $n \geq 2$	A000027	Theorem 2.12
$\{132, 213, 312\}$			Theorem 2.13
$\{123, 231, 312\}$	n, for $n \geq 3$	A000027	Theorem 2.14
$\{123, 213, 231\}$	$4(n-1)$, for $n \geq 2$	A008586	Theorem 2.15
$\{123, 132, 213, 231\}$	4, for $n \geq 2$	N/A	Theorem 2.16
$\{123, 132, 231, 312\}$	2, for $n \geq 3$	N/A	Theorem 2.17
$\{132, 213, 231, 312\}$			Theorem 2.18
$\{123, 132, 213, 231, 312\}$	1, for $n \geq 3$	N/A	Theorem 2.19

Tab. 1: A summary of the enumeration of nonnesting permutations avoiding subsets of $\mathcal{S}_3$ of size at least 2, as well as two classes of nonnesting permutations avoiding a single pattern. The formulas are valid for $n \geq 1$ unless otherwise stated. OEIS refers to the Online Encyclopedia of Integer Sequences OEIS Foundation Inc. (2023), OGF stands for ordinary generating function, and F_n denotes the nth Fibonacci number.

2.1 Avoiding one pattern

By reversal and complementation, the enumeration of nonnesting permutations avoiding a single pattern in $\mathcal{S}_3$ reduces to the enumeration of the sets $\mathcal{C}_n(123)$ and $\mathcal{C}_n(132)$. After we stated these as open problems in a previous version of this article posted online, a functional equation for the generating function for $c_n(132)$ has been found very recently by Archer and Laudone (2025). We still do not have a formula for $c_n(123)$. For comparison, the enumeration of quasi-Stirling permutations avoiding a simgle pattern in $\mathcal{S}_3$ was also left as an open problem in Archer et al. (2019), and recently solved in Archer and Laudone (2025) for the pattern 132.

It is easier, however, to enumerate nonnesting permutations avoiding a pattern of length 3 with repeated letters. Disregarding the trivial case of the pattern 111, which is avoided by all nonnesting permutations, reversal and complementation reduces this problem to the enumeration of the sets $\mathcal{C}_n(112)$ and $\mathcal{C}_n(121)$, which are treated below.

Theorem 2.1. *For all $n \geq 1$, we have $c_n(112) = C_n$.*

Proof: For any $1 \leq i < j \leq n$, the subsequence consisting of entries i and j in $\pi \in \mathcal{C}_n(112)$ must be either $jjii$ or $jiji$, since otherwise they would either create a nesting or an occurrence of 112. It follows that, in the interpretation of nonnesting permutations as nonnesting matchings with labeled arcs, a permutation avoids 112 if and only if the arcs are labeled so that their left endpoints (namely, the first occurrence of each value) appear in decreasing order. Thus, there is exactly one possible labeling of each nonnesting matching. It follows that $c_n(112)$ is simply the number of nonnesting matchings of $[2n]$, which is C_n. □

Theorem 2.2. *For all $n \geq 1$, we have $c_n(121) = n!$.*

Proof: For any $1 \leq i < j \leq n$, the subsequence consisting of entries i and j in $\pi \in \mathcal{C}_n(121)$ must be either $jjii$ or $iijj$, since otherwise they would either create a nesting or an occurrence of 121. This forces repeated entries in π to occur next to each other. Thus π is obtained from a permutation in $\mathcal{S}_n$ by simply duplicating each entry. This leaves $n!$ possibilities. □

Note that $\mathcal{C}_n(112) = \mathcal{C}_n(122)$, and that $\mathcal{C}_n(121) = \mathcal{C}_n(212)$. The latter set is the intersection of nonnesting permutations and Stirling permutations.

2.2 Avoiding two patterns

We start with the simple case where the two avoided patterns are monotonic.

Theorem 2.3. *For all $k, \ell \geq 2$ and $n \geq (k-1)(\ell-1)+1$, we have $c_n(12\dots k, \ell\dots 21) = 0$.*

Proof: In any element of $\mathcal{C}_n$, the subsequence obtained by deleting one copy of each entry is a permutation in $\mathcal{S}_n$. By Erdős and Szekeres (1935), if $n \geq (k-1)(\ell-1)+1$, every permutation in $\mathcal{S}_n$ must contain either an increasing subsequence of length k or a decreasing subsequence of length ℓ, that is, one of the patterns $12\dots k$ or $\ell\dots 21$. It follows that $c_n(12\dots k, \ell\dots 21) = 0$. □

The following result is an immediate consequence of the above theorem. It will save us some work when classifying nonnesting permutations avoiding larger sets of patterns.

Corollary 2.4. *For any $\Lambda \subseteq \mathcal{S}_3$ such that $\{123, 321\} \subseteq \Lambda$, we have $c_n(\Lambda) = 0$ for all $n \geq 5$.*

Theorem 2.5. *For all $n \geq 2$, we have*

$$c_n(123, 231) = \frac{n^2+5n-6}{2}.$$

Proof: Let $\Lambda = \{123, 231\}$. We can write $\pi \in \mathcal{C}_n(\Lambda)$ uniquely as $\pi = \alpha 1 \beta 1 \gamma$ for some α, β, γ. Since π avoids 123 and 231, the words α, β and γ must be weakly decreasing, and we must have $\alpha \geq \beta$ and $\beta \geq \gamma$. In particular, $|\mathrm{Set}(\alpha) \cap \mathrm{Set}(\beta)| \leq 1$ and $|\mathrm{Set}(\beta) \cap \mathrm{Set}(\gamma)| \leq 1$. Note also that, since π is nonnesting, β cannot have repeated entries, and $\mathrm{Set}(\alpha) \cap \mathrm{Set}(\gamma) = \emptyset$. It follows that β must have length at most 2, leaving four cases:

(1) $\pi = \alpha 11\gamma$,

(2) $\pi = \alpha' i1i1\gamma$ for some $i \in \{2, 3, \ldots, n\}$,

(3) $\pi = \alpha 1i1i\gamma'$ for some $i \in \{2, 3, \ldots, n\}$,

(4) $\pi = \alpha'(i+1)1(i+1)i1i\gamma'$ for some $i \in \{2, 3, \ldots, n-1\}$.

In case (1), both α and γ must consist of decreasing sequences of double entries, and π is uniquely determined by $\mathrm{Set}(\gamma)$, since $\mathrm{Set}(\alpha) = \{2, 3, \ldots, n\} \setminus \mathrm{Set}(\gamma)$. Additionally, in order for π to avoid 231, the elements of $\mathrm{Set}(\gamma)$ must be consecutive. Thus, either γ is empty, or $\mathrm{Set}(\gamma) = \{i+1, i+2, \ldots, j\}$ for some $1 \leq i < j \leq n$. It follows that there are $1 + \binom{n}{2}$ permutations in case (1).

In case (2), i must be smaller than all the entries in α' and larger than all the entries in γ. Thus, π is determined by the choice of $i \in \{2, 3, \ldots, n\}$, so there are $n-1$ permutations in this case. A similar argument shows that there are $n-1$ permutations in case (3).

In case (4), π is determined by the choice of $i \in \{2, 3, \ldots, n-1\}$, so there are $n-2$ permutations in this case.

Adding the number of permutations in all four cases, we obtain

$$c_n(\Lambda) = 1 + \binom{n}{2} + (n-1) + (n-1) + (n-2) = \frac{n^2 + 5n - 6}{2}.$$

□

We will use F_n to denote the nth Fibonacci number, with the convention $F_0 = F_1 = 1$.

Theorem 2.6. *For all $n \geq 0$, we have $c_n(132, 213) = F_n^2$.*

Proof: Let $\Lambda = \{132, 213\}$. Writing permutations $\pi \in \mathcal{C}_n(\Lambda)$ as $\pi = \alpha n \beta n \gamma$ for some words α, β, γ, we can separate them into four cases:

(1) $\alpha = \beta = \varepsilon$,

(2) $\alpha = \varepsilon$ and $\beta \neq \varepsilon$,

(3) $\alpha \neq \varepsilon$ and $\beta = \varepsilon$,

(4) $\alpha \neq \varepsilon$ and $\beta \neq \varepsilon$.

Denote the number of permutations in each case by $a_n^{(1)}$, $a_n^{(2)}$, $a_n^{(3)}$ and $a_n^{(4)}$, respectively, so that $c_n(\Lambda) = a_n^{(1)} + a_n^{(2)} + a_n^{(3)} + a_n^{(4)}$.

To obtain recurrence relations for these numbers, we consider the possible ways to generate a permutation in $\mathcal{C}_{n+1}(\Lambda)$ by inserting two entries $n+1$ in a permutation in $\mathcal{C}_n(\Lambda)$ from each of the above four cases.

In case (1), there are four ways to insert two entries $n+1$ in $nn\gamma$ without creating nestings or occurrences of 213, namely $(n+1)(n+1)nn\gamma$, $(n+1)n(n+1)n\gamma$, $nn(n+1)(n+1)\gamma$ and $n(n+1)n(n+1)\gamma$, yielding a permutation in each of cases (1), (2), (3) and (4), respectively.

In case (2), each permutation $n\beta n\gamma$ generates two permutations $(n+1)(n+1)n\beta n\gamma$ and $(n+1)n(n+1)\beta n\gamma$, which belong to cases (1) and (2), respectively.

In case (3), each permutation $\alpha nn\gamma$ generates two permutations $(n+1)(n+1)\alpha nn\gamma$ and $\alpha nn(n+1)(n+1)\gamma$, which belong to cases (1) and (3), respectively.

In case (4), each permutation $\alpha n\beta n\gamma$ generates one permutation $(n+1)(n+1)\alpha n\beta n\gamma$, in case (1). Indeed, inserting an $n+1$ anywhere after the first entry of α and before the second n would create a 132, whereas inserting it anywhere after the second n would create a 213.

Keeping track of how many permutations in $\mathcal{C}_{n+1}(\Lambda)$ of each type are generated in each case, we conclude that

$$\begin{aligned}
a_{n+1}^{(1)} &= a_n^{(1)} + a_n^{(2)} + a_n^{(3)} + a_n^{(4)} = c_n(\Lambda),\\
a_{n+1}^{(2)} &= a_n^{(1)} + a_n^{(2)},\\
a_{n+1}^{(3)} &= a_n^{(1)} + a_n^{(3)},\\
a_{n+1}^{(4)} &= a_n^{(1)} = c_{n-1}(\Lambda),
\end{aligned}$$

from where

$$\begin{aligned}
c_{n+1}(\Lambda) &= a_{n+1}^{(1)} + a_{n+1}^{(2)} + a_{n+1}^{(3)} + a_{n+1}^{(4)}\\
&= 4a_n^{(1)} + 2a_n^{(2)} + 2a_n^{(3)} + a_n^{(4)}\\
&= 2(a_n^{(1)} + a_n^{(2)} + a_n^{(3)} + a_n^{(4)}) + 2a_n^{(1)} - a_n^{(4)}\\
&= 2c_n(\Lambda) + 2c_{n-1}(\Lambda) - c_{n-2}(\Lambda),
\end{aligned}$$

with initial conditions $c_0(\Lambda) = c_1(\Lambda) = 1$. This is the same recurrence satisfied by the squared Fibonacci numbers:

$$\begin{aligned}
F_{n+1}^2 &= (F_n + F_{n-1})^2\\
&= F_n^2 + 2F_nF_{n-1} + F_{n-1}^2\\
&= F_n^2 + (F_{n-1} + F_{n-2})F_{n-1} + F_n(F_n - F_{n-2}) + F_{n-1}^2\\
&= 2F_n^2 + 2F_{n-1}^2 + F_{n-2}F_{n-1} - F_nF_{n-2}\\
&= 2F_n^2 + 2F_{n-1}^2 - F_{n-2}^2.
\end{aligned}$$

□

Theorem 2.7. *For all $n \geq 2$, we have $c_n(132, 231) = 2^n$.*

Proof: Let $\Lambda = \{132, 231\}$. The formula clearly holds for $n = 2$, since $c_2(\Lambda) = 4$, so let us assume that $n \geq 3$. To generate an element in $\mathcal{C}_n(\Lambda)$ from $\pi \in \mathcal{C}_{n-1}(\Lambda)$, the only locations to insert n without creating an occurrence of 132 or 231 are at the very beginning or at the very end. If we also want to avoid nestings, both entries n have to be inserted in the same location. This gives the recurrence $c_n(\Lambda) = 2c_{n-1}(\Lambda)$, which implies $c_n(\Lambda) = 2^n$. □

Theorem 2.8. *For all $n \geq 2$, we have $c_n(132, 312) = 4 \cdot 3^{n-2}$.*

Proof: Let $\Lambda = \{132, 312\}$, and assume that $n \geq 3$. Any permutation in $\mathcal{C}_n(\Lambda)$ must end with 1 or n, otherwise the last entry would be part of an occurrence of 132 or 312. Since complementation respects avoidance of Λ, it gives a bijection between permutations in $\mathcal{C}_n(\Lambda)$ that end with n, and those that end with 1. It follows that the number of permutations in each of the two cases is $c_n(\Lambda)/2$.

If $\pi \in \mathcal{C}_n(\Lambda)$ ends with n, we can write $\pi = \alpha n \beta n$. Avoidance of 132 forces $\alpha \geq \beta$, and the nonnesting condition prevents β from having repeated entries. It follows that $\beta = \varepsilon$ or $\beta = 1$. In the first case, α can be any element of $\mathcal{C}_{n-1}(\Lambda)$, so there are $c_{n-1}(\Lambda)$ such permutations. In the second case, we can write $\pi = \alpha_1 1 \alpha_2 n 1 n$. Removing the two copies of n gives a bijection between such permutations and the set of permutations in $\mathcal{C}_{n-1}(\Lambda)$ that end with a 1, so there are $c_{n-1}(\Lambda)/2$ such permutations.

We obtain the recurrence

$$c_n(\Lambda)/2 = c_{n-1}(\Lambda) + c_{n-1}(\Lambda)/2,$$

from where $c_n(\Lambda) = 3c_{n-1}(\Lambda)$. Using the initial condition $c_2(\Lambda) = 4$, the result follows. □

Theorem 2.9. *For all $n \geq 2$, we have $c_n(123, 213) = 4 \cdot 3^{n-2}$.*

Proof: Let $\Lambda = \{123, 213\}$, and let $n \geq 2$. Write $\pi \in \mathcal{C}_n(\Lambda)$ as $\pi = \alpha n \beta n \gamma$. To avoid both 123 and 213, we must have $|\mathrm{Set}(\alpha) \cup \mathrm{Set}(\beta)| \leq 1$. This condition, together with the fact that β cannot have repeated letters to avoid a nesting, leaves four cases:

(1) $\pi = nn\gamma$,

(2) $\pi = iinn\gamma$ for some $i \in [n-1]$,

(3) $\pi = inin\gamma$ for some $i \in [n-1]$,

(4) $\pi = nin\gamma$ for some $i \in [n-1]$.

Denote the number of permutations in each case by $a_n^{(1)}$, $a_n^{(2)}$, $a_n^{(3)}$ and $a_n^{(4)}$, respectively, so that $c_n(\Lambda) = a_n^{(1)} + a_n^{(2)} + a_n^{(3)} + a_n^{(4)}$. To obtain recurrence relations, we look at the possible ways that a permutation in each of these cases could be generated by inserting two entries n into a permutation in $\mathcal{C}_{n-1}(\Lambda)$.

In case (1), the word γ is an arbitrary permutation in $\mathcal{C}_{n-1}(\Lambda)$, so

$$a_n^{(1)} = c_{n-1}(\Lambda). \tag{1}$$

In case (2), after removing the entries n, the permutation $ii\gamma$ is an arbitrary permutation in $\mathcal{C}_{n-1}(\Lambda)$ starting with a double letter, that is, any permutation from cases (1) and (2). It follows that

$$a_n^{(2)} = a_{n-1}^{(1)} + a_{n-1}^{(2)}. \tag{2}$$

The same is true in case (3), so $a_n^{(3)} = a_n^{(2)}$.

In case (4), the word $i\gamma$ obtained after removing the entries n is an arbitrary permutation in $\mathcal{C}_{n-1}(\Lambda)$, so $a_n^{(4)} = c_{n-1}(\Lambda)$.

From equation (1) and the fact that $a_n^{(1)} = a_n^{(4)}$ and $a_n^{(2)} = a_n^{(3)}$, we get

$$a_n^{(1)} = c_{n-1}(\Lambda) = a_{n-1}^{(1)} + a_{n-1}^{(2)} + a_{n-1}^{(3)} + a_{n-1}^{(4)} = 2a_{n-1}^{(1)} + 2a_{n-1}^{(2)} \tag{3}$$

for $n \geq 3$. Combining this with equation (2), we see that $a_n^{(1)} = 2a_n^{(2)}$ for $n \geq 3$. Using this fact in equation (3), we obtain

$$a_n^{(1)} = 2a_{n-1}^{(1)} + a_{n-1}^{(1)} = 3a_{n-1}^{(1)}$$

for $n \geq 4$, or equivalently,

$$c_{n-1}(\Lambda) = 3c_{n-2}(\Lambda).$$

The stated result follows from this recurrence, using the initial condition $c_2(\Lambda) = 4$. $\square$

Remark 2.10. A natural bijection between noncrossing and nonnesting permutations of $[n] \sqcup [n]$ is obtained by applying the bijection between noncrossing and nonnesting matchings described in Athanasiadis (1998), which preserves the left endpoints of the arcs, and labeling the arcs according to these left endpoints. It can be shown that avoidance of the pair of patterns $\{321, 312\}$ is preserved by this bijection, and so Theorem 2.9 above is equivalent to (Archer et al., 2019, Thm. 4.4).

2.3 Avoiding three patterns

For avoidance of sets of three patterns of length 3, the case analysis is often similar to the proofs in the previous subsection.

Theorem 2.11. *The ordinary generating function for nonnesting permutations that avoid* $\{123, 132, 213\}$ *is*

$$\sum_{n\geq 0} c_n(123, 132, 213)\, x^n = \frac{1-x}{1-2x-2x^2+2x^3}.$$

Proof: Let $\Lambda = \{123, 132, 213\}$, and let $n \geq 2$. Write $\pi \in \mathcal{C}_n(\Lambda)$ as $\pi = \alpha n \beta n \gamma$. As in the proof of Theorem 2.9, avoidance of 123 and 213 implies that $\mathrm{Set}(\alpha) \cup \mathrm{Set}(\beta)$ is either empty or consists of one element, which must be $n-1$ in order for π to also avoid 132. We have the same four cases as in the proof of Theorem 2.9, but now we set $i = n-1$ in all of them.

In case (1), γ can be any permutation in $\mathcal{C}_{n-1}$, and in each of cases (2) and (3), γ can be any permutation in $\mathcal{C}_{n-2}(\Lambda)$. Let $A_n^{(4)}$ be the set of permutations in case (4), and let $a_n^{(4)} = |A_n^{(4)}|$. By the above decomposition,

$$c_n(\Lambda) = c_{n-1}(\Lambda) + 2c_{n-2}(\Lambda) + a_n^{(4)}. \tag{4}$$

Removing the two copies of n from $\pi \in A_n^{(4)}$ produces a bijection between $A_n^{(4)}$ and the set of permutations in $\mathcal{C}_{n-1}(\Lambda)$ that start with the largest entry, namely those from cases (1) or (4). Since the number of elements in $\mathcal{C}_{n-1}(\Lambda)$ in case (1) are counted by $c_{n-2}(\Lambda)$, it follows that

$$a_n^{(4)} = c_{n-2}(\Lambda) + a_{n-1}^{(4)}. \tag{5}$$

Shifting the indices in equation (4) down by one and solving for $a_{n-1}^{(4)}$, we get $a_{n-1}^{(4)} = c_{n-1}(\Lambda) - c_{n-2}(\Lambda) - 2c_{n-3}(\Lambda)$. Substituting in equation (5), we obtain

$$a_n^{(4)} = c_{n-1}(\Lambda) - 2c_{n-3}(\Lambda).$$

Finally, using this expression in equation (4) yields the recurrence

$$c_n(\Lambda) = 2c_{n-1}(\Lambda) + 2c_{n-2}(\Lambda) - 2c_{n-3}(\Lambda)$$

for $n \geq 3$, with initial conditions $c_0(\Lambda) = c_1(\Lambda) = 1$ and $c_2(\Lambda) = 4$. This recurrence is equivalent to the stated generating function. □

The sequence $c_n(123, 132, 213)$ appears in OEIS Foundation Inc. (2023) as sequence A052528, although with a very different interpretation. Specifically, as shown by Hoang and Mütze (2021), it counts vertex-transitive cover graphs of lattice quotients of essential lattice congruences of the weak order on $\mathcal{S}_{n+1}$.

Theorem 2.12. *For all $n \geq 2$, we have $c_n(123, 213, 312) = n + 2$.*

Proof: Let $\Lambda = \{123, 213, 312\}$, and let $n \geq 2$. Write $\pi \in \mathcal{C}_n(\Lambda)$ as $\pi = \alpha n \beta n \gamma$. To avoid both 123 and 213, we must have $|\mathrm{Set}(\alpha) \cup \mathrm{Set}(\beta)| \leq 1$. Additionally, to avoid 312, γ must be weakly decreasing, and $\beta \geq \gamma$. Combined with the nonnesting condition, this leaves four possibilities:

$$\begin{aligned}
&\pi = nn(n-1)(n-1)\dots 11,\\
&\pi = n(n-1)n(n-1)\,(n-2)(n-2)(n-3)(n-3)\dots 11,\\
&\pi = (n-1)n(n-1)n\,(n-2)(n-2)(n-3)(n-3)\dots 11, \text{ or}\\
&\pi = ii\,nn\,(n-1)(n-1)(n-2)(n-2)\dots\widehat{ii}\dots 11
\end{aligned}$$

for some $i \in [n-1]$, where we use $\widehat{ii}$ to indicate that we are skipping these entries. We conclude that

$$c_n(\Lambda) = 1 + 1 + 1 + (n-1) = n + 2.$$

□

Theorem 2.13. *For all $n \geq 2$, we have $c_n(132, 213, 312) = n + 2$.*

Proof: Let $n \geq 2$, and write $\pi \in \mathcal{C}_n(132, 213, 312)$ as $\pi = \alpha n \beta n \gamma$. To avoid 213, α and β must be weakly increasing, and $\alpha \leq \beta$. To avoid 312, β and γ must be weakly decreasing, and $\beta \geq \gamma$. Additionally, for π to avoid 132, we must have $\alpha \geq \beta$ and $\alpha > \gamma$, where the strictness comes from the nonnesting condition. The requirement that β is both weakly increasing and weakly decreasing, along with the nonnesting condition, implies that β has length at most one.

If $\beta = \varepsilon$, the above conditions on α and γ imply that

$$\pi = ii(i+1)(i+1)\dots nn\,(i-1)(i-1)(i-2)(i-2)\dots 11$$

for some $i \in [n]$, giving n different permutations.

If β has length one, then the requirements $\alpha \leq \beta$ and $\alpha \geq \beta$ imply that $\alpha = \varepsilon$ or $\alpha = \beta$. Additionally, the condition $\beta \geq \gamma$ implies that $\beta = n - 1$ in this case. Since γ is weakly decreasing, this leaves the two possibilities

$$\begin{aligned}
&\pi = n(n-1)n(n-1)\,(n-2)(n-2)(n-3)(n-3)\dots 11,\\
&\pi = (n-1)n(n-1)n\,(n-2)(n-2)(n-3)(n-3)\dots 11,
\end{aligned}$$

for a total of $n + 2$ permutations. □

Theorem 2.14. *For all $n \geq 3$, we have $c_n(123, 231, 312) = n$.*

Proof: Let $n \geq 2$, and write $\pi \in \mathcal{C}_n(123, 231, 312)$ as $\pi = \alpha n \beta n \gamma$. To avoid 123, α and β must be weakly decreasing, and $\alpha \geq \beta$. To avoid 312, β and γ must be weakly decreasing, and $\beta \geq \gamma$. To avoid 231, we must have $\alpha \leq \beta$, $\beta \leq \gamma$, and $\alpha < \gamma$, using also the nonnesting condition.

If $\beta = \varepsilon$, the fact that α and γ are weakly decreasing, along with the inequality $\alpha < \gamma$, implies that

$$\pi = (i-1)(i-1)(i-2)(i-2)\dots 11\, nn(n-1)(n-1)\dots ii$$

for some $i \in [n]$, giving n different permutations.

If $\beta \neq \varepsilon$, let i be an entry in β, and note that the nonnesting condition requires that the other copy of i appears in α or γ. This forces $\mathrm{Set}(\alpha) \cup \mathrm{Set}(\beta) \cup \mathrm{Set}(\gamma) = \{i\}$, because if some $j \neq i$ was in this set, then one of the conditions $\alpha \leq \beta$, $\alpha \geq \beta$, $\beta \leq \gamma$, or $\beta \geq \gamma$ would be violated. This can only happen if $n = 2$, and π must be one of the permutations 1212 or 2121 in this case. □

Theorem 2.15. *For all $n \geq 2$, we have $c_n(123, 213, 231) = 4(n-1)$.*

Proof: Let $\Lambda = \{123, 213, 231\}$ and let $n \geq 3$. Write $\pi \in \mathcal{C}_n(\Lambda)$ as $\pi = \alpha n \beta n \gamma$. As in the proof of Theorem 2.9, avoidance of 123 and 213 implies that $\mathrm{Set}(\alpha) \cup \mathrm{Set}(\beta)$ is either empty or consists of one element, which must be 1 in order for π to also avoid 231. This leaves the four cases from the proof of Theorem 2.9, where now we set $i = 1$.

In case (1), γ can be any element of $\mathcal{C}_{n-1}(\Lambda)$, so there are $c_{n-1}(\Lambda)$ permutations.

In cases (2) and (3), avoidance of 123 forces γ to be weakly decreasing, resulting in the two permutations

$$\pi = 11nn(n-1)(n-1)\dots 22,$$
$$\pi = 1n1n(n-1)(n-1)\dots 22.$$

In case (4), we can write $\pi = n1n\gamma_1 1\gamma_2$, where $\gamma_1\gamma_2$ is weakly decreasing, since π avoids 123. The nonnesting condition prevents γ_1 from having repeated letters, so $\gamma_1 = \varepsilon$ or $\gamma_1 = n-1$ (the latter assumes that $n \geq 3$), resulting in the two permutations

$$\pi = n1n1(n-1)(n-1)(n-2)(n-1)\dots 22,$$
$$\pi = n1n(n-1)1(n-1)\,(n-2)(n-2)\dots 22.$$

Combining all the cases, we obtain the recurrence

$$c_n(\Lambda) = c_{n-1}(\Lambda) + 4$$

for $n \geq 3$. Using the initial condition $c_2(\Lambda) = 4$, the result follows. □

2.4 Avoiding four or five patterns

There are three cases of sets $\Lambda \subseteq \mathcal{S}_3$ of size 4 and one case of size 5 that are not covered by Corollary 2.4. In all of them, the number of nonnesting permutations of $[n] \sqcup [n]$ avoiding Λ is constant for $n \geq 3$.

Theorem 2.16. *For all $n \geq 2$, we have $c_n(123, 132, 213, 231) = 4$.*

Proof: Let $\Lambda = \{123, 132, 213, 231\}$. For $n \geq 3$, any $\pi \in \mathcal{C}_n(\Lambda)$ must be of the form $nn\alpha$, since the avoidance condition requires that in any subsequence $\pi_i\pi_j\pi_k$ of distinct letters, π_i must be the largest. Therefore, $c_n(\Lambda) = c_{n-1}(\Lambda)$ for $n \geq 3$. Since $c_2(\Lambda) = 4$, the result follows. □

Theorem 2.17. *For all $n \geq 3$, we have $c_n(123, 132, 231, 312) = 2$.*

Proof: Let $\Lambda = \{123, 132, 231, 312\}$ and let $n \geq 3$. Avoidance of 132 and 231, together with the nonnesting condition, implies that any $\pi \in \mathcal{C}_n(\Lambda)$ must be of the form αnn or $nn\gamma$. Additionally, avoidance of 123 and 312 forces α and γ to be weakly decreasing. Thus, for $n \geq 3$,

$$\mathcal{C}_n(\Lambda) = \{(n-1)(n-1)(n-2)(n-2)\dots 11nn, nn(n-1)(n-1)\dots 11\}.$$

□

Theorem 2.18. *For all $n \geq 3$, we have $c_n(132, 213, 231, 312) = 2$.*

Proof: Let $\Lambda = \{132, 213, 231, 312\}$. Any subsequence of $\pi \in \mathcal{C}_n(\Lambda)$ of length 3 with distinct entries must be increasing or decreasing. Hence, for $n \geq 3$, we have $\mathcal{C}_n(\Lambda) = \{1122\dots nn, nn\dots 2211\}$. □

Theorem 2.19. *For all $n \geq 3$, we have $c_n(123, 132, 213, 231, 312) = 1$.*

Proof: Any subsequence of $\pi \in \mathcal{C}_n(123, 132, 213, 231, 312)$ of length 3 with distinct entries must be weakly decreasing. Hence, for $n \geq 3$, the only possibility is $\pi = nn\dots 2211$. □

3 Some patterns of length 4

In this section we give a few results about nonnesting permutations avoiding sets of patterns of length 4. We do not systematically analyze all sets, but rather we introduce some tools and provide a sample of results for which the enumeration sequences are interesting. We focus on patterns where one letter is repeated, and often appearing in adjacent positions. Tables 2 and 3 list sets Λ of patterns with repeated letters (up to reversal and complementation) for which we have found a formula for $c_n(\Lambda)$. Patterns where the repeated letters are adjacent are colored according to the permutation in $\mathcal{S}_3$, up to reverse-complement, obtained when removing one of the repeated letters: red for 123, orange for 321, blue for 132 and 213, and violet for 231 and 312.

To prove some of these formulas, it will be convenient to view permutations $\pi \in \mathcal{C}_n$ as labeled nonnesting matchings of $[2n]$, where there is an arc between i and j with label ℓ if $\pi_i = \pi_j = \ell$. The nonnesting condition guarantees that the order in which the left endpoints of the arcs appear is the same as the order in which their right endpoints appear, so there is a natural ordering of the arcs from left to right. The permutation in $\mathcal{S}_n$ obtained when reading the labels of the arcs from left to right will be called the *underlying permutation* of π, and denoted by $\hat{\pi}$. Note that $\hat{\pi}$ is the subsequence of π obtained by taking the left copy of each letter, or alternatively by taking the right copy of each letter. For example, if $\pi = 1521352434 \in \mathcal{C}_5$ (whose matching appears on the left of Figure 1), then its underlying permutation is $\hat{\pi} = 15234 \in \mathcal{S}_5$.

Λ	Formula for $c_n(\Lambda)$	OEIS code	Result in the paper
$\{1223\}$	C_n^2	A001246	Theorem 3.3
$\{1332\}$			
$\{\sigma, \tau\}$, where $\sigma \in \{1123, 1223, 1233\}$, $\tau \in \{3321, 3221, 3211\}$	0, for $n \geq 5$	N/A	Theorem 3.5
$\{1223, 1332\}$	$2^{n-1} C_n$	A003645	Theorem 3.3
$\{1332, 2113\}$			
$\{1332, 2331\}$			
$\{1332, 3112\}$			
$\{1123, 1132\}$			Theorem 3.7
$\{1322, 3122\}$			
$\{1223, 2331\}$	$\left(\binom{n}{2} + 1\right) C_n$	N/A	Theorem 3.3
$\{1322, 2231\}$	$\binom{2n}{n} - 2^{n-1}$	A085781	Theorem 3.30
$\{1231, 1321\}$	EGF: $\dfrac{2}{3 - e^{2x}}$	A122704	Theorem 3.33

Tab. 2: A summary of our results enumerating nonnesting permutations avoiding some sets of one or two patterns patterns of length 4. EGF stands for exponential generating function.

3.1 Patterns whose repeated letters are in the middle

In some cases, it is possible to describe pattern-avoiding nonnesting permutations by imposing restrictions on the underlying permutation, whereas the (unlabeled) nonnesting matching is arbitrary. When this happens, the resulting formulas have a factor of C_n to account for the possible nonnesting matchings.

The next lemma will be useful when avoiding patterns of length 4 with a repeated letter in the middle, as it translates this restriction to an avoidance condition on the underlying permutation.

Lemma 3.1. *Let $\pi \in \mathcal{C}_n$, let $\hat{\pi} \in \mathcal{S}_n$ be its underlying permutation, and let $ijk \in \mathcal{S}_3$. Then π avoids $ijjk$ if and only if $\hat{\pi}$ avoids ijk.*

Proof: If π contains $ijjk$, then the subsequence of π consisting of the left copy of each letter must contain ijk. Conversely, if $\hat{\pi}$ contains ijk, the nonnesting condition forces the right copy of (the letter playing the role of[(i)]) j to appear before the right copy of k. It follows that π contains $ijjk$. □

For a set $\Sigma \subseteq \mathcal{S}_3$, we denote by $\mathcal{S}_n(\Sigma)$ the set of permutations in $\mathcal{S}_n$ that avoid all the patterns in Σ, and let $s_n(\Sigma) = |\mathcal{S}_n(\Sigma)|$. The next lemma reduces the enumeration of nonnesting permutations avoiding patterns of length 4 with a repeated letter in the middle to the enumeration of permutations in $\mathcal{S}_n$ avoiding patterns of length 3, which was done by Simion and Schmidt (1985); see also Knuth (1997).

Lemma 3.2. *Let $\Sigma \subseteq \mathcal{S}_3$, and let $\Lambda = \{\sigma_1\sigma_2\sigma_2\sigma_3 : \sigma \in \Sigma\}$. Then, for any $n \geq 1$, $c_n(\Lambda) = s_n(\Sigma)\, C_n$.*

[(i)] When j is an entry in a pattern, we will often refer to "copies of j" in a permutation to mean copies of the letter playing the role of j in an occurrence of the pattern.

Λ	Formula for $c_n(\Lambda)$	OEIS code	Result in the paper
$\{1223, 1332, 2113\}$	F_nC_n	A098614	Theorem 3.3
$\{1123, 1132, 2133\}$			Theorem 3.7
$\{1223, 1332, 2331\}$	$\binom{2n}{n-1}$	A001791	Theorem 3.3
$\{1332, 2113, 2331\}$			
$\{1223, 1332, 3112\}$			
$\{1223, 2331, 3112\}$			
$\{1123, 1132, 2331\}$			Theorem 3.7
$\{1123, 1132, 3122\}$			
$\{1332, 2213, 2231\}$			
$\{1123, 1322, 2331\}$	$C_{n+1} - 1$	A001453	Theorem 3.9
$\{1132, 2213, 2231\}$			Theorem 3.14
$\{1233, 1322, 3122\}$			Theorem 3.21
$\{1322, 2213, 2231\}$			Theorem 3.15
$\{1223, 2231, 3112\}$			Theorem 3.23
$\{1123, 1132, 2311\}$	$\frac{n^3+9n^2-10n}{6}$, for $n \geq 2$	A060488	Theorem 3.10
$\{1123, 1322, 2311\}$	$\frac{n^3+6n^2-7n+6}{6}$	A027378	Theorem 3.11
$\{1132, 2213, 2311\}$			Theorem 3.16
$\{1233, 1132, 2311\}$	n^2+n-1, for $n \geq 3$	A028387	Theorem 3.12
$\{1233, 1322, 2311\}$	n^2	A000290	Theorem 3.13
$\{1322, 2133, 2311\}$			Theorem 3.20
$\{1123, 2231, 3312\}$			Theorem 3.29
$\{1332, 2133, 2311\}$	$2n^2-3n+2$	A084849	Theorem 3.18
$\{1123, 2331, 3312\}$			Theorem 3.26
$\{1132, 2133, 2311\}$	n^2+n-2, for $n \geq 2$	A028552	Theorem 3.19
$\{1123, 2311, 3122\}$			Theorem 3.25
$\{1123, 1132, 3312\}$	$\frac{7n^2-17n+14}{2}$, for $n \geq 2$	A140065 (values differ by 1)	Theorem 3.22
$\{1123, 2311, 3112\}$	$\frac{n^3+3n^2+8n-12}{6}$, for $n \geq 2$	A341209 (values differ by 1)	Theorem 3.24
$\{1123, 2311, 3312\}$	$\frac{n^2+7n-10}{2}$, for $n \geq 2$	A183905	Theorem 3.27
$\{1223, 2231, 3312\}$	$\frac{n^3+2n}{3}$	A006527	Theorem 3.28
$\{1132, 3112, 3121\}$	$5 \cdot 3^{n-2} - 1$, for $n \geq 2$	A198643	Theorem 3.31
$\{1231, 1321, 2113\}$	OGF: $\frac{1+2x-\sqrt{1-8x+4x^2}}{6x}$	A007564	Theorem 3.34
$\{1231, 1321, 2132, 2312, 3123, 3213\}$	$n!F_n$	A005442	Theorem 3.32

Tab. 3: A summary of our results enumerating nonnesting permutations avoiding some sets of three or more patterns patterns of length 4.

Proof: Let $\sigma \in \mathcal{S}_3$. By Lemma 3.1, $\pi \in \mathcal{C}_n$ avoids $\sigma_1\sigma_2\sigma_2\sigma_3$ if and only if the underlying permutation $\hat{\pi} \in \mathcal{S}_n$ avoids σ. Thus, $\pi \in \mathcal{C}_n(\Lambda)$ is determined by first choosing a nonnesting matching on $[2n]$, of which there are C_n, and then an underlying permutation $\hat{\pi} \in \mathcal{S}_n(\Sigma)$, of which there are $s_n(\Sigma)$. □

Theorem 3.3. *For all $n \geq 1$, we have*

(a) $c_n(ijjk) = C_n^2$ *for every* $ijk \in \mathcal{S}_3$*,*

(b) $c_n(1223, 1332) = c_n(1332, 2113) = c_n(1332, 2331) = c_n(1332, 3112) = 2^{n-1}C_n$*,*

(c) $c_n(1223, 2331) = \left(\binom{n}{2} + 1\right) C_n$*,*

(d) $c_n(1223, 1332, 2113) = F_n C_n$*,*

(e) $c_n(1223, 1332, 2331) = c_n(1332, 2113, 2331) = c_n(1223, 1332, 3112) = c_n(1223, 2331, 3112) = nC_n = \binom{2n}{n-1}$.

Proof: These results follow from Lemma 3.2, along with the following formulas:

(a) $s_n(ijk) = C_n$, as shown in Knuth (1997);

(b) $s_n(123, 132) = s_n(132, 213) = s_n(132, 231) = s_n(132, 312) = 2^{n-1}$, as shown in (Simion and Schmidt, 1985, Prop. 7–10);

(c) $s_n(123, 231) = \binom{n}{2} + 1$, as shown in (Simion and Schmidt, 1985, Prop. 11);

(d) $s_n(123, 132, 213) = F_n$, by (Simion and Schmidt, 1985, Prop. 15);

(e) $s_n(123, 132, 231) = s_n(132, 213, 231) = s_n(123, 132, 312) = s_n(123, 231, 312) = n$, by (Simion and Schmidt, 1985, Prop. 16 and 16*).

□

Lemma 3.1 would not hold if we replaced $ijjk$ with $iijk$. For example, $\pi = 113232$ contains 1123 but its underlying permutation $\hat{\pi} = 132$ avoids 123. To enumerate permutations avoiding patterns where the repeated letter is not in the middle, the next lemma will be useful.

Lemma 3.4. *Let $\pi \in \mathcal{C}_n$, and let $ijk \in \mathcal{S}_3$. If π avoids either $iijk$ or $ijkk$, then π avoids $ijjk$.*

Proof: We prove the contrapositive statement. Suppose that π contains $ijjk$. Since π is nonnesting, the other copy of i must occur before the right copy of j. This creates an occurrence of $iijk$. A symmetric argument shows that π also contains $ijkk$. □

Theorem 3.5. *Let $\sigma \in \{1123, 1223, 1233\}$ and $\tau \in \{3321, 3221, 3211\}$. Then, for all $n \geq 5$, we have $c_n(\sigma, \tau) = 0$.*

Proof: By Lemma 3.4, any $\pi \in \mathcal{C}_n(\sigma, \tau)$ must avoid 1223 and 3221. But then, by Lemma 3.1, its underlying permutation $\hat{\pi} \in \mathcal{S}_n$ must avoid 123 and 321. We know by Erdős and Szekeres (1935) that $s_n(123, 321) = 0$ for all $n \geq 5$. Therefore, $c_n(\sigma, \tau) = 0$ for all $n \geq 5$. □

The following lemma is a partial converse of Lemma 3.4.

Lemma 3.6. *Let $ijk \in \mathcal{S}_3$, let $\Lambda = \{iikj, ikkj, ikjj, ijkj, ikjk\}$, and let $\pi \in \mathcal{C}_n(ijjk)$. If π avoids some $\sigma \in \Lambda$, then π avoids $iijk$.*

Proof: Again, we prove the contrapositive statement. Let $\pi \in \mathcal{C}_n(ijjk)$, and suppose that π contains $iijk$. The other copy of j must be to the right of this k, in order to avoid $ijjk$, so π contains $iijkj$. Now, the other copy of k must be to the left of the first copy of j. Therefore, π must contain either $ikijkj$ or $iikjkj$. In both cases, π contains all the patterns in Λ. $\square$

Lemmas 3.4 and 3.6 provide bijections between many sets of pattern-avoiding nonnesting permutations, allowing us to derive from Theorem 3.3 some formulas for patterns where the repeated letter is not in the middle. The next theorem gives a sample of some such results, which is by no means exhaustive.

Theorem 3.7. *For all $n \geq 1$, we have*

(a) $c_n(1123, 1132) = c_n(1322, 3122) = 2^{n-1}C_n$,

(b) $c_n(1123, 1132, 2133) = F_nC_n$,

(c) $c_n(1123, 1132, 2331) = c_n(1123, 1132, 3122) = c_n(1332, 2213, 2231) = \binom{2n}{n-1}$.

Proof: We claim that $\mathcal{C}_n(1123, 1132) = \mathcal{C}_n(1223, 1332)$. The inclusion to the right follows from Lemma 3.4. For the reverse inclusion, suppose that $\pi \in \mathcal{C}_n(1223, 1332)$. Lemma 3.6 with $ijk = 123$ implies that π avoids 1123, and the same lemma with $ijk = 132$ implies that π avoids 1132. A similar argument shows that $\mathcal{C}_n(1322, 3122) = \mathcal{C}_n(1332, 3112)$. Part (a) now follows from Theorem 3.3(b).

For part (b), one can similarly show that $\mathcal{C}_n(1123, 1132, 2133) = \mathcal{C}_n(1223, 1332, 2113)$ using Lemmas 3.4 and 3.6, and then apply Theorem 3.3(d).

For part (c), Lemmas 3.4 and 3.6 imply the equalities $\mathcal{C}_n(1123, 1132, 2331) = \mathcal{C}_n(1223, 1332, 2331)$, $\mathcal{C}_n(1123, 1132, 3122) = \mathcal{C}_n(1223, 1332, 3112)$, and $\mathcal{C}_n(1332, 2213, 2231) = \mathcal{C}_n(1332, 2113, 2331)$. The enumeration of these sets is given in Theorem 3.3(e). $\square$

3.2 Other patterns whose repeated letters are adjacent

The restrictions that we consider in this subsection no longer translate into restrictions for the underlying permutations. These enumerative results often have more complicated proofs that require separating the permutations into different cases. We will often decompose permutations as follows.

Lemma 3.8. *Any $\pi \in \mathcal{C}_n$ can be written as $\pi = \alpha 1 \beta 1 \gamma$, where β has no repeated entries, and $\mathrm{Set}(\alpha) \cap \mathrm{Set}(\gamma) = \emptyset$. Thus, we have a disjoint union $\{2, 3, \dots, n\} = A \sqcup B_1 \sqcup B_2 \sqcup C$, where*

$$A = \mathrm{Set}(\alpha) \setminus \mathrm{Set}(\beta),\ B_1 = \mathrm{Set}(\alpha) \cap \mathrm{Set}(\beta),\ B_2 = \mathrm{Set}(\gamma) \cap \mathrm{Set}(\beta),\ C = \mathrm{Set}(\gamma) \setminus \mathrm{Set}(\beta). \quad (6)$$

Additionally, elements of B_1 (resp. B_2) must appear in the same order in β as in α (resp. γ). If α is weakly monotone, then it consists of the elements of A (each of which is duplicated) followed by the elements of B_1. Similarly, if γ is weakly monotone, it consists of the elements of B_2 followed by the elements of C (each of which is duplicated).

Proof: The positions of the 1s and the nonnesting condition guarantee that β has no repeated entries, and that no entry appears in both α and γ. Entries in β that have their other copy in α (resp. γ) must appear in the same order in both subwords because of the nonnesting condition. In the special case that α is weakly monotone, the nonnesting condition prevents duplicated entries (those in A) to appear after entries in B_1, and similarly when γ is weakly monotone. $\square$

We will use the notation from Lemma 3.8 throughout this section. Additionally, we let β_1 and β_2 be the subsequences of β consisting of the elements of B_1 and B_2, respectively.

The next five theorems deal with subsets of $\mathcal{C}_n(1223, 1332, 2331)$, the first set in Theorem 3.3(e). Figure 2 shows the containment relationships between these sets as a Hasse diagram.

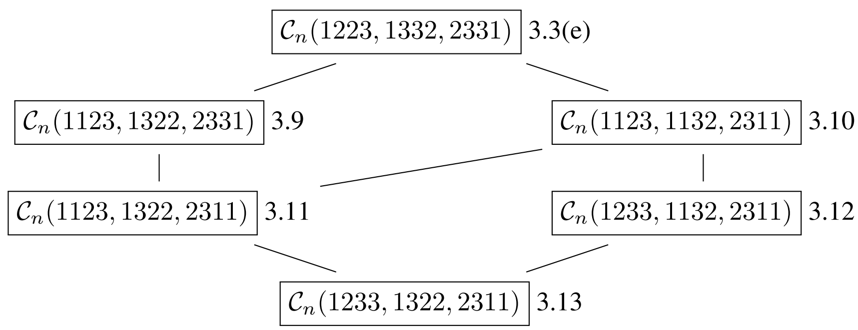

Fig. 2: The subsets of $\mathcal{C}_n(1223, 1332, 2331)$ enumerated in this section, along with the theorem number.

Theorem 3.9. *For all $n \geq 1$, we have $c_n(1123, 1322, 2331) = C_{n+1} - 1$.*

Proof: We decompose $\pi \in \mathcal{C}_n(1123, 1322, 2331)$ as in Lemma 3.8. Since π avoids 1322, it must also avoid 1332 by Lemma 3.4. Now, Lemma 3.6, together with avoidance of 1123, implies that π avoids 1132 as well. Avoidance of both 1123 and 1132 implies that $|\mathrm{Set}(\gamma)| \leq 1$. And since π avoids 2331, $\alpha\beta_1$ must avoid 122 which, as in the proof of Theorem 2.1, is equivalent to its underlying permutation being decreasing. Since β_1 has no repeated letters, $\alpha 1 \beta_1 1$ avoids 122 as well.

If $\gamma = \varepsilon$, then $\pi = \alpha 1 \beta_1 1$ is an arbitrary permutation in $\mathcal{C}_n(122)$. Indeed, avoidance of 122 implies avoidance of 1322 and 2331, and it is equivalent to avoidance of 112, which implies avoidance of 1123. By Theorem 2.1, there are C_n permutations in this case.

Now suppose that $\mathrm{Set}(\gamma) = \{k\}$ for some $2 \leq k \leq n$. Avoidance of 1322 requires that, in α, any entries larger than k must be to the left of any entries smaller than k. Thus, we can write $\alpha = \alpha_1\alpha_2$, where α_1 and α_2 consist of entries larger and smaller than k, respectively.

Consider first the case when all the elements in B_1 are smaller than k. Then all the entries greater than k are in α_1, and $\mathrm{st}(\alpha_1)$ is an arbitrary permutation in $\mathcal{C}_{n-k}(122)$. Similarly, $\mathrm{st}(\alpha_2 1 \beta_1 1)$ is an arbitrary permutation in $\mathcal{C}_{k-1}(122)$. It follows that $\alpha_2 1 \beta 1 \gamma$ is an arbitrary permutation in $\mathcal{C}_k$ whose underlying permutation is $(k-1)(k-2)\dots 1k$. Indeed, this condition on $\alpha_2 1 \beta 1 \gamma$ and the fact that $\mathrm{st}(\alpha_1) \in \mathcal{C}_{n-k}(122)$ guarantee that $\pi = \alpha_1\alpha_2 1 \beta 1 \gamma$ does not contain any of the patterns $1123, 1322, 2331$. Since there are C_k permutations in $\mathcal{C}_k$ with a fixed underlying permutation (one for each nonnesting matching), the number of permutations π in this case is

$$\sum_{k=2}^{n} C_{n-k}C_k = C_{n+1} - C_n - C_{n-1}.$$

Finally, consider the case when some element $b \in B_1$ is greater than k. Since the underlying permutation of $\alpha\beta_1$ is decreasing and β_1 has no repeated letters, β_1 is decreasing, so we can assume that b is the first entry in β_1. Since π avoids 1322, the first copy of k must appear before b, so $\beta = k\beta_1$.

Let us show that $\alpha_2 = (k-1)(k-2)\dots 2$. First, α_2 cannot have repeated letters; otherwise, together with k and b, they would form an occurrence of 1123. Second, α_2 contains the first occurrence of each letter in $\{2,\dots k-1\}$, and they must appear in decreasing order because otherwise π would contain 2331.

We claim that, in fact,

$$\pi = \alpha_1\,(k-1)(k-2)\dots 21\,k\,b(b-1)\dots(k+1)\,(k-1)(k-2)\dots 21\,k,$$

where $\mathrm{st}(\alpha_1 1\,b(b-1)\dots(k+1)\,1)$ is an arbitrary permutation in $\mathcal{C}_{n-k+1}(122)$ not ending with 11. Clearly, this permutation avoids 122 (because π avoids 2331) and does not end with 11 (because $b > k$). To see that it is arbitrary, note that avoidance of 122 in this permutation guarantees that π avoids the three patterns $1123, 1322, 2331$. Permutations in $\mathcal{C}_{n-k+1}(122)$ that do end with 11, by removing 11 and standardizing, are in bijection with permutations in $\mathcal{C}_{n-k}(122)$. We deduce that the number of permutations π in this case is

$$\sum_{k=2}^{n}(C_{n-k+1} - C_{n-k}) = C_{n-1} - 1.$$

Summing up all the cases, we have

$$c_n(1123, 1322, 2331) = C_n + (C_{n+1} - C_n - C_{n-1}) + (C_{n-1} - 1) = C_{n+1} - 1.$$

□

Theorem 3.10. *For all $n \geq 2$, we have*

$$c_n(1123, 1132, 2311) = \frac{n^3 + 9n^2 - 10n}{6}.$$

Proof: We decompose $\pi \in \mathcal{C}_n(1123, 1132, 2311)$, for $n \geq 2$, as in Lemma 3.8. In order for π to avoid 2311, α must be weakly decreasing, and so the elements of B_1 must be decreasing in β. To avoid both 1123 and 1132, we must have $|\mathrm{Set}(\gamma)| \leq 1$, leaving the following three possibilities for γ.

If $\gamma = \varepsilon$, it follows that

$$\pi = nn(n-1)(n-1)\dots(i+1)(i+1)\,i(i-1)\dots 1\,i(i-1)\dots 1$$

for some $i \in [n]$, giving n permutations.

Suppose now that $\gamma = jj$ for some $j \in \{2,\dots,n\}$. If all the elements of B_1 are smaller than j, we have

$$\pi = nn(n-1)(n-1)\dots\widehat{jj}\dots(i+1)(i+1)\,i(i-1)\dots 1\,i(i-1)\dots 1\,jj \tag{7}$$

for some $1 \leq i < j \leq n$, giving $\binom{n}{2}$ permutations. Otherwise, in order to avoid 2311, only one element of B_1 can be bigger than j, so

$$\pi = nn(n-1)(n-1)\dots(j+2)(j+2)\,(j+1)(j-1)\dots 1\,(j+1)(j-1)\dots 1\,jj \tag{8}$$

with $2 \leq j \leq n-1$, giving $n-2$ permutations in this case. Adding these cases, the number of permutations where $\gamma = jj$ for some j equals

$$\binom{n}{2} + n - 2.$$

Finally, suppose that $\gamma = j$ for some $j \in \{2, \dots, n\}$, which forces the other copy of j to appear in β. Recall that the other entries in β (that is, the elements of B_1) are decreasing. If all these elements are smaller than j, then the are no restrictions on the position of j inside β, and π is obtained from equation (7) by moving the first copy of j and inserting it in β, in one of the i available positions. Thus, the number of permutations in this case is

$$\sum_{1 \leq i < j \leq n} i = \binom{n+1}{3}.$$

If some elements of B_1 are larger than j, consider two subcases. If j is the first entry in β, then

$$\pi = nn(n-1)(n-1)\dots(i+1)(i+1)\,i(i-1)\dots\widehat{j}\dots 1\,j\,i(i-1)\dots\widehat{j}\dots 1\,j, \tag{9}$$

for some $2 \leq j < i \leq n$, giving $\binom{n-1}{2}$ permutations. Otherwise, in order to avoid 2311, there can be only one element of B_1 that is larger than j. In this case, π is obtained from equation (8) by moving the first copy of j and inserting it in β, in any of the $j-1$ positions other than the first one, giving

$$\sum_{j=2}^{n-1} (j-1) = \binom{n-1}{2} \tag{10}$$

permutations.

By adding all the cases, we have

$$c_n(1123, 1132, 2311) = n + \binom{n}{2} + n - 2 + \binom{n+1}{3} + \binom{n-1}{2} + \binom{n-1}{2} = \frac{n^3 + 9n^2 - 10n}{6}.$$

$\square$

Theorem 3.11. *For all $n \geq 1$, we have*

$$c_n(1123, 1322, 2311) = \frac{n^3 + 6n^2 - 7n + 6}{6}.$$

Proof: Let us first show that

$$\mathcal{C}_n(1123, 1322, 2311) = \mathcal{C}_n(1123, 1322, 2331) \cap \mathcal{C}_n(1123, 1132, 2311), \tag{11}$$

that is, the intersection of the sets from Theorems 3.9 and 3.10.

Let $\pi \in \mathcal{C}_n(1123, 1322, 2311)$. Since π avoids 2311, it also avoids 2331 by Lemma 3.4. On the other hand, as in the proof of Theorem 3.9, avoidance of 1123 and 1322 implies avoidance of 1132. This proves the inclusion to the right in equation (11). Conversely, a permutation in the intersection of the two sets on the right-hand side must avoid the three patterns $1123, 1322, 2311$.

We will adapt the proof of Theorem 3.10 by removing the two cases where the permutation π contains 1322. One is when π is given by equation (8), accounting for $n-2$ permutations. The other the case counted in equation (10), namely, when $\gamma = j$, there is an element in B_1 larger than j, and j is not the first entry in β.

Adding the remaining cases, we get

$$c_n(1123, 1132, 2311) = n + \binom{n}{2} + \binom{n+1}{3} + \binom{n-1}{2} = \frac{n^3 + 6n^2 - 7n + 6}{6}.$$

□

Theorem 3.12. *For all $n \geq 3$, we have $c_n(1233, 1132, 2311) = n^2 + n - 1$.*

Proof: Let $\pi \in \mathcal{C}_n(1233, 1132, 2311)$. Since π avoids 1233, it must also avoid 1223 by Lemma 3.4. Now, Lemma 3.6, together with avoidance of 1132, implies that π also avoids 1123. It follows that $\mathcal{C}_n(1233, 1132, 2311) \subseteq \mathcal{C}_n(1123, 1132, 2311)$, the set considered in Theorem 3.10.

Let us show how to modify the proof of this theorem to eliminate the cases where π contains 1233. The case $\gamma = \varepsilon$ does not change and contributes n permutations. In the case $\gamma = jj$, the permutation in equation (7) avoids 1233 only if $i = 1$, giving $n-1$ permutations. The permutation in equation (8) avoids 1233 only if $j = 2$, giving 1 permutation, if we use the assumption $n \geq 3$.

In the case $\gamma = j$, if the elements of B_1 are smaller than j, then the other copy of j has to be the first entry in β in order to avoid 1233, giving $\binom{n}{2}$ permutations of the form

$$\pi = nn(n-1)(n-1)\dots\widehat{jj}\dots(i+1)(i+1)\,i(i-1)\dots1\,j\,i(i-1)\dots1\,j \tag{12}$$

for $1 \leq i < j \leq n$. If some element of B_1 is larger than j, we get the $\binom{n-1}{2}$ permutations from equation (9) where j is the first entry in β. If j is not the first entry, then

$$\pi = nn(n-1)(n-1)\dots(j+2)(j+2)\,(j+1)(j-1)\dots1\,(j+1)j(j-1)\dots1\,j$$

for some $2 \leq j \leq n-1$, giving $n-2$ permutations.

Adding up all the cases, we get

$$c_n(1233, 1132, 2311) = n + (n-1) + 1 + \binom{n}{2} + \binom{n-1}{2} + (n-2) = n^2 + n - 1.$$

□

Theorem 3.13. *For all $n \geq 1$, we have $c_n(1233, 1322, 2311) = n^2$.*

Proof: Let us first show that

$$\mathcal{C}_n(1233, 1322, 2311) = \mathcal{C}_n(1123, 1322, 2311) \cap \mathcal{C}_n(1233, 1132, 2311), \tag{13}$$

that is, the intersection of the sets from Theorems 3.11 and 3.12.

Let $\pi \in \mathcal{C}_n(1233, 1322, 2311)$. Since π avoids 1233, it must also avoid 1223 by Lemma 3.4. But avoidance of 1223 and 1322 implies avoidance of 1123 by Lemma 3.6. This shows that $\mathcal{C}_n(1233, 1322, 2311) \subseteq \mathcal{C}_n(1123, 1322, 2311)$. Similarly, since π avoids 1322, it must also avoid 1332 by Lemma 3.4. But avoidance of 1332 and 1233 implies avoidance of 1132 by Lemma 3.6. This shows that $\mathcal{C}_n(1233, 1322, 2311) \subseteq \mathcal{C}_n(1233, 1132, 2311)$. Conversely, if a permutation is in the intersection on the right-hand side of equation (13), then it clearly avoids the patterns $1233, 1322, 2311$.

To find $c_n(1233, 1322, 2311)$, we follow the proofs of Theorems 3.11 and 3.12, and take the permutations that appear in both. When $\gamma = \varepsilon$, we get the same n permutations. When $\gamma = jj$, we get the $n-1$ permutation from equation (7) with $i = 1$. When $\gamma = j$, we get the $\binom{n}{2}$ permutations from equation (12) and the $\binom{n-1}{2}$ permutations from equation (9).

Adding up all the cases, we get

$$c_n(1233, 1322, 2311) = n + (n-1) + \binom{n}{2} + \binom{n-1}{2} = n^2.$$

□

In the next six theorems, we consider subsets of $\mathcal{C}_n(1332, 2113, 2331)$, which is the second set in Theorem 3.3(e). Figure 3 shows the containment relationships between these sets.

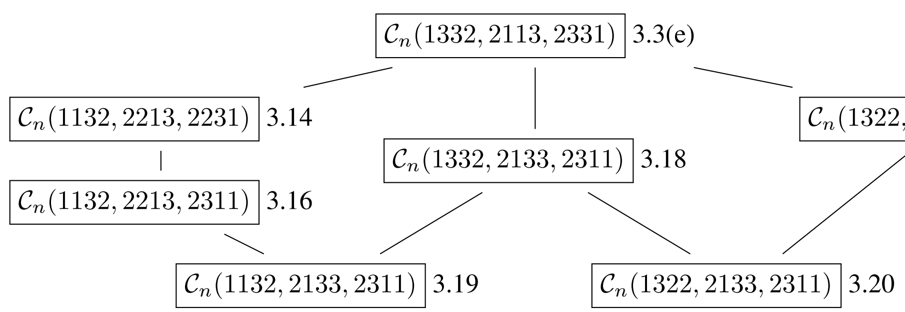

Fig. 3: The subsets of $\mathcal{C}_n(1332, 2113, 2331)$ enumerated in this section.

In the next proof, we let $\mathcal{D}_n$ be the set of Dyck words of length $2n$, that is, words consisting of n us and n ds with the property that no prefix contains more ds than us. It is well known (see Stanley (2015)) that $|\mathcal{D}_n| = C_n$.

Theorem 3.14. *For all $n \geq 1$, we have $c_n(1132, 2213, 2231) = C_{n+1} - 1$.*

Proof: We decompose $\pi \in \mathcal{C}_n(1132, 2213, 2231)$ as in Lemma 3.8. Avoidance of 2213 implies avoidance of 2113 by Lemma 3.4, which requires $\alpha > \gamma$. Avoidance of 1132 forces γ to be weakly increasing, so we can write $\beta_2 = 23 \dots i$ and $\gamma = 23 \dots i\,(i+1)(i+1)(i+2)(i+2) \dots jj$ for some $1 \leq i \leq j \leq n$. Avoidance of 2231 forces $\alpha\beta_1$ to avoid 112, which implies that its underlying permutation is decreasing, so in particular β_1 is decreasing, since it consists of second copies of entries. It follows that $\mathrm{st}(\alpha 1 \beta_1 1) \in \mathcal{C}_{n-j+1}(112)$.

In fact, the above are the only restrictions on α, β_1, β_2, and γ, in the sense that any choice of $1 \leq i \leq j \leq n$, any choice of $\mathrm{st}(\alpha 1\beta_1 1) \in \mathcal{C}_{n-j+1}(112)$, and any way to interleave the entries of $\beta_2 = 23\dots i$ with the entries of β_1 determines a (unique) permutation $\pi \in \mathcal{C}_n(1132, 2213, 2231)$. To count these choices, we will describe a bijection between such permutations and certain Dyck words. Given $\pi \in \mathcal{C}_n(1132, 2213, 2231)$ decomposed as above, construct a Dyck word as follows.

1. Start with the Dyck word $w_0 \in \mathcal{D}_{n-j+1}$ obtained from the permutation $\mathrm{st}(\alpha 1\beta_1 1)$ by simply replacing the first copy of each entry with a u and the second copy with a d. Viewing $\mathrm{st}(\alpha 1\beta_1 1)$ as a nonnesting matching, this is the standard bijection between nonnesting matchings and Dyck words.

2. Insert a ud right after the last u of w_0. This is the u corresponding to the first copy of 1, since β_1 consists only of second copies of entries. Let w_0' be the resulting word in $\mathcal{D}_{n-j+2}$. Note that each of the ds after this inserted ud corresponds to an element of $\beta_1 1$.

3. For each entry of β_2 that is interleaved with $1\beta_1 1$ in π, insert a ud in the corresponding location within the last run of ds in w_0'. Specifically, elements of β_2 that lie between the first 1 and β_1 become uds inserted right after the ud from step 2, and elements of β_2 that lie between β_1 and the second 1 become uds inserted right before the last d of w_0'. This step inserts a total of $i-1$ uds, producing a word in $\mathcal{D}_{n-j+i+1}$.

4. Finally, append $|\mathrm{Set}(\gamma)| = j - i$ uds to the end of the word, to obtain a word $w \in \mathcal{D}_{n+1}$.

We claim that the map $\pi \mapsto w$ is a bijection between $\mathcal{C}_n(1132, 2213, 2231)$ and $\mathcal{D}_{n+1} \setminus \{(\mathtt{ud})^{n+1}\}$, from which it will follow that $c_n(1132, 2213, 2231) = C_{n+1} - 1$.

First, it is clear by construction that $w \in \mathcal{D}_{n+1}$, and that $w \neq (\mathtt{ud})^{n+1}$, since w has the two consecutive us that were created in step 2. To see that it is a bijection, let us show that, given an arbitrary $w \in \mathcal{D}_{n+1} \setminus \{(\mathtt{ud})^{n+1}\}$, we can uniquely recover the permutation π that it came from. We start by finding the last two consecutive us in w, which must exist because $w \neq (\mathtt{ud})^{n+1}$. Then, the word w_0 obtained from w by removing all the pairs ud to the right of the first of these two us, determines the permutation $\mathrm{st}(\alpha 1\beta_1 1)$ by simply reversing step 1. The location of the removed pairs ud determine the positions of the entries in β_2 relative to those of β_1, and the number of removed pairs ud at the end of w determine $|\mathrm{Set}(\gamma)|$. This information uniquely determines the permutation $\pi \in \mathcal{C}_n(1132, 2213, 2231)$. $\square$

As an example of the above bijection, let

$$\pi = 12\,11\,12\,10\,9\,8\,7\,11\,1\,2\,10\,3\,4\,9\,8\,5\,7\,1\,2\,3\,4\,5\,6\,6 \in \mathcal{C}_{12}(1132, 2213, 2231),$$

which has $i = 5$ and $j = 6$. In step 1, we have

$$\mathrm{st}(\alpha 1\beta_1 1) = \mathrm{st}(12\,11\,12\,10\,9\,8\,7\,11\,1\,10\,9\,8\,7\,1) = 7\,6\,7\,5\,4\,3\,2\,6\,1\,5\,4\,3\,2\,1,$$

which gives the Dyck word $w_0 = \mathtt{uuduuuududdddd}$. In step 2, we obtain

$$w_0' = \mathtt{uuduuuuduudddddd},$$

where the five d steps after the inserted ud correspond to $\beta_1 1 = 10\,9\,8\,7\,1$. After steps 3 and 4, we get

$$w = \mathtt{uuduuuuduududdududuudddudddud}.$$

Theorem 3.15. *For all $n \geq 1$, we have $c_n(1322, 2213, 2231) = C_{n+1} - 1$.*

Proof: We decompose $\pi \in \mathcal{C}_n(1322, 2213, 2231)$ as in Lemma 3.8. Since π avoids 1322, $\beta_2\gamma$ avoids 211, and since π avoids 2231, $\alpha\beta_1$ avoids 112. As in the previous proof, avoidance of 2213 implies that $\alpha > \gamma$. Let $k = |\mathrm{Set}(\alpha)|$.

If $B_1 = \emptyset$, then $\mathrm{st}(\alpha)$ is an arbitrary permutation in $\mathcal{C}_k(112)$, and $\mathrm{st}(1\beta_2 1\gamma)$ is an arbitrary permutation in $\mathcal{C}_{n-k}(112)$. By Theorem 2.1, these sets are enumerated by the Catalan numbers. Summing over k, we get

$$\sum_{k=0}^{n-1} C_k C_{n-k} = C_{n+1} - C_n$$

permutations.

Now suppose $B_1 \neq \emptyset$. Since $\alpha > \gamma$, avoidance of 1322 implies that $C = \emptyset$, and that $\beta = \beta_2\beta_1$, that is, the elements of B_2 are to the left of those of B_1. It follows that

$$\pi = \alpha\, 12 \dots (n-k)\, \beta_1\, 12 \dots (n-k),$$

where $\mathrm{st}(\alpha 1 \beta_1 1)$ is an arbitrary permutation in $\mathcal{C}_{k+1}(112)$ not ending with 11. By Theorem 2.1, these are counted by $C_{k+1} - C_k$. Summing over k, we get

$$\sum_{k=1}^{n-1} C_{k+1} - C_k = C_n - C_1$$

permutations.

Adding up the permutations in both cases, we have

$$c_n(1322, 2213, 2231) = (C_{n+1} - C_n) + (C_n - C_1) = C_{n+1} - 1.$$

□

Theorem 3.16. *For all $n \geq 1$, we have*

$$c_n(1132, 2213, 2311) = \frac{n^3 + 6n^2 - 7n + 6}{6}.$$

Proof: We have $\mathcal{C}_n(1132, 2213, 2311) \subseteq \mathcal{C}_n(1132, 2213, 2231)$. This is because avoidance of 2311 implies avoidance of 2331 by Lemma 3.4, which in turn, using that π avoids 2213, implies avoidance of 2231 by Lemma 3.6.

We decompose $\pi \in \mathcal{C}_n(1132, 2213, 2311)$ as in Lemma 3.8. As in the proof of Theorem 3.14, avoidance of 1132 requires γ to be weakly increasing, and avoidance of 2213 implies that $\alpha > \gamma$. Additionally, avoidance of 2311 now requires α to be weakly decreasing,

By Lemma 3.8, elements of B_1 form decreasing subsequences in both α and β, whereas elements of B_2 form increasing subsequences in both β and γ. Additionally, since α and γ are weakly monotone, it follows from Lemma 3.8 that there exist $1 \leq i \leq j \leq k \leq n$ such that $B_2 = \{2, 3, \dots, i\}$, $C = \{i+1, i+2, \dots, j\}$, $B_1 = \{j+1, j+2, \dots, k\}$, and $A = \{k+1, k+2, \dots, n\}$. Let us consider three cases depending on the cardinality of B_1.

If $|B_1| \geq 2$ (that is, $k-j \geq 2$), the property $\alpha > \gamma$, together with avoidance of 2311, forces $C = \emptyset$ (that is, $i = j$). Avoidance of 2311 also requires that, in β, the elements of B_2 appear to the left of the elements of B_1. Therefore, any choice of j, k satisfying $1 \leq j < k-1 \leq n-1$ determines the permutation

$$\pi = nn(n-1)(n-1)\ldots(k+1)(k+1)\,k(k-1)\ldots(j+1)\,12\ldots j\,k(k-1)\ldots(j+1)\,12\ldots j \quad (14)$$

This leaves $\binom{n-1}{2}$ permutations in this case.

If $B_1 = \emptyset$ (that is, $j = k$), π is uniquely determined by the values i, j such that $1 \leq i \leq j \leq n$, leaving $\binom{n+1}{2}$ permutations.

If $|B_1| = 1$ (that is, $B_1 = \{j+1\} = \{k\}$), there are no restrictions on the position of this entry in β. Thus, π is determined by the values i, k such that $1 \leq i < k \leq n$, and the choice of the position of the entry k in β, for which we have $|\beta| = i$ choices. This leaves

$$\sum_{1 \leq i < k \leq n} i = \binom{n+1}{3} \quad (15)$$

permutations.

Adding up the three cases, we obtain

$$c_n(1132, 2213, 2311) = \binom{n-1}{2} + \binom{n+1}{2} + \binom{n+1}{3} = \frac{n^3 + 6n^2 - 7n + 6}{6}.$$

□

We note that Lemmas 3.4 and 3.6 imply that $\mathcal{C}_n(1132, 2213, 2311) = \mathcal{C}_n(1132, 2113, 2311)$.

The next lemma will be useful in some of the upcoming proofs. Since avoidance of 221 is equivalent to avoidance of 211, we have $\mathcal{C}_n(221, 2133) = \mathcal{C}_n(211, 2133)$.

Lemma 3.17. *For all $n \geq 1$, permutations $\pi \in \mathcal{C}_n(221, 2133)$ are those of the form*

$$\pi = 1122\ldots ii\,(i+1)(i+2)\ldots n\,(i+1)(i+2)\ldots n$$

for some $0 \leq i < n$. In particular, $|\mathcal{C}_n(221, 2133)| = n$.

Proof: Viewing $\pi \in \mathcal{C}_n$ as a nonnesting matching, avoidance of 221 is equivalent to the labels of the arcs being increasing from left to right, similarly to the proof of Theorem 2.1. Additionally, for any three arcs labeled $a_1 < a_2 < a_3$ from left to right, if the arcs a_1 and a_2 cross each other, then the arc a_3 must cross both of them; otherwise τ would contain the subsequence $a_2a_1a_3a_3$, which is an occurrence of 2133. This forces π to have the stated form, and it is clear that such a permutation avoids 2133. □

Theorem 3.18. *For all $n \geq 1$, we have $c_n(1332, 2133, 2311) = 2n^2 - 3n + 2$.*

Proof: We decompose $\pi \in \mathcal{C}_n(1332, 2133, 2311)$ as in Lemma 3.8. Avoidance of 2311 implies that α is weakly decreasing. Avoidance of 2133 implies avoidance of 2113 by Lemma 3.4, which forces $\alpha > \gamma$. Since π avoids 1332, $\beta_2\gamma$ must avoid 221, and since β_2 has no repeated entries, $\tau := 1\beta_2 1\gamma$ avoids 221 as well. By Lemma 3.17,

$$\tau = 1122\ldots ii\,(i+1)(i+2)\ldots j\,(i+1)(i+2)\ldots j \quad (16)$$

for some $0 \leq i < j \leq n$.

If $|B_1| \geq 2$, avoidance of 2311 requires that $i = 0$, and that, in β, the elements of B_2 appear to the left of those in B_1. Thus, we get the same $\binom{n-1}{2}$ permutations as in equation (14).

If $B_1 = \emptyset$, we have $\pi = nn(n-1)(n-1)\dots(j+1)(j+1)\,\tau$, with τ as in equation (16), giving $\binom{n+1}{2}$ permutations.

If $|B_1| = 1$, we must have $B_1 = \{j+1\}$, and $\alpha = nn(n-1)(n-1)\dots(j+2)(j+2)\,(j+1)$. For any $1 \leq i < j \leq n-1$ in equation (16), we can insert the other copy of $j+1$ in between the two 1s, giving $\binom{n-1}{2}$ permutations. If $i = 0$ in equation (16), we have $\tau = 12\dots j\,12\dots j$ for some $1 \leq j \leq n-1$, and we can insert the entry $j+1$ in j possible positions, giving $\sum_{j=1}^{n-1} j = \binom{n}{2}$ permutations.

Adding up all the cases, we get

$$c_n(1332, 2133, 2311) = \binom{n-1}{2} + \binom{n+1}{2} + \binom{n-1}{2} + \binom{n}{2} = 2n^2 - 3n + 2.$$

□

Theorem 3.19. *For all $n \geq 2$, we have $c_n(1132, 2133, 2311) = n^2 + n - 2$.*

Proof: We claim that $\mathcal{C}_n(1132, 2133, 2311) = \mathcal{C}_n(1132, 2213, 2311) \cap \mathcal{C}_n(1332, 2133, 2311)$, the sets from Theorems 3.16 and 3.18. For the inclusion to the right, note that avoidance of 1132 implies avoidance of 1332 by Lemma 3.4. By the same lemma, avoidance of 2133 implies avoidance of 2113, which then implies avoidance of 2213 by Lemma 3.6, using the fact that π avoids 2311. Inclusion to the left is trivial.

We will follow the proof of Theorem 3.16 and count only permutations that avoid 2133. In the case $|B_1| \geq 2$, the $\binom{n-1}{2}$ permutations from equation (14) avoid 2133.

If $B_1 = \emptyset$, avoidance of 2133 requires that either $i = 1$, giving n permutations (one for each $1 \leq j \leq n$), or that $2 \leq i = j \leq n$, giving $n-1$ permutations.

If $|B_1| = 1$, avoidance of 2133 requires that either $i = 1$, giving $n-1$ permutations (one for each $1 < k \leq n$), or that $2 \leq i = k-1$, giving $\sum_{i=2}^{n-1} i = \binom{n}{2} - 1$ permutations, assuming that $n \geq 2$, by changing equation (15) accordingly.

In total, we have

$$c_n(1132, 2133, 2311) = \binom{n-1}{2} + n + (n-1) + (n-1) + \binom{n}{2} - 1 = n^2 + n - 2.$$

□

Theorem 3.20. *For all $n \geq 1$, we have $c_n(1322, 2133, 2311) = n^2$.*

Proof: Let us show that $\mathcal{C}_n(1322, 2133, 2311) = \mathcal{C}_n(1332, 2133, 2311) \cap \mathcal{C}_n(1322, 2213, 2231)$, which are the sets from Theorems 3.18 and 3.15. Indeed, avoidance of 1322 implies avoidance of 1332 by Lemma 3.4. On the other hand, avoidance of 2133 and 2311 implies avoidance of 2113 and 2331 by Lemma 3.4, which imply avoidance of 2213 and 2231 by Lemma 3.6. Inclusion to the left is straightforward.

Let us follow the proof of Theorem 3.18 and count only permutations that also avoid 1322. In the cases $|B_1| \geq 2$ and $B_1 = \emptyset$, the same $\binom{n-1}{2} + \binom{n+1}{2}$ permutations avoid 1322. In the case $|B_1| = 1$, avoidance of 1322 requires $i=0$, and inserting the entry $j+1$ in the rightmost available position (i.e., as the last entry of β), giving $n-1$ permutations (one for each $1\le j\le n-1$).

In total, we have

$$c_n(1332,2133,2311)=\binom{n-1}{2}+\binom{n+1}{2}+n-1=n^2.$$

□

The next two theorems consider subsets of $\mathcal{C}_n(1223,1332,3112)$, the third set in Theorem 3.3(e). Neither of these subsets is contained in the other.

Theorem 3.21. *For all $n\ge 1$, we have $c_n(1233,1322,3122)=C_{n+1}-1$.*

Proof: By taking the reverse-complement, we have $c_n(1233,1322,3122)=c_n(1123,2213,2231)$. We will enumerate permutations $\pi\in\mathcal{C}_n(1123,2213,2231)$ by decomposing them as in Lemma 3.8. As in the proof of Theorem 3.14, avoidance of 2213 implies that $\alpha>\gamma$, and avoidance of 2231 implies that $\alpha\beta_1$ avoids 112. The only difference is that now the third avoided pattern is 1123 instead of 1132, so now γ has to be weakly decreasing instead of weakly increasing. We can write $\beta_2=j(j-1)\dots(j-i+2)$ and $\gamma=j(j-1)\dots(j-i+2)\,(j-i+1)(j-i+1)(j-i)(j-i)\dots 22$ for some $1\le i\le j\le n$. These are the only restrictions on α, β_1, β_2 and γ, in the sense that they guarantee that π avoids the three patterns $1123,2213,2231$.

Thus, if we take the complement of $\beta_2\gamma$, by replacing each entry $b\in\{2,3,\dots,j\}$ with $j+2-b$, and keep all the other entries unchanged, the decomposition of the resulting permutation π' satisfies precisely the restrictions given in the proof of Theorem 3.14 when characterizing permutations that avoid 1132, 2213 and 2231. Thus, the map $\pi\mapsto\pi'$ is a bijection from $\mathcal{C}_n(1123,2213,2231)$ to $\mathcal{C}_n(1132,2213,2231)$. In particular, by Theorem 3.14, we have

$$c_n(1123,2213,2231)=c_n(1132,2213,2231)=C_{n+1}-1.$$

□

Theorem 3.22. *For all $n\ge 2$, we have*

$$c_n(1123,1132,3312)=\frac{7n^2-17n+14}{2}.$$

Proof: Let $\Lambda=\{1123,1132,3312\}$ and let $n\ge 3$. We decompose $\pi\in\mathcal{C}_n(\Lambda)$ as in Lemma 3.8. Avoidance of 1123 and 1132 implies that $|\mathrm{Set}(\gamma)|\le 1$.

Consider the first the case $|\mathrm{Set}(\gamma)|=1$. Avoidance of 3312 implies that $\mathrm{Set}(\gamma)=\{n\}$. Since π avoids 1123 and ends with n, the permutation π' obtained from π by removing the two copies of n must avoid 112. Since π also avoids 3312, Lemma 3.17 implies that

$$\pi'=(n-1)(n-2)\dots(i+1)\,(n-1)(n-2)\dots(i+1)\,ii(i-1)(i-1)\dots 11$$

for some $0\le i\le n-2$. If $i=0$, then there are n possible positions for the first copy of n, namely immediately before the second copy of j for any $1\le j\le n$, giving n permutations. For each $1\le i\le$

$n-2$, there are two possible positions for the first copy of n, namely immediately before or after the second 1, giving $2(n-2)$ permutations.

Suppose now that $\gamma = \varepsilon$. If $A = \emptyset$, then $\pi = \beta 1 \beta 1$, where β, after subtracting 1 from each entry, is an arbitrary permutation in $\mathcal{S}_{n-1}(123, 132, 312)$. Since there are $n-1$ such permutations (Simion and Schmidt, 1985, Prop. 16 and 16*), this gives $n-1$ possibilities for π. If $A \neq \emptyset$, avoidance of 3312 requires that $A < B_1$. Now avoidance of 1132 implies that $|B_1| \leq 1$. We consider two cases.

If $B_1 = \emptyset$, then $\pi = \alpha 11$, where $\mathrm{st}(\alpha)$ is an arbitrary permutation in $\mathcal{C}_{n-1}(\Lambda)$. Thus, there are $c_{n-1}(\Lambda)$ permutations of this form.

If $|B_1| = 1$, the condition $A < B_1$ implies that $B_1 = \{n\}$, and so $\beta = n$. In this case, if α' is the permutation obtained by removing the copy of n from α, then $\mathrm{st}(\alpha')$ is an arbitrary permutation in $\mathcal{C}_{n-2}(112, 3312)$. Indeed, α' must avoid 112 because π avoids 1123, and one can check that if α' avoids 112 and 3312, then π avoids the three patterns in Λ. By Lemma 3.17,

$$\alpha' = (n-1)(n-2)\dots(i+1)\,(n-1)(n-2)\dots(i+1)\,ii(i-1)(i-1)\dots 22$$

for some $1 \leq i \leq n-2$. If $i = 1$, then the first copy of n can be inserted in $n-1$ positions in α', namely immediately before of the second copy of j for any $2 \leq j \leq n-1$, or at the end. If $2 \leq i \leq n-2$, then the first copy of n can be inserted two positions, namely immediately before or after the second 2, giving $2(n-3)$ permutations.

Combining all the cases, we get the recurrence

$$c_n(\Lambda) = n + 2(n-2) + (n-1) + c_{n-1}(\Lambda) + (n-1) + 2(n-3) = c_{n-1}(\Lambda) + 7n - 12.$$

Using the initial condition $c_2(\Lambda) = 4$, we deduce the stated formula for $c_n(\Lambda)$. □

In the next seven theorems, we consider subsets of $\mathcal{C}_n(1223, 2331, 3112)$, the fourth set in Theorem 3.3(e). Figure 4 shows the containment relationships between these sets.

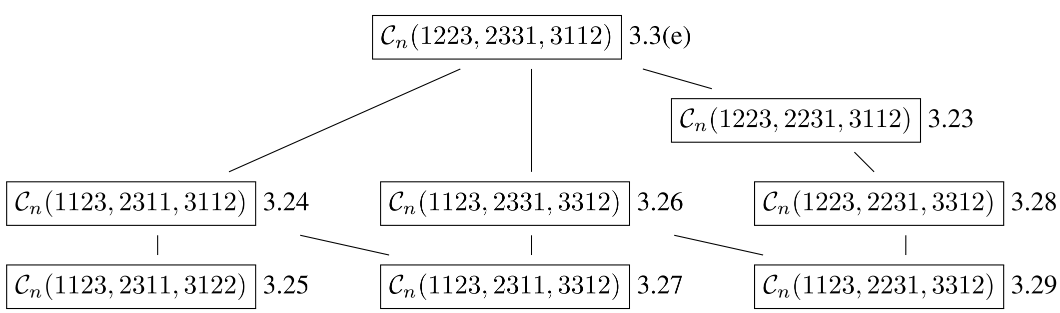

Fig. 4: The subsets of $\mathcal{C}_n(1223, 2331, 3112)$ enumerated in this section.

Theorem 3.23. *For all $n \geq 1$, we have $c_n(1223, 2231, 3112) = C_{n+1} - 1$.*

Proof: We decompose $\pi \in \mathcal{C}_n(1223, 2231, 3112)$ as in Lemma 3.8. Avoidance of 3112 implies that $\alpha < \gamma$. Let $k \in [n]$ be such that $\mathrm{Set}(\alpha) = \{2, 3, \dots, k\}$ and $\mathrm{Set}(\gamma) = \{k+1, k+2, \dots, n\}$. Since π avoids 2231, $\alpha\beta_1$ must avoid 112, and since π avoids 1223, $\beta_2\gamma$ must avoid 112 as well. Thus, the underlying permutations of $\alpha\beta_1$ and $\beta_2\gamma$ are decreasing, which implies that β_1 and β_2 are decreasing, since β_1 consists of only right copies of entries, and β_2 consists of only left copies. Additionally, avoidance of 2231 forces β_2 to be to the left of β_1; otherwise, if $b_1 \in B_1$ appears to the left of $b_2 \in B_2$ within β, the subsequence $b_1b_1b_21$ (where the first copy of b_1 is in α) would be an occurrence of 2231.

If $\beta_2 = \varepsilon$, then $\alpha 1 \beta 1$ is an arbitrary permutation in $\mathcal{C}_k(112)$, and $\mathrm{st}(\gamma)$ is an arbitrary permutation in $\mathcal{C}_{n-k}(112)$. Thus, by Theorem 2.1, there are

$$\sum_{k=1}^{n} C_k C_{n-k} = C_{n+1} - C_n$$

possibilities for π in this case.

If $\beta_2 \neq \varepsilon$, then avoidance of 2231, together with the fact that $\alpha < \beta$, forces $A = \emptyset$. In this case, we have

$$\pi = k(k-1)\dots 1\, \beta_2\, k(k-1)\dots 1\, \gamma,$$

where $\mathrm{st}(1\beta_2 1\gamma)$ is an arbitrary permutation in $\mathcal{C}_{n-k+1}$ whose underlying permutation is $1(k+1)k\dots 2$ and does not start with 11. Indeed, $\mathrm{st}(1\beta_2 1\gamma)$ has these properties because $\beta_2\gamma$ has a decreasing underlying permutation, and $\beta_2 \neq \varepsilon$. Additionally, these properties guarantee that π avoids the patterns $1223, 2231, 3112$. Since the number of permutations in $\mathcal{C}_{n-k+1}$ with a given underlying permutation is C_{n-k+1} and the number of those that start with 11 is C_{n-k}, the total number of possibilities for π in this case is

$$\sum_{k=1}^{n-1} (C_{n-k+1} - C_{n-k}) = C_n - C_1.$$

Adding up both cases, we obtain

$$c_n(1223, 2231, 3112) = (C_{n+1} - C_n) + (C_n - C_1) = C_{n+1} - 1.$$

□

Theorem 3.24. *For all $n \geq 2$, we have*

$$c_n(1123, 2311, 3112) = \frac{n^3 + 3n^2 + 8n - 12}{6}.$$

Proof: Let $n \geq 2$, and decompose $\pi \in \mathcal{C}_n(1123, 2311, 3112)$ as in Lemma 3.8. Avoidance of 1123 and 2311 forces γ and α to be weakly decreasing, respectively. Avoidance of 3112 requires $\alpha < \gamma$.

If $B_2 = \emptyset$, we must have

$$\pi = jj(j-1)(j-1)\dots(i+1)(i+1)\, i(i-1)\dots 1\, i(i-1)\dots 1\, nn(n-1)(n-1)\dots(j+1)(j+1) \quad (17)$$

for some $1 \leq i \leq j \leq n$, giving $\binom{n+1}{2}$ permutations.

If $|B_2| = 1$, Lemma 3.8, along with the fact that $\alpha < \gamma$ and γ is weakly decreasing, imply that $B_2 = \{n\}$. In this case, π has a form similar to equation (17), with $1 \leq i \leq j \leq n-1$, but where the first n is instead inserted in β (i.e., between the two copies of 1), in one of the i available positions. The number of permutations of this form is

$$\sum_{1\le i\le j\le n-1} i = \binom{n+1}{3}. \tag{18}$$

Finally, consider the case $|B_2| \geq 2$. Avoidance of 2311 forces $C = \emptyset$, and avoidance of 1123 forces $A = \emptyset$. Therefore,

$$\pi = i(i-1)\dots 1\, n(n-1)\dots 1\, n(n-1)\dots (i+1) \tag{19}$$

for some $1 \leq i \leq n-2$, giving $n-2$ permutations.

Adding up the three cases,

$$c_n(1123, 2311, 3112) = \binom{n+1}{2} + \binom{n+1}{3} + n - 2 = \frac{n^3+3n^2+8n-12}{6}.$$

□

Theorem 3.25. *For all $n \geq 2$, we have $c_n(1123, 2311, 3122) = n^2 + n - 2$.*

Proof: Since π avoids 3122, it also avoids 3112 by Lemma 3.4. It follows that $\mathcal{C}_n(1123, 2311, 3122) \subseteq \mathcal{C}_n(1123, 2311, 3112)$, the set that we enumerated in Theorem 3.24. In the proof of this theorem, the only case where π may contain the pattern 3122 is when $B_2 = \{n\}$. In this case, we must have $j = n-1$ in order to avoid 3122. Therefore, equation (18) becomes

$$\sum_{i=1}^{n-1} i = \binom{n}{2},$$

and adding up the three cases, we now get

$$c_n(1123, 2311, 3122) = \binom{n+1}{2} + \binom{n}{2} + n - 2 = n^2 + n - 2.$$

□

Theorem 3.26. *For all $n \geq 1$, we have $c_n(1123, 2331, 3312) = 2n^2 - 3n + 2$.*

Proof: We decompose $\pi \in \mathcal{C}_n(1123, 2331, 3312)$ as in Lemma 3.8. Avoidance of 1123 forces γ to be weakly decreasing. By Lemma 3.4, avoidance of 3312 implies avoidance of 3112, which requires $\alpha < \gamma$.

Since π avoids 2331, $\alpha\beta_1$ avoids 122, and so does $\tau \coloneqq \alpha 1 \beta_1 1$. Since τ also avoids 3312, Lemma 3.17 applied to the reversal of τ implies that

$$\tau = j(j-1)\dots(i+1)\, j(j-1)\dots(i+1)\, ii(i-1)(i-1)\dots 11 \tag{20}$$

for some $0 \leq i < j \leq n$.

If $B_2 = \emptyset$, then $\gamma = nn(n-1)(n-1)\dots(j+1)(j+1)$, and any choice of $0 \leq i < j \leq n$ gives a valid π, producing $\binom{n+1}{2}$ permutations.

If $|B_2| = 1$, then $B_2 = \{n\}$ and $\gamma = n\,(n-1)(n-1)\dots(j+1)(j+1)$. If $i = 0$, then $\tau = j(j-1)\dots 1\,j(j-1)\dots 1$, and we can insert the other n in any of the j positions in β, giving $\sum_{j=1}^{n-1} j = \binom{n}{2}$ possibilities for π. If $i \geq 1$ (that is, $B_1 = \emptyset$), then the other n is the only entry in β, so we get $\binom{n-1}{2}$ possibilities, one for each choice of $1 \leq i < j \leq n-1$.

Finally, if $|B_2| \geq 2$, avoidance of 1123 implies that $A = \emptyset$ and that, in β, the elements of B_2 are to the left of those of B_1. It follows that

$$\pi = j(j-1)\dots 1\,n(n-1)\dots(k+1)\,j(j-1)\dots 1\,n(n-1)\dots(k+1)\,kk(k-1)(k-1)\dots(j+1)(j+1)$$

for some $1 \leq j \leq k \leq n-2$, giving $\binom{n-1}{2}$ permutations.

Adding up all the cases,

$$c_n(1123, 2331, 3312) = \binom{n+1}{2} + \binom{n}{2} + \binom{n-1}{2} + \binom{n-1}{2} = 2n^2 - 3n + 2.$$

□

Theorem 3.27. *For all $n \geq 2$, we have*

$$c_n(1123, 2311, 3312) = \frac{n^2 + 7n - 10}{2}.$$

Proof: We have that $\mathcal{C}_n(1123, 2311, 3312) = \mathcal{C}_n(1123, 2311, 3112) \cap \mathcal{C}_n(1123, 2331, 3312)$, the sets from Theorems 3.24 and 3.26. Indeed, by Lemma 3.4, avoidance of 3312 implies avoidance of 3112, and avoidance of 2311 implies avoidance of 2331. Inclusion to the left is trivial.

Let us follow the proof of Theorem 3.24 and count only permutations that avoid 3312. If $B_2 = \emptyset$, the permutation in equation (17) avoids 3312 only if $1 = i \leq j \leq n$, giving n permutations, or if $2 \leq i = j \leq n$, giving $n-1$ permutations. If $B_2 = \{n\}$, the copy of n in β can be inserted in one position if $1 = i \leq j \leq n-1$, giving $n-1$ permutations, and in i positions if $2 \leq i = j \leq n$, giving $\sum_{i=2}^{n-1} i = \binom{n}{2} - 1$ permutations. If $|B_2| \geq 2$, all $n-2$ permutation in equation (19) avoid 3312.

Adding up the three cases,

$$c_n(1123, 2311, 3312) = n + (n-1) + (n-1) + \binom{n}{2} - 1 + (n-2) = \frac{n^2 + 7n - 10}{2}.$$

□

Theorem 3.28. *For all $n \geq 1$, we have*

$$c_n(1223, 2231, 3312) = \frac{n^3 + 2n}{3}.$$

Proof: We have $\mathcal{C}_n(1223, 2231, 3312) \subseteq \mathcal{C}_n(1223, 2231, 3112)$ by Lemma 3.4. As in the proof of Theorem 3.23, decomposing $\pi \in \mathcal{C}_n(1223, 2231, 3312)$ as in Lemma 3.8, we have $\alpha < \gamma$, and the elements of β_2 are to the left of those of β_1. Thus, we can write $\pi = \alpha 1 \beta_2 \beta_1 1 \gamma$.

Since π avoids 2231, $\alpha\beta_1$ avoids 112, and so the word $\tau_1 \coloneqq \alpha 1 \beta_1 1$ is a nonnesting permutation that avoids 112 and 3312. By Lemma 3.17 applied to the reversal of τ_1, we have

$$\tau_1 = j(j-1)\dots(i+1)\,j(j-1)\dots(i+1)\,ii(i-1)(i-1)\dots 11$$

for some $0 \le i < j \le n$. On the other hand, the word $\tau_2 \coloneqq \beta_2\gamma$, after standardizing (by subtracting j from each entry), is also a nonnesting permutation that avoids 112 (since π avoids 1223) and 3312, so again we must have

$$\tau_2 = n(n-1)\dots(k+1)\,n(n-1)\dots(k+1)\,kk(k-1)(k-1)\dots(j+1)(j+1)$$

for some $j \le k \le n$.

Now let us analyze how τ_1 and τ_2 can overlap with each other. If $i = 0$, the above conditions imply that

$$\begin{aligned}\pi = j(j-1)\dots 1\,n(n-1)\dots(\ell+1)\,j(j-1)\dots 1 \\ \ell(\ell-1)\dots(k+1)\,n(n-1)\dots(k+1)\,kk(k-1)(k-1)\dots(j+1)(j+1),\end{aligned}$$

for some $1 \le j \le k \le \ell \le n$. However, to avoid double-counting, we do not count the case when $k = n-1$ and $\ell = n$, since, for any given $j \in [n-1]$, such indices would produce the same permutation π as when $k = \ell = n$. This gives $\binom{n+2}{3} - (n-1)$ different permutations.

If $i \ge 1$, then avoidance of 2231 forces $\pi = \tau_1\tau_2$. In this case, we get a permutation for each choice of indices $1 \le i < j \le k \le n$, but again, to avoid double-counting, we do not allow $k = n-1$. This gives $\binom{n+1}{3} - \binom{n-1}{2}$ different permutations.

Adding the two cases,

$$c_n(1223, 2231, 3312) = \binom{n+2}{3} - (n-1) + \binom{n+1}{3} - \binom{n-1}{2} = \frac{n^3+2n}{3}.$$

$\square$

Theorem 3.29. *For all $n \ge 1$, we have $c_n(1123, 2231, 3312) = n^2$.*

Proof: We have that $\mathcal{C}_n(1123, 2231, 3312) = \mathcal{C}_n(1123, 2331, 3312) \cap \mathcal{C}_n(1223, 2231, 3312)$, the sets from Theorems 3.26 and 3.28. This is because, by Lemma 3.4, avoidance of 2231 implies avoidance of 2331, and avoidance of 1123 implies avoidance of 1223.

Let us follow the proof of Theorem 3.26 and consider only permutations that avoid 2231. In the cases $B_2 = \emptyset$ and $|B_2| \ge 2$, all $\binom{n+1}{2} + \binom{n-1}{2}$ permutations in that proof avoid 2231.

In the case $B_2 = \{n\}$, if $i = 0$, then n has to be inserted in the first position of β in order for π to avoid 2231, giving $n-1$ permutations coming from the choices of $1 \le j \le n-1$ in equation (20). If $i \ge 1$, all permutations contain 2231, so we do not count them here.

Adding up all the cases,

$$c_n(1123, 2231, 3312) = \binom{n+1}{2} + \binom{n-1}{2} + n - 1 = n^2.$$

$\square$

The last result in this subsection concerns a set of nonnesting permutations avoiding two patterns. Despite the simple formula, the proof is a more technical than the above ones. We define a grand Dyck word of length $2n$ to be a sequence of n $\mathtt{u}$s and n $\mathtt{d}$s with no other restrictions. It is well know that the the number of grand Dyck words of length $2n$ is $\binom{2n}{n}$, and that its generating function is $\sum_{n\geq 0}\binom{2n}{n} = \frac{1}{\sqrt{1-4x}}$.

Theorem 3.30. *For all $n \geq 1$, we have*

$$c_n(1322, 2231) = \binom{2n}{n} - 2^{n-1}.$$

Proof: We decompose $\pi \in \mathcal{C}_n(1322, 2231)$ as in Lemma 3.8. Since π avoids 1322, $\beta_2\gamma$ avoids 211, so its underlying permutation is increasing. Similarly, since π avoids 2231, $\alpha\beta_1$ avoids 112, so its underlying permutation is decreasing. It follows that β_2 is increasing and β_1 is decreasing.

Let us show that $\beta = \beta_2\beta_1$, that is the elements in B_2 are to the left of the elements of B_1 in β. Suppose for contradiction that $b_1 \in B_1$ and $b_2 \in B_2$ and that b_1 is to the left of b_2 within β. Then β contains the subsequence $b_11b_1b_21b_2$. If $b_1 < b_2$, then $b_1b_1b_21$ is an occurrence of 2231, and if $b_1 > b_2$, then $1b_1b_2b_2$ is an occurrence of 1322, which is a contradiction in both cases.

If $A \cup C = \emptyset$, we have $\pi = \beta_11\beta_2\beta_11\beta_2$, and π is determined by which elements from $\{2, 3, \ldots, n\}$ are in B_1, giving 2^{n-1} permutations if $n \geq 1$. The corresponding ordinary generating function is

$$1 + \sum_{n\geq 1} 2^{n-1}x^n = \frac{1-x}{1-2x}. \tag{21}$$

Suppose now that $A \cup C \neq \emptyset$, and let us show that $B < A \cup C$. Indeed, if there were elements $b \in B$ and $a \in A$ such that $a < b$, then $aab1$ would an occurrence of 2231. Similarly, if there was a $c \in C$ such that $c < b$, then $1bcc$ would an occurrence of 1322. Let us assume that the smallest element in $A \cup C$ is in A; the case where it is in C is symmetric.

Since π avoids 1322, for each $c \in C$, all the elements of α larger than c must come before those smaller than c. Similarly, since π avoids 2231, for each $a \in A$, all the elements of γ smaller than a must come before those larger than a. This means that we have disjoint unions $A = A_1 \sqcup A_2 \sqcup \ldots$ and $C = C_1 \sqcup C_2 \sqcup \ldots$ where the A_i and C_i are nonempty intervals (i.e., sets of consecutive integers) such that

$$B < A_1 < C_1 < A_2 < C_2 < \ldots, \tag{22}$$

all the elements of A_{i+1} appear to the left of those of A_i, and all the elements of C_i appear to the left of those of C_{i+1}, for all i.

Denoting the restriction of π to the elements in each A_i by $\pi|_{A_i}$, the word $\mathrm{st}(\pi|_{A_i})$ is an arbitrary nonnesting permutation avoiding 112, so it can be encoded as a Dyck word by replacing the first copy of each entry with a $\mathtt{u}$ and the second copy with a $\mathtt{d}$. Denote this word by $w(\pi|_{A_i})$. Similarly, $\mathrm{st}(\pi|_{C_i})$ forms an arbitrary nonnesting permutation avoiding 211, which can be encoded as a Dyck word $w(\pi|_{C_i})$. It follows that the restriction of π to $A \cup C$ can be encoded as a grand Dyck word

$$\ldots w(\pi|_{C_2})^r\, w(\pi|_{A_2})\, w(\pi|_{C_1})^r\, w(\pi|_{A_1}), \tag{23}$$

that is, for each of the sets A_i and C_i in the opposite order from equation (22) and we consider their associated Dyck words, reversing the ones coming from the sets C_i. Viewing words as lattice paths with steps $\mathtt{u} = (1,1)$ and $\mathtt{d} = (1,-1)$ starting at the origin, the reversed Dyck words correspond to portions of the path below the x-axis.

If $\beta = \varepsilon$, we can recover π uniquely from the above grand Dyck word, which has length $2(n-1)$. However, to deal with arbitrary β, we have to modify the last portion $w(\pi|_{A_1})$ of the above word, to take into account how the elements of A_1 and B_1 may be interleaved. Let $a = \min A_1$, which is the rightmost entry of $\pi|_{A_1}$, and write $B_1 = B_1^L \sqcup B_1^R$, where B_1^L (resp. B_1^R) are the elements whose first copy appears before (resp. after) the second copy of a. Note that $B_1^L > B_1^R$, since β_1 is decreasing, and that $B_1^L > B_2$, since otherwise π would contain 1322 (with a playing the role of 3). Therefore, $B_1^R \cup B_2 = \{2,3,\dots,k+1\}$ for some $0 \le k \le n-2$. Note also that, in γ, the elements of B_2 must appear to the left of the elements of C, since otherwise, any $c \in C$ to the left of $b_2 \in B_2$ would create a subsequence $aacb_2$, which is an occurrence of 2231.

The restriction of π to $A_1 \cup B_1^L \cup \{1\}$, after standardizing, is a nonnesting permutation avoiding 112, which can be encoded as a Dyck word $w(\pi|_{A_1\cup B_1^L\cup\{1\}})$. However, this is not an arbitrary Dyck word, but rather one with the property that the rightmost $\mathtt{u}$ (which corresponds to the first copy of 1) is preceded by a $\mathtt{d}$ (which corresponds to the second copy of a). Given a Dyck word with this property, we not only can recover $\mathrm{st}(\pi|_{A_1\cup B_1^L\cup\{1\}})$, by we also know that $|B_1^L|+1$ is precisely the number of $\mathtt{d}$s after the last $\mathtt{u}$. Dyck words whose rightmost $\mathtt{u}$ is preceded by a $\mathtt{d}$, by turning this pair $\mathtt{du}$ into $\mathtt{ud}$, are in bijection with Dyck words ending in $\mathtt{dd}$. Let $w'(\pi|_{A_1\cup B_1^L\cup\{1\}})$ be the Dyck word obtained after this transformation. Replacing $w(\pi|_{A_1})$ with $w'(\pi|_{A_1\cup B_1^L\cup\{1\}})$ on the right of equation (23), we obtain a grand Dyck word ending with $\mathtt{dd}$. Denote this word by $g(\pi)$.

The remaining piece of information needed to determine π is which elements of $\{2,3,\dots,k+1\}$ belong to B_2, since the rest must belong to B_1^R.

The map $\pi \mapsto (g(\pi), B_2)$ is a bijection from permutations $\pi \in \mathcal{C}_n(1322, 2231)$ where the smallest element of $A \cup C$ is in A, to pairs consisting of a grand Dyck word $g(\pi)$ of semilength $n-k$ (for some $0 \le k \le n-2$) ending with $\mathtt{dd}$, and a subset $B_2 \subseteq \{2,3,\dots,k+1\}$. For permutations $\pi \in \mathcal{C}_n(1322, 2231)$ where the smallest element of $A \cup C$ is in C, a symmetric construction produces a pair $(g(\pi), B_1)$, where the Dyck word $g(\pi)$ ends with $\mathtt{uu}$, and $B_1 \subseteq \{2,3,\dots,k+1\}$.

The generating function for grand Dyck words ending with $\mathtt{dd}$ or $\mathtt{uu}$, or equivalently, nonempty grand Dyck words not ending with $\mathtt{ud}$ or $\mathtt{du}$, is

$$\frac{1}{\sqrt{1-4x}} - 1 - \frac{2x}{\sqrt{1-4x}}.$$

On the other hand, the generating function for subsets of $\{2,3,\dots,k+1\}$ is $\frac{1}{1-2x}$.

Multiplying these and adding equation (21), we deduce that

$$\sum_{n\ge 0} c_n(1322, 2231)\, x^n = \left(\frac{1-2x}{\sqrt{1-4x}} - 1\right)\frac{1}{1-2x} + \frac{1-x}{1-2x} = \frac{1}{\sqrt{1-4x}} - \frac{x}{1-2x},$$

and so, extracting coefficients,

$$c_n(1322, 2231) = \binom{2n}{n} - 2^{n-1}.$$

□

As an example of the construction in the proof of Theorem 3.30, let

$$\pi = 17\,16\,17\,16\,15\,15\,12\,11\,10\,12\,9\,8\,11\,7\,10\,5\,4\,1\,2\,3\,6\,9\,8\,7\,5\,4\,1\,2\,3\,6\,13\,14\,13\,14 \in \mathcal{C}_{17}(1322, 2231),$$

which has $A_2 = \{15, 16, 17\}$, $C_1 = \{13, 14\}$, $A_1 = \{10, 11, 12\}$, $B_1^L = \{7, 8, 9\}$, $B_1^R = \{4, 5\}$, $B_2 = \{2, 3, 6\}$, and $k = 5$. Then $w(\pi|_{A_2}) = \mathtt{uuddud}$, $w(\pi|_{C_1})^r = \mathtt{dduu}$,

$$w(\pi|_{A_1 \cup B_1^L \cup \{1\}}) = w(12\,11\,10\,12\,9\,8\,11\,7\,10\,1\,9\,8\,7\,1) = \mathtt{uuuduudududdddd},$$

and $w'(\pi|_{A_1 \cup B_1^L \cup \{1\}}) = \mathtt{uuuduuduudddd d}$. Concatenating these words, we get

$$g(\pi) = \mathtt{uuddud}\,\mathtt{dduu}\,\mathtt{uuuduuduuddddd}.$$

3.3 Patterns with non-adjacent repeated letters

In this subsection we consider sets of patterns of length 4 that include patterns with repeated letters in non-adjacent positions. For the sets we consider, the number of nonnesting permutations avoiding them is still given by nice formulas.

Theorem 3.31. *For all $n \geq 2$, we have $c_n(1132, 3112, 3121) = 5 \cdot 3^{n-2} - 1$.*

Proof: Let $\Lambda = \{1132, 3112, 3121\}$, and decompose $\pi \in \mathcal{C}_n(\Lambda)$ as in Lemma 3.8. Avoidance of 1132 requires γ to be weakly increasing, avoidance of 3112 requires $\alpha < \gamma$, and avoidance of 3121 requires $\alpha \leq \beta$, so in particular $|B_1| \leq 1$. By Lemma 3.8, the entries in B_2 form an increasing sequence in both β and γ, and γ consists of the elements of B_2 followed by the elements of C, each of which is duplicated. In particular, π ends with n if and only if $\gamma \neq \varepsilon$. Let $c_n = c_n(\Lambda)$, and let r_n denote the number of permutations in $\mathcal{C}_n(\Lambda)$ that end with an n, so that $c_n - r_n$ is the number of those that do not.

Let us first focus on permutations that do not end with an n, namely those with $\gamma = \varepsilon$, and suppose that $n \geq 2$. If $\beta = \varepsilon$, these are permutations of the form $\pi = \alpha 11$, where $\mathrm{st}(\alpha)$ is an arbitrary element of $\mathcal{C}_{n-1}(\Lambda)$. If $\beta \neq \varepsilon$, then the condition $\alpha \leq \beta$ and the nonnesting property implies that $\pi = \alpha 1 n 1$, where $\mathrm{st}(\alpha n)$ is an arbitrary element of $\mathcal{C}_{n-1}(\Lambda)$ ending with its largest entry. It follows that

$$c_n - r_n = c_{n-1} + r_{n-1}. \tag{24}$$

Now consider permutations in $\mathcal{C}_n(\Lambda)$ that end with an n, and suppose that $n \geq 3$. If $C \neq \emptyset$, these permutations end in fact with nn, and removing this pair of entries yields an arbitrary permutation in $\mathcal{C}_{n-1}(\Lambda)$, so these are counted by c_{n-1}. Suppose now that $C = \emptyset$, which requires $B_2 \neq \emptyset$ for the permutation to end with n. Consider two cases depending on the cardinality of A, with subcases depending on whether $|B_1|$ equals 0 or 1.

- Case $A \neq \emptyset$. We must have $|B_2| \leq 1$ in this case; otherwise, taking $i, i' \in B_2$ with $i < i'$, the subsequence $22i'i$, where i' is an entry in β and i is an entry in γ, would be an occurrence of 1132. Combined with the above conditions, this forces $B_2 = \{n\}$.

  If $|B_1| = 0$, we must have $\pi = \alpha 1 n 1 n$, where $\mathrm{st}(\alpha)$ is an arbitrary element of $\mathcal{C}_{n-2}(\Lambda)$, giving c_{n-2} permutations.

  If $|B_1| = 1$, we must have $B_1 = \{n-1\}$, where $n-1$ is also the largest entry in α, and so $\pi = \alpha 1 (n-1) n 1 n$, where $\mathrm{st}(\alpha(n-1))$ is an arbitrary element of $\mathcal{C}_{n-2}(\Lambda)$ ending with its largest entry. Indeed, if this word avoids 1132, 3112 and 3121, then so does π. This subcase contributes r_{n-2} permutations, except when $n = 3$, in which case the resulting permutation $\pi = 212313$ has $A = \emptyset$, so it will be counted in the next case instead.

- Case $A = \emptyset$. If $|B_1| = 0$, the only possibility is $\pi = 12 \dots n123 \dots n$.

  If $|B_1| = 1$, we must have $B_1 = \{2\}$ and $B_2 = \{3, 4, \dots, n\}$. If $n \geq 4$, avoidance of 1132 requires that 2 appears at the end of β; otherwise π would contain the subsequence $22n3$, where n is an entry in β and 3 is an entry in γ. This forces $\pi = 21\,34 \dots n\,21\,34 \dots n$, except when $n = 3$, where we get the additional permutation $\pi = 2\,1\,23\,1\,3$ which was not counted in the previous case.

Combining both cases for permutations ending with an n, we obtain

$$r_n = c_{n-1} + c_{n-2} + r_{n-2} + 2 \tag{25}$$

for $n \geq 3$.

Adding equations (24) and (25), and then using the equality $r_{n-1} + r_{n-2} = c_{n-1} - c_{n-2}$, which follows from equation (24) with the index shifted by one, we obtain the recurrence

$$c_n = 2c_{n-1} + r_{n-1} + c_{n-2} + r_{n-2} + 2 = 3c_{n-1} + 2$$

for $n \geq 3$. From this recurrence, along with the initial condition $c_2 = 4$, we can prove the formula $c_n = 5 \cdot 3^{n-2} - 1$ by induction. □

Theorem 3.32. *For all $n \geq 1$, we have $c_n(1231, 1321, 2132, 2312, 3123, 3213) = n!F_n$.*

Proof: Let $\Lambda = \{1231, 1321, 2132, 2312, 3123, 3213\}$. Patterns in Λ are precisely those of the form $ijki$, where $ijk \in \mathcal{S}_3$. It follows that $\pi \in \mathcal{C}_n(\Lambda)$ if and only every arc in its associated matching connects adjacent entries, or entries having only one entry in between.

Let a_n be the number of such matchings of $[2n]$. Such a matching either has an arc $(2n-1, 2n)$, giving rise to a matching of $[2n-2]$ on the remaining vertices, or it has arcs $(2n-3, 2n-1)$ and $(2n-2, 2n)$, giving rise to a matching of $[2n-4]$. Therefore, $a_n = a_{n-1} + a_{n-2}$, with initial conditions $a_1 = 1$ and $a_2 = 2$, implying that $a_n = F_n$.

Each matching can be labeled in $n!$ ways to form a permutation in $\mathcal{C}_n(\Lambda)$, proving the stated formula. □

Our last two results are proved using exponential and ordinary generating functions, respectively.

Theorem 3.33. *The exponential generating function for nonnesting permutations that avoid $\{1231, 1321\}$ is*

$$\sum_{n \geq 0} c_n(1231, 1321)\, \frac{x^n}{n!} = \frac{2}{3 - e^{2x}}$$

Proof: Let $\Lambda = \{1231, 1321\}$. We will find a differential equation satisfied by $A(x) = \sum_{n \geq 0} c_n(\Lambda)\, \frac{x^n}{n!}$.

The coefficient of $\frac{x^n}{n!}$ in the derivative $A'(x)$ counts permutations $\pi \in \mathcal{C}_{n+1}(\Lambda)$. As in Lemma 3.8, such permutations can be written as $\pi = \alpha 1 \beta 1 \gamma$, where β has no repeated entries, and $\mathrm{Set}(\alpha) \cap \mathrm{Set}(\gamma) = \emptyset$. Additionally, β is either empty or has length 1, since two distinct values in β would create an occurrence of 1231 or 1321.

If $\beta = \varepsilon$, then the standardized words $\mathrm{st}(\alpha)$ and $\mathrm{st}(\gamma)$ are arbitrary Λ-avoiding nonnesting permutations.

If $\beta = b$ for some $b \in \{2, 3, \ldots, n+1\}$, and the other copy of b appears in α, then $\mathrm{st}(\alpha b)$ and $\mathrm{st}(\gamma)$ are again arbitrary Λ-avoiding nonnesting permutations (with the caveat that $\mathrm{st}(\alpha b)$ is nonempty). If the other copy of b appears in γ, the same is true for $\mathrm{st}(\alpha)$ and $\mathrm{st}(b\gamma)$.

It follows that $\pi \in \mathcal{C}_{n+1}(\Lambda)$ equals one of the following:

(1) $\alpha 1 1 \gamma$, where $\mathrm{st}(\alpha) \in \mathcal{C}_k(\Lambda)$ and $\mathrm{st}(\gamma) \in \mathcal{C}_{n-k}(\Lambda)$ for some $0 \le k \le n$,

(2) $\alpha 1 b 1 \gamma$, where $\mathrm{st}(\alpha b) \in \mathcal{C}_k(\Lambda)$ and $\mathrm{st}(\gamma) \in \mathcal{C}_{n-k}(\Lambda)$ for some $1 \le k \le n$,

(3) $\alpha 1 b 1 \gamma$, where $\mathrm{st}(b\gamma) \in \mathcal{C}_k(\Lambda)$ and $\mathrm{st}(\alpha) \in \mathcal{C}_{n-k}(\Lambda)$ for some $1 \le k \le n$.

Summing over $n \ge 0$, case (1) contributes $A(x)^2$ to the exponential generating function, since $\mathrm{Set}(\alpha)$ and $\mathrm{Set}(\gamma)$ form an arbitrary ordered partition of $\{2, 3, \ldots, n+1\}$ into two nonempty sets, see e.g. (Stanley, 1999, Prop. 5.1.1). Each of the cases (2) and (3) contributes $(A(x)-1)A(x)$ because one of the blocks is nonempty. This gives the differential equation

$$A'(x) = A(x)^2 + 2(A(x)-1)A(x) = 3A(x)^2 - 2A(x),$$

with initial condition $A(0) = 1$. Solving this equation, we deduce that $A(x) = \frac{2}{3-e^{2x}}$. □

Theorem 3.34. *The ordinary generating function for nonnesting permutations that avoid* $\{1231, 1321, 2113\}$ *is*

$$\sum_{n\ge 0} c_n(1231, 1321, 2113)\, x^n = \frac{1 + 2x - \sqrt{1 - 8x + 4x^2}}{6x}.$$

Proof: Let $\Lambda = \{1231, 1321, 2113\}$. Decomposing permutations $\pi \in \mathcal{C}_{n+1}(\Lambda)$ as in the proof of Theorem 3.33, the additional condition of avoiding 2113 requires $\alpha > \gamma$. Thus, $\mathrm{Set}(\gamma) = \{2, 3, \ldots, k+1\}$ and $\mathrm{Set}(\alpha) = \{k+2, k+3, \ldots, n+1\}$ for some $0 \le k \le n$, which makes the use of ordinary generating functions suitable in this case.

Letting $B(x) = \sum_{n\ge 0} c_n(\Lambda)\, x^n$, the same three cases as in the proof of Theorem 3.33, plus the empty permutation, give the equation

$$B(x) = 1 + xB(x)^2 + 2x(B(x)-1)B(x).$$

Solving for $B(x)$, we obtain the stated expression for the generating function. □

4 Further research

In a preprint version of this article, we proposed the open problem of finding a formula for the number of noncrossing permutations avoiding a single pattern in $\mathcal{S}_3$. This problem has recently been solved for the pattern 132 in Archer and Laudone (2025). The question remains open for the pattern 123, as it does in the noncrossing case studied in Archer et al. (2019).

Problem 1. *Find an expression for* $c_n(123)$.

The values of $c_n(123)$ for $1 \leq n \leq 8$ are $1, 4, 17, 82, 406, 2070, 10729, 56394, \ldots$. This sequence does not appear in the Online Encyclopedia of Integer Sequences OEIS Foundation Inc. (2023) at the time of writing this paper.

For nonnesting permutations avoiding sets of patterns of length 4, we have presented some results in Section 3, but there are many other sets to be considered. In Table 4 we list some cases that seem to give interesting enumeration sequences. All the conjectures have been checked for n up to 8.

Λ	Conjecture for $c_n(\Lambda)$	OEIS code
$\{1322\}$	$\frac{1}{n}\sum_{k=0}^{n-1}\binom{3n}{k}\binom{2n-k-2}{n-1}$	A007297
$\{1132, 2213\}$	OGF: $\frac{(1-x)^2-\sqrt{(1-x)^4-4x(1-x)^2}}{2x}$	A006319
$\{1233, 1322\}$		
$\{1132, 3312\}$	$3^n - 3\cdot 2^{n-1} + 1$	A168583
$\{1231, 1312, 2231, 3221\}$	OGF: $\frac{1-3x+2x^2}{(1-3x)(1-x-x^2)}$	A099159

Tab. 4: Some conjectures on the enumeration of nonnesting permutations avoiding other patterns.

We also note that for some of the sets of patterns in Table 3 we arrived at the same enumeration formulas, such as $C_{n+1} - 1$, using different proof methods. It would be interesting to find direct bijections explaining these Wilf equivalences. In the same vein, we wonder if there is a simple bijective proof of Theorem 2.6, namely, a bijection between $\mathcal{C}_n(132, 213)$ and pairs of Fibonacci objects of the same size.

Tables 1, 2 and 3 show that some of the enumeration sequences of pattern-avoiding nonnesting permutations are constant, others are polynomials, others grow exponentially, and others grow factorially. It would be interesting to understand the possible asymptotic behaviors of these sequences, the nature of their generating functions, and how these are determined by the properties of the avoided patterns.

Finally, in Elizalde (2024), it is shown that the polynomial enumerating all nonnesting permutations with respect to the number of descents has an unexpectedly simple factorization, and that its coefficients are palindromic. This suggests the study of the distribution of the number of descents on pattern-avoiding nonnesting permutations. Combining (Elizalde, 2024, Thm. 2.6) with our Lemma 3.1, we obtain similar factorizations for the descent polynomials of nonnesting permutations avoiding patterns of the form $ijjk$. Specifically, denoting the number of descents by $\mathrm{des}(\alpha_1 \ldots \alpha_k) = |\{i : \alpha_i > \alpha_{i+1}\}|$ and the Narayana polynomials by $N_n(t) = \sum_{d=0}^{n-1} \frac{1}{n}\binom{n}{d}\binom{n}{d+1}t^d$, we have the following refinement of Lemma 3.2.

Theorem 4.1. *Let $\Sigma \subseteq \mathcal{S}_3$, and let $\Lambda = \{\sigma_1\sigma_2\sigma_2\sigma_3 : \sigma \in \Sigma\}$. Then, for any $n \geq 1$,*

$$\sum_{\pi \in \mathcal{C}_n(\Lambda)} t^{\mathrm{des}(\pi)} = N_n(t) \sum_{\hat{\pi} \in \mathcal{S}_n(\Sigma)} t^{\mathrm{des}(\hat{\pi})}.$$

It follows, for example, that $\sum_{\pi \in \mathcal{C}_n(1332)} t^{\mathrm{des}(\pi)} = N_n(t)^2$. In particular, the distribution of the number of descents over $\mathcal{C}_n(1332)$ is symmetric. It is not hard to show that this distribution is also symmetric over $\mathcal{C}_n(132, 231)$, $\mathcal{C}_n(121)$ and $\mathcal{C}_n(112)$. It would be interesting to determine which sets of patterns have this property.

Acknowledgements

We thank Kate Kucharczuk for useful observations in the early stages of this work, Robert Dougherty-Bliss and Ryan Maguire for their help with some computations, and the referees for helpful comments.

References

K. Archer and R. P. Laudone. Pattern avoidance in non-crossing and non-nesting permutations, 2025. arXiv:2502.13309.

K. Archer, A. Gregory, B. Pennington, and S. Slayden. Pattern restricted quasi-Stirling permutations. *Australas. J. Combin.*, 74:389–407, 2019.

C. A. Athanasiadis. On noncrossing and nonnesting partitions for classical reflection groups. *Electron. J. Combin.*, 5:Research Paper 42, 16, 1998.

O. Bernardi. Deformations of the braid arrangement and trees. *Adv. Math.*, 335:466–518, 2018.

S. Elizalde. Descents on quasi-Stirling permutations. *J. Combin. Theory Ser. A*, 180:Paper No. 105429, 35, 2021.

S. Elizalde. Descents on nonnesting multipermutations. *European J. Combin.*, 121:Paper No. 103846, 23, 2024.

S. Elizalde. Canon permutations and generalized descents of standard Young tableaux. *Enumer. Comb. Appl.*, 5(2):Paper No. S2R13, 10, 2025.

P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935.

I. Gessel and R. P. Stanley. Stirling polynomials. *J. Combinatorial Theory Ser. A*, 24(1):24–33, 1978.

H. P. Hoang and T. Mütze. Combinatorial generation via permutation languages. II. Lattice congruences. *Israel J. Math.*, 244(1):359–417, 2021.

D. E. Knuth. *The art of computer programming. Vol. 1*. Addison-Wesley, Reading, MA, third edition, 1997. ISBN 0-201-89683-4. Fundamental algorithms.

P. A. MacMahon. *Combinatory Analysis*. Cambridge University Press, 1915.

OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences, 2023. Published electronically at `http://oeis.org`.

R. Simion and F. W. Schmidt. Restricted permutations. *European J. Combin.*, 6(4):383–406, 1985.

R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. ISBN 0-521-56069-1; 0-521-78987-7.

R. P. Stanley. *Catalan numbers*. Cambridge University Press, New York, 2015. ISBN 978-1-107-42774-7; 978-1-107-07509-2.

S. H. F. Yan, L. Yang, Y. Huang, and X. Zhu. Statistics on quasi-Stirling permutations of multisets. *J. Algebraic Combin.*, 55(4):1265–1277, 2022.