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Ramsey goodness of stars and fans for the Hajós graph

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Given two graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ denotes the smallest integer N such that any red-blue coloring of the edges of K_N contains either a red G_1 or a blue G_2. Let G_1 be a graph with chromatic number χ and chromatic surplus s, and let G_2 be a connected graph with n vertices. The graph G_2 is said to be Ramsey-good for the graph G_1 (or simply G_1-good) if, for $n \geq s$,

$$R(G_1, G_2) = (\chi - 1)(n - 1) + s.$$

The G_1-good property has been extensively studied for star-like graphs when G_1 is a graph with $\chi(G_1) \geq 3$, as seen in works by Burr-Faudree-Rousseau-Schelp (*J. Graph Theory*, 1983), Li-Rousseau (*J. Graph Theory*, 1996), Lin-Li-Dong (*European J. Combin.*, 2010), Fox-He-Wigderson (*Adv. Combin.*, 2023), and Liu-Li (*J. Graph Theory*, 2025), among others. However, all prior results require G_1 to have chromatic surplus 1. In this paper, we extend this investigation to graphs with chromatic surplus 2 by considering the Hajós graph H_a. For a star $K_{1,n}$, we prove that $K_{1,n}$ is H_a-good if and only if n is even. For a fan F_n with $n \geq 111$, we prove that F_n is H_a-good.

Keywords: Ramsey goodness, the Hajós graph, fan

1 Introduction

The Ramsey number is a bivariate function that assigns a positive integer $R(G_1, G_2)$ to every pair of simple graphs G_1 and G_2. It is defined as the smallest integer N such that any graph Γ on N vertices contains G_1 as a subgraph, or its complement $\overline{\Gamma}$ contains G_2 as a subgraph.

The study of Ramsey numbers for sparse graphs has flourished since the 1970s. At that time, Burr (1981) established a general lower bound for the Ramsey number of any pair of graphs. Suppose G_2 is a connected graph, and let $\chi(G_1)$ and $s(G_1)$ denote the chromatic number and the chromatic surplus of G_1, respectively, where the chromatic surplus refers to the smallest size of a color class over all proper $\chi(G_1)$-colorings of G_1. Then, if $|G_2| \geq s(G_1)$,

$$R(G_1, G_2) \geq (\chi(G_1) - 1)(|G_2| - 1) + s(G_1). \tag{1}$$

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When equality holds in (1), the graph G_2 is said to be G_1-*good*, or equivalently, G_2 is *Ramsey-good* for G_1. In the special case where G_1 is the complete graph K_k, such graphs G_2 are referred to as k-*good* by Burr and Erdős (1983).

When G_1 is the complete $(k+1)$-partite graph $K_{1,m_1,m_2,\ldots,m_k}$ and G_2 is a star, Burr et al. (1983) proved that for $m = \min\{m_i \mid i \in [k]\}$ and sufficiently large n,

$$R(K_{1,m_1,m_2,\ldots,m_k}, K_{1,n}) = \begin{cases} k\cdot(n+m-2)+1 & \text{if both } m \text{ and } n \text{ are even,} \\ k\cdot(n+m-1)+1 & \text{otherwise.} \end{cases}$$

It follows that $K_{1,n}$ is $K_{1,m_1,m_2,\ldots,m_k}$-good only when $m_1 = 1$, or when $m_1 = 2$ and n is even.

A frequently studied class of star-like graphs is the fan. A fan F_n is the graph formed by n triangles sharing a common vertex; see Figure 1. The concept of the fan was first introduced into extremal graph theory by Erdős et al. (1995), who studied its extremal graph and Turán number. For results on the Ramsey numbers of fans, see Chen et al. (2021); Dvořák and Metrebian (2023); Lin and Li (2009); Zhang et al. (2015).

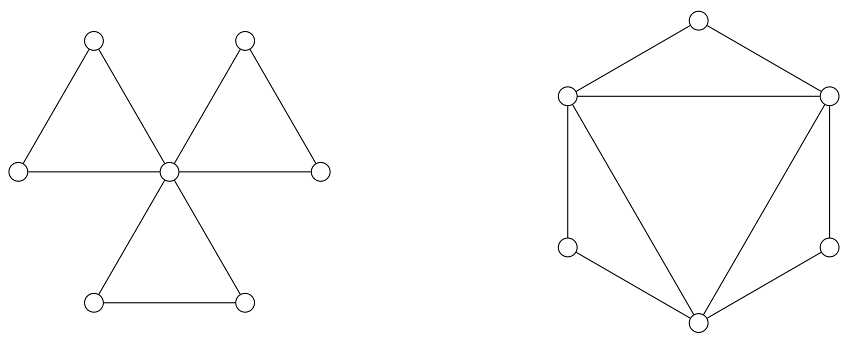

Fig. 1: The fan F_3 (left) and the Hajós graph (right)

The concept of a fan can be extended to that of a generalized fan. Given a graph H, let nH denote the disjoint union of n copies of H, and let $H_1 + H_2$ denote the join of two disjoint graphs H_1 and H_2, obtained by adding all edges between $V(H_1)$ and $V(H_2)$. The graph $K_1 + nH$ is called a generalized fan. In particular, if H is K_1, then $K_1 + nH$ is the star $K_{1,n}$; if H is K_2, then $K_1 + nH$ is the fan F_n.

Li and Rousseau (1996) investigated the Ramsey-goodness of generalized fans. For any graphs H and G, they proved that $K_1 + nH$ is Ramsey-good for $K_2 + G$ when n is sufficiently large.

Observe that in any proper coloring of the graph K_2+G, there are two color classes containing only one vertex each. What happens if we require only one color class to have a single vertex? Let $K_1 + K_k(m)$ denote the complete $(k+1)$-partite graph in which one partite set has size 1 and each of the remaining k partite sets has size m. Lin et al. (2010) showed that for $k \geq 2$, if m is odd or if $n|H|$ is odd, then

$$R(K_1 + K_k(m), K_1 + nH) = k(n|H| + m - 1) + 1.$$

It follows that whenever $m \geq 2$, the graph $K_1 + nH$ is not Ramsey-good for $K_1 + K_k(m)$ in this setting.

Subsequently, without relying on the Erdős-Stone-Simonovits stability theorem, Chung and Lin (2025) determined the exact value of $R(K_{1,m_1,m_2,\ldots,m_k}, K_1 + nH)$ for large n in all cases. Let $m = \min\{m_i \mid$

$i \in [k]\}$. Then

$$r(K_{1,m_1,m_2,\ldots,m_k}, K_1 + nH) = \begin{cases} k \cdot (n|H| + m - 2) + 1 & \text{if both } m \text{ and } n|H| \text{ are even,} \\ k \cdot (n|H| + m - 1) + 1 & \text{otherwise.} \end{cases}$$

Another frequently studied star-like graph is the book $B_{k,n}$, which is the graph $K_k + (n-k)K_1$. Equivalently, $B_{k,n}$ can be viewed as the graph obtained by blowing up the center of the star $K_{1,n-k}$ into a clique K_k, while preserving the adjacency between each vertex of K_k and the remaining vertices. For results concerning the Ramsey goodness of $B_{k,n}$, we refer the reader to Fox et al. (2023); Nikiforov and Rousseau (2004); Liu and Li (2025). For results on the Ramsey non-goodness of $B_{k,n}$, see Fan and Lin (2023); Fan et al. (2024); Lin and Liu (2021).

It is easy to observe that in all the aforementioned works, when G_2 is a star or a star-like graph, the corresponding graph G_1 is always required to have chromatic surplus 1. We extend these results by considering G_1 to be the Hajós graph H_a, which has chromatic surplus 2. The Hajós graph, named after the Hungarian mathematician György Hajós, is a graph consisting of six vertices and nine edges. It is constructed by starting with a triangle and, for each of its edges, adding a new vertex and joining it to both endpoints of that edge; see Figure 1.

In this paper, we establish two main results concerning Ramsey goodness for the Hajós graph.

Our first result establishes that the star $K_{1,n}$ is H_a-good if and only if n is even.

Theorem 1. $R(H_a, K_{1,n}) = \begin{cases} 2n+2 & \textit{for even } n \geq 2, \\ 2n+3 & \textit{for odd } n \geq 3. \end{cases}$

Our second result determines that F_n is H_a-good for $n \geq 111$.

Theorem 2. $R(H_a, F_n) = 4n + 2$ *for* $n \geq 111$.

We present the proofs of the two theorems in Section 2 and Section 3, respectively. The proof of Theorem 2 is more involved, and the lower bound $n \geq 111$ is not the best possible; rather, it arises from the limitations of our method. Therefore, identifying a smaller positive integer n such that F_n is H_a-good is a worthwhile direction for further research.

At the end of this section, we introduce some concepts and notation that will be used throughout. We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. For a graph G and a vertex subset $X \subseteq V(G)$, the subgraph induced by X is denoted by $G[X]$, whose vertex set is X and whose edge set consists of all edges in G with both endpoints in X. For $u \in V(G)$ and $X \subseteq V(G)$, let $N_X(u)$ denote the set of neighbors of u in X, and let $|N_X(u)|$ denote the number of such neighbors, also written as $d_X(u)$. When $X = V(G)$, we write $N_X(u)$ and $d_X(u)$ simply as $N(u)$ and $d(u)$, respectively. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Notation and terminology not explicitly defined follow Bondy and Murty (2008).

2 Proof of Theorem 1

When n is even, the lower bound can be derived from inequality (1):

$$R(H_a, K_{1,n}) \geq (\chi(H_a) - 1)(|K_{1,n}| - 1) + s(H_a) = 2n + 2.$$

When n is odd, let $\ell = (n+1)/2$. Consider the graph $(\ell K_2) + (\ell K_2)$, which consists of two disjoint copies of a matching with ℓ edges, where the edges between the two copies form a complete bipartite graph. It is straightforward to verify that this graph does not contain the Hajós graph as a subgraph, and that the maximum degree in its complement is $n-1$. Therefore, $R(H_a, K_{1,n}) \geq 2n+3$ for odd n.

For the upper bound, we first consider the case $n = 2$, namely, proving that $R(H_a, K_{1,2}) \leq 6$. For any graph H of order 6, if its complement $\overline{H}$ does not contain $K_{1,2}$ as a subgraph, then the edge set of $\overline{H}$ forms a matching. Consequently, H must contain $K_{2,2,2}$ as a subgraph, which in turn contains H_a as a subgraph.

Now suppose $n \geq 3$. Let G be an arbitrary graph with $2n+2+\mathbf{1}_{\text{odd}}(n)$ vertices, where $\mathbf{1}_{\text{odd}}(n)$ is an indicator function: $\mathbf{1}_{\text{odd}}(n) = 0$ if n is even, and $\mathbf{1}_{\text{odd}}(n) = 1$ if n is odd. Suppose $\overline{G}$ contains no copy of $K_{1,n}$. We now prove that G must contain the Hajós graph as a subgraph.

Since $\Delta(\overline{G}) \leq n-1$, it follows that $\delta(G) \geq n+2+\mathbf{1}_{\text{odd}}(n)$. We proceed by considering three cases.

Case 1. *The graph G does not contain K_4 as a subgraph.*

By the classical result of Chvátal (1977), we have $R(K_3, K_{1,n}) = 2n+1$. Therefore, the graph G must contain a triangle; without loss of generality, let its vertices be u_1, u_2, u_3. For $1 \leq i < j \leq 3$, the minimum degree condition implies

$$d(u_i) + d(u_j) \geq 2\delta(G) \geq 2n + 4 + 2 \times \mathbf{1}_{\text{odd}}(n) \geq |G| + 2.$$

Thus, vertices u_1 and u_2 share a common neighbor distinct from u_3, denoted u_4; similarly, u_1 and u_3 share a common neighbor $u_5 \neq u_2$, and u_2 and u_3 share a common neighbor $u_6 \neq u_1$. Since G does not contain K_4 as a subgraph, the vertices u_4, u_5, u_6 must be pairwise distinct. Consequently, these six vertices induce a Hajós graph.

Case 2. *The graph G contains $K_5 - e$ as a subgraph.*

Let us denote the five vertices of one such $K_5 - e$ by v_1, v_2, v_3, v_4, v_5, and let the set of remaining vertices be W. Then we have $|W| = 2n - 3 + \mathbf{1}_{\text{odd}}(n)$. Note that $K_5 - e$ contains a clique of four vertices; without loss of generality, assume that v_1, v_2, v_3, v_4 induce a K_4. By the minimum degree condition, for each $i \in [4]$, the vertex v_i has at least $n - 2 + \mathbf{1}_{\text{odd}}(n)$ neighbors in W. Since $n \geq 3$, it follows that

$$4(n - 2 + \mathbf{1}_{\text{odd}}(n)) > 2n - 3 + \mathbf{1}_{\text{odd}}(n) = |W|.$$

Therefore, among v_1, v_2, v_3, v_4, there must exist two vertices, say v_1 and v_2, which share a common neighbor in W, denoted by v_6. Because the vertex v_5 has at least two neighbors in $\{v_2, v_3, v_4\}$, we may assume without loss of generality that v_5 is adjacent to both v_2 and v_3. Thus, we obtain a Hajós graph, where the vertex subsets $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_6\}$, $\{v_2, v_3, v_5\}$, and $\{v_1, v_3, v_4\}$ each induce a triangle.

Case 3. *The graph G does not contain $K_5 - e$ as a subgraph, but contains K_4 as a subgraph.*

Let w_1, w_2, w_3, w_4 be the vertices of a K_4 in G.

We first consider the case when n is even. Since $\delta(G) \geq n+2$, each vertex w_i for $i \in [4]$ has at least $n-1$ neighbors in $V(G) \setminus \{w_1, w_2, w_3, w_4\}$. As $|V(G) \setminus \{w_1, w_2, w_3, w_4\}| = 2n-2$, there must exist two vertices, say w_1 and w_2, that share a common neighbor in $V(G) \setminus \{w_1, w_2, w_3, w_4\}$, denoted by w_5. Since G does not contain $K_5 - e$ as a subgraph, neither w_3 nor w_4 is adjacent to w_5.

Let $X = V(G) \setminus \{w_1, w_2, w_3, w_4, w_5\}$. Then $|X| = 2n - 3$. By the minimum degree condition,

$$d_X(w_1) \geq n-2, d_X(w_2) \geq n-2, d_X(w_3) \geq n-1, \text{and } d_X(w_4) \geq n-1.$$

If w_1 and w_3 have a common neighbor in X, denoted by w_6, then we can construct a Hajós graph in which the vertex subsets $\{w_1, w_2, w_3\}$, $\{w_1, w_2, w_5\}$, $\{w_2, w_3, w_4\}$, and $\{w_1, w_3, w_6\}$ each induce a triangle. Hence, $N_X(w_1) \cap N_X(w_3) = \emptyset$. By symmetry, we have $N_X(w_2) \cap N_X(w_3) = \emptyset$.

Since $|X| = 2n - 3$, $d_X(w_1) \geq n - 2$ and $d_X(w_3) \geq n - 1$, the sets $N_X(w_1)$ and $N_X(w_3)$ form a partition of X, and we have $|N_X(w_1)| = n - 2$ and $|N_X(w_3)| = n - 1$. Given that $d_X(w_2) \geq n - 2$ and $N_X(w_2) \cap N_X(w_3) = \emptyset$, it follows that $N_X(w_1) = N_X(w_2)$. Therefore,

$$N(w_1) \cap N(w_2) = N_X(w_1) \cup \{w_3, w_4, w_5\} \text{ and } |N(w_1) \cap N(w_2)| = n + 1.$$

Since n is even, by the classical result of Harary (1972),

$$R(K_{1,2}, K_{1,n}) = n + 1 = |N(w_1) \cap N(w_2)|.$$

By assumption, $\overline{G}$ does not contain $K_{1,n}$ as a subgraph. Hence, the graph $K_{1,2}$ appears within the common neighborhood of w_1 and w_2, and together with w_1 and w_2 induces a subgraph isomorphic to $K_5 - e$. This contradicts the assumption that G does not contain $K_5 - e$ as a subgraph.

Now consider the case where n is odd. Let $Y = V(G) \setminus \{w_1, w_2, w_3, w_4\}$. Since $\delta(G) \geq n + 3$, each vertex w_i for $i \in [4]$ has at least n neighbors in Y. Given that $|Y| = 2n - 1$, any two vertices among w_1, w_2, w_3, w_4 share at least one common neighbor in Y. Without loss of generality, assume that w_1 and w_2 share a common neighbor w_5 in Y, and w_2 and w_3 share a common neighbor w_6 in Y. As G does not contain $K_5 - e$ as a subgraph, vertices w_5 and w_6 must be distinct. Consequently, we obtain a Hajós graph in which the vertex subsets $\{w_1, w_2, w_3\}$, $\{w_1, w_2, w_5\}$, $\{w_2, w_3, w_6\}$, and $\{w_1, w_3, w_4\}$ each induce a triangle. □

3 Proof of Theorem 2

The proof of Theorem 2 relies on the following key lemma, which was established by Zeng et al. (2025).

Lemma 1. $R(W_4, F_n) = 4n + 1$ *for* $n \geq 111$.

Now we begin the proof of Theorem 2.

Proof of Theorem 2: The lower bound follows directly from inequality (1):

$$R(H_a, F_n) \geq (\chi(H_a) - 1)(|F_n| - 1) + s(H_a) = (3 - 1)(2n + 1 - 1) + 2 = 4n + 2.$$

To prove the upper bound, we proceed by contradiction. Assume that a graph G with $4n + 2$ vertices contains neither H_a as a subgraph nor F_n as a subgraph in $\overline{G}$. We will derive contradictions in the following two cases.

Case 1. $\Delta(\overline{G}) \geq 2n + 2$

Let u be a vertex in $\overline{G}$ with the maximum degree. Denote by $H^{'}$ the subgraph of $\overline{G}$ induced by the set of vertices adjacent to u in $\overline{G}$, i.e., $H^{'} = \overline{G}[N_{\overline{G}}(u)]$. Let M be a maximum matching in $H^{'}$. Then M must contain exactly $n - 1$ edges. This is because if M has at least n edges, these n edges together with u would form an F_n in $\overline{G}$, a contradiction. Conversely, if M has at most $n - 2$ edges, the number of vertices in $H^{'}$ not covered by M (i.e., the remaining vertices after removing the vertices of M) is at least $2n + 2 - 2(n - 2) = 6$. These vertices form a complete graph in G, which clearly contains H_a as a subgraph, leading to a contradiction.

Since $\Delta(\overline{G}) \geq 2n+2$, it follows that $|V(H^{'}) \setminus V(M)| \geq 4$, and the vertices in $V(H^{'}) \setminus V(M)$ induce a complete subgraph in G.

- If $|V(H^{'}) \setminus V(M)| \geq 6$, it is evident that G contains H_a as a subgraph.
- If $|V(H^{'}) \setminus V(M)| = 5$, consider any edge $y_1y_2 \in M$. At least one of y_1 or y_2 is adjacent to at least four vertices in $V(H^{'}) \setminus V(M)$ in G. Otherwise, $H^{'}$ would contain a matching larger than M, contradicting the maximality of M. Without loss of generality, assume y_1 is adjacent to at least four vertices in $V(H^{'}) \setminus V(M)$. Then the subgraph of G induced by $(V(H^{'}) \setminus V(M)) \cup \{y_1\}$ must contain H_a as a subgraph, leading to a contradiction.
- If $|V(H^{'}) \setminus V(M)| = 4$, consider any two edges y_1y_2 and $y_3y_4 \in M$. For each edge, at least one endpoint must be adjacent to at least three vertices in $V(H^{'}) \setminus V(M)$ in G. Otherwise, $H^{'}$ would contain a matching larger than M, again contradicting the maximality of M. Without loss of generality, assume y_1 and y_3 are each adjacent to at least three vertices in $V(H^{'}) \setminus V(M)$. Then the subgraph of G induced by $(V(H^{'}) \setminus V(M)) \cup \{y_1, y_3\}$ must contain H_a as a subgraph, leading to a contradiction.

Case 2. $\delta(G) \geq 2n$

According to Lemma 1, the graph G contains W_4 as a subgraph. Let the center of this W_4 be u_0, with the corresponding cycle $u_1u_2u_3u_4u_1$. We have the following claim.

Claim 1. *The edges* $u_1u_3, u_2u_4 \notin E(G)$.

Proof: We only need to prove that $u_1u_3 \notin E(G)$; the other case can be proven similarly. Suppose $u_1u_3 \in E(G)$. Since $\delta(G) \geq 2n$, the set $\{u_1, u_2, u_3\}$ must contain two vertices, say u_1 and u_3, that have a common neighbor in $V(G) \setminus V(W_4)$. Otherwise, this would contradict the fact that G has $4n+2$ vertices. Let u_5 be a common neighbor of u_1 and u_3 in $V(G) \setminus V(W_4)$. Then the subgraph induced by the vertex set $\{u_0, u_1, u_2, u_3, u_4, u_5\}$ in G contains H_a as a subgraph, a contradiction. □

If u_1 and u_2 have a common neighbor distinct from u_0, denoted by u_5, then the subgraph induced by the vertex set $\{u_0, u_1, u_2, u_3, u_4, u_5\}$ contains H_a as a subgraph. Therefore, the only common neighbor of u_1 and u_2 can be u_0. Furthermore, since $\delta(G) \geq 2n$, u_2 must have at least $2n-1$ neighbors in $V(G) \setminus N(u_1)$. Similarly, u_4 must also have at least $2n-1$ neighbors in $V(G) \setminus N(u_1)$. Hence, the number of common neighbors of u_2 and u_4 in $V(G) \setminus N(u_1)$ is at least

$$2(2n-1) - |V(G) \setminus N(u_1)| \geq 2(2n-1) - (4n+2-2n) = 2n-4.$$

We denote the set of common neighbors of vertices u_2 and u_4, excluding u_0, as U_1. Then, $|U_1| \geq 2n-4$. Similarly, let U_2 represent the set of common neighbors of vertices u_1 and u_3, excluding u_0. By symmetry, we have $|U_2| \geq 2n-4$. Since u_1 and u_2 have only one common neighbor, u_0, it follows that $U_1 \cap U_2 = \emptyset$. It is easy to verify that $u_1, u_3 \in U_1$ and $u_2, u_4 \in U_2$. To avoid the appearance of the graph H_a, for $i = 1, 3$ and $j = 2, 4$, the only common neighbor of u_i and u_j must be u_0. Therefore, the vertices u_1 and u_3 are isolated in the graph $G[U_1]$, and the vertices u_2 and u_4 are isolated in the graph $G[U_2]$. Furthermore, we have the following claim.

Claim 2. *Every connected component of the graph* $G[U_1 \setminus \{u_1, u_3\}]$ *is either an isolated vertex or a star.*

Proof: First, we prove that the graph $G[U_1 \setminus \{u_1, u_3\}]$ does not contain P_4 as a subgraph. We proceed by contradiction. Suppose $G[U_1 \setminus \{u_1, u_3\}]$ contains a P_4 as a subgraph, with vertices x_1, x_2, x_3, x_4. The subgraph induced by the vertex set $\{u_2, u_4, x_1, x_2, x_3, x_4\}$ in G contains H_a as a subgraph, which leads to a contradiction. Thus, each connected component of the graph $G[U_1 \setminus \{u_1, u_3\}]$ is either an isolated vertex, a star, or a triangle. Note that when a connected component is K_2, it is also considered a star.

Next, we only need to prove that the graph $G[U_1 \setminus \{u_1, u_3\}]$ does not contain a triangle as a subgraph. Again, we proceed by contradiction. Suppose $G[U_1 \setminus \{u_1, u_3\}]$ contains a triangle with vertices x_1, x_2, x_3. There must exist two vertices among $\{x_1, x_2, x_3\}$, say x_1 and x_3, that have a common neighbor in $V(G) \setminus \{x_1, x_2, x_3, u_2, u_4\}$, otherwise this would contradict the fact that the number of vertices in G is $4n + 2$. Let x_4 be a common neighbor of x_1 and x_3 in $V(G) \setminus \{x_1, x_2, x_3, u_2, u_4\}$. Then the subgraph induced by the vertex set $\{x_1, x_2, x_3, x_4, u_2, u_4\}$ in G contains H_a as a subgraph, which leads to a contradiction. □

By symmetry, it can also be shown that each connected component of the graph $G[U_2 \setminus \{u_2, u_4\}]$ is either an isolated vertex or a star. Hence, the following claim holds.

Claim 3. $|U_1| \leq 2n$ *and* $|U_2| \leq 2n$*.*

Proof: We proceed by contradiction. Assume $|U_1| \geq 2n + 1$ or $|U_2| \geq 2n + 1$. By symmetry, we may assume $|U_1| \geq 2n + 1$. It suffices to find a fan F_n centered at u_1 in $\overline{G}$, leading to a contradiction.

By the previous claim, each connected component of the graph $G[U_1 \setminus \{u_1\}]$ is either an isolated vertex or a star. Let the nontrivial (non-isolated) connected components of this graph be $C_1, \ldots, C_k$. Note that in each connected component, at most one vertex has degree greater than 1 in $G[U_1 \setminus \{u_1\}]$.

When $k \geq 2$, for each $1 \leq i \leq k - 1$, the center vertex of C_i and a leaf vertex of C_{i+1} form an edge in $\overline{G}$. Similarly, the center vertex of C_k and a leaf vertex of C_1 form an edge in $\overline{G}$. These k edges form a matching in $\overline{G}$. The remaining vertices in $\overline{G}[U_1 \setminus \{u_1\}]$ induce a complete subgraph, allowing us to find a matching of n edges in $\overline{G}[U_1 \setminus \{u_1\}]$. This gives a fan F_n centered at u_1 in $\overline{G}$, a contradiction.

When $k = 1$, the graph $G[U_1 \setminus \{u_1\}]$ has only one nontrivial connected component, which must be a star. Let v_0 be the center of this star. Then $v_0 u_3 \in E(\overline{G})$. Since the remaining vertices in $\overline{G}[U_1 \setminus \{u_1\}]$ induce a complete subgraph, $v_0 u_3$ can be extended to a matching of n edges in $\overline{G}[U_1 \setminus \{u_1\}]$. This gives a fan F_n centered at u_1 in $\overline{G}$, a contradiction.

When $k = 0$, the graph $G[U_1 \setminus \{u_1\}]$ is an empty graph, so $\overline{G}[U_1 \setminus \{u_1\}]$ contains a matching of n edges. This also gives a fan F_n centered at u_1 in $\overline{G}$, a contradiction. □

We denote the set of vertices outside $U_1 \cup U_2 \cup \{u_0\}$ by W. Then,

$$|W| \leq 4n + 2 - 2(2n - 4) - 1 = 9.$$

Furthermore, we partition W into four subsets: W_1, W_2, W_3, and W_4. If a vertex in W is adjacent to u_2 or u_4, it belongs to W_1. If a vertex in W is adjacent to u_1 or u_3, it belongs to W_2. If a vertex $w \in W$ is not adjacent to any of u_1, u_2, u_3, or u_4, and the number of its neighbors in U_2 is at least the number of its neighbors in U_1, i.e., $d_{U_2}(w) \geq d_{U_1}(w)$, then it belongs to W_3. Otherwise, if $d_{U_2}(w) < d_{U_1}(w)$, it belongs to W_4. To avoid the appearance of the graph H_a, we have $W_1 \cap W_2 = \emptyset$. Consequently, W_1, W_2, W_3, and W_4 form a partition of W.

Note that the neighbors of u_2 must belong to $W_1 \cup U_1 \cup \{u_0\}$. Since $\delta(G) \geq 2n$, we have $|W_1| \geq 2n - 1 - |U_1|$. By symmetry, $|W_2| \geq 2n - 1 - |U_2|$. Hence, $|W_3| + |W_4| \leq (4n+2) - 1 - 2(2n-1) = 3$.

By the pigeonhole principle, either $|U_1| + |W_1| + |W_3| \geq 2n + 1$, or $|U_2| + |W_2| + |W_4| \geq 2n + 1$. Without loss of generality, assume the former holds. Next, we will show that if G does not contain H_a as a subgraph, then $\overline{G}[U_1 \cup W_1 \cup W_3]$ must contain a fan F_n centered at u_1.

Note that every vertex in $(U_1 \cup W_1 \cup W_3) \setminus \{u_1, u_3\}$ is not adjacent to u_1 or u_3. Therefore, it suffices to find a matching of n edges in $\overline{G}[(U_1 \cup W_1 \cup W_3) \setminus \{u_1\}]$, which yields a fan F_n centered at u_1 in $\overline{G}$.

From $W_1 \cup W_3$, select $2n+1-|U_1|$ vertices, denoted by $w_1, \dots, w_t$, where $t = 2n+1-|U_1|$. Since $|U_1| \geq 2n-4$, it follows that $t \leq 5$. We first find a matching M_1 in $\overline{G}[\{w_1, \dots, w_t\} \cup (U_1 \setminus \{u_1, u_3\})]$ such that each edge of M_1 has at least one endpoint in $\{w_1, \dots, w_t\}$ and $V(M_1)$ includes as many vertices as possible from $\{w_1, \dots, w_t\}$. The following claim holds:

Claim 4. *At most one vertex in $\{w_1, \dots, w_t\}$ does not belong to $V(M_1)$.*

Proof: Suppose at least three vertices in $\{w_1, \dots, w_t\}$ are not in $V(M_1)$. Without loss of generality, assume $w_1, w_2, w_3 \notin V(M_1)$. It follows that w_1, w_2, w_3 form a triangle in G. To see this, if any edge is missing from G, say $w_1w_2 \in E(\overline{G})$, we could add the edge w_1w_2 to M_1, forming a new matching that includes more vertices from $\{w_1, \dots, w_t\}$, contradicting the choice of M_1. Additionally, by the choice of M_1, w_1, w_2, and w_3 must each be adjacent to every vertex in $U_1 \setminus (\{u_1, u_3\} \cup V(M_1))$. Since $t \leq 5$, at most two vertices of $V(M_1)$ are in U_1. Thus, $|U_1 \setminus (\{u_1, u_3\} \cup V(M_1))| \geq (2n-4)-(2+2) = 2n-8$. Selecting any three vertices from $U_1 \setminus (\{u_1, u_3\} \cup V(M_1))$, together with w_1, w_2, w_3, induces a complete multipartite graph $K_{1,1,1,3}$ in G, which contains H_a as a subgraph, a contradiction.

If exactly two vertices in $\{w_1, \dots, w_t\}$ are not in $V(M_1)$, assume $w_1, w_2 \notin V(M_1)$. Then $w_1w_2 \in E(G)$ by the choice of M_1. Moreover, w_1 and w_2 must be adjacent to every vertex in $U_1 \setminus (\{u_1, u_3\} \cup V(M_1))$. Since at most three vertices in $V(M_1)$ are in U_1, we have $|U_1 \setminus (\{u_1, u_3\} \cup V(M_1))| \geq |U_1| - 5 \geq 2n-9$.

First, consider $w_1, w_2 \in W_1$. If w_1 and w_2 are both adjacent to either u_2 or u_4, without loss of generality, assume $w_1u_2, w_2u_2 \in E(G)$. Selecting any three vertices from $U_1 \setminus (\{u_1, u_3\} \cup V(M_1))$, together with w_1, w_2, and u_2, induces a complete multipartite graph $K_{1,1,1,3}$ in G, which contains H_a as a subgraph, leading to a contradiction. If w_1 and w_2 are adjacent to different vertices in $\{u_2, u_4\}$, assume $w_1u_2, w_2u_4 \in E(G)$. Selecting any two vertices u_x, u_y from $U_1 \setminus (\{u_1, u_3\} \cup V(M_1))$, the four sets $\{w_1, w_2, u_x\}$, $\{w_1, w_2, u_y\}$, $\{w_1, u_2, u_y\}$, and $\{w_2, u_4, u_y\}$ each induce a triangle. Thus, G contains H_a as a subgraph, a contradiction.

Next, consider $w_1, w_2 \in W_3$. By the definition of W_3, w_1 and w_2 each have at least $2n-9$ neighbors in U_2. Thus, w_1 and w_2 share at least $2(2n-9) - |U_2| \geq 2n-18$ common neighbors in U_2, where this inequality follows from Claim 3.

If each common neighbor of w_1 and w_2 in U_2 has degree at least 2 in $G[U_2]$, note that each connected component in $G[U_2]$ has at most one vertex of degree at least 2. Hence, $G[U_2]$ must have at least $2n-18$ components, each containing a vertex of degree at least 2, contradicting $|U_2| \leq 2n$. Thus, there exists a common neighbor u_z of w_1 and w_2 in U_2 with degree at most 1 in $G[U_2]$. Then the vertex u_z has at least $2n-2-|W| \geq 2n-11$ neighbors in U_1. Consequently, u_z, w_1, and w_2 share at least $(2n-11)+(|U_1|-5)-|U_1| \geq 2n-16$ common neighbors in U_1. Selecting three such common neighbors together with u_z, w_1, w_2 induces a complete multipartite graph $K_{1,1,1,3}$, which contains H_a as a subgraph, a contradiction.

If w_1 and w_2 belong to W_1 and W_3, respectively, let $w_1 \in W_1$ and $w_2 \in W_3$. Then w_1 must be adjacent to either u_2 or u_4. By symmetry, assume $w_1u_2 \in E(G)$. According to the selection of W_3, w_2 has at least $2n-9$ neighbors in U_2. Therefore, w_2 must have a neighbor u_z in U_2 whose degree in $G[U_2]$ is at most 1; otherwise, this would contradict $|U_2| \leq 2n$. Consequently, the number of neighbors of u_z in U_1 is at least $2n-2-|W| \geq 2n-11$. Thus, the number of common neighbors of u_z, w_1, and w_2 in U_1 is at least $(2n-11)+(|U_1|-5)-|U_1| \geq 2n-16$. Selecting two common neighbors, together with u_z, w_1, w_2, and u_2, forms a subgraph in G that contains H_a as a subgraph, leading to a contradiction. □

Based on Claim 2, every connected component in the graph $G[U_1 \setminus (\{u_1, u_3\} \cup V(M_1))]$ is either an isolated vertex or a star. Denote the connected components of this graph as $D_1, \ldots, D_k$. Note that each connected component contains at most one vertex with degree greater than 1 in $G[U_1 \setminus (\{u_1, u_3\} \cup V(M_1))]$. Let $H = G[(U_1 \cup \{w_1, \ldots, w_t\}) \setminus (\{u_1\} \cup V(M_1))]$. We establish the following claim.

Claim 5. *There exists a matching M_2 in $\overline{H}$ that covers all vertices in H with degree greater than* 1.

Proof: When $k \geq 2$, for $i \in [k]$ and $D_{k+1} := D_1$, if D_i contains a vertex with degree greater than 1, then this vertex has no edge to a vertex with degree at most 1 in D_{i+1}. That is, there exists an edge between these two vertices in $\overline{H}$. Add this edge to M_2. If exactly one vertex in $\{w_1, \ldots, w_t\}$ is not in $V(M_1)$, assume without loss of generality that $w_1 \notin V(M_1)$. Then $w_1u_3 \in E(\overline{G})$, and we add the edge w_1u_3 to M_2. If $\{w_1, \ldots, w_t\} \subseteq V(M_1)$, the edge w_1u_3 does not need to be included in M_2. Clearly, M_2 covers all vertices in H with degree greater than 1.

When $k = 1$, the graph $G[U_1 \setminus (\{u_1, u_3\} \cup V(M_1))]$ has only one connected component, which must be a star. Let the center of this star be v_0, and let two of its leaves be v_1 and v_2.

If $\{w_1, \ldots, w_t\} \subseteq V(M_1)$, add the edge v_0u_3 to M_2. Since v_0 is the only vertex in H with degree greater than 1, M_2 covers all such vertices.

If exactly one vertex in $\{w_1, \ldots, w_t\}$ is not in $V(M_1)$, assume without loss of generality that $w_1 \notin V(M_1)$. By the choice of M_1, w_1 is adjacent to all vertices in $U_1 \setminus (\{u_1, u_3\} \cup V(M_1))$, including v_0, v_1, and v_2. This implies that w_1 has at least $2n - 11$ neighbors in U_1.

If $w_1 \in W_3$, by the definition of W_3, w_1 has at least $2n - 11$ neighbors in U_2. Since v_2 has exactly one neighbor in U_1, it must have at least $2n - 2 - |W| \geq 2n - 11$ neighbors in U_2. Thus, w_1 and v_2 share at least $2(2n - 11) - |U_2| \geq 2n - 22$ common neighbors in U_2. Let u' be one such common neighbor. The sets $\{w_1, v_0, v_1\}$, $\{w_1, v_0, v_2\}$, $\{w_1, v_2, u'\}$, and $\{u_2, v_0, v_2\}$ each induce a triangle. Therefore, G contains H_a as a subgraph, a contradiction.

If $w_1 \in W_1$, then either $w_1u_2 \in E(G)$ or $w_1u_4 \in E(G)$. By symmetry, assume $w_1u_2 \in E(G)$. The sets $\{w_1, v_0, v_1\}$, $\{w_1, v_0, v_2\}$, $\{w_1, v_2, u_2\}$, and $\{u_4, v_0, v_2\}$ each induce a triangle. Hence, G contains H_a as a subgraph, a contradiction. □

In the subgraph $G[(U_1 \cup \{w_1, \ldots, w_t\}) \setminus (\{u_1\} \cup V(M_1) \cup V(M_2))]$, each vertex has degree at most 1, and this subgraph contains at least $2(n - |M_1| - |M_2|)$ vertices. Below, we demonstrate that $n - |M_1| - |M_2| \geq 2$.

Since in the graph $G[U_1 \setminus (\{u_1, u_3\} \cup V(M_1))]$, at most one-third of the vertices have degree greater than 1, the vertex set $V(M_2)$ can contain at most two-thirds of this portion. Therefore, the remaining vertices number at least $|U_1 \setminus (\{u_1, u_3\} \cup V(M_1))|/3 \geq (2n - 11)/3$. Combining this with $n \geq 111$, the subgraph $G[(U_1 \cup \{w_1, \ldots, w_t\}) \setminus (\{u_1\} \cup V(M_1) \cup V(M_2))]$ has at least $(222 - 11)/3 \geq 70$ vertices.

Hence, in the graph $\overline{G}[(U_1 \cup \{w_1, \ldots, w_t\}) \setminus (\{u_1\} \cup V(M_1) \cup V(M_2))]$, we can identify a matching consisting of $n - |M_1| - |M_2|$ edges, denoted by M_3. In $\overline{G}$, the edge set $M_1 \cup M_2 \cup M_3$ forms a matching of n edges, and every vertex in this matching is adjacent to the vertex u_1. Consequently, $\overline{G}$ contains F_n as a subgraph, leading to a final contradiction. □

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Data Availability Statement

No data was used or generated in this research.

Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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