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Low complexity binary words avoiding $(5/2)^+$-powers

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Rote words are infinite words that contain $2n$ factors of length n for every $n \geq 1$. Shallit and Shur, as well as Ollinger and Shallit, showed that there are Rote words that avoid $(5/2)^+$-powers and that this is best possible. In this note we give a structure theorem for the Rote words that avoid $(5/2)^+$-powers, confirming a conjecture of Ollinger and Shallit.

Keywords: Rote word, factor complexity, $(5/2)^+$-power, structure theorem

1 Introduction

Two central concepts in combinatorics on words are *power avoidance* and *factor complexity*. Recently, Shallit and Shur (2019) initiated the systematic investigation of the interplay between these two concepts. They examined two dual problems: 1) Given a particular power avoidance constraint, determine the range of possible factor complexities among all infinite words avoiding the specified power; and, 2) Given a class of words with specified factor complexities, determine the powers that are avoided by at least one word in this class.

A well-known classical result provides a solution for the latter problem for the class of *Sturmian words*, i.e., the class of infinite words that contain $n + 1$ factors of length n for every $n \geq 1$: the *Fibonacci word* avoids $(5 + \sqrt{5})/2$-powers, and this is best possible among all Sturmian words (Carpi and de Luca (2000)). The Sturmian words are the aperiodic infinite words with the least possible factor complexity function; Shallit and Shur (2019) and Ollinger and Shallit (2024) studied another class of infinite words with low complexity, namely, the *Rote words*, which are the infinite words that contain $2n$ factors of length n for every $n \geq 1$ (Rote (1994)). Each paper gives an example of an infinite Rote word that avoids $(5/2)^+$-powers and shows that this is best possible among all Rote words. Ollinger and Shallit end their paper by observing that the Rote words that avoid $(5/2)^+$-powers appear to have a certain rigid structure

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reminiscent of the famous structure theorem of Restivo and Salemi (1985) for the class of overlap-free words. In this note we obtain the precise structure theorem.

To state our first structure theorem, we introduce *proper words* and *antiproper words*, which are ternary words.

Definition 1. For $u \in \Sigma_3^*$, denote the Parikh vector of u by $\pi(u)$so that $\pi(u) = [|u|_0, |u|_1, |u|_2]$. For $x, y \in \Sigma_3^*$, we say that $\pi(x) > \pi(y)$ if:

1. We have $|x|_i \geq |y|_i$ for all $i \in \Sigma_3$.
2. For at least one $i \in \Sigma_3$ we have $|x|_i > |y|_i$.

Call a word $u \in \Sigma_3^*$ *proper* if:

1. Word u has no factor $xyxyx$ where $\pi(x) > \pi(y)$.
2. None of the words 00, 11, 22, 20, 10101, 2121, or 10210210 is a factor of u.

Call a word $\boldsymbol{u} \in \Sigma_3^\omega$ proper if all of its finite factors are proper.

Call a word $u \in \Sigma_3^*$ *antiproper* if its reverse u^R is proper. Call a word $\boldsymbol{u} \in \Sigma_3^\omega$ antiproper if all of its finite factors are antiproper.

Proper words obey a structure theorem similar to that of Restivo and Salemi, as do antiproper words. We introduce a morphism $f : \Sigma_3^* \to \Sigma_3^*$ and its reverse $h : \Sigma_3^* \to \Sigma_3^*$, given by:

$$\begin{aligned} f(0) &= 0121 & \qquad h(0) &= 1210 \\ f(1) &= 021 & \qquad h(1) &= 120 \\ f(2) &= 01 & \qquad h(2) &= 10. \end{aligned}$$

Theorem 1. *(First Structure Theorem)*

1. *Let $\boldsymbol{u} \in \Sigma_3^\omega$ be proper. Then a final segment of $\boldsymbol{u}$ has the form $f(\boldsymbol{v})$ for some proper $\boldsymbol{v} \in \Sigma_3^\omega$.*
2. *Let $\boldsymbol{u} \in \Sigma_3^\omega$ be antiproper. Then a final segment of $\boldsymbol{u}$ has the form $h(\boldsymbol{v})$ for some antiproper $\boldsymbol{v} \in \Sigma_3^\omega$.*

For our second structure theorem we consider the length-4 factors of a Rote word.

Definition 2. Let F be the set

$$F = \{0110, 1001, 0011, 1100, 0010, 0100, 1101, 1010\}.$$

Let $g : \Sigma_3^* \to \Sigma_2^*$ be the morphism given by

$$\begin{aligned} g(0) &= 011 \\ g(1) &= 0 \\ g(2) &= 01. \end{aligned}$$

We denote the complement of a binary word w by $\overline{w}$; thus $\overline{1101} = 0010$. We extend this notation to binary languages in the usual way. We denote the reversal of a word w by w^R; thus $1101^R = 1011$. We extend this notation to languages in the usual way.

We can now characterize the structure of Rote words that avoid $(5/2)^+$-powers.

Theorem 2. *(Second Structure Theorem) Let $\boldsymbol{w}$ be a Rote word that avoids $(5/2)^+$-powers. There are four cases:*

1. *The set of length-4 factors of $\boldsymbol{w}$ is F. For every positive integer n, a final segment of $\boldsymbol{w}$ has the form $g(f^n(\boldsymbol{u}))$ for some proper $\boldsymbol{u} \in \Sigma_3^\omega$.*

2. *The set of length-4 factors of $\boldsymbol{w}$ is $\bar{F}$. For every positive integer n, a final segment of $\boldsymbol{w}$ has the form $\overline{g(f^n(\boldsymbol{u}))}$ for some proper $\boldsymbol{u} \in \Sigma_3^\omega$.*

3. *The set of length-4 factors of $\boldsymbol{w}$ is F^R. For every positive integer n, a final segment of $\boldsymbol{w}$ has the form $g(h^n(\boldsymbol{u}))$ for some proper $\boldsymbol{u} \in \Sigma_3^\omega$.*

4. *The set of length-4 factors of $\boldsymbol{w}$ is $\overline{F^R}$. For every positive integer n, a final segment of $\boldsymbol{w}$ has the form $\overline{g(h^n(\boldsymbol{u}))}$ for some proper $\boldsymbol{u} \in \Sigma_3^\omega$.*

2 Preliminaries

For a positive integer n, let $\Sigma_n = \{0, 1, \ldots, n-1\}$ and let Σ_n^* denote the set of all finite words over Σ_n. By a binary word we mean a (finite or infinite) word over Σ_2. Let w be a word and write $w = xyz$. Then the word x is a *prefix* of w, the word y is a *factor* of w, and the word z is a *suffix* of w. A map $f : \Sigma_m^* \to \Sigma_n^*$ is a *morphism* if $f(xy) = f(x)f(y)$ for all x and y.

Let w have length ℓ and smallest period p. The *exponent of* w is the quantity $k = \ell/p$ and w is called a *k-power*. A *k^+-power* is a word with exponent $> k$. A 2-power is a *square* and a 2^+-power is an *overlap*. A word w *avoids k-powers* (resp. *k^+-powers*) if none of its factors are k'-powers for any $k' >= k$ (resp. $k' > k$); we also say that w is *k-power-free* (resp. *k^+-power-free*).

Let $\mathbf{x}$ be an infinite word. The *factor complexity of* $\mathbf{x}$ is the function of n that associates each length n with the number of factors of $\mathbf{x}$ of length n. A *Sturmian word* is any infinite word with factor complexity $n+1$; a *Rote word* is any infinite word with factor complexity $2n$.

Recently the morphisms f and g have proved useful in several constructions (Currie et al. (2023); Dvořáková et al. (2024); Ollinger and Shallit (2024)). Dvořáková et al. (2024) showed that $g(f^\omega(0))$ avoids $(5/2)^+$-powers, while Ollinger and Shallit (2024) showed that the factor complexity of this word is $2n$ (i.e., that it is a Rote word). The latter authors conjectured that there is a structure theorem involving f and g for the class of $(5/2)^+$-power-free Rote words.

A prototypical structure theorem of this type was obtained by Restivo and Salemi (1985) for the class of binary overlap-free words. (See also Fife (1980); Shur (1996); Shallit (2011).) In this case, the structure is specified using the Thue–Morse morphism

$$\mu(0) = 01$$
$$\mu(1) = 10.$$

Restivo and Salemi showed the following:

Theorem 3. *Let $\boldsymbol{w} \in \Sigma_2^\omega$ be overlap-free. Then a final segment of $\boldsymbol{w}$ has the form $\mu(\boldsymbol{u})$ where $\boldsymbol{u}$ is overlap-free.*

Karhumäki and Shallit (2004) later showed that the same structure theorem holds for the class of binary $(7/3)^+$-power-free words. In this note we establish a similar structure theorem for the class of $(5/2)^+$-power-free Rote words in terms of the morphisms f and g given above. The reader may also compare the present structure theorem with the main result of Currie et al. (2020–2021), which establishes a structure theorem for the class of infinite $14/5$-power-free binary rich words (*rich* means that every factor of length n contains n distinct non-empty palindromes), and which also involves a sub-family of Rote words, namely the *complementary symmetric Rote words*.

3 Obtaining the structure theorem

Lemma 1. *Let $\boldsymbol{w}$ be an infinite binary word which avoids $\frac{5}{2}^+$ powers. Then both of the words* 0110 *and* 1001 *are factors of* $\boldsymbol{w}$.

Proof: A backtrack search shows that the longest $\frac{5}{2}^+$ power free binary word not containing 0110 has length 14. Thus 0110 (and symmetrically, 1001) is a factor of $\boldsymbol{w}$. □

Lemma 2. *Let $\boldsymbol{w}$ be an infinite binary word which avoids $\frac{5}{2}^+$ powers. At least 3 of the words in*

$$C = \{0010, 0100, 1011, 1101\}$$

are factors of $\boldsymbol{w}$.

Proof: Six backtrack searches (one for each pair) show that the longest $\frac{5}{2}^+$ power free binary word omitting a pair of these factors has length 44. □

Lemma 3. *Let $\boldsymbol{w}$ be an infinite binary word with factor complexity at most $2n$, which avoids $\frac{5}{2}^+$ powers. Both of* 0011 *and* 1100 *are factors of* $\boldsymbol{w}$.

Proof: Consider the set of seven binary words

$$A = \{0010, 0100, 0101, 1010, 1011, 1101, 1100\}.$$

For each word $a \in A$, a backtrack search shows that the longest $\frac{5}{2}^+$ power free binary word containing neither of a and 0011 as a factor has length no more than 31.

It follows that if 0011 is not a factor of $\boldsymbol{w}$, then $\boldsymbol{w}$ contains the seven words of A as length 4 factors. By Lemma 1, it also contains 0110 and 1001 as factors. However, now $\boldsymbol{w}$ contains 9 factors of length 4, contradicting the fact that its factor complexity is at most $2n$.

We conclude that 0011 (and symmetrically, 1100) is a factor of $\boldsymbol{w}$. □

Lemma 4. *Let $\boldsymbol{w}$ be an infinite binary word with factor complexity at most $2n$, which avoids $\frac{5}{2}^+$ powers. At least one of* 0101 *and* 1010 *is a factor of* $\boldsymbol{w}$.

Proof: Consider the set D containing 17 binary words of length 9 given by

$$D = \{00100110, 01001100, 10011001, 00110010, 01100100, 11001001,$$

$$10010011, 00110011, 01100110, 11001101, 10011011, 00110110,$$

$$01101100, 11011001, 10110010, 10110011, 11001100\}.$$

For each word $d \in D$, a backtrack search shows that the longest $\frac{5}{2}^+$ power free binary word containing none of d, 0101, and 1010 as a factor has length no more than 88. It follows that if neither of 0101 and 1010 is a factor of $\boldsymbol{w}$, then every word of D is a factor. However, this would imply that $\boldsymbol{w}$ contained 17 factors of length 8, contradicting the fact that its factor complexity is at most $2n$. We conclude that at least one of 0101 and 1010 is a factor of $\boldsymbol{w}$. □

Theorem 4. *Let $\boldsymbol{w}$ be an infinite binary word with factor complexity at most $2n$, which avoids $\frac{5}{2}^+$ powers. Up to binary complement and/or reversal, the set of length 4 factors of $\boldsymbol{w}$ is*

$$\{0110, 1001, 0011, 1100, 0010, 0100, 1101, 1010\}.$$

Proof: By Lemma 1, the set of length 4 factors includes 0110 and 1001. By Lemma 3, the set of length 4 factors includes 0011 and 1100. Combining Lemmas 2 and 4 with the fact that $\boldsymbol{w}$ has at most 8 length 4 factors, the set of length 4 factors contains exactly 3 words from $C = \{0100, 0010, 1011, 1101\}$ and exactly one word from $\{0101, 1010\}$. Since each word of C maps to each of the others under complement and/or reversal, assume without loss of generality that the 3 words from C are 0010, 0100, and 1101. Thus $\boldsymbol{w}$ does not contain the factor 1011.

A backtrack search shows that the longest $\frac{5}{2}^+$ power free binary word not containing 1011 or 1010 as a factor has length 20. We conclude that $\boldsymbol{w}$ contains the factor 1010, so that set of length 4 factors of $\boldsymbol{w}$ is

$$\{0110, 1001, 0011, 1100, 0010, 0100, 1101, 1010\}.$$

□

Lemma 5. *Suppose that $u \in \Sigma_3^*$, $w \in \Sigma_2^*$, and $g : \Sigma_3^* \to \Sigma_2^*$ is a non-erasing morphism. If $g(u) = w$, and w avoids $\frac{5}{2}^+$ powers, then u has no factor $xyxyx$ where $\pi(x) > \pi(y)$.*

Proof: Suppose u has a factor $xyxyx$ where $\pi(x) > \pi(y)$. Then $|g(x)| > |g(y)|$, and w contains the $\frac{5}{2}^+$ power $g(x)g(y)g(x)g(y)g(x)$. □

Lemma 6. *Suppose that $u, v \in \Sigma_3^*$, and $f : \Sigma_3^* \to \Sigma_3^*$ is a non-erasing morphism. If $f(v) = u$, and u has no factor $xyxyx$ where $\pi(x) > \pi(y)$, then v has no factor $XYXYX$ where $\pi(X) > \pi(Y)$.*

Proof: Suppose v has a factor $XYXYX$ where $\pi(X) > \pi(Y)$. Let $x = f(X)$ and $y = f(Y)$. Then $\pi(x) > \pi(y)$, and u contains the factor $xyxyx$. □

Lemma 7. *Let $\boldsymbol{w}$ be an infinite binary word which avoids $\frac{5}{2}^+$ powers. Suppose that the set of length 4 factors of $\boldsymbol{w}$ is*

$$F = \{0110, 1001, 0011, 1100, 0010, 0100, 1101, 1010\}.$$

Then a final segment of $\boldsymbol{w}$ has the form $g(\boldsymbol{u})$ for some proper $\boldsymbol{u} \in \Sigma_3^\omega$.

Proof: Since 111 is not a factor of $\boldsymbol{w}$, any final segment of $\boldsymbol{w}$ beginning with 0 has the form $g(\boldsymbol{u})$ for some $\boldsymbol{u} \in \Sigma_3^\omega$. We will show, replacing $\boldsymbol{u}$ by one of its final segments if necessary, that $\boldsymbol{u}$ is proper.

The fact that $\boldsymbol{u}$ has no factor $xyxyx$ where $\pi(x) > \pi(y)$ follows from Lemma 5.

We conclude by showing that none of the words 00, 11, 22, 20, 10101, 2121, or 10210210 is a factor of $\boldsymbol{u}$:

Word 00**:** If 00 is a factor of $\boldsymbol{u}$, then $\boldsymbol{w}$ contains the factor $g(00) = 011011$. However, then $\boldsymbol{w}$ contains 1011, which is not in F.

Word 11**:** If 11 is a factor of $\boldsymbol{u}$, then $11a$ is a factor of $\boldsymbol{u}$ for some $a \in \Sigma_3$. Then $\boldsymbol{w}$ contains the factor $g(11a)$, which starts with $g(1)g(1)0 = 000$, a $\frac{5}{2}^+$ power.

Word 22**:** If 22 is a factor of $\boldsymbol{u}$, then $\boldsymbol{w}$ contains the factor $g(22) = 0101 \notin F$.

Word 20**:** If 20 is a factor of $\boldsymbol{u}$, then $\boldsymbol{w}$ contains the factor $g(20) = 01011$, which starts with $0101 \notin F$.

Word 10101**:** If 10101 is not a factor of $\boldsymbol{u}$ more than once, replace $\boldsymbol{u}$ by one of its final segments not containing 10101.

On the other hand, if 10101 is a factor of $\boldsymbol{u}$ more than once, then $a10101b$ is a factor of $\boldsymbol{u}$ for some $a, b \in \Sigma_3$. Since 11 is not a factor of $\boldsymbol{u}$, we can in fact specify that $a \in \Sigma_3 - \{1\}$. This implies that 1 is the last letter of $g(a)$. Also, 0 is the first letter of $g(b)$. Then $\boldsymbol{w}$ contains the factor $g(a10101b)$, which contains $1g(10101)0 = 10011001100$, a $\frac{5}{2}^+$ power.

Word 2121**:** If 2121 is a factor of $\boldsymbol{u}$, then $2121a$ is a factor of $\boldsymbol{u}$ for some $a \in \Sigma_3 - \{1\}$. Thus 01 is a prefix of $g(a)$, and $\boldsymbol{w}$ contains the factor $g(2121)01 = 01001001$, a $\frac{5}{2}^+$ power.

Word 10210210**:** If 10210210 is a factor of $\boldsymbol{u}$, then $\boldsymbol{w}$ contains the factor

$$g(10210210) = 0011010011010011,$$

a $\frac{5}{2}^+$ power. □

We can now prove Theorem 1:

Proof of Theorem 1: Assume without loss of generality, replacing $\boldsymbol{u}$ by a final segment if necessary, that $\boldsymbol{u}$ starts with the letter 0. Write $\boldsymbol{u}$ as a concatenation of 0-blocks, i.e., words which start with 0, and contain the letter 0 exactly once. Since $\boldsymbol{u}$ is proper, it does not contain a factor 00 or 20. It follows that its 0-blocks end with 1. Since every occurrence of 2 in a proper word can only be followed by a 1, $\boldsymbol{u}$ cannot contain the factor 212; otherwise it would contain the forbidden factor 2121. We conclude that the 0-blocks of $\boldsymbol{u}$ are among 01, 021 and 0121. Notice that the last two of these words are always followed by a 0 in $\boldsymbol{u}$. We thus conclude that a final segment of $\boldsymbol{u}$ has the form $f(\boldsymbol{v})$ for some $\boldsymbol{v} \in \Sigma_3^\omega$. We will show, replacing $\boldsymbol{v}$ by one of its final segments if necessary, that $\boldsymbol{v}$ is proper.

The fact that $\boldsymbol{v}$ has no factor $xyxyx$ where $\pi(x) > \pi(y)$ follows from Lemma 6.

We conclude by showing that a final segment of $\boldsymbol{v}$ contains none of the words 22, 20, 00, 11, 10101, 2121, or 10210210 as a factor:

Word 22**:** If 22 is not a factor of $\boldsymbol{v}$ more than once, then replace $\boldsymbol{v}$ by one of its final segments not containing 22. Otherwise, 22 is a factor of $\boldsymbol{v}$ more than once, so that $\boldsymbol{u}$ contains a factor $1f(22) = 10101$. This is impossible, since $\boldsymbol{u}$ is proper.

Word 20**:** If 20 is not a factor of $\boldsymbol{v}$ more than once, then replace $\boldsymbol{v}$ by one of its final segments not containing 20. Otherwise, 20 is a factor of $\boldsymbol{v}$ more than once, so that $\boldsymbol{u}$ contains a factor $1f(20) = 1010121$, which starts with 10101. This is impossible, since $\boldsymbol{u}$ is proper.

Word 00**:** Suppose that 00 is a factor of $\boldsymbol{v}$. If 000 is a factor of $\boldsymbol{v}$, then $f(000) = 012101210121$ is a factor of $\boldsymbol{u}$. However $012101210121 = xyxyx$ where $x = 0121$, $y = \epsilon$, and cannot be a factor of $\boldsymbol{u}$. It follows that 000 is not a factor of $\boldsymbol{v}$.

If 00 is not a factor of $\boldsymbol{v}$ more than once, then replace $\boldsymbol{v}$ by one of its final segments not containing 00. Otherwise, 00 is a factor of $\boldsymbol{v}$ more than once, so that $100a$ is a factor of $\boldsymbol{v}$ for some $a \in \Sigma_3$. Since $f(a)$ starts with 0, this implies that $\boldsymbol{u}$ has a factor $f(100a)$, starting with 021012101210. This contains the word $xyxyx$ where $x = 210$, $y = 1$, which is impossible, since $\boldsymbol{u}$ is proper.

Word 11**:** If 11 is not a factor of $\boldsymbol{v}$ more than once, then replace $\boldsymbol{v}$ by one of its final segments not containing 11. Otherwise, 11 is a factor of $\boldsymbol{v}$ more than once, so that $\boldsymbol{u}$ contains a factor $1f(11)0 = 10210210$, which is impossible, since $\boldsymbol{u}$ is proper.

Word 10101**:** If 10101 is not a factor of $\boldsymbol{v}$ more than once, then replace $\boldsymbol{v}$ by one of its final segments not containing 10101. Otherwise, 10101 is a factor of $\boldsymbol{v}$ more than once, so that $\boldsymbol{u}$ contains a factor $1f(10101)0 = 1021012102101210210 = xyxyx$, where $x = 10210$ and $y = 12$. This is impossible, since $\boldsymbol{u}$ is proper.

Word 2121**:** If 2121 is not a factor of $\boldsymbol{v}$ more than once, then replace $\boldsymbol{v}$ by one of its final segments not containing 2121. Otherwise, 2121 is a factor of $\boldsymbol{v}$ more than once. Since 22 is not a factor of v, $a2121b$ is a factor of $\boldsymbol{v}$ for some $a, b \in \Sigma_3$ where $a \neq 2$. Then $f(a)$ ends in 21 and $f(b)$ begins with 0. Thus $\boldsymbol{u}$ contains the factor $21f(2121)0 = 2101021010210 = xyxyx$ where $x = 210$, and $y = 10$, which is impossible, since $\boldsymbol{u}$ is proper.

Word 10210210**:** If 10210210 is a factor of $\boldsymbol{v}$, then $\boldsymbol{u}$ contains the factor

$$f(10210210) = 021012101021012101021012 1 = xyxyx,$$

where $x = 0210121$, and $y = 01$, which is impossible, since $\boldsymbol{u}$ is proper.

The proof for antiproper words is the same, *mutatis mutandi*. □

We can now prove Theorem 2:

Proof of Theorem 2: The first case follows from Theorem 1 and Lemma 7 by induction. The other cases follow, *mutatis mutandi*. □

As an example of the third case of the theorem, Shallit and Shur (2019) consider a word $\tau(\mathbf{G})$, where $\tau : \Sigma_3^* \to \Sigma_2^*$ is the morphism given by

$$\tau(0) = 0$$
$$\tau(1) = 01$$
$$\tau(2) = 011$$

and $\mathbf{G}$ is the fixed point of θ, where $\theta : \Sigma_3^* \to \Sigma_3^*$ is the morphism given by

$$\begin{aligned}\theta(0) &= 01\\ \theta(1) &= 2\\ \theta(2) &= 02.\end{aligned}$$

Letting σ be the permutation $0 \to 1 \to 2 \to 0$, one checks that $\tau = g\sigma$ and $\theta^2 = \sigma^{-1}h\sigma$. It follows that

$$\begin{aligned}\tau(\mathbf{G}) &= \tau(\theta^{2n}(\theta^\omega(0)))\\ &= g\sigma((\sigma^{-1}h\sigma)^{n-1}(\sigma^{-1}h\sigma(\theta^\omega(0)))\\ &= gh^{n-1}(\boldsymbol{u})),\end{aligned}$$

where $\boldsymbol{u} = h(\sigma(\theta^\omega(0)))$, which is antiproper.

Many open problems remain concerning the relationship between low factor complexity and avoidable powers. Moving to a ternary alphabet, Shallit and Shur (2019) showed that the word

$$\mathbf{G} = 0120201020120102012\cdots$$

has critical exponent $2.4808627\cdots$ and factor complexity $2n+1$ for all $n \geq 1$. They conjectured that this exponent is minimal among all infinite ternary words with complexity $2n+1$. This conjecture was recently confirmed by Currie (2025). It would thus be natural to explore the possibility of a structure theorem for this class of words.

4 Appendix: Python code and output

The backtrack searches mentioned in the paper run quickly in Python. Here is our code and its output:

```
def fhpf(w): #Word w is 5/2+ power suffix free
    p=1
    while (5*p<2*len(w)):
        if (w[(-(p+1)//2)-p:]==w[(-(p+1)//2)-2*p:-p]):
            return(False)
        p=p+1
    return(True)

def good(w): # Word w has no suffix which is a 5/2+ power, or is in the
             # set Factors.
    for f in Factors:
        k=len(f)
        if ((len(w)>= k)and(w[-k:]==f)):
            return(False)
    return(fhpf(w))

def search(target): # This returns the lexicographically least word not
                    # containing a 5/2+ power or a word in the set Factors
    w=''
    Max=0
    while (len(w)<=target):
        if (good(w)):
            Max=max(Max,len(w))
```

```
            if (len(w)==target):
                return(w)
            w+='0'
        else:
            while((len(w)>0) and (w[-1]=='1')):
                w=w[:-1]
            if(len(w)==0):
                print('Longest 5/2+-power-free word with no factor in ',Factors,'
                    has length ',Max)
                return()
            c=chr(ord(w[-1])+1)
            w=w[:-1]
            w+=c
    return()


# Lemma 1

print('=============')
print('Computations for Lemma 1')
print(' ')

Factors=['0110']
search(200)

# Lemma 2

print(' ')
print('=============')
print('Computations for Lemma 2')
print(' ')

C=['0010','0100','1011','1101']
for i in range(4):
    for j in range(i,4):
        Factors=[C[i]]
        Factors.append(C[j])
        search(200)


# Lemma 3

print(' ')
print('=============')
print('Computations for Lemma 3')
print(' ')

A=['0010','0100','0101','1010','1011','1101','1100']
for j in A:
    Factors=['0011']
    Factors.append(j)
    search(200)


# Lemma 4
```

```
print(' ')
print('=============')
print('Computations for Lemma 4')
print(' ')

D=['00100110','01001100','10011001','00110010',
 '01100100','11001001','10010011','00110011',
 '01100110','11001101','10011011','00110110',
 '01101100','11011001','10110010','10110011','11001100']
for j in D:
    Factors=['0101','1010']
    Factors.append(j)
    search(200)

# Theorem 4

print(' ')
print('=============')
print('Computations for Theorem 4')
print(' ')

Factors=['1011','1010']
search(200)
```

```
==============================================================================

=============
Computations for Lemma 1

Longest 5/2+-power-free word with no factor in  ['0110']  has length  14

=============
Computations for Lemma 2

Longest 5/2+-power-free word with no factor in  ['0010', '0100']  has length  44
Longest 5/2+-power-free word with no factor in  ['0010', '1011']  has length  28
Longest 5/2+-power-free word with no factor in  ['0010', '1101']  has length  13
Longest 5/2+-power-free word with no factor in  ['0100', '1011']  has length  13
Longest 5/2+-power-free word with no factor in  ['0100', '1101']  has length  28
Longest 5/2+-power-free word with no factor in  ['1011', '1101']  has length  44

=============
Computations for Lemma 3

Longest 5/2+-power-free word with no factor in  ['0011', '0010']  has length  15
Longest 5/2+-power-free word with no factor in  ['0011', '0100']  has length  31
Longest 5/2+-power-free word with no factor in  ['0011', '0101']  has length  12
Longest 5/2+-power-free word with no factor in  ['0011', '1010']  has length  18
Longest 5/2+-power-free word with no factor in  ['0011', '1011']  has length  15
Longest 5/2+-power-free word with no factor in  ['0011', '1101']  has length  31
Longest 5/2+-power-free word with no factor in  ['0011', '1100']  has length  30

=============
Computations for Lemma 4
```

```
Longest 5/2+-power-free word with no factor in  ['0101', '1010', '00100110']  has length  24
Longest 5/2+-power-free word with no factor in  ['0101', '1010', '01001100']  has length  50
Longest 5/2+-power-free word with no factor in  ['0101', '1010', '10011001']  has length  33
Longest 5/2+-power-free word with no factor in  ['0101', '1010', '00110010']  has length  50
Longest 5/2+-power-free word with no factor in  ['0101', '1010', '01100100']  has length  24
Longest 5/2+-power-free word with no factor in  ['0101', '1010', '11001001']  has length  24
Longest 5/2+-power-free word with no factor in  ['0101', '1010', '10010011']  has length  24
Longest 5/2+-power-free word with no factor in  ['0101', '1010', '00110011']  has length  52
Longest 5/2+-power-free word with no factor in  ['0101', '1010', '01100110']  has length  33
Longest 5/2+-power-free word with no factor in  ['0101', '1010', '11001101']  has length  50
Longest 5/2+-power-free word with no factor in  ['0101', '1010', '10011011']  has length  24
Longest 5/2+-power-free word with no factor in  ['0101', '1010', '00110110']  has length  24
Longest 5/2+-power-free word with no factor in  ['0101', '1010', '01101100']  has length  24
Longest 5/2+-power-free word with no factor in  ['0101', '1010', '11011001']  has length  24
Longest 5/2+-power-free word with no factor in  ['0101', '1010', '10110010']  has length  88
Longest 5/2+-power-free word with no factor in  ['0101', '1010', '10110011']  has length  50
Longest 5/2+-power-free word with no factor in  ['0101', '1010', '11001100']  has length  52

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Computations for Theorem 4

Longest 5/2+-power-free word with no factor in  ['1011', '1010']  has length  20
()
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