Discrete Mathematics and Theoretical Computer Science ....
.... **vol. 27:3 #25 (2025)**

Spanning trees of claw-free graphs with few leaves and branch vertices

Pham Hoang Ha Nguyen Gia Hien

Department of Mathematics, Hanoi National University of Education, Hanoi, Vietnam

revisions 17th Feb. 2025, 5th Nov. 2025; accepted 8th Nov. 2025.

Let T be a tree. A vertex of degree one is a *leaf* of T and a vertex of degree at least three is a *branch vertex* of T. A graph is said to be claw-free if it does not contain $K_{1,3}$ as an induced subgraph. In this paper, we study the spanning trees with a bounded number of leaves and branch vertices of claw-free graphs. Applying the main results, we also give some improvements of previous results on the spanning trees with few branch vertices for the case of claw-free graphs.

Keywords: spanning tree; leaf; branch vertex; independence number; degree sum

1 Introduction

Let G be a finite, simple graph with no loops. The set of vertices and the set of edges of G are denoted by $V(G)$ and $E(G)$, respectively. For each vertex v of $V(G)$, we denote the set of vertices which are adjacent to v in G by $N_G(v)$ and the degree of v in G by $\deg_G(v)$. We define $G - uv$ and $G + uv$ to be the graphs obtained by subtracting and adding the edge uv to G, respectively. For every subset X of $V(G)$, the subgraph of G induced by X is denoted by $G[X]$. A graph is called $K_{1,r}$-free if it does not have $K_{1,r}$ as an induced subgraph. A $K_{1,3}$-free graph is also called a claw-free graph.

For a graph G, $X \subset V(G)$ is an independent set of G if no two vertices of X are adjacent in G. We denote the largest size of independent sets of G by $\alpha(G)$. For $k \geq 1$, we define

$$\sigma_k(G) = \begin{cases} +\infty & \text{if } \alpha(G) < k, \\ \min\{\sum_{i=1}^{k} \deg_G(v_i) | \{v_1, \ldots, v_k\} \text{ is an independent set of G}\} & \text{if } \alpha(G) \geq k. \end{cases}$$

Let T be a spanning tree of G. A vertex is called a leaf of T if it has degree one in T. A vertex is called a branch vertex of T if it has degree strictly greater than two in T. The set of leaves and the set of branch vertices of T are denoted by $L(T)$ and $B(T)$, respectively. For each positive integer k, let $B_k(T)$ ($B_{\leq k}(T)$) be the set of branch vertices in T with degree k (at most k, respectively).

There are many conditions for a graph G to have a spanning tree T with a bounded number of leaves or branch vertices. We refer the readers to [1], [7], [10], [23] for examples.

For claw-free graphs, Gargano et al. [8] proved the following theorem.

ISSN 1365–8050

Theorem 1.1 (Gargano et al. [8]) *Let $k \geq 0$ be an integer and let G be a connected claw-free graph. If $\sigma_{k+3}(G) \geq |G| - k - 2$, then there exists a spanning tree T of G such that $|B(T)| \leq k$.*

In 2020, Gould and Shull [9] proved a conjecture on the spanning tree of a claw-free graph with few branch vertices proposed by Matsuda et al. [20].

Theorem 1.2 (Gould and Shull [9]) *Let $k \geq 0$ be an integer and let G be a connected claw-free graph. If $\sigma_{2k+3}(G) \geq |G| - 2$, then there exists a spanning tree T of G such that $|B(T)| \leq k$.*

Moreover, many researchers studied the case of $K_{1,r}$-free graphs ($r \geq 4$) , see [2], [3], [4],[14], [15], [17], [18] for examples.
Regarding the conditions for a graph to have a bounded number of leaves and branch vertices, Nikoghosyan [21], Saito and Sano [22] independently proved the following.

Theorem 1.3 (Nikoghosyan [21], Saito and Sano [22]) *Let $k \geq 2$ be an integer. If a connected graph G satisfies $\sigma_2(G) \geq |G| - k + 1$, then there exists a spanning tree T of G such that $|L(T)| + |B(T)| \leq k + 1$.*

In 2019, Maezawa et al. [19] gave an improvement of the above result. They proved the following theorem.

Theorem 1.4 (Maezawa et al. [19]) *Let $k \geq 2$ be an integer and let G be a connected graph. Suppose that G satisfies $\max\{\deg_G(x), \deg_G(y)\} \geq \dfrac{|G| - k + 1}{2}$ for every two vertices x, y such that $xy \notin E(G)$, then there exists a spanning tree T of G such that $|L(T)| + |B(T)| \leq k + 1$.*

For the case of claw-free graphs, Hanh [13] proved the following theorem.

Theorem 1.5 (Hanh [13]) *Let G be a connected claw-free graph. If $\sigma_5(G) \geq |G| - 2$, then G has a spanning tree with at most 5 leaves and branch vertices.*

In the case of $K_{1,4}$-free, Ha [11] stated the following result.

Theorem 1.6 (Ha [11]) *Let G be a connected $K_{1,4}$-free graph and k, m be two non-negative integers with $m \leq k + 1$. If $\sigma_{m+2}(G) \geq |G| - k$, then there exists a spanning tree T of G such that $|L(T)| + |B(T)| \leq k + m + 2$.*

In the case of $K_{1,5}$-free graphs, two results were introduced as the followings.

Theorem 1.7 (Ha and Trang [12]) *Let G be a connected $K_{1,5}$-free graph. If $\sigma_4(G) \geq |G| - 1$, then there exists a spanning tree T of G such that $|L(T)| + |B(T)| \leq 5$.*

Theorem 1.8 (Diep et al. [5]) *Let G be a connected $K_{1,5}$-free graph. If $\sigma_5(G) \geq |G| - 2$, then G contains a spanning tree with $|L(T)| + |B(T)| \leq 7$.*

In this paper, we continue to study some sufficient conditions for a connected claw-free graph to have a spanning tree with few leaves and branch vertices. The main purpose of this paper is to prove the following theorem.

Theorem 1.9 *Let m, n be two positive integers ($n \geq 2$). Let G be a connected claw-free graph. If $\sigma_{m+1}(G) \geq |G| - n + m - 1$ and $m \leq \lceil \dfrac{2n}{3} \rceil$, then G has a spanning tree with at most n leaves and branch vertices. Here, the notation $\lceil r \rceil$ stands for the smallest integer not less than the real number r.*

It is easy to see that we directly gain the above result of Hanh [13] by Theorem 1.9 with $m = 4$ and $n = 5$.

Using Theorem 1.9 with $m = 1$ and $n = k + 1$ for a positive integer k, then we have the following corollary.

Corollary 1.10 *Let k be a positive integer and let G be a connected claw-free graph. If $\sigma_2(G) \geq |G| - k - 1$, then G has a spanning tree with at most $k + 1$ leaves and branch vertices.*

This is an improvement of Theorem 1.3 in the case of claw-free graphs.

On the other hand, since $|L(T)| \geq |B(T)| + 2$ for each tree T, we obtain that if a tree T has at most $2k + 3$ leaves and branch vertices, then $|B(T)| \leq k$. By motivating this fact, we give some sufficient conditions for a claw-free graph to have a spanning tree with few branch vertices.

Let k be an arbitrary positive integer and $n = 2k + 3$. Using the same technique of proof of Theorem 1.9, we gain the following result.

Theorem 1.11 *Let k, m be two positive integers such that $k + 3 \leq m \leq k + \dfrac{k-1}{3} + 3$ and let G be a connected claw-free graph. If $\sigma_{m+1}(G) \geq |G| - 2k + m - 4$, then G has a spanning tree with at most $2k + 3$ leaves and branch vertices.*

In Theorem 1.11, consider the case $m = k + 3$, we obtain a stronger result of Gargano et al. [8].

Corollary 1.12 *Let k be a positive integer and let G be a connected claw-free graph. If $\sigma_{k+4}(G) \geq |G| - k - 1$, then G has a spanning tree with at most k branch vertices.*

2 Definitions and Notations

In this section, we recall some definitions in [9] which are needed for the proof of main results. We refer to [6] for terminology and notation not defined here.

Definition 2.1 ([9]) *Let T be a tree and let e be an edge of T. For any two vertices of T, say u and v, are joined by a unique path, denoted by $P_T[u, v]$. We also denote $u_v = V(P_T[u, v]) \cap N_T(u)$ and e_v as the vertex incident to e in the direction toward v.*

Definition 2.2 ([9]) *Let T be a spanning tree of a graph G and let $v \in V(G)$ and $e \in E(T)$. Denote $g(e, v)$ as the vertex incident to e farthest away from v in T. We say v is an oblique neighbor of e with respect to T if $vg(e, v) \in E(G)$. Let $X \subseteq V(G)$. The edge e has an oblique neighbor in the set X if there exists a vertex of X which is an oblique neighbor of e with respect to T.*

Definition 2.3 ([9]) *Let T be a spanning tree of a graph G. Two vertices are pseudoadjacent with respect to T if there is some $e \in E(T)$ which has them both as oblique neighbors. Similarly, a vertex set is pseudoindependent with respect to T if no two vertices in the set are pseudoadjacent with respect to T.*

Definition 2.4 ([9]) *Let T be a spanning tree of a graph G with $|B(T)| > 0$ and let $r \in B(T)$ be a root of T. Then, each branch vertex b has a distance $d(b, r)$ and a degree $\deg_T(b)$. We define a sequence, denoted by (T, r), on the set $B(T)$, which contains the distance-degree pairs of all vertices of $B(T)$ to r in lexicographically increasing order. That is, shortest distance first, and smallest degree first given the same distance.*

Definition 2.5 ([9]) *Given two sequences (T_1, r_1) and (T_2, r_2). We define (T_1, r_1) to be smaller than (T_2, r_2) if the distance-degree pair of (T_1, r_1) is smaller than that of (T_2, r_2) at the first different entry between two sequences.*

3 Proof of Theorem 1.9

We prove the theorem by contradiction. Suppose that G has no spanning trees with at most total n leaves and branch vertices. Then, for all spanning trees T of G, we have $|L(T)| + |B(T)| \geq n + 1$. If $|B(T)| = 0$, then $|L(T)| = 2$. So $|L(T)| + |B(T)| = 2 < n + 1$. This is a contradiction. Hence, $|B(T)| \geq 1$. Choose a spanning tree T of G such that:
(C1) $|B(T)|$ is as small as possible.

We consider two cases as follows.

Case 1. $|B_3(T)| = 0$ for all spanning trees T satisfying the condition (C1).

In this case, we choose a spanning tree T of G such that:
(C2) $|L(T)|$ is as small as possible, subject to (C1).
We have

$$|L(T)| = 2 + \sum_{b \in B(T)} (\deg_T(b) - 2) \geq 2 + 2|B(T)|.$$

So

$$3|L(T)| \geq 2 + 2|L(T)| + 2|B(T)| \geq 2 + 2(n + 1) = 2n + 4.$$

Hence, $|L(T)| \geq \dfrac{2n + 4}{3}$. Since $|L(T)|$ is an integer and the assumptions of Theorem 1.9, we conclude that $|L(T)| \geq m + 1$.

Let $r \in B(T)$ be the root of T.

Claim 3.1 *For $b \in B(T)$, if $b_1 b_2 \in E(G)$ and b_1, b_2 are children of b, then bb_1 and bb_2 have no oblique neighbors in $L(T)$.*

Proof. Suppose the assertion of the claim is false. Then, there exists $z \in L(T)$ such that z is pseudoadjacent to bb_1. If $b_1 \in V(P_T[b, z])$, then $T' = T - \{bb_1, bb_2\} + \{bz, b_1 b_2\}$ violates the assumption of Case 1 if $b \in B_4(T)$ or condition (C2) otherwise. If $b \in V(P_T[b_1, z])$, then $T' = T - \{bb_1\} + \{zb_1\}$ violates the assumption of Case 1 if $b \in B_4(T)$ or condition (C2) otherwise. The case for bb_2 is done by symmetry. This completes the proof of Claim 3.1. □

Claim 3.2 *$L(T)$ is an independent set.*

Proof. Suppose that there are two leaves u, v of T such that $uv \in E(G)$. Let t be the nearest branch vertice of u. Then $T' = T - \{tt_u\} + \{uv\}$ violates either the assumption of Case 1 if $t \in B_4(T)$ or the condition (C2) for otherwise. Therefore, Claim 3.2 is proved. □

Claim 3.3 *$L(T)$ is a pseudoindependent set.*

Proof. Suppose to the contrary that there exists $\{u, v\} \subset L(T)$ and an edge e of T such that $ug(e, u) \in E(G)$ and $vg(e, v) \in E(G)$.
If $g(e, u) = g(e, v) = a$. Denote $\{t\} = V(P_T[u, a]) \cap V(P_T[a, v]) \cap V(P_T[v, u])$. Since $G[a, e_t, u, v]$ is not a claw and $uv \notin E(G)$, we obtain either $ue_t \in E(G)$ or $ve_t \in E(G)$. Without loss of generality, we may assume that $ue_t \in E(G)$. Then the spanning tree $T' = T - \{e, tt_u\} + \{ue_t, va\}$ contradicts either the assumption of Case 1 if $t \in B_4(T)$ or the condition (C2) if not.

If $g(e,u) = e_v$ and $g(e,v) = e_u$, then $e \in P_T[u,v]$. Denote $\{s\} = V(P_T[u,r]) \cap V(P_T[r,v]) \cap V(P_T[v,u])$. Without loss of generality, we may assume that $s \in V(P_T[e_u,u])$. Then $T' = T - \{e, ss_u\} + \{ue_v, ve_u\}$ violates the assumption of Case 1 if $s \in B_4(T)$ or the condition (C2) otherwise. We conclude that $L(T)$ is a pseudoindependent set. □

Let Q be a subset of $L(T)$ such that $|Q| = m+1$. Let s be the number of edges in T which have no oblique neighbors in Q.

By Claim 3.2, we deduce that for each leaf $l \in L(T) \setminus Q$, ll_r has no oblique neighbors in Q. Let A be the set of all such edges ll_r in T.

On the other hand, for each $b \in B(T)$, $\deg_T(b) \geq 4$ and the fact that G is claw-free, we obtain that there exist two children b_1, b_2 of b such that $b_1b_2 \in E(G)$. Then bb_1 and bb_2 have no oblique neighbors in Q from the fact of Claim 3.1. We denote by B the set of all such edges bb_1, bb_2 in T.

We will prove that A and B are disjoint. Suppose that there exists an edge e of T such that $e \in A \cap B$. We deduce that $e_r \in B(T)$, $g(e,r) \in L(T)$ and $g(e,r)$ is adjacent to a child $z \neq g(e,r)$ of e_r. Then $T' = T - \{e_r z\} + \{zg(e,r)\}$ violates the assumption of Case 1 if $e_r \in B_4(T)$ or condition (C2) otherwise. We conclude that A and B are disjoint sets.

Therefore, we have

$$\begin{aligned} s &\geq |A| + |B| \geq |L(T)| - |Q| + 2|B(T) \setminus \{r\}| + 2 = |L(T)| - (m+1) + 2|B(T)| \\ &\geq n + 1 - m. \end{aligned}$$

On the other hand, for any $x, y \in V(T)$, we have $xy \in E(G)$ if and only if x is an oblique neighbor of yy_x. Therefore, the number of edges of T with x as an oblique neighbor equals the degree of x in G. Therefore, combining with Claim 3.3, we obtain that

$$\sigma_{m+1}(G) \leq \sum_{t \in Q} \deg_G(t) \leq |E(T)| - s \leq (|G| - 1) - (n - m + 1) = |G| - n + m - 2.$$

This contradicts with the assumption of Theorem 1.9.

Case 2. There exists at least one spanning tree T of G such that $|B_3(T)| > 0$.

Let $r \in B_3(T)$ be a root of T.

In this case, in all spanning trees satisfying the condition (C1) and having at least one branch vertex of degree 3, we choose a spanning tree T with the root r such that:

(C3) $\sum_{v \in B_{\geq 5}(T)} (\deg_T(v) - 4)$ is as small as possible.

(C4) (T, r) is lexicographically as small as possible, subject to (C3).

Thus $\deg_T(r) = 3$.

We have

$$|L(T)| = 2 + \sum_{b \in B(T)} (\deg_T(b) - 2) \geq 2 + |B_3(T)| + 2|B_{\geq 4}(T)|.$$

So

$$|L(T)| + |B_3(T)| \geq 2 + 2|B_3(T)| + 2|B_{\geq 4}(T)| = 2(1 + |B(T)|) \geq 2(1 + n + 1 - |L(T)|).$$

This implies

$$3|L(T)| + 3|B_3(T)| \geq 2n + 4 + 2|B_3(T)| \geq 2n + 6.$$

Therefore, we obtain

$$|L(T)| + |B_3(T)| \geq \frac{2n+6}{3} \geq m + 2.$$

Since $|L(T)| + |B_3(T)|$ is an integer and the assumptions of Theorem 1.9, we obtain $|L(T)| + |B_3(T)| \geq m + 2$. Let $H = L(T) \cup B_3(T) \setminus \{r\}$. Then $|H| \geq m + 1$. We now have the following claims.

Claim 3.4 *If $u \in B(T) \setminus \{r\}$ and a is a child of u, then a is adjacent to at least one neighbor of u.*

Proof. Assume that $ab \notin E(G)$ for all $b \in N_T(u) \setminus \{a\}$. Then let c be a child of u which is different from a. Since $G[u, a, c, u_r]$ is claw-free and $au_r, ac \notin E(G)$, we have $cu_r \in E(G)$. This concludes that $bu_r \in E(G)$ for every $b \in N_T(u) \setminus \{a, u_r\}$. Then the spanning tree $T' = T - \{ub | b \in N_T(u) \setminus \{a, u_r\}\} + \{u_r b | b \in N_T(u) \setminus \{a, u_r\}\}$ violates either the condition (C1) if $u_r \in B(T)$ or the condition (C4) if $u_r \notin B(T)$. So the claim holds. □

Claim 3.5 *If $u \in B_3(T) \setminus \{r\}$ and a, b are two children of u, then $ab \in E(G)$.*

Proof. Suppose for a contradiction that $ab \notin E(G)$, since $G[u, a, b, u_r]$ is not a claw and $ab \notin E(G)$, we obtain either $au_r \in E(G)$ or $bu_r \in E(G)$. Without loss of generality, we may assume that $au_r \in E(G)$. Then the spanning tree $T' = T - \{au\} + \{au_r\}$ contradicts either the condition (C1) if $u_r \in B(T)$ or the condition (C4) if $u_r \notin B(T)$. This completes the proof of Claim 3.5. □

Claim 3.6 *H is an independent set.*

Proof. Suppose this is false. Then there exists $\{u, v\} \subset H$ such that $uv \in E(G)$. If $u \in V(P_T[r, v])$, then $\deg_T(u) = 3$. We denote $\{u^*\} = N_T(u) \setminus \{u_r, u_v\}$. By Claim 3.5, we obtain that $u^* u_v \in E(G)$. Then $T' = T - \{uu^*, uu_v\} + \{u^* u_v, uv\}$ violates the condition (C1). The case $v \in V(P_T[r, u])$ is done by symmetry. Otherwise, we have $\{w\} = V(P_T[r, u]) \cap V(P_T[u, v]) \cap V(P_T[v, r]) \not\subset \{u, v\}$. Consider the spanning tree $T' = T - \{ww_u\} + \{uv\}$. This contradicts either the condition (C1) if $w \in B_3(T)$, or the condition (C3) if $w \in B_{\geq 5}(T)$ or the condition (C4) if $w \in B_4(T)$. So we conclude that H is an independent set. □

Claim 3.7 *H is a pseudoindependent set.*

Proof. Assume that there exists $\{u, v\} \subset H$ and an edge $e \in E(T)$ such that $ug(e, u) \in E(G)$ and $vg(e, v) \in E(G)$. Let x be a leaf or branch vertex which is nearest to e in the direction away from r. Denote $\{w\} = V(P_T[u, v]) \cap V(P_T[v, r]) \cap V(P_T[r, u])$.

If $e \in P_T[u, v]$, then $g(e, u) \neq g(e, v)$. Now if $u \in V(P_T[r, v])$, then $\deg_T(u) = 3$ and we denote $\{u^*\} = N_T(u) \setminus \{u_v, u_r\}$. By Claim 3.5, we may obtain that $u^* u_v \in E(G)$. Then $T' = T - \{uu_v, uu^*, e\} + \{ug(e, u), vg(e, v), u^* u_v\}$ violates the condition (C1). The case $v \in V(P_T[r, u])$ is done by symmetry. If $w \notin \{u, v\}$, we consider the spanning tree

$$T' = \begin{cases} T - \{e, ww_u\} + \{ug(e, u), vg(e, v)\}, & \text{if } e \neq ww_u, \\ T - \{e\} + \{vw_u\}, & \text{if } e = ww_u. \end{cases}$$

Then T' violates either the condition (C1) if $w \in B_3(T)$, or the condition (C3) if $w \in B_{\geq 5}(T)$ or the condition (C4) if $w \in B_4(T)$.

If $e \notin P_T[u,v]$, denote $\{p\} = V(P_T[x,u]) \cap V(P_T[u,r]) \cap V(P_T[r,x])$ and $\{q\} = V(P_T[x,v]) \cap V(P_T[v,r]) \cap V(P_T[r,x])$. We consider four cases as follows.

Case 3.7.1: Suppose $r \in V(P_T[w,e_x])$, then $p = r$ or $q = r$. Without loss of generality, we may assume that $p = r$, thus $g(e,u) = g(e,v) = e_x$. Since $G[e_x, e_r, u, v]$ is claw-free, we obtain either $ue_r \in E(G)$ or $ve_r \in E(G)$.

If $ue_r \in E(G)$, we consider the spanning tree

$$T' = \begin{cases} T - \{e, rr_x\} + \{ue_r, ve_x\}, & \text{if } e \neq rr_x, \\ T - \{e\} + \{ve_x\}, & \text{if } e = rr_x. \end{cases}$$

Then T' violates the condition (C1).

If $ve_r \in E(G)$, we will prove that $e_x \neq x$. Assume that $e_x = x$, then $T' = T - \{rr_u\} + \{ue_x\}$ violates the condition (C1), thus e_{xx} exists. Since $G[e_x, e_{xx}, u, v]$ is claw-free, we obtain either $ue_{xx} \in E(G)$ or $ve_{xx} \in E(G)$. If $ve_{xx} \in E(G)$, then $T' = T - \{e_x e_{xx}, rr_u\} + \{ue_x, ve_{xx}\}$ violates the condition (C1). If $ue_{xx} \in E(G)$, then $T' = T - \{e_x e_{xx}, qq_x\} + \{ve_x, ue_{xx}\}$ violates either the condition (C1) if $q \in B_3(T)$, or the condition (C3) if $q \in B_{\geq 5}(T)$ or the condition (C4) if $q \in B_4(T)$.

Case 3.7.2: Suppose $e_x \in V(P_T[r,w])$, then $g(e,u) = g(e,v) = e_r$. Since $G[e_r, e_x, u, v]$ is claw-free, we obtain either $ue_x \in E(G)$ or $ve_x \in E(G)$. If $u \in V(P_T[r,v])$, then $T' = T - \{uu_v\} + \{ve_r\}$ violates either the condition (C1) if $e_r \in B(T)$ or the condition (C4) otherwise. The case $v \in V(P_T[r,u])$ is done by symmetry. Now we consider the case $w \notin \{u,v\}$. Without loss of generality, we may assume that $ue_x \in E(G)$. If $w \in B_3(T)$, then $T' = T - \{ww_u\} + \{ue_r\}$ violates the condition (C1) if $e_r \in B(T)$ or condition (C4) if not. If $w \in B_4(T)$, then $T' = T - \{ww_u, ww_v\} + \{ue_r, ve_r\}$ violates the condition (C1) if $e_r \in B(T)$ or condition (C4) if not. If $w \in B_{\geq 5}(T)$, then $T' = T - \{e, ww_u\} + \{ue_x, ve_r\}$ violates the condition (C3).

Case 3.7.3: Suppose $w \in V(P_T[r,e_r]) \setminus \{r\}$ and $e \notin P_T[u,v]$, then $p = w$ or $q = w$. Without loss of generality, we may assume that $p = w$, thus $g(e,u) = g(e,v) = e_x$. We consider two cases as follows.

Subcase 1: $w \in \{u,v\}$. By symmetry, we also assume that $w = u$. This implies $u \in V(P_T[r,v])$. We will prove that $e_x \neq x$. Denote $\{u^*\} = N_T(u) \setminus \{u_r, u_x\}$. By Claim 3.5, we obtain that $u^* u_x \in E(G)$. If $e_x = x$, then $T' = T - \{uu^*, uu_x\} + \{u^* u_x, ux\}$ violates the condition (C1), so e_{xx} exists. Since $G[e_x, e_{xx}, u, v]$ is not a claw graph and $uv \notin E(G)$, we may obtain either $ue_{xx} \in E(G)$ or $ve_{xx} \in E(G)$. If $ve_{xx} \in E(G)$, then $T' = T - \{e_x e_{xx}, uu_x, uu^*\} + \{u_x u^*, ve_{xx}, ue_x\}$ violates the condition (C1). If $ue_{xx} \in E(G)$, we now consider the degree and position of q.

If $q = v$, denote $\{v^*\} = N_T(v) \setminus \{v_x, v_r\}$. By Claim 3.5, we obtain that $v^* v_x \in E(G)$. Then $T' = T - \{uu^*, uu_x, vv^*, vv_x\} + \{u^* u_x, v^* v_x, ue_x, ve_x\}$ violates the condition (C1).

If $q = u$, since $G[u, u_r, u_x, e_{xx}]$ is claw-free and $u_r u_x \notin E(G)$, we obtain either $u_r e_{xx} \in E(G)$ or $u_x e_{xx} \in E(G)$. If $u_r e_{xx} \in E(G)$, then $T' = T - \{e_x e_{xx}, uu^*\} + \{u_r e_{xx}, ve_x\}$ violates either the condition (C1) if $u_r \in B(T)$ or the condition (C4) if $u_r \notin B(T)$. If $u_x e_{xx} \in E(G)$, then $T' = T - \{e_x e_{xx}, uu_x\} + \{u_x e_{xx}, ve_x\}$ violates the condition (C1).

Now if $q \notin \{u,v\}$. Then $T' = T - \{qq_v, e_x e_{xx}\} + \{ue_{xx}, ve_x\}$ violates either the condition (C1) if $q \in B_3(T)$ or the condition (C3) if $q \in B_{\geq 5}(T)$. So we only have to consider the case $q \in B_4(T)$. By Claim 3.4, q_v is adjacent to a vertex in $N_T(q) \setminus \{q_v\}$. Denote $\{q^*\} = N_T(q) \setminus \{q_x, q_v, q_r\}$. We consider the spanning tree

$$T' = \begin{cases} T - \{e_x e_{xx}, qq_v, qq_r\} + \{q_v q_r, ve_x, ue_{xx}\}, & \text{if } q_v q_r \in E(G), \\ T - \{e_x e_{xx}, qq_v, qq_x\} + \{q_v q_x, ue_x, ue_{xx}\}, & \text{if } q_v q_x \in E(G), \\ T - \{e_x e_{xx}, qq_v, qq^*\} + \{q_v q^*, ve_x, ue_{xx}\}, & \text{if } q_v q^* \in E(G). \end{cases}$$

Then T' violates the condition (C1).

Subcase 2: $w \notin \{u,v\}$. Since $G[e_x, e_r, u, v]$ is claw-free, we obtain either $ue_r \in E(G)$ or $ve_r \in E(G)$.

If $p = q = w$, without loss of generality, we may assume that $ue_r \in E(G)$. We consider the spanning tree

$$T' = \begin{cases} T - \{e, ww_x\} + \{ue_r, ve_x\}, & \text{if } e \neq ww_x, \\ T - \{e\} + \{ve_x\}, & \text{if } e = ww_x. \end{cases}$$

Then T' violates the condition (C1) if $w \in B_3(T)$, the condition (C3) if $w \in B_{\geq 5}(T)$ or the condition (C4) if $w \in B_4(T)$.

If $q = v$, denote $\{v^*\} = N_T(v) \setminus \{v_x, v_r\}$. We will prove that $e_x \neq x$. Suppose that $e_x = x$, by Claim 3.5, we obtain that $v^*v_x \in E(G)$. Then $T' = T - \{vv_x, vv^*\} + \{ve_x, v_xv^*\}$ violates the condition (C1), so $e_x \neq x$. Now $T' = T - \{ww_x, vv_x, vv^*\} + \{v^*v_x, ue_x, ve_x\}$ violates the condition (C1) if $w \in B_3(T)$, the condition (C3) if $w \in B_{\geq 5}(T)$ or the condition (C4) if $w \in B_4(T)$.

If $q \neq w$ and $q \neq v$, we will prove that $e_x \neq x$. Suppose that $e_x = x$, by Claim 3.6, we deduce that $\deg_T(x) \geq 4$. Then $T' = T - \{qq_v\} + \{ve_x\}$ violates either the condition (C1) if $q \in B_3(T)$ or the condition (C3) if $q \in B_{\geq 5}(T)$. So we only have to consider the case $q \in B_4(T)$. By Claim 3.4, q_v is adjacent to a vertex in $N_T(q) \setminus \{q_v\}$. Denote $\{q^*\} = N_T(q) \setminus \{q_x, q_v, q_r\}$. We consider the spanning tree

$$T' = \begin{cases} T - \{qq_v, qq_r\} + \{q_vq_r, ve_x\}, & \text{if } q_vq_r \in E(G), \\ T - \{qq_v, qq_x\} + \{q_vq_x, ue_x\}, & \text{if } q_vq_x \in E(G), \\ T - \{qq_v, qq^*\} + \{q_vq^*, ve_x\}, & \text{if } q_vq^* \in E(G). \end{cases}$$

Then T' violates the condition (C1). So $e_x \neq x$, thus e_{xx} exists.

Since $G[e_x, e_{xx}, u, v]$ is claw-free and $uv \notin E(G)$, we may obtain either $ue_{xx} \in E(G)$ or $ve_{xx} \in E(G)$. If $ve_{xx} \in E(G)$, then $T' = T - \{e_xe_{xx}, ww_x\} + \{ue_x, ve_{xx}\}$ contradicts either the condition (C1) if $w \in B_3(T)$ or the condition (C3) if $w \in B_{\geq 5}(T)$ or the condition (C4) if $w \in B_4(T)$. If $ue_{xx} \in E(G)$, then $T' = T - \{e_xe_{xx}, qq_v\} + \{ve_x, ue_{xx}\}$ violates either the condition (C1) if $q \in B_3(T)$ or the condition (C3) if $q \in B_{\geq 5}(T)$. Now we consider the case $q \in B_4(T)$. By Claim 3.4, q_v is adjacent to a vertex in $N_T(q) \setminus \{q_v\}$. Denote $\{q^*\} = N_T(q) \setminus \{q_x, q_v, q_r\}$. We consider the spanning tree

$$T' = \begin{cases} T - \{e_xe_{xx}, qq_v, qq_r\} + \{q_vq_r, ve_x, ue_{xx}\}, & \text{if } q_vq_r \in E(G), \\ T - \{e_xe_{xx}, qq_v, qq_x\} + \{q_vq_x, ue_x, ue_{xx}\}, & \text{if } q_vq_x \in E(G) \text{ and } u \in B_3(T), \\ T - \{e_xe_{xx}, qq_v, qq^*\} + \{q_vq^*, ve_x, ue_{xx}\}, & \text{if } q_vq^* \in E(G). \end{cases}$$

Then T' violates the condition (C1). If $q_vq_x \in E(G)$ and $u \in L(T)$, then $T' = T - \{e_xe_{xx}, qq_v\} + \{ve_x, ue_{xx}\}$ violates the condition (C1) if $q_v \in B_3(T)$, the condition (C3) if $q_v \in B_{\geq 5}(T)$ or the condition (C4) otherwise.

Case 3.7.4: If no vertex in the set $\{w, r, e_x\}$ is between the other two vertices, then $ue_x \in E(G), ve_x \in E(G)$ and $p = q$. Since $G[e_x, e_r, u, v]$ is claw-free, we obtain either $ue_r \in E(G)$ or $ve_r \in E(G)$. Without loss of generality, we may assume that $ue_r \in E(G)$. We consider the spanning tree

$$T' = \begin{cases} T - \{e, qq_x\} + \{ue_r, ve_x\}, & \text{if } e \neq qq_x, \\ T - \{e\} + \{ue_x\}, & \text{if } e = qq_x. \end{cases}$$

Then T' violates the condition (C1) if $q \in B_3(T)$, the condition (C3) if $q \in B_{\geq 5}(T)$ or the condition (C4) if $q \in B_4(T)$.

So we conclude that H is a pseudoindependent set. $\square$

Claim 3.8 *If r_1, r_2 are adjacent to r in T and $r_1r_2 \in E(G)$, then rr_1 and rr_2 have no oblique neighbors in H.*

Proof. Suppose that there exists $z \in H$ and z is pseudoadjacent to rr_1. We consider the spanning tree

$$T' = \begin{cases} T - \{rr_1, rr_2\} + \{rz, r_1r_2\}, & \text{if } r_1 \in V(P_T[r,z]), \\ T - \{rr_1\} + \{zr_1\}, & \text{if } r \in V(P_T[r_1,z]). \end{cases}$$

Then T' violates condition (C1). Hence rr_1 has no oblique neighbors in H. By symmetry, rr_2 has no oblique neighbors in H. □

Since $|H| \geq m+1$, we choose a subset M of H such that $|M| = m+1$.

Claim 3.9 *For $b \in B(T) \setminus (M \cup \{r\})$, if $b_1b_2 \in E(G)$ and b_1, b_2 are children of b, then bb_1 or bb_2 has no oblique neighbors in M.*

Proof. Suppose to the contrary that there exists some vertex $b \in B(T) \setminus (M \cup \{r\})$ with two children b_1, b_2 and $z, t \in M$ such that $b_1b_2 \in E(G)$ and z, t are pseudoadjacent to bb_1, bb_2, respectively. We consider two cases as follows.

Case 3.9.1: Suppose $b_1 \in V(P_T[b,z])$. Then $T' = T - \{bb_1, bb_2\} + \{bz, b_1b_2\}$ violates either the condition (C1) if $b \in B_3(T)$, or the condition (C3) if $b \in B_{\geq 5}(T)$ or the condition (C4) if $b \in B_4(T)$.

Case 3.9.2: Suppose $b \in V(P_T[b_1,z])$. Then $T' = T - \{bb_1\} + \{zb_1\}$ violates either the condition (C1) if $b \in B_3(T)$ or the condition (C3) if $b \in B_{\geq 5}(T)$. Now we consider the case $b \in B_4(T)$ and $b \in V(P_T[b_1,z]) \cap V(P_T[b_2,t])$. If $z \neq t$, $b_1 \notin V(P_T[t,b])$ and $b_2 \notin V(P_T[z,b])$, then $T' = T - \{bb_1, bb_2\} + \{zb_1, tb_2\}$ violates the condition (C1). Now if $z \neq t$ and $b_1 \in V(P_T[t,b])$, then $T' = T - \{bb_2\} + \{tb_2\}$ violates the condition (C4). If $z = t$ and $d(b,r) \geq d(z,r)$, then $T' = T - \{bb_1, bb_2\} + \{zb_1, zb_2\}$ violates either the condition (C1) if $z \in B_3(T)$ or the condition (C4) if $z \in L(T)$, as $\deg_{T'}(z) = 3 < 4 = \deg_T(b)$ and $d(b,r) \geq d(z,r)$. If $z = t$ and $d(b,r) < d(z,r)$, then $T' = T - \{bb_1\} + \{zb_1\}$ violates the condition (C4). This completes the proof of Claim 3.9. □

Now we conclude that:

If $u \in B_3(T) \setminus (M \cup \{r\})$, combining with Claims 3.5 and 3.9, then there exists at least one child a of u such that the edge au has no oblique neighbors in M. If $u = r$, there exist at least two neighbors r_1, r_2 of r such that rr_1, rr_2 have no oblique neighbors in M by Claim 3.8. Let C be the set of all such edges in T.

If $b \in B_{\geq 4}(T)$. Since G is claw-free, b has two children b_1, b_2 such that $b_1b_2 \in E(G)$. Combining with Claim 3.9, we obtain that there exists at least one child c of b such that the edge bc has no oblique neighbors in M and c is adjacent to at least one different child of b. Let D be the set of all such edges in T.

By Claim 3.6, we deduce that for each leaf $l \in L(T) \setminus M$, ll_r has no oblique neighbors in M. Let E be the set of all such edges in T.

We will prove that C, D and E are disjoint sets. The fact that $C \cap D = \emptyset$ is trivial, so suppose to the contrary that there exists $e \in E(T), e \in (C \cup D) \cap E$. So, we deduce that $e_r \in B(T), g(e,r) \in L(T)$ and $g(e,r)$ is adjacent to a child $z \neq g(e,r)$ of e_r. We consider the tree $T' = T - \{e_rz\} + \{zg(e,r)\}$. This spanning tree violates either the condition (C1) if $e_r \in B_3(T)$, or the condition (C3) if $e_r \in B_{\geq 5}(T)$ or the condition (C4) otherwise. Then we conclude that C, D and E are distinct.

Therefore, if we set h to be the number of edges in T which have no oblique neighbors in M, then we get

$$\begin{aligned} h &\geq |C| + |D| + |E| \\ &\geq 2 + |(B(T) \setminus \{r\}) \setminus M)| + |L(T) \setminus M| \\ &= |B(T)| + 1 + |L(T)| - |M| = |L(T)| + |B(T)| - m \\ &\geq n + 1 - m. \end{aligned}$$

So, we obtain that

$$\sigma_{m+1}(G) \leq \sum_{t \in M} \deg_G(t) \leq |E(T)| - h \leq |G| - n + m - 2.$$

This gives a contradiction with the assumption of Theorem 1.9. The proof of Theorem 1.9 is completed.

References

[1] H. Broersma and H. Tuinstra, Independence trees and Hamilton cycles, *J. Graph Theory* **29**:227-237,1998.

[2] G. Chen, Y. Chen, Z. Hu and S. Zhang, Spanning trees with at most k leaves in 2-connected $K_{1,r}$-free graphs, *Appl. Math. Comput.* **445**:127842, 2023.

[3] Y. Chen, G. Chen and Z. Hu, Spanning 3-ended trees in k-connected $K_{1,4}$-free graphs, *Sci. China Math.* **57**:1579-1586, 2014.

[4] Y. Chen, P. H. Ha and D. D. Hanh, Spanning trees with at most 4 leaves in $K_{1,5}$-free graphs, *Discrete Math.* **342**:2342-2349, 2019.

[5] P. N. Diep, P. H. Ha and B. N. Tu, Spanning trees of a $K_{1,5}$-free graph with at most seven leaves and branch vertices in total, *Studia Sci. Math. Hungar.* 2025. https://doi.org/10.1556/012.2025.04334.

[6] R. Diestel, Graph Theory, 3rd Edition, Springer, Berlin, 2005.

[7] E. Flandrin, T. Kaiser, R. Kuzel, H. Li and Z. Ryjácek, Neighborhood unions and extremal spanning trees, *Discrete Math.* **308**:2343-2350, 2008.

[8] L. Gargano, M. Hammar, P. Hell, L. Stacho and U. Vaccaro, Spanning spiders and light - splitting swithches, *Discrete Math.* **285**:83-95, 2004.

[9] R. Gould and W. Shull, On spanning trees with few branch vertices, *Discrete Math.* **343** (1):111581, 2020.

[10] P. H. Ha, A note on the independence number, connectivity and k-ended tree, *Discrete Appl. Math.* **305**:142-144, 2021.

[11] P. H. Ha, Spanning trees of $K_{1,4}$-free graphs with a bounded number of leaves and branch vertices, *Acta Math. Vietnam* **50**:263–271, 2025.

[12] P. H. Ha and N. H. Trang, Spanning trees with at most 5 leaves and branch vertices in total of $K_{1,5}$-free graphs, *Taiwanese J. Math.* **28** (5):847-855, 2024.

[13] D. D. Hanh, Degree conditions for claw-free graphs to have spanning trees with at most five branch vertices and leaves in total, *Studia Sci. Math. Hungar.* **59**:58-66, 2022.

[14] D. D. Hanh, Spanning trees with a bounded number of branch vertices in a $K_{1,4}$-free graph, *Discuss. Math. Graph Theory* **43**:1195-1202, 2023.

[15] Z. Hu and P. Sun, Spanning 5-ended trees in $K_{1,5}$-free graphs, *Bull. Malays. Math.Sci. Soc.* **43**:2565-2586, 2020.

[16] M. Kano, A. Kyaw, H. Matsuda, K. Ozeki, A. Saito and T. Yamashita, Spanning trees with a bounded number of leaves in a claw-free graph, *Ars Combin.* **103**:137-154, 2012.

[17] A. Kyaw, Spanning trees with at most 3 leaves in $K_{1,4}$-free graphs, *Discrete Math.* **309**:6146-6148, 2009.

[18] A. Kyaw, Spanning trees with at most k leaves in $K_{1,4}$-free graphs, *Discrete Math.* **311**:2135-2142, 2011.

[19] S. Maezawa, R. Matsubara and H. Matsuda, Degree conditions for graphs to have spanning trees with few branch vertices and leaves, *Graphs Combin.* **35**:231-238, 2019.

[20] H. Matsuda, K. Ozeki and T. Yamashita, Spanning trees with a bounded number of branch vertices in a claw-free graph, *Graphs Combin.* **30**:429-437, 2014.

[21] Zh. G. Nikoghosyan, Spanning trees with few branch and end vertices, *Math. Probl. Comput. Sci.* **46**:18-25, 2016.

[22] A. Saito and K. Sano, Spanning trees homeomorphic to a small tree, *Discrete Math.* **339**:677-681, 2016.

[23] S. Win, On a conjecture of Las Vergnas concerning certain spanning trees in graphs. *Resultate Math.* **2**:215-224, 1979.