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*Degree Realization by Bipartite Multigraphs**

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The problem of realizing a given degree sequence by a multigraph can be thought of as a relaxation of the classical degree realization problem (where the realizing graph is simple). This paper concerns the case where the realizing multigraph is required to be bipartite.

The problem of characterizing degree sequences that can be realized by a bipartite (simple) graph has two variants. In the simpler one, termed BDR^P, the partition of the degree sequence into two sides is given as part of the input. A complete characterization for realizability in this variant was given by Gale and Ryser over sixty years ago. However, the variant where the partition is not given, termed BDR, is still open.

For bipartite multigraph realizations, there are also two variants. For BDR^P, where the partition is given as part of the input, a complete characterization was known for determining whether there is a multigraph realization whose underlying graph is bipartite, such that the *maximum* number of copies of an edge is at most r. We present a complete characterization for determining if there is a bipartite multigraph realization such that the *total* number of excess edges is at most t. We show that optimizing these two measures may lead to different realizations, and that optimizing by one measure may increase the other substantially. As for the variant BDR, where the partition is not given, we show that determining whether a given (single) sequence admits a bipartite multigraph realization is NP-hard. Moreover, we show that this hardness result extends to any graph family which is a sub-family of bipartite graphs and a super-family of paths. On the positive side, we provide an algorithm that computes optimal realizations for the case where the number of balanced partitions is polynomial, and present sufficient conditions for the existence of bipartite multigraph realizations that depend only on the largest degree of the sequence.

Keywords: Degree Sequences, Graph Realization, Bipartite Graphs, Graphic Sequences, Bigraphic Sequences, Multigraph Realization.

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1 Introduction

1.1 Background and Motivation

Degree realization. This paper concerns a classical network design problem known as the GRAPHIC DEGREE REALIZATION problem (GDR). The number of neighbors or connections of a vertex in a graph is called its *degree*, and it provides information on its centrality and importance. For the entire graph, the sequence of vertex-degrees is a significant characteristic which has been studied for over sixty years. The graphic degree realization problem asks if a given non-increasing sequence of positive integers $d = (d_1, ..., d_n)$ is *graphic*, i.e., if it is the sequence of vertex-degrees of some graph. Erdös and Gallai [12] gave a characterization for graphic sequences, though not a method for finding a *realizing* graph. Havel and Hakimi [16, 15] proposed an algorithm that either generates a realizing graph or proves that the sequence is not graphic. Degree realization problems have found several interesting applications, most notably in network design, and also in the study of social networks [6, 9, 11, 19], chemical networks [24], and network evolution [18].

Relaxed degree realization by multigraphs. An interesting direction in the study of realization problems involves *relaxed* (or approximate) realizations (cf. [1]). Such realizations are well-motivated by applications in two wider contexts. In scientific contexts, a given sequence may represent (noisy) data resulting from an experiment, and the goal is to find a model that fits the data. In such situations, it may happen that *no* graph fits the input degree sequence exactly, and consequently it may be necessary to search for the graph "closest" to the given sequence. In an engineering context, a given degree sequence constitutes constraints for the design of a network. It might happen that satisfying all of the desired constraints *simultaneously* is not feasible, or causes other issues, e.g., unreasonably increasing the costs. In such cases, relaxed solutions bypassing the problem may be relevant.

In the current paper we focus on a specific type of relaxed realizations where the graph is allowed to have parallel edges, namely, the realization may be a *multigraph.* It is easy to verify that if *(multiple) self-loops* are allowed, then *every* sequence $d = (d_1, \ldots, d_n)$ whose sum $\sum_i d_i$ is even has a realization by a multigraph. Hence, we focus on the case where self-loops are not allowed.

The problem of degree realization by multigraphs has been studied in the past as well. Owens and Trent [21] gave a condition for the existence of a multigraph realization. Will and Hulett [26] studied the problem of finding a multigraph realization of a given sequence such that the underlying graph of the realization contains as few edges as possible. They proved that such a realization is composed of components, each of which is either a tree or a tree with a single odd cycle. Hulett, Will, and Woeginger [17] showed that this problem is strongly NP-hard.

Degree realization by bipartite graphs. The BIGRAPHIC DEGREE REALIZATION problem (BDR) is a natural variant of GDR, where the realizing graph is required to be bipartite. The problem has a sub-variant, denoted BDR^P, in which *two* sequences are given as input, representing the vertex-degree sequences of the two sides of a bipartite realizing graph. (In contrast, in the general problem, a *single* sequence is given as input, and the goal is to find a realizing bipartite graph based on some partition of the given sequence.) BDR^P was solved by Gale and Ryser [13, 23] even before Erdös and Gallai's characterization of graphic sequences. However, the general problem – mentioned as an open problem over forty years ago [22] – remains unsolved today.

A (non-increasing) sequence of integers $d = (d_1, ..., d_n)$ can only be *bigraphic*, i.e., the vertex-degree sequence of a bipartite graph, if it can be partitioned into two sub-sequences or *blocks* of equal total sum.

The latter problem is known as the *partition problem* and it is solvable in polynomial time assuming that $d_1 < n$ (which is a necessary condition for d to be bigraphic). Yet, BDR bears two obstacles. First, a sequence may have several partitions of which some are bigraphic and others are not. Second, the number of partitions may be exponentially large in n. Recent attempts on the BDR problem (see [2, 5, 4]) try to identify a small set of partitions, which are suitable to decide BDR for the whole sequence. Each partition in the small set is tested using the Gale-Ryser characterization. In case all of them fail the test, it is conjectured that no partition of the sequence is bigraphic. The conjecture was shown to be true in case there exists a special partition that (perfectly) splits the degrees into small and large ones.

Paralleling the above discussion concerning relaxed degree realizations by *general* multigraphs, one may look for relaxed degree realizations by *bipartite multigraphs*. This question is our main interest in the current paper.

1.2 Our Contribution

In this paper, we consider the problem of finding relaxed *bipartite multigraph* realizations for a given degree sequence or a given partition. That is, the relaxed realizations must fulfill the degree constraints exactly but are allowed to have parallel edges. (Self-loops are disallowed.)

To evaluate the quality of a realization by a multigraph, we use two measures:

(i) The *total multiplicity* of the multigraph, i.e., the number of parallel edges.
(ii) The *maximum multiplicity* of the multigraph, i.e., the maximum number of edges between any two of its vertices.

As shown later, these measures are non-equivalent, in the sense that there are examples for sequences where realizations optimizing one measure are sub-optimal in the other, and vice-versa.

Section 2 introduces formally the basic notions and measures under study. For relaxed realizations by *general* multigraphs, it follows from the characterizations given, respectively, by Owens and Trent [21] and Chungphaisan [8], how to optimize the two measurements and find the respective multigraph realizations. For relaxed realization by *bipartite* multigraphs, finding a realization for BDR^P (the given partition variant) that minimizes the *maximum* multiplicity follows from the characterization presented by Berge [20].

In Section 3 we provide a characterization for bipartite multigraphs based on a given partition (BDR^P). More specifically, we present results on multigraph realizations with bounded *total* multiplicity for BDR^P.

In Section 4 we show that optimizing total multiplicity and maximum multiplicity may lead to different realizations. Moreover, optimizing by one measure may increase the other substantially.

One necessary condition for a (non-increasing) sequence $d = (d_1, \ldots, d_n)$ to be bigraphic is that it can be *partitioned*. If $d_1 < n$, this problem can be decided in polynomial time. However, for a multigraph realization to exist, the inequality $d_1 < n$ is not a necessary condition, and it turns out that BDR is NP-hard. We review this matter in greater detail in Section 5 and show that this hardness results extends to any graph family which is a sub-family of bipartite graphs and a super-family of paths. We discuss an output sensitive algorithm to generate all partitions of a given sequence. In case the number of partitions of a sequence is small, the algorithm allows us to find optimal realizations with respect to both criteria.

In Section 6, we discuss sufficient conditions for the existence of approximate bipartite realizations that depend only on the largest degree of a given sequence.

2 Preliminaries

Let $d = (d_1, d_2, \ldots, d_n)$ be a sequence of positive integers in non-increasing order. (All sequences that we consider are assumed to be of positive integers and in a non-increasing order.) The *volume* of d is $\sum d = \sum_{i=1}^{n} d_i$. For a graph G, denote the sequence of its vertex-degrees by $\deg(G)$. Sequence d is *graphic* if there is a graph G such that $\deg(G) = d$. We say that G is a *realization* of d. Note that every realization of d has $m = \sum d/2$ edges. Consequently, a graphic sequence must have even volume. In turn, we call a sequence of positive integers with even volume a *degree sequence*. We use the operator $\circ$ to define $d \circ d'$ as the concatenation of two degree sequences d and d' (in non-increasing order).

2.1 Multigraphs as Approximate Realizations

Let $H = (V, E)$ be a multigraph without loops. In this case, E is a multiset. Denote by $E_H(v, u)$ the multiset of edges connecting $v, u \in V$. If $|E_H(v, u)| > 1$, we say that the edge (v, u) has $|E_H(v, u)| - 1$ *excess* copies. Let E' be the set that is obtained by removing excess edges from E. The graph $G = (V, E')$ is called the *underlying graph* of H.

We view multigraphs as *approximate* realizations of sequences that are not graphic. Owens and Trent [21] gave a condition for the existence of a multigraph realization.

Theorem 1 (Owens and Trent [21])**.** *A degree sequence d can be realized by a multigraph if and only if $d_1 \leq \sum_{i=2}^{n} d_i$.*

To measure the quality of an approximate realization we introduce two metrics. First, the *maximum multiplicity* of a multigraph H is the maximum number of copies of an edge, namely

$$\mathsf{MaxMult}(H) \triangleq \max_{(v,w)\in E} (|E_H(v, w)|) ,$$

and for a sequence d define

$$\mathsf{MaxMult}(d) \triangleq \min\{\mathsf{MaxMult}(H) \mid H \text{ realizes } d\} .$$

We say that a sequence d is *r-max-graphic* if $\mathsf{MaxMult}(d) \leq r$, for a positive integer r.

Second, the *total multiplicity* of a multigraph H is the total number of excess copies, namely

$$\mathsf{TotMult}(H) \triangleq \sum_{(v,w)\in E} (|E_H(v, w)| - 1) = |E| - |E'| ,$$

where E' is the edge set of the underlying graph of H. For a sequence d define

$$\mathsf{TotMult}(d) \triangleq \min\{\mathsf{TotMult}(H) \mid H \text{ realizes } d\} .$$

We say a sequence d is *t-tot-graphic* if $\mathsf{TotMult}(d) \leq t$, for a positive integer t.

2.2 General Multigraphs

Given a degree sequence d, our goal is to compute $\mathsf{MaxMult}(d)$ and $\mathsf{TotMult}(d)$.

We note that the best realization in terms of maximum multiplicity is not necessarily the same as the best one in terms of total multiplicity. See more on this issue in Section 4.

Next, we iterate the characterization of Erdös and Gallai [12] for graphic sequence.

Theorem 2 (Erdös-Gallai [12]). *A degree sequence d is graphic if and only if, for $\ell = 1, \ldots, n$,*

$$\sum_{i=1}^{\ell} d_i \leq \ell(\ell-1) + \sum_{i=\ell+1}^{n} \min\{\ell, d_i\} . \tag{1}$$

Theorem 2 implies an $\mathcal{O}(n)$ algorithm to verify whether a sequence is graphic. Chungphaisan [8] extended the above characterization to multigraphs with bounded maximum multiplicity as follows.

Theorem 3 (Chungphaisan [8]). *Let r be a positive integer. A degree sequence d is r-max-graphic if and only if, for $\ell = 1, \ldots, n$,*

$$\sum_{i=1}^{\ell} d_i \leq r\ell(\ell-1) + \sum_{i=\ell+1}^{n} \min\{r\ell, d_i\} . \tag{2}$$

Notice the similarity to the Erdös-Gallai equations. Moreover, since $\mathsf{MaxMult}(d) \leq d_1$, it follows that $\mathsf{MaxMult}(d)$ can be computed in $\mathcal{O}(n \cdot \log(d_1))$.

The problem of finding a multigraph realization with low total multiplicity was solved by Owens and Trent [21]. They showed that the minimum total multiplicity is equal to the minimum number of degree 2 vertices that should be added to make the sequence graphic. We provide a simpler proof of their result.

Theorem 4 (Owens and Trent [21]). *Let d be a degree sequence such that $d_1 \leq \sum_{i=2}^{n} d_i$. Then, d is t-tot-graphic if and only if $d \circ 2^t$ is graphic.*

Proof: Let d be a degree sequence such that $d_1 \leq \sum_{i=2}^{n} d_i$. First, assume that d can be realized by a multigraph H with $\mathsf{TotMult}(H) \leq t$. Let F be the set of excess edges in H. Construct a simple graph G by replacing each edge $f = (x, y) \in F$ with two edges (x, v_f) and (y, v_f), where v_f is a new vertex. Clearly, this does not change the degrees of x and y and adds a vertex v_f of degree 2. Hence the degree sequence of G is $d \circ 2^{|F|}$. Also, G is simple. If $|F| < t$, then one may replace any edge in G with a path containing $t - |F|$ edges, yielding a graph with degree sequence $d \circ 2^t$.

Conversely, suppose the sequence $d \circ 2^t$ is graphic. Let G be a simple graph that realizes the sequence. Pick a degree 2 vertex v with neighbors x and y, replace the edges (v, x) and (v, y) with the edge (x, y), and remove v from G. This transformation eliminates one degree 2 vertex from G without changing the remaining degrees. But it may increase the number of excess edges by one (if the edge (x, y) already exists in G). Performing this operation for t times, we obtain a multigraph H with $\mathsf{TotMult}(H) \leq t$ and degree sequence d. □

The next corollary follows readily with Theorems 2 and 4.

Corollary 5. *Let t be a positive integer, and let $d' = d \circ 2^t$. A sequence d is t-tot-graphic if and only if, for $\ell = 1, \ldots, n+t$,*

$$\sum_{i=1}^{\ell} d'_i \leq \ell(\ell-1) + \sum_{i=\ell+1}^{n+t} \min\{\ell, d'_i\} . \tag{3}$$

Owens and Trent [21] implicitly suggest to compute $\mathsf{TotMult}(d)$ by computing the minimum t such that $d \circ 2^t$ is graphic. Using binary search would lead to a running time of $\mathcal{O}(n \cdot \log(\mathsf{TotMult}(d)))$.

Several authors [25, 27] noticed that the equations of Theorem 2 are not minimal. For a degree sequence d where $d_1 > 1$, let $\text{box}(d) = max\{i \mid d_i > i\}$. If Equation (1) holds for the index $\ell = \text{box}(d)$, then it holds for index $\ell + 1$. To see this, consider the equations for the two indices and compare the change in the left hand side (LHS) and right hand side (RHS). Observe that the RHS increases at least by $(\ell+1)\cdot\ell - \ell\cdot(\ell-1) = 2\ell$ while the LHS only increases by $d_{\ell+1} \leq \ell$. It follows that Equation (1) does not have to be checked for indices $\ell > \text{box}(d)$. If $d_1 = 1$, we define $\text{box}(d) = 0$. Note that in this case d is realized by a matching graph.

Observation 6 ([25, 27])**.** *A degree sequence* d *is graphic if and only if, for* $\ell = 1, \ldots, box(d)$*,*

$$\sum_{i=1}^{\ell} d_i \leq \ell(\ell-1) + \sum_{i=\ell+1}^{n} \min\{\ell, d_i\} \,. \tag{4}$$

On a side note, it is also known that only up to k many equations have to be checked where k is the number of different degrees of a sequence (cf. [20, 25, 27]).

Observation 6 helps to simplify Corollary 5.

Corollary 7. *Let* t *be a positive integer. A degree sequence* d *is* t*-tot-graphic if and only if, for* $\ell = 1, \ldots, box(d)$*,*

$$\sum_{i=1}^{\ell} d_i \leq \ell(\ell-1) + \sum_{i=\ell+1}^{n} \min\{\ell, d_i\} + t \cdot \min\{\ell, 2\} \,. \tag{5}$$

Proof: Let d and t be as in the corollary. In case $d_1 = 1$, the sequence d is graphic, i.e., it is t-tot-graphic for any positive integer t.

Hence, assume that $d_1 > 1$. Also, let $d' = d \circ 2^t$. One can verify that Equations (5) are the (reduced) Erdös-Gallai inequalities of Observation 6 for d'. Moreover, $\text{box}(d) = \text{box}(d')$, and the claim follows. $\square$

Corollary 7 implies a simple algorithm to compute $\mathsf{TotMult}(d)$. Let

$$\Delta_\ell(d) = \sum_{i=1}^{\ell} d_i - (\ell(\ell-1) + \sum_{i=\ell+1}^{n} \min\{\ell, d_i\}),$$

for $\ell = 1, \ldots, n$, be the Erdös-Gallai differences of a sequence d. Hence, $\mathsf{TotMult}(d) = \max\{\Delta_1, \Delta_{\mathsf{max}}/2\}$, where $\Delta_{\mathsf{max}}(d) = \max_{2 \leq \ell \leq \text{box}(d)} \Delta_\ell(d)$, implying a $\mathcal{O}(n)$ algorithm to calculate $\mathsf{TotMult}(d)$.

2.3 Bipartite Multigraphs

In this section, we start investigating whether a degree sequence has a bipartite realization, i.e., if it is *bigraphic* or not. Particularly, we are interested in multigraph realizations where the underlying graph is bipartite.

Let d be a degree sequence such that $\sum d = 2m$ for some integer m. A *block* of d is a subsequence a such that $\sum a = m$. Define the set of blocks as $B(d)$. For each $a \in B(d)$ there is a disjoint $b \in B(d)$ such that $d = a \circ b$. We call such a pair $a, b \in B(d)$ a balanced *partition* of d since $\sum a = \sum b$. Denote the set of all balanced partitions of d by $\mathtt{BP}(d) = \{\{a, b\} \mid a, b \in B(d),\ a \circ b = d\}$. We say a partition $(a, b) \in \mathtt{BP}(d)$ is *bigraphic* if there is a bipartite realization $G = (A, B, E)$ of d such that $\deg(A) = a$ and $\deg(B) = b$ are the vertex-degree sequences of A and B, respectively.

Observe that, as in the case of general graphs, the best realization in terms of maximum multiplicity is not necessarily the same as the best one in terms of total multiplicity. See Section 4.

Note that not every graphic sequence has a balanced partition. Yet, if d is bigraphic, then $\mathtt{BP}(d)$ is not empty. The Gale-Ryser theorem characterizes when a partition is bigraphic.

Theorem 8 (Gale-Ryser [13, 23])**.** *Let* d *be a degree sequence and partition* $(a,b) \in \mathtt{BP}(d)$ *where* $a = (a_1, a_2, \ldots, a_p)$ *and* $b = (b_1, b_2, \ldots, b_q)$*. The partition* (a,b) *is bigraphic if and only if, for* $\ell = 1, \ldots, p$*,*

$$\sum_{i=1}^{\ell} a_i \leq \sum_{i=1}^{q} \min\{\ell, b_i\} \ . \tag{6}$$

We point out that Theorem 8 does not characterize bigraphic degree sequences. Indeed, if the partition is not specified, it is not known how to determine whether a graphic sequence is bigraphic or not. There are sequences where some partitions are bigraphic while others are not. Moreover, $|\mathtt{BP}(d)|$ might be exponentially large in the input size n.

We turn back to approximate realizations by bipartite multigraphs. A multigraph is bipartite if its underlying graph is bipartite. Analogue to above, we use the maximum and total multiplicity to measure the quality of a realization. Naturally, let

$$\mathsf{MaxMult}^{bi}(d) \triangleq \min\{\mathsf{MaxMult}(H) \mid H \text{ is bipartite and realizes } d\} \ .$$

For a partition $(a,b) \in \mathtt{BP}(d)$, we define

$$\mathsf{MaxMult}^{bi}(a,b) \triangleq \min\{\mathsf{MaxMult}(H) \mid H = (A,B,E) \text{ s.t. } \deg(A) = a \text{ and } \deg(B) = b\} \ .$$

Let r be a positive integer. If there is a bipartite multigraph $H = (A,B,E)$ where $\mathsf{MaxMult}(H) \leq r$, we say that d is r-max-bigraphic. Moreover, we say that the partition $(a,b) \in \mathtt{BP}(d)$, where $a = \deg(A)$ and $b = \deg(B)$, is r-max-bigraphic. Miller [20] cites the following result of Berge characterizing r-max-bigraphic partitions.

Theorem 9 (Berge [20])**.** *Let* r *be a positive integer. Consider a degree sequence* d *and a partition* $(a,b) \in \mathtt{BP}(d)$*, where* $a = (a_1, \ldots, a_p)$ *and* $b = (b_1, \ldots, b_q)$*. Then* (a,b) *is* r*-max-bigraphic if and only if, for* $\ell = 1, \ldots, p$*,*

$$\sum_{i=1}^{\ell} a_i \leq \sum_{i=1}^{q} \min\{\ell r, b_i\} \ . \tag{7}$$

Note the similarity to the Gale-Ryser theorem. Theorem 9 implies that $\mathsf{MaxMult}^{bi}(a,b)$ can be computed in $\mathcal{O}(n \cdot \log(d_1))$ using binary search.

For the second approximation criterion, we bound the total multiplicity of a bipartite multigraph realization. Define

$$\mathsf{TotMult}^{bi}(d) \triangleq \min\{\mathsf{TotMult}(H) \mid H \text{ is bipartite and realizes } d\} \ .$$

Additionally, for a partition $(a,b) \in \mathtt{BP}(d)$, we define

$$\mathsf{TotMult}^{bi}(a,b) \triangleq \min\{\mathsf{TotMult}(H) \mid H = (A,B,E) \text{ s.t. } \deg(A) = a \text{ and } \deg(B) = b\} \ .$$

We present our results on determining $\mathsf{TotMult}^{bi}(a,b)$ in the next section. In Sections 5 and 6, we consider $\mathsf{MaxMult}^{bi}(d)$ and $\mathsf{TotMult}^{bi}(d)$.

3 Multigraph Realizations of Bi-sequences

In this section, we are interested in bipartite multigraph realizations with low total multiplicity, assuming that we are given a sequence and a specific balanced partition. First, we provide a characterization similar to Theorem 4 for bipartite multigraph realizations for a given partition.

Theorem 10. *Let d be a degree sequence and t be a positive integer. Then, d is t-tot-bigraphic if and only if there exists a partition $(a,b) \in \mathtt{BP}(d)$ such that $(a \circ (1^t), b \circ (1^t))$ is bigraphic.*

Proof: Let d, t be as in the theorem. Assume that there is a bipartite multigraph $H = (L,R,E)$ with $\mathsf{TotMult}(H) \leq t$. Hence, there is a partition $(a,b) \in \mathtt{BP}(d)$ where $\deg(L) = a$ and $\deg(R) = b$. Let F be the set of excess edges in H. Construct a bipartite graph G by applying the following transformation. For every excess edge $(x,y) \in F$, add a new vertex x_e to A and a new vertex y_e to B, and replace (x,y) by the two edges (x,y_e) and (x_e,y). Note that x_e and y_e are placed on opposite partitions of G. Since there are t excess edges, G realizes $(a \circ 1^t, b \circ 1^t)$ without excess edges.

For the other direction, assume that there exists a partition $(a,b) \in \mathtt{BP}(d)$ such that $(a \circ (1^t), b \circ (1^t))$ is realized by a bipartite graph $G = (L,R,E)$. Let $x_1,\ldots,x_t$ and $y_1,\ldots,y_t$ be some vertices of degree one in L and R, respectively. Also, for every i, let y'_i (respectively, x'_i) be the only neighbor of x_i (resp., y_i). Construct a bipartite multigraph H by replacing the edges (x_i,y'_i) and (x'_i,y_i) with the edge (x'_i,y'_i) and discarding the vertices x_i and y_i, for every i. Since this transformation may add up to t excess edges, we have that $\mathsf{TotMult}(H) \leq t$. □

The above characterization leads to extended Gale-Ryser conditions for total multiplicity.

Theorem 11. *Let d be a degree sequence with partition $(a,b) \in \mathtt{BP}(d)$, where $a = (a_1,\ldots,a_p)$ and $b = (b_1,\ldots,b_q)$, and let t be a positive integer. The partition (a,b) is t-tot-bigraphic if and only if, for all $\ell = 1,\ldots,p$,*

$$\sum_{i=1}^{\ell} a_i \leq \sum_{i=1}^{q} \min\{\ell, b_i\} + t \ . \tag{8}$$

Proof: Consider (a,b) and t as in the theorem. One can verify that the following equations are the Gale-Ryser conditions of Theorem 8 for the partition $(a \circ (1^t), b \circ (1^t))$ of Theorem 10: For all $\ell = 1,\ldots,p$,

$$\sum_{i=1}^{\ell} a_i \leq \sum_{i=1}^{q} \min\{\ell, b_i\} + t \ , \tag{9}$$

and for all $h = 1,\ldots,t$,

$$\sum_{i=1}^{p} a_i + h \leq \sum_{i=1}^{q} \min\{p+h, b_i\} + t \ . \tag{10}$$

To finish the proof, we argue that Equation (10) holds for any $h \in \{0,\ldots,t\}$ if Equation (9) holds for $\ell = p$. Recall that $\sum_{i=1}^{q} \min\{p+h, b_i\} = \sum_{i=1}^{q} b_i$ if $p+h \geq b_1$. It follows that Equation (10) holds for indices $h \geq b_1 - p$.

Observe that $\sum_{i=1}^{q} \min\{p+h+1, b_i\} - \sum_{i=1}^{q} \min\{p+h, b_i\} \geq 1$ for $p+h < b_1$, i.e., the RHS of Equation (10) grows by at least 1 when moving from index $p+h$ to index $p+h+1$. By assumption, Equation (9) holds for $\ell = p$, implying that Equation (10) holds for $h = 0$. Since the LHS of Equation (10) grows by 1 exactly, Equation (10) holds for indices $h < b_1 - p$. □

Given a degree sequence d with partition $(a, b) \in \texttt{BP}(d)$, Theorem 11 implies that

$$\mathsf{TotMult}^{bi}(a,b) = \max_{1 \le \ell \le p} \left(\sum_{i=1}^{\ell} a_i - \sum_{i=1}^{q} \min\{\ell, b_i\} \right) .$$

It follows that $\mathsf{TotMult}^{bi}(a, b)$ can be computed in time $\mathcal{O}(n)$.

4 Total Multiplicity vs. Maximum Multiplicity

In this section we show that the measures of total multiplicity and maximum multiplicity sometimes display radically different behavior. Specifically, we show that there are sequences such that a realization that minimizes the total multiplicity may be far from achieving minimum maximum multiplicity, and vice versa.

First, we notice that by definition of $\mathsf{TotMult}$ and $\mathsf{TotMult}^{bi}$, in order to minimize the total multiplicity one needs to use a maximum number of edges, or to maximize the number of edges in the underlying graph.

Observation 12. *Let d be a sequence and let $H = (V, E)$ be a multigraph that realizes d. Let $G' = (V, E')$ be the underlying graph of H. Then,*

- $\mathsf{TotMult}(H) = \mathsf{TotMult}(d)$ *if and only if* $|E'|$ *is maximized.*
- $\mathsf{TotMult}^{bi}(H) = \mathsf{TotMult}^{bi}(d)$ *if and only if* $|E'|$ *is maximized.*

4.1 General Graphs

Let $n \geq 5$, and consider the sequence

$$\hat{d} = \left((2n-2)^2, (n-1)^{n-2}\right) .$$

Observe that $\sum \hat{d} = (n-1)(n+2)$. The following two lemmas show that a realization of $\hat{d}$ attaining minimum total multiplicity is far from obtaining minimum maximum multiplicity, and vise versa.

Lemma 13. $\mathsf{TotMult}(\hat{d}) = n - 1$*, and if H realizes $\hat{d}$ such that* $\mathsf{TotMult}(H) = \mathsf{TotMult}(\hat{d})$*, then* $\mathsf{MaxMult}(H) = n$*.*

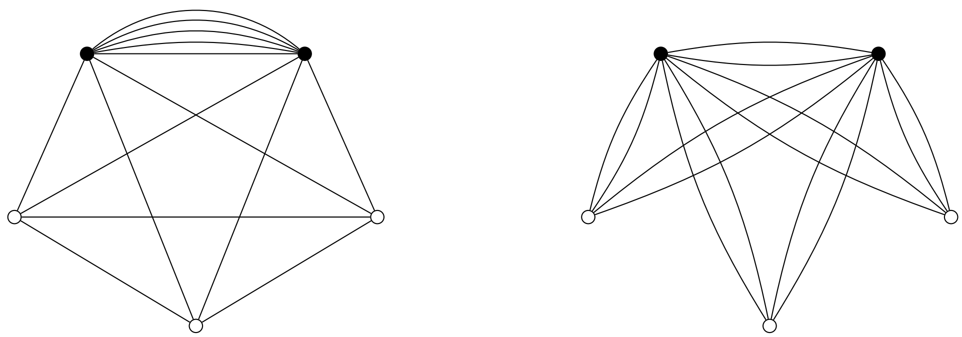
Proof: Consider a mutligraph H which is composed of a full graph and $n - 1$ copies of the edge $(1, 2)$. More formally, let $H = (V, E)$ be a multigraph where

$$E = \{(i, j) \mid 1 \le i < j \le n\} \uplus \biguplus_{t=1}^{n-1} \{(1, 2)\} .$$

Recall that E is a multiset. See example in Figure 1a.

We have that $\deg(1) = \deg(2) = (n - 1) + (n - 1) = 2n - 2$, and $\deg(i) = (n - 1)$, for $i > 2$, as required. Thus, H realizes $\hat{d}$. Observe that $|E'| = \binom{n}{2}$, hence by Observation 12 we have that

$$\mathsf{TotMult}(\hat{d}) = \mathsf{TotMult}(H) = (n-1) .$$



(a) Optimal TotMult realization H_1.

(b) Optimal MaxMult realization H_2.

Fig. 1: Optimal multigraph realizations for the sequence $\hat{d} = (8^2, 4^3)$ ($n = 5$). On the left we have $\mathsf{TotMult}(H_1) = 4$ and $\mathsf{MaxMult}(H_1) = 5$, while on the right we have $\mathsf{TotMult}(H_2) = 7$ and $\mathsf{MaxMult}(H_2) = 2$.

Let $\bar{H}$ be a realization such that $\mathsf{TotMult}(\bar{H}) = \mathsf{TotMult}(\hat{d})$. Hence, $\left|\bar{E}'\right| = \binom{n}{2}$. Consider a vertex i, such that $i > 2$. All edges touching i must be used at least once. Since $\hat{d}_i = n-1$, all edge adjacent to i must be used exactly once. It follows that all excess edges are connected to the vertices $\{1,2\}$. It follows that $\bar{H} = H$. Hence, H minimizes the maximum multiplicity, and $\mathsf{MaxMult}(H) = n$. □

Lemma 14. $\mathsf{MaxMult}(\hat{d}) = 2$*, and if* H *realizes* $\hat{d}$ *such that* $\mathsf{MaxMult}(H) = \mathsf{MaxMult}(\hat{d})$*, then* $\mathsf{TotMult}(H) \geq 2n-3$.

Proof: Consider a vertex 1 (or 2). To minimize its load, its degree requirement should be distributed equally among the rest of the vertices. This leads to a realization H in which each edge of 1 and 2 has two copies. The degree requirements of the rest of the vertices are obtained by removing a cycle from a complete graphs (this is the reason for requiring $n \geq 5$). Formally, $H = (V,E)$ is define as follows:

$$E = \{(1,i),(1,i) \mid i \geq 2\} \uplus \{(2,i),(2,i) \mid i \geq 3\} \uplus \{(i,j) \mid 2 < i, j \neq i+1 \text{ and } (i,j) \neq (3,n)\} \ .$$

See example in Figure 1b.

We have that $\deg(1) = \deg(2) = 2(n-1)$, and $\deg(i) = 2+(n-1-2) = n-1$, for $i > 2$, as required. Thus, H realizes $\hat{d}$. Moreover, each edges has at most two copies, which means that $\mathsf{MaxMult}(\hat{d}) = 2$.

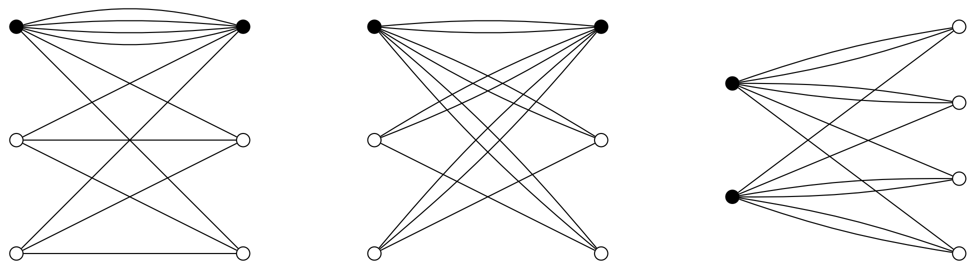
Observe that an edge (i,j), where $i,j > 2$ has at most a single copy. Hence, H minimizes the total multiplicity. In addition, $\mathsf{TotMult}(H) = (n-1)+(n-2) = 2n-3$. □

Corollary 15. *Let* $n \geq 5$*. There exists a sequence* $\hat{d}$ *of length* n *such that* $\mathsf{TotMult}(H_2) - \mathsf{TotMult}(H_1) = n-2$ *and* $\mathsf{MaxMult}(H_1) - \mathsf{MaxMult}(H_2) = n-2$*, for any with two realizations* H_1 *and* H_2 *of* $\hat{d}$ *such that* $\mathsf{TotMult}(H_1) = \mathsf{TotMult}(\hat{d})$ *and* $\mathsf{MaxMult}(H_2) = \mathsf{MaxMult}(\hat{d})$.

4.2 Bipartite Graphs

Let n be an even integer such that $n \geq 4$, and consider the sequence

$$\tilde{d} = \left(n^2, (n/2)^{n-2}\right) \ .$$



(a) Optimal $\mathsf{TotMult}^{bi}$ realization H_1. (b) Optimal $\mathsf{MaxMult}^{bi}$ realization H_2. (c) Optimal $\mathsf{MaxMult}^{bi}$ realization H_3.

Fig. 2: Multigraph bipartite realizations for the sequence $\tilde{d} = (6^2, 3^4)$. On the left we have $\mathsf{TotMult}^{bi}(H_1) = 3$ and $\mathsf{MaxMult}^{bi}(H_1) = 4$; In the center we have $\mathsf{TotMult}^{bi}(H_2) = 5$ and $\mathsf{MaxMult}^{bi}(H_2) = 2$; On the right we have $\mathsf{TotMult}^{bi}(H_3) = 4$ and $\mathsf{MaxMult}^{bi}(H_3) = 2$.

Lemma 16. $\mathsf{TotMult}^{bi}(\tilde{d}) = n/2$*, and if* H *realizes* $\tilde{d}$ *such that* $\mathsf{TotMult}^{bi}(H) = \mathsf{TotMult}^{bi}(\tilde{d})$*, then* $\mathsf{MaxMult}^{bi}(H) \geq n/2 + 1$.

Proof: Let $a = b = (n, (\frac{n}{2})^{(n-2)/2})$. We construct a multigraph H that realizes (a, b), which consist of a complete bipartite graph and $n/2$ copies of the edge $(1, 1)$. Formally, $H = (A, B, E)$, where

$$E = \{(i, j) \mid 1 \leq i, j \leq n/2\} \uplus \biguplus_{t=1}^{n/2} \{(1, 1)\} \ .$$

See example in Figure 2a.

On both sides we have that $\deg(1) = \frac{n}{2} + \frac{n}{2} = n$, and $\deg(i) = \frac{n}{2}$, for $i > 1$, as required. Thus, H realizes (a, b). Observe that $|E'| = \frac{n}{2} \cdot \frac{n}{2}$, hence by Observation 12 we have that

$$\mathsf{TotMult}^{bi}(\tilde{d}) = \mathsf{TotMult}^{bi}(a, b) = \mathsf{TotMult}^{bi}(H) = n/2 \ .$$

Let $\bar{H}$ be a realization such that $\mathsf{TotMult}^{bi}(\bar{H}) = \mathsf{TotMult}^{bi}(a, b)$. It follows that $\left|\bar{E}'\right| = \frac{n}{2} \cdot \frac{n}{2}$. Consider a vertex i, such that $i > 2$. All edges touching i must be used at least once. Since $\tilde{d}_i = \frac{n}{2}$, all edges touching i must be used exactly once. It follows that all excess edges are connected to the vertices $\{1, 2\}$. Hence, $\bar{H} = H$. Also, $\mathsf{MaxMult}^{bi}(H) = n/2 + 1$. □

Lemma 17. $\mathsf{MaxMult}^{bi}(\tilde{d}) = 2$*, and if* H *realizes* $\tilde{d}$ *such that* $\mathsf{MaxMult}(H) = \mathsf{MaxMult}(\tilde{d})$*, then* $\mathsf{TotMult}(H) \geq n - 2$.

Proof: There are two possible partitions for $\tilde{d}$:

(P1) $a = b = \left(n, (\frac{n}{2})^{(n-2)/2}\right)$, and

(P2) $a' = \left(n^2, (\frac{n}{2})^{n/2-3}\right)$ and $b' = \left((\frac{n}{2})^{n/2+1}\right)$.

First examine partition (P1). Consider a vertex $1 \in A$ (or $1 \in B$). To minimize its load, its degree requirement should be distributed equally among the rest of the vertices on the other side of the partition. This leads to a realization H, where vertex $1 \in A$ is connected to all the vertices in B by two copies, while vertex $1 \in B$ is connected to all the vertices in A by two copies. The requirement of the other vertices is obtained using a complete bipartite graph minus a perfect matching. Hence, $H = (A, B, E)$, where

$$E = \{(1,1),(1,1)\} \uplus \{(1,i),(1,i),(i,1),(i,1) \mid i \geq 2\} \uplus \{(i,j) \mid i,j \geq 2, j \neq i\} \ .$$

See example in Figure 2b.

We have that $\deg(1) = 2(n/2) = n$, and $\deg(i) = 2 + (n/2 - 2) = n/2$, for $i > 2$, on both sides, as required. Thus, H realizes (a, b). Moreover, $\mathsf{MaxMult}^{bi}(a, b) = \mathsf{MaxMult}^{bi}(H) = 2$, since each edge has at most two copies. Observe that an edge (i, j), where $i, j > 1$ has at most a single copy. Hence, H minimizes the maximum multiplicity. In addition, $\mathsf{TotMult}(H) = n/2 + n/2 - 1 = n - 1$.

Next, consider partition (P2). We construct a realization $H'' = (V'', E'')$ as follows:

$$E'' = \{(i,j) \mid i \leq 2 \text{ or } i > 2 \text{ and } j \neq i\} \uplus \{(1,j) \mid j \leq n/2-1\} \uplus \{(2,j) \mid j \geq 3\} \ .$$

See example in Figure 2c.

On the left side, we have that $\deg(1) = \deg(2) = n/2+1+n/2-1 = n$, and $\deg(i) = n/2+1-1 = n/2$, for $i > 3$. On the right side, $\deg(j) = n/2 - 1 + 2$, for $i \in \{3, \ldots, n/2-1\}$, and $\deg(j) = n/2 + 1$, for $i \in \{1, 2, n/2, n/2+1\}$. Thus, H'' realizes (a', b'). Moreover, $\mathsf{MaxMult}^{bi}(a', b') = \mathsf{MaxMult}^{bi}(H'') = 2$, since each edge has at most two copies. Furthermore, $\mathsf{TotMult}(H'') = 2(n/2-1) = n - 2$. □

Corollary 18. *Let n be an even integer such that $n \geq 4$. There exists a sequence $\tilde{d}$ of length n such that* $\mathsf{TotMult}^{bi}(H_2) - \mathsf{TotMult}^{bi}(H_1) = n/2 - 2$ *and* $\mathsf{MaxMult}^{bi}(H_1) - \mathsf{MaxMult}^{bi}(H_2) = n/2 - 1$, *for any with two realizations H_1 and H_2 of $\tilde{d}$ such that* $\mathsf{TotMult}^{bi}(H_1) = \mathsf{TotMult}^{bi}(\tilde{d})$ *and* $\mathsf{MaxMult}^{bi}(H_2) = \mathsf{MaxMult}^{bi}(\tilde{d})$.

5 Bipartite Realization of a Single Sequence

In this section, we study the following question: given a degree sequence d, can it be realized as a multigraph whose underlying graph is bipartite? Also, if there exists such a realization, we would like to find one which minimizes the maximum or the total multiplicity.

5.1 Hardness Result

Given a sequence and a balanced partition one may construct a bipartite multigraph realization by assigning edges in an arbitrary manner.

Observation 19. *Let d be a sequence and let $(\ell, r) \in \mathtt{BP}(d)$ be a partition of d. Then, there exists a bipartite multigraph realization of (ℓ, r).*

It follows that deciding whether a degree sequence d can it be realized as a multigraph whose underlying graph is bipartite is NP-hard.

Theorem 20. *Deciding if a degree sequence d admits a bipartite multigraph realization is NP-hard.*

Proof: We prove the theorem using a reduction from the PARTITION problem. Recall that PARTITION contains all sequences $(a_1, \ldots, a_n)$ such that there exists an index set $S \subseteq [1, n]$ for which $\sum_{i \in S} a_i =$

$\sum_{i \notin S} a_i$ (see. e.g., [14]). Observation 19 implies that d is a PARTITION instance if and only if d admits a bipartite mulitgraph realization. □

Observation 19 also implies a reduction from bipartite multigraph realization to PARTITION. Since PARTITION admits a pseudo-polynomial time algorithm, we have the following.

Theorem 21. *Deciding if a sequence* d *admits a bipartite multigraph realization can be done in pseudo-polynomial time.*

We note that there may be an exponential number of balanced partitions of a sequence d even if $d_1 < n$ (see, e.g., [5]). In bipartite multigraph realization it is enough to find any balanced partition. However, in BDR the partition should also satisfy the Gale-Ryser conditions.

Next, we show that deciding whether a given sequence has a multigraph realization whose underlying graph belongs to a graph family is hard for any family which is a subfamily of bipartite graphs and a super family of paths. We start be defining the following variant of PARTITION we refer to as PARTITION$'$. A sequence of integers $b = (b_1, \ldots, b_n)$ is in PARTITION$'$ if and only if

1. n is even.
2. There exists $B > 0$ such that $b_i \geq 2B$, for every $i \in \{1, \ldots, n-2\}$, $b_{n-1} = b_n = B$, and $\sum_{i=1}^{n} b_i = (2n-1)B$.
3. There exists an index set S such that $\sum_{i \in S} b_i = \sum_{i \notin S} b_i$.

Observation 22. *Let* d *be a sequence that satisfies the first two conditions of* PARTITION$'$*, and let* $S \subseteq \{1, \ldots, n-2\}$*. Then,* $\sum_{i \in S} b_i \leq B(2|S|+1)$.

Proof:

$$\sum_{i \in S} b_i = \sum_{i=1}^{n} b_i - \sum_{i \notin S, i < n-1} b_i - b_{n-1} - b_n \leq (2n-1) \cdot B - (n-2-|S|) \cdot 2B - 2B = B(2|S|+1) .$$

□

Observation 23. *Let* $b \in$ PARTITION$'$*, and let* S *be an index set such that* $\sum_{i \in S} b_i = \sum_{i \notin S} b_i$*. Then,* $|S| = n/2$ *and* $|S \cap \{n-1, n\}| = 1$.

Proof: Assume that $|S| \leq n/2 - 1$, and let $S' = S \cap \{1, \ldots, n-2\}$ and $S'' = S \cap \{n-1, n\}$. By Observation 22 we have that

$$\sum_{i \in S} b_i \leq B(|S'| \cdot 2 + 1) + B|S"| \leq B(|S| \cdot 2 + 1) \leq B(n-2+1) = B(n-1) .$$

A contradiction. A similar argument works for the case where $|S| \geq n/2 + 1$.

Let $|S| = n/2$ and assume that $\{n-1, n\} \subseteq S$. It follows that

$$\sum_{i \in S} b_i = 2B + \sum_{i \in S, i \leq n-2} b_i \leq 2B + (n/2-2)2B + B = (n-1)B .$$

A contradiction. A similar argument works for the case where $\{n-1, n\} \cap S = \emptyset$. □

We show that this variant of PARTITION is NP-hard.

Lemma 24. PARTITION$'$ *is NP-hard.*

Proof: We prove the theorem by a reduction from PARTITION. Given a sequence $a = (a_1, \ldots, a_n)$, where $B = \sum_i a_i$, we construct the following degree sequence b as follows:

$$b_j = \begin{cases} 2B + a_j & j \in \{1, \ldots, n\}\,, \\ 2B & j \in \{n+1, \ldots, 2n\}\,, \\ B & j \in \{2n+1, 2n+2\}\,. \end{cases}$$

The length of b is $2n+2$ which is even. Observe that $b_i \geq 2B$, for every $i \in \{1, \ldots, 2n\}$, and $b_{2n+1} = b_{2n+2} = B$. Also,

$$\sum_{i=1}^{2n+2} b_i = 4nB + \sum_i a_i + 2B = (4n+3)B = (2(2n+2)-1)B\,.$$

Hence it remains to show that $a \in$ PARTITION if and only if there exists an index set S such that $\sum_{i \in S} b_i = \sum_{i \notin S} b_i$ and $|S| = n+1$.

Suppose that $a \in$ PARTITION and let T be an index set such that $\sum_{i \in T} a_i = \sum_{i \notin T} a_i$. Let $S = T \cup \{i+n : i \notin T\} \cup \{2n+1\}$. Observe that $|S| = n+1$ and

$$\sum_{i \in S} b_i = \sum_{i \in T} (2B + a_i) + (n - |T|)2B + B = (2n+1)B + B/2\,,$$

As required.

On the other hand, assume that $b \in$ PARTITION$'$ and let S be an index set such that $\sum_{i \in S} b_i = \sum_{i \notin S} b_i$ and $|S| = n+1$. By Observation 23 we may assume, without loss of generality, that $2n+1 \in S$ and $2n+2 \notin S$. Let $T = S \cap \{1, \ldots, n\}$. We have that

$$\sum_{i \in T} a_i = \sum_{i \in T} (b_i - 2B) = \sum_{i \in S} b_i - |T|\,2B - (n - |T|)2B - B = (2n+1)B + B/2 - 2nB - B = B/2\,.$$

□

It is said that a sequence b has a *sound* permutation if the following conditions hold:

1. $\sum_{i=1}^{n} (-1)^i b_{\pi(i)} = 0$.
2. $\sum_{i=1}^{k} (-1)^{i-1} b_{\pi(k-i+1)} > 0$, for all $k < n$.

Next, we show that a sequence $b \in$ PARTITION$'$ has a sound permutation.

Lemma 25. *If* $b \in$ PARTITION$'$, *then* b *admits a sound permutation.*

Proof: Let S be an index set such that $\sum_{i \in S} b_i = \sum_{i \notin S} b_i$ and $|S| = n/2$. By Observation 23 we may assume, without loss of generality, that $n-1 \in S$ and $n \notin S$. We define permutation π as follows. First, let $\pi(1) = n-1$, and $\pi(n) = n$. Also, assign the remaining $n/2 - 1$ members of S to odd indices. The remaining $n/2 - 1$ non-members of S are assigned to even indices.

Condition 1 is satisfied, since

$$\sum_{i:\pi(i) \text{ is odd}} b_i = \sum_{i \in S} b_i = \sum_{i \notin S} b_i = \sum_{i:\pi(i) \text{ is even}} b_i \ .$$

It remains to prove that Condition 2 is satisfied. If k is odd, we have that

$$\sum_{i=1}^{k} (-1)^{i-1} b_{\pi(k-i+1)} = \sum_{j=1}^{(k+1)/2} b_{2j-1} - \sum_{j=1}^{(k-1)/2} b_{2j} \geq [B + (k-1)/2 \cdot 2B] - [(k-1)/2 \cdot 2B + B/2] \geq B/2 \ .$$

Otherwise,

$$\sum_{i=1}^{k} (-1)^{i-1} b_{\pi(k-i+1)} = -\sum_{j=1}^{k/2} b_{2j-1} + \sum_{j=1}^{k/2} b_{2j} \geq -[B + (k/2-1) \cdot 2B + B/2] + [k/2 \cdot 2B] \geq B/2 \ .$$

The lemma follows. □

We are now ready for the Hardness result regarding bipartite multigraphs.

Theorem 26. *Let $\mathcal{F}$ be a family of bipartite graphs which contains all paths. It is NP-hard to decide if a degree sequence d admits a multigraph realization whose underlying graph is in $\mathcal{F}$.*

Proof: We prove the theorem using a reduction f from PARTITION'. The reduction is as follows: $d = f(b) = b$, if n is even and there exists $B > 0$ such that $b_i \geq 2B$, for every $i \in \{1, \ldots, n-2\}$, $b_{n-1} = b_n = B$, and $\sum_{i=1}^{n} b_i = (2n-1)B$. Otherwise, $d = f(b) = d_0$, where d_0 is a sequence that cannot be realized. Hence, we need to show that $b \in$ PARTITION$'$ if and only if d is realizable using an underlying graph from $\mathcal{F}$.

First assume that $b \in$ PARTITION$'$. In this case, $d = b$. Hence, there exists a sound permutation π for d. Define the following multigraph H whose underlying graph G is a path. The number of edges between $v_{\pi(k)}$ and $v_{\pi(k+1)}$ is

$$\sum_{i=1}^{k} (-1)^{i-1} d_{\pi(k-i+1)} \ .$$

Since π is sound, these numbers are positive. It is not hard to verify that H realizes d.

Now assume that d realizable by a graph G from $\mathcal{F}$. It follows that $d = f(b) = b$. It follows that n is even and there exists $B > 0$ such that $b_i \geq 2B$, for every $i \in \{1, \ldots, n-2\}$, $b_{n-1} = b_n = B$, and $\sum_{i=1}^{n} b_i = (2n-1)B$. Since $G \in \mathcal{F}$, it is bipartite, and thus there are two partitions L and R, such that

$$\sum_{j \in L} d_j = \sum_{j \in R} d_j \ .$$

Hence, $b \in$ PARTITION$'$. □

Corollary 27. *It is NP-hard to decide if a degree sequence d admits a multigraph realization whose underlying graph is a path, a caterpillar, a bounded-degree tree, a tree, a forest, and a connected bipartite graph.*

5.2 Computing all Balanced Partitions of a Degree Sequence

We describe an algorithm that given a degree sequence d, computes all balanced partitions of d. The algorithm relies on the self-reducibility of the SUBSET-SUM problem. Recall that in SUBSET-SUM the input is a sequence of numbers $(a_1, \ldots, a_n)$ and an additional number t, and the question is whether there is a subset S such that $\sum_{i \in S} a_i = t$. Let Subset-Sum-DP be a dynamic programming algorithm for SUBSET-SUM whose running time is denoted by $T_{DP}(a, t)$ (see, e.g., [10]). The running time of the dynamic programming algorithm can be bounded by $O(n \cdot \min\{\sum a, |\mathtt{BP}(a)|\})$.

In this section we abuse notation by presenting a sequence d as a sequence of q blocks, namely $d = (d_1^{n_1}, d_2^{n_2}, \ldots, d_q^{n_q})$. The algorithm for computing all balanced partitions of a sequence d is recursive, and it works as follows. The input is a suffix of d, i.e., $(d_k^{n_k}, \ldots, d_q^{n_q})$, represented by d and k, and a partition (L, R) of the prefix $(d_1^{n_1}, \ldots, d_{k-1}^{n_{k-1}})$. If the current suffix is empty, then it checks whether the current partition is balanced, and if it is balanced, then the partitioned is returned. Otherwise, it checks whether the current partition can be completed to a balanced partition. If the answer is YES, then the algorithm is invoked for the $n_k + 1$ options of adding the n_k copies of d_k to (L, R). The initial call is $(d, 1, \emptyset, \emptyset)$.

Algorithm 1: Partitions(d, k, L, R)

if $k = q + 1$ **then**
  if $(L, R) \in \mathtt{BP}(d)$ **then return** $\{(L, R)\}$;
else
  if *Subset-Sum-DP*$((d_k^{n_k}, \ldots, d_q^{n_q}), \sum L - \sum R) =$ *NO* **then return** $\emptyset$;
$\mathcal{P} \leftarrow \emptyset$
for $i = 0$ *to* n_k **do**
  $L' \leftarrow L \circ (d_k^i)$
  $R' \leftarrow L \circ (d_k^{n_k - i})$
  $\mathcal{P} \leftarrow \mathcal{P} \cup$ Partitions$(d, k+1, L', R')$
return $\mathcal{P}$

Lemma 28. *Algorithm Partitions returns all balanced partitions of d.*

Proof: Observe that each recursive call of the algorithm corresponds to a partition of a prefix of d. We prove that, given a prefix partition, the algorithm returns all of its balanced completions.

At the recursion base, if (L, R) is a partition of d, then it is returned if and only if $(L, R) \in \mathtt{BP}(d)$. For the inductive step, let (L, R) be a partition of the prefix $(d_1^{n_1}, \ldots, d_{k-1}^{n_{k-1}})$. If (L, R) gets a NO from Subset-Sum-DP, then it cannot be completed to a balanced partition, and indeed no partition that corresponds to the prefix (L, R) is returned. If (L, R) gets a YES, then all possible partitions of $(d_1^{n_1}, \ldots, d_k^{n_k})$ are checked. By the inductive hypothesis the algorithm returns all balanced partitions that complete $(d_1^{n_1}, \ldots, d_{k-1}^{n_{k-1}})$. □

The complexity of Algorithm Partitions is dominated by the total time spent on the invocations of Subset-Sum-DP, therefore we need to bound the number of invocations of Subset-Sum-DP. More specifically, we show the following bound.

Lemma 29. *Algorithm Partition invokes Subset-Sum-DP at most* $2n \cdot |\mathtt{BP}(d)|$ *times.*

Proof: Let us illustrate the recursive execution of the algorithm on d by a computation tree T consisting of $q+1$ levels. Each node in the tree is labeled by a triple (L, R, A), where $A \in \{YES, NO\}$, The first two entries in the label corresponds to the prefix partition (L, R) in the invocation, and the third corresponds to whether the partition can be completed to a balanced partition. Note that each such node corresponds to a single invocation of Subset-Sum-DP (or alternatively checking whether $\sum L = \sum R$ in level $q+1$). It follows that the number of invocations of Subset-Sum-DP is bounded by the size of the computation tree, not including level $q+1$.

We refer to a node as a YES-node (NO-node) if its label end with a YES (NO). Observe that all NO-nodes are leaves. On the other hand, there may be internal YES-nodes. If a YES-node is a leaf, then it corresponds to a balanced partition (L, R). Clearly, the number of YES-nodes in level $k+1$ of the tree is no less than the number of YES-nodes in level k. Moreover, a NO-node must have a YES-node as a sibling, hence the number of NO-nodes in level $k \le q$ is at most n_k times the number of YES-nodes in level k. Adding it all up we get:

$$\sum_{k=1}^{q} |\mathtt{BP}(d)| (n_k + 1) = (n+q) |\mathtt{BP}(d)| \le 2n \cdot |\mathtt{BP}(d)| \ .$$

□

Hence, the lemma allows us to get an upper bound on the time complexity of the algorithm.

Corollary 30. *The running time of Algorithm Partition is* $O(n^2 |\mathtt{BP}(d)| \min\{\sum d, |\mathtt{BP}(d)|\})$.

Clearly, the above running time becomes polynomial, if $|\mathtt{BP}(d)|$ is polynomial.

Due to Theorem 9, the minimum r such that a given partition is r-max-bigraphic can be computed efficiently implying the following result.

Corollary 31. *Let* d *be a degree sequence of length* n *such that* $|\mathtt{BP}(d)| = \mathcal{O}(n^c)$*, for some constant* c*. Then,* $\mathsf{MaxMult}^{bi}(d)$ *can be computed in polynomial time.*

Similarly, Theorem 11 implies the following.

Corollary 32. *Let* d *be a degree sequence of length* n *such that* $|\mathtt{BP}(d)| = \mathcal{O}(n^c)$*, for some constant* c*. Then,* $\mathsf{TotMult}^{bi}(d)$ *can be computed in polynomial time.*

We remark that a useful special subclass consists of sequences with a *constant* number of different degrees, since such a sequence can have at most polynomially many different partitions.

Corollary 33. *Let* q *be some constant and* $d = (d_1^{n_1}, d_2^{n_2}, \ldots, d_q^{n_q})$ *be a degree sequence, where* $n = \sum_{i=1}^{q} n_i$*. Then,* $\mathtt{BP}(d) = \mathcal{O}(n^c)$*, for some constant* c*.*

6 Small Maximum Degree Sequences

Towards attacking the realizability problem of general bigraphic sequences, we look at the question of bounding the total deviation of a nonincreasing sequence $d = (d_1, \ldots, d_n)$ as a function of its maximum degree, denoted $\Delta = d_1$.

Burstein and Rubin [7] consider the realization problem for directed graphs with loops, which is equivalent to BDR^P. (Directed edges go from the first partition to the second.) They give the following sufficient condition for a pair of sequences to be the in- and out-degrees of a directed graph with loops.

Theorem 34 (Burstein and Rubin [7]). *Consider a degree sequence d with a partition $(a,b) \in \mathtt{BP}(d)$ assuming that a and b have the same length p. Let $\sum a = \sum b = pc$ where c is the average degree. If $a_1 b_1 \le pc + 1$, then d is realizable by a directed graph with loops.*

In what follows we make use of the following straightforward technical claim which slightly strengthens a similar claim from [2].

Observation 35. *Consider a nonincreasing integer sequence $d = (d_1, \dots, d_k)$ of total sum $\sum d = D$. Then, $\sum_{i=1}^{\ell} d_i \ge \lceil \ell D/k \rceil$, for every $1 \le \ell \le k$.*

Proof: Since d is nonincreasing, $\frac{1}{\ell}\sum_{i=1}^{\ell} d_i \ge \frac{1}{k-\ell}\sum_{i=\ell+1}^{k} d_i$. Consequently,

$$D = \sum_{i=1}^{k} d_i = \sum_{i=1}^{\ell} d_i + \sum_{i=\ell+1}^{k} d_i \le \sum_{i=1}^{\ell} d_i + \frac{k-\ell}{\ell}\sum_{i=1}^{\ell} d_i = \frac{k}{\ell}\sum_{i=1}^{\ell} d_i \ ,$$

implying the claim. □

6.1 Bounding the Maximum Multiplicity

Theorem 34 is extended to bipartite multigraphs with bounded maximum multiplicity, i.e., to r-max-bigraphic sequences. The following is a slightly stronger version of Lemma 14 from [2].

Lemma 36. *Let r be a positive integer. Consider a degree sequence d of length n with a partition $(a,b) \in \mathtt{BP}(d)$. If $a_1 \cdot b_1 \le r \cdot \sum d/2 + r$, then (a,b) is r-max-bigraphic.*

Proof: Let r, d and (a,b) as in the lemma where $a = (a_1, a_2, \dots, a_p)$ and $b = (b_1, b_2, \dots, b_q)$. Moreover, let $X = \sum a = \sum b = \sum d/2$. To prove the claim, we assume that $a_1 \cdot b_1 \le r \cdot X + r$, and show that Equation (7) holds for a fixed index $\ell \in [p]$. The lemma then follows due to Theorem 9.

First, we consider the case where $b_1 \le \ell r$. Then, $\sum_{i=1}^{q} \min\{\ell r, b_i\} = X \ge \sum_{i=1}^{\ell} a_i$, and Equation (7) holds.

In the following, we assume that $\ell r < b_1$. Define the *conjugate* sequence of b as $\widetilde{b}_j = |\{b_i \mid b_i \ge j\}|$, for every j. Note that the conjugate sequence $\widetilde{b}$ of b is nonincreasing, and that $\sum_{j=1}^{\ell r} \widetilde{b}_j = \sum_{i=1}^{q} \min\{\ell r, b_i\}$. By Observation 35,

$$\sum_{i=1}^{q} \min\{\ell r, b_i\} \ge \lceil \ell r X/b_1 \rceil \ge \lceil \ell(a_1 b_1 - r)/b_1 \rceil = \lceil \ell a_1 - \ell r/b_1 \rceil = \ell a_1 \ .$$

As a is nonincreasing, we have that $\sum_{i=1}^{\ell} a_i \le \ell a_1 \le \sum_{i=1}^{q} \min\{\ell r, b_i\}$. The lemma follows. □

Lemma 37. *There exists a degree sequence d with a partition $(a,b) \in \mathtt{BP}(d)$, such that $a_1 \cdot b_1 = r \cdot \sum d/2 + r$, which is r-max-bigraphic, but not $(r-1)$-max-bigraphic.*

Proof: Consider the sequence $d = (q^{2k-1}, (q-1)^2)$ for positive integers q, k such that $q = r \cdot k$. This sequence has a unique partition $(a,b) \in \mathtt{BP}(d)$, where $a = b = (q^{k-1}, (q-1))$. One can verify that $a_1 b_1 = q^2$, while

$$r \cdot \sum d/2 + r = r(qk-1) + r = rqk = q^2 \ .$$

The partition (a, b) is r-max-bigraphic, but no better. □

Lemma 36 is stated for a given partition (BDR^P). For BDR, we immediately have the following which is a slight improvement over Corollary 16 form [2].

Corollary 38. *Let r be a positive integer and d be a partitionable degree sequence. If $d_1^2 \le r \cdot \sum d/2 + r$, then any partition $(a, b) \in \mathtt{BP}(d)$ is r-max-bigraphic.*

6.2 Bounding the Total Multiplicity

In this section, we establish results for total multiplicity analogous to those obtained in the previous section for the maximum multiplicity.

Lemma 39. *Let t be a positive integer. Consider a degree sequence d of length n with a partition $(a, b) \in \mathtt{BP}(d)$. If $a_1 \cdot b_1 \le \sum d/2 + t + 1$, then (a, b) is t-tot-bigraphic.*

Proof: Let t, d and (a, b) as in the lemma where $a = (a_1, a_2, \ldots, a_p)$ and $b = (b_1, b_2, \ldots, b_q)$, and let $X = \sum a = \sum b = \sum d/2$. To prove the claim, we assume that $a_1 \cdot b_1 \le X + t + 1$, and show that Equation (8) holds for every index $\ell \in [p]$. The lemma then follows due to Theorem 11.

First, consider the case where $\ell \ge b_1$. In this case,

$$\sum_{i=1}^{q} \min\{\ell, b_i\} = \sum b = X \ge \sum_{i=1}^{\ell} a_i ,$$

and Equation (8) holds.

Next, assume that $\ell < b_1$. Note that the conjugate sequence $\widetilde{b}$ of b is nonincreasing, and that $\sum_{j=1}^{\ell} \widetilde{b}_j = \sum_{i=1}^{q} \min\{\ell, b_i\}$. By Observation 35,

$$\sum_{i=1}^{q} \min\{\ell, b_i\} + t \ge \left\lceil \frac{\ell X}{b_1} \right\rceil + t \ge \left\lceil \frac{\ell(a_1 b_1 - t - 1)}{b_1} \right\rceil + t = \left\lceil \ell a_1 - \frac{\ell(t+1)}{b_1} \right\rceil + t \ge \ell a_1 .$$

As a is nonincreasing, we have that $\sum_{i=1}^{\ell} a_i \le \ell a_1 \le \sum_{i=1}^{q} \min\{\ell, b_i\} + t$. The lemma follows. □

The following lemma shows that the above bound it tight.

Lemma 40. *There exists a degree sequence d with a partition $(a, b) \in \mathtt{BP}(d)$, such that $a_1 \cdot b_1 = \sum d/2 + t + 2$, and (a, b) is not t-tot-bigraphic.*

Proof: Consider the sequence $d = (k^{2(k-1)}, 1^2)$, for a positive integer $k > 1$. This sequence has only one partition $(a, b) \in \mathtt{BP}(d)$, where $a = b = (k^{k-1}, 1)$. Observe that $a_1 b_1 = k^2$, while $\sum d/2 = k(k-1) + 1$.

Assume that $t = k - 2$. Hence, $a_1 b_1 = \sum d/2 + t + 1$. For every $\ell < k$, we have that

$$\sum_{i=1}^{k} \min\{\ell, b_i\} + t = k + (\ell - 1)(k - 1) + k - 2 = \ell k - \ell - 1 + k \ge \ell k = \sum_{i=1}^{\ell} a_i .$$

For $\ell = k$, we have

$$\sum_{i=1}^{k} \min\{\ell, b_i\} + t \ge \sum d/2 = \sum_{i=1}^{k} a_i .$$

Now assume that $t = k - 3$. Hence, $a_1 b_1 = \sum d/2 + t + 2$. For every $\ell < k$, we have that

$$\sum_{i=1}^{k} \min\{\ell, b_i\} + t = k + (\ell - 1)(k - 1) + k - 3 = \ell k - \ell - 2 + k \ .$$

If $\ell = k - 1$, we get that

$$\sum_{i=1}^{k} \min\{\ell, b_i\} + t = (k-1)k - (k-1) - 2 + k = (k-1)k - 1 < \sum_{i=1}^{\ell} a_i \ ,$$

which means that (a, b) is not t-tot-bigraphic. □

Similar to above, Lemma 40 (stated for BDR^P) implies the following for BDR.

Corollary 41. *Let* t *be a positive integer and* d *be a partitionable degree sequence. If* $d_1^2 \leq \sum d/2 + t + 1$*, then any partition* $(a, b) \in \mathtt{BP}(d)$ *is* t*-tot-bigraphic.*

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