

# Vertex-transitive nut graph order–degree existence problem

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A nut graph is a nontrivial simple graph whose adjacency matrix has a simple eigenvalue zero such that the corresponding eigenvector has no zero entries. It is known that the order  $n$  and degree  $d$  of a vertex-transitive nut graph satisfy  $4 \mid d$ ,  $d \geq 4$ ,  $2 \mid n$  and  $n \geq d + 4$ ; or  $d \equiv 2 \pmod{4}$ ,  $d \geq 6$ ,  $4 \mid n$  and  $n \geq d + 6$ . Here, we prove that for each such  $n$  and  $d$ , there exists a  $d$ -regular Cayley nut graph of order  $n$ . As a direct consequence, we obtain all the pairs  $(n, d)$  for which there is a  $d$ -regular vertex-transitive (resp. Cayley) nut graph of order  $n$ .

**Keywords:** nut graph, vertex-transitive graph, Cayley graph, circulant graph, regular graph, dihedral group, cyclic polynomial

## 1 Introduction

We consider all the graphs to be undirected, finite and simple, and use  $V(G)$  to denote the vertex set of a graph  $G$ . A *nut graph* is a nontrivial graph such that its adjacency matrix has a simple eigenvalue zero with the corresponding eigenvector having no zero entries. The nut graphs were introduced as a mathematical curiosity in the 1990s by Sciriha and Gutman [44–47, 53], while the chemical justification for studying these graphs was later discovered through a series of papers [25, 27, 51, 52]. An algorithm for generating nonisomorphic nut graphs was subsequently implemented by Coolsaet, Fowler and Goedgebeur [11], while the notion of nut graph was generalized to signed graphs [4] and directed graphs [8]. For more results on nut graphs, the reader is referred to [48, 49] and the monograph [50] by Sciriha and Farrugia.

A *vertex-transitive graph* is a graph  $G$  whose automorphism group acts transitively on  $V(G)$ . For any group  $\Gamma$  with the identity  $e$  and a subset  $C \subseteq \Gamma \setminus \{e\}$  closed under inversion, by  $\text{Cay}(\Gamma, C)$  we denote the graph  $G$  such that:

- (i)  $V(G) = \Gamma$ ; and
- (ii) any two vertices  $u, v \in \Gamma$  are adjacent if and only if  $vu^{-1} \in C$ .

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In this context, we refer to  $C$  as the corresponding connection set. A *Cayley graph* is a graph that is isomorphic to  $\text{Cay}(\Gamma, C)$  for some finite group  $\Gamma$  and connection set  $C$ . A *circulant graph* is a graph that has an automorphism with a single orbit, or equivalently, it is a Cayley graph where the group  $\Gamma$  is cyclic.

Here, we consider several realizability problems concerning the existence of  $d$ -regular nut graphs of order  $n$  belonging to a certain class, for given parameters  $d$  and  $n$ . To this end, for any  $d \in \mathbb{N}_0$ , let  $\mathfrak{N}_d^{\text{reg}}$  be the set of all the  $n \in \mathbb{N}$  for which there exists a  $d$ -regular nut graph of order  $n$ . Similarly, let  $\mathfrak{N}_d^{\text{VT}}$  (resp.  $\mathfrak{N}_d^{\text{Cay}}$ ,  $\mathfrak{N}_d^{\text{circ}}$ ) be the set of all the orders attainable by a  $d$ -regular vertex-transitive (resp. Cayley, circulant) nut graph. Clearly,

$$\mathfrak{N}_d^{\text{circ}} \subseteq \mathfrak{N}_d^{\text{Cay}} \subseteq \mathfrak{N}_d^{\text{VT}} \subseteq \mathfrak{N}_d^{\text{reg}}$$

holds for each  $d \in \mathbb{N}_0$ . We also trivially observe that  $\mathfrak{N}_0^{\text{reg}} = \mathfrak{N}_1^{\text{reg}} = \mathfrak{N}_2^{\text{reg}} = \emptyset$ .

The study of regular nut graphs was initiated by Gauci, Pisanski and Sciriha through the following order–degree existence problem.

**Problem 1.1** ([30, Problem 12]). For each degree  $d$ , determine the set  $\mathfrak{N}_d^{\text{reg}}$ .

In the same paper, the next initial result was obtained.

**Theorem 1.2** ([30, Theorems 2 and 3]). *The following holds:*

$$\mathfrak{N}_3^{\text{reg}} = \{12\} \cup \{n \in \mathbb{N} : n \text{ is even and } n \geq 18\} \quad \text{and} \quad \mathfrak{N}_4^{\text{reg}} = \{8, 10, 12\} \cup \{n \in \mathbb{N} : n \geq 14\}.$$

This result was subsequently extended by Fowler, Gauci, Goedgebeur, Pisanski and Sciriha as follows.

**Theorem 1.3** ([26, Theorem 7]). *The following statements hold:*

- (i)  $\mathfrak{N}_5^{\text{reg}} = \{n \in \mathbb{N} : n \text{ is even and } n \geq 10\}$ ;
- (ii)  $\mathfrak{N}_6^{\text{reg}} = \{n \in \mathbb{N} : n \geq 12\}$ ;
- (iii)  $\mathfrak{N}_7^{\text{reg}} = \{n \in \mathbb{N} : n \text{ is even and } n \geq 12\}$ ;
- (iv)  $\mathfrak{N}_8^{\text{reg}} = \{12\} \cup \{n \in \mathbb{N} : n \geq 14\}$ ;
- (v)  $\mathfrak{N}_9^{\text{reg}} = \{n \in \mathbb{N} : n \text{ is even and } n \geq 16\}$ ;
- (vi)  $\mathfrak{N}_{10}^{\text{reg}} = \{n \in \mathbb{N} : n \geq 15\}$ ;
- (vii)  $\mathfrak{N}_{11}^{\text{reg}} = \{n \in \mathbb{N} : n \text{ is even and } n \geq 16\}$ .

Later on, the set  $\mathfrak{N}_{12}^{\text{reg}}$  was also determined by Bašić, Knor and Škrekovski.

**Theorem 1.4** ([5, Theorem 1.3]).  $\mathfrak{N}_{12}^{\text{reg}} = \{n \in \mathbb{N} : n \geq 16\}$ .

Fowler, Gauci, Goedgebeur, Pisanski and Sciriha initiated the vertex-transitive nut graph order–degree existence problem by posing the next question.

**Problem 1.5** ([26, Question 9]). For what pairs  $(n, d)$  does a vertex-transitive nut graph of order  $n$  and degree  $d$  exist?

In the same paper, the following necessary condition for Problem 1.5 was proved.

**Theorem 1.6** ([26, Theorem 10]). *Let  $G$  be a vertex-transitive nut graph on  $n$  vertices, of degree  $d$ . Then  $n$  and  $d$  satisfy the following conditions. Either  $d \equiv 0 \pmod{4}$ , and  $n \equiv 0 \pmod{2}$  and  $n \geq d + 4$ ; or  $d \equiv 2 \pmod{4}$ , and  $n \equiv 0 \pmod{4}$  and  $n \geq d + 6$ .*

The circulant and Cayley nut graphs, which both form a subclass of the vertex-transitive nut graphs, were then investigated through a series of papers [16–20], leading to the following two results.

**Theorem 1.7** ([17, Theorem 1.8]). *For each  $d \in \mathbb{N}_0$ , the set  $\mathfrak{N}_d^{\text{circ}}$  is given by*

$$\mathfrak{N}_d^{\text{circ}} = \begin{cases} \emptyset, & \text{if } d = 0 \text{ or } 4 \nmid d, \\ \{n \in \mathbb{N} : n \text{ is even and } n \geq d + 4\}, & \text{if } d \equiv 4 \pmod{8}, \\ \{14\} \cup \{n \in \mathbb{N} : n \text{ is even and } n \geq 18\}, & \text{if } d = 8, \\ \{n \in \mathbb{N} : n \text{ is even and } n \geq d + 6\}, & \text{if } 8 \mid d \text{ and } d \geq 16. \end{cases}$$

**Theorem 1.8** ([19, Corollaries 8 and 9]). *For each  $d \in \mathbb{N}$  such that  $4 \mid d$ , the sets  $\mathfrak{N}_d^{\text{VT}}$  and  $\mathfrak{N}_d^{\text{Cay}}$  are given by*

$$\mathfrak{N}_d^{\text{VT}} = \mathfrak{N}_d^{\text{Cay}} = \{n \in \mathbb{N} : n \text{ is even and } n \geq d + 4\}.$$

The closely related polycirculant nut graphs were studied in [1, 21, 22]. For other recent results concerning the automorphisms of nut graphs, the reader is referred to [2, 3, 6].

The following result on the degrees of regular and Cayley nut graphs was recently obtained.

**Theorem 1.9** ([7]). *The set  $\mathfrak{N}_d^{\text{reg}}$  is infinite for any  $d \geq 3$ , while the set  $\mathfrak{N}_d^{\text{Cay}}$  is infinite for any even  $d \geq 4$ .*

Here, we completely solve Problem 1.5 through a constructive approach by using Cayley nut graphs, thereby extending Theorems 1.8 and 1.9. Additionally, this gives an inverse result to Theorem 1.6 in the sense of showing that the necessary condition is sharp. Our main result is embodied in the following theorem.

**Theorem 1.10.** *For each  $d \in \mathbb{N}_0$ , the sets  $\mathfrak{N}_d^{\text{VT}}$  and  $\mathfrak{N}_d^{\text{Cay}}$  are given by*

$$\mathfrak{N}_d^{\text{VT}} = \mathfrak{N}_d^{\text{Cay}} = \begin{cases} \emptyset, & \text{if } d \text{ is odd or } d < 4, \\ \{n \in \mathbb{N} : n \text{ is even and } n \geq d + 4\}, & \text{if } 4 \mid d \text{ and } d \geq 4, \\ \{n \in \mathbb{N} : 4 \mid n \text{ and } n \geq d + 6\}, & \text{if } d \equiv 2 \pmod{4} \text{ and } d \geq 6. \end{cases}$$

As it turns out, the necessary condition from Theorem 1.6 for the existence of a  $d$ -regular vertex-transitive nut graph of order  $n$  is also sufficient, apart from the trivial case when  $d = 0$  or  $d = 2$ .

In the rest of the paper, our main focus is to prove Theorem 1.10. In Section 2, we overview the theory necessary to carry out the proof. Afterwards, in Section 3, we obtain several results on the divisibility of four auxiliary families of polynomials by the cyclotomic polynomials. Finally, in Section 4, we rely on constructions of Cayley nut graphs based on dihedral groups to complete the proof of Theorem 1.10 and end the paper with a brief conclusion in Section 5. The proof of several results from Section 3 is completed through a computer-assisted approach by using the `Python` and `SageMath` [54] scripts that can be found in [15].

## 2 Preliminaries

For any graph  $G$ , let  $A(G)$  denote the adjacency matrix of  $G$  and let  $\sigma(G)$  be the spectrum of  $A(G)$ , regarded as a multiset. Also, let  $\overline{G}$  denote the complement of a graph  $G$ . We will need the following well-known result; for the proof, see the standard literature on spectral graph theory [9, 10, 12–14].

**Lemma 2.1.** *Let  $G$  be a regular graph of order  $n$  with  $\sigma(G) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ , where  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . Then*

$$\sigma(\overline{G}) = \{n-1-\lambda_1, -1-\lambda_n, -1-\lambda_{n-1}, \dots, -1-\lambda_2\}.$$

Given a graph  $G$ , let  $\eta(G)$  denote the multiplicity of zero as an eigenvalue of  $A(G)$ . The following property of vertex-transitive graphs is well known and follows directly from [12, p. 135].

**Lemma 2.2.** *Let  $G$  be a vertex-transitive graph. Then  $\eta(G) = 1$  if and only if  $G$  is a nut graph.*

For each  $n \geq 3$ , we use  $\text{Dih}(n)$  to denote the dihedral group of order  $2n$ , i.e.,

$$\text{Dih}(n) = \langle r, s \mid r^n = s^2 = e, srs = r^{-1} \rangle.$$

Here,  $e$ ,  $r$  and  $s$  signify the identity, the rotation by  $\frac{2\pi}{n}$  and a reflection symmetry, respectively. Besides, for any  $n \in \mathbb{N}_0$ , we denote the identity matrix of order  $n$  by  $I_n$ , and for any  $m, n \in \mathbb{N}_0$ , we denote the zero matrix with  $m$  rows and  $n$  columns by  $O_{m,n}$ . When the matrix size is clear from context, we may drop the subscripts and write  $I$  or  $O$  for short. We resume with the next lemma.

**Lemma 2.3.** *For some  $n \in \mathbb{N}$  and each  $j = 0, 1, 2, 3$ , let  $A^{(j)}$  be the circulant matrix*

$$A^{(j)} = \begin{bmatrix} a_0^{(j)} & a_1^{(j)} & a_2^{(j)} & \cdots & a_{n-1}^{(j)} \\ a_{n-1}^{(j)} & a_0^{(j)} & a_1^{(j)} & \cdots & a_{n-2}^{(j)} \\ a_{n-2}^{(j)} & a_{n-1}^{(j)} & a_0^{(j)} & \cdots & a_{n-3}^{(j)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{(j)} & a_2^{(j)} & a_3^{(j)} & \cdots & a_0^{(j)} \end{bmatrix}.$$

Then the matrix given in the block form

$$\begin{bmatrix} A^{(0)} & A^{(1)} \\ A^{(2)} & A^{(3)} \end{bmatrix} \tag{1}$$

is similar to the direct sum

$$\bigoplus_{\zeta} \begin{bmatrix} P_0(\zeta) & P_1(\zeta) \\ P_2(\zeta) & P_3(\zeta) \end{bmatrix},$$

where

$$P_j(x) = a_0^{(j)} + a_1^{(j)}x + a_2^{(j)}x^2 + \cdots + a_{n-1}^{(j)}x^{n-1} \quad (j = 0, 1, 2, 3),$$

and  $\zeta$  ranges over the  $n$ -th roots of unity.

**Proof:** Let  $\omega = e^{2\pi i/n}$  and let  $U \in \mathbb{C}^{n \times n}$  be defined as

$$U_{k,\ell} = \omega^{(k-1)(\ell-1)} \quad (k, \ell = 1, 2, \dots, n).$$

Observe that  $UU^* = U^*U = nI_n$  and  $A^{(j)}U = UD^{(j)}$ , where

$$D^{(j)} = \text{diag}(P_j(1), P_j(\omega), P_j(\omega^2), \dots, P_j(\omega^{n-1})) \quad (j = 0, 1, 2, 3).$$

Therefore,

$$\begin{bmatrix} A^{(0)} & A^{(1)} \\ A^{(2)} & A^{(3)} \end{bmatrix} \begin{bmatrix} U & O \\ O & U \end{bmatrix} = \begin{bmatrix} A^{(0)}U & A^{(1)}U \\ A^{(2)}U & A^{(3)}U \end{bmatrix} = \begin{bmatrix} UD^{(0)} & UD^{(1)} \\ UD^{(2)} & UD^{(3)} \end{bmatrix} = \begin{bmatrix} U & O \\ O & U \end{bmatrix} \begin{bmatrix} D^{(0)} & D^{(1)} \\ D^{(2)} & D^{(3)} \end{bmatrix},$$

which implies that the matrix (1) is similar to

$$\begin{bmatrix} P_0(1) & 0 & \cdots & 0 & P_1(1) & 0 & \cdots & 0 \\ 0 & P_0(\omega) & \cdots & 0 & 0 & P_1(\omega) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_0(\omega^{n-1}) & 0 & 0 & \cdots & P_1(\omega^{n-1}) \\ P_2(1) & 0 & \cdots & 0 & P_3(1) & 0 & \cdots & 0 \\ 0 & P_2(\omega) & \cdots & 0 & 0 & P_3(\omega) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_2(\omega^{n-1}) & 0 & 0 & \cdots & P_3(\omega^{n-1}) \end{bmatrix}. \quad (2)$$

The result now follows by simultaneously rearranging the rows and columns of (2) in the order  $1, n+1, 2, n+2, 3, n+3, \dots, n, 2n$ .  $\square$

The *connection set* of a binary circulant matrix  $C \in \mathbb{R}^{n \times n}$  is the set comprising the integers  $j \in \mathbb{Z}_n$  such that  $C_{1,1+j} = 1$ , with the index addition being done modulo  $n$ . As a direct consequence to Lemmas 2.2 and 2.3, we obtain the following result on the nut property of Cayley graphs for the dihedral group.

**Lemma 2.4.** *For some  $n \geq 3$ , let  $G$  be the graph  $\text{Cay}(\text{Dih}(n), \{r^{a_1}, r^{a_2}, \dots, r^{a_k}, r^{b_1}s, r^{b_2}s, \dots, r^{b_\ell}s\})$ , where  $k, \ell \in \mathbb{N}_0$ ,  $1 \leq a_1 < a_2 < \dots < a_k < n$  and  $0 \leq b_1 < b_2 < \dots < b_\ell < n$ . Also, for each  $n$ -th root of unity  $\zeta$ , let*

$$A_\zeta = \begin{bmatrix} \sum_{j=1}^k \zeta^{a_j} & \sum_{j=1}^\ell \zeta^{-b_j} \\ \sum_{j=1}^\ell \zeta^{b_j} & \sum_{j=1}^k \zeta^{a_j} \end{bmatrix}.$$

*Then  $G$  is a nut graph if and only if exactly one of the  $A_\zeta$  matrices has a simple eigenvalue zero, while all the others are invertible.*

**Proof:** Observe that if we arrange the vertices of  $G$  as  $e, r, r^2, \dots, r^{n-1}, s, r^{-1}s, r^{-2}s, \dots, r^{-(n-1)}s$ , then  $A(G)$  has the form

$$\begin{bmatrix} C_0 & C_1 \\ C_2 & C_0 \end{bmatrix},$$

where  $C_0, C_1$  and  $C_2$  are the binary circulant matrices with the connection sets

$$\{a_1, a_2, \dots, a_k\}, \quad \{-b_1, -b_2, \dots, -b_\ell\} \quad \text{and} \quad \{b_1, b_2, \dots, b_\ell\},$$

respectively. Therefore, Lemma 2.3 implies that  $A(G)$  is similar to  $\bigoplus_{\zeta} A_{\zeta}$ , where  $\zeta$  ranges over the  $n$ -th roots of unity. By Lemma 2.2, we conclude that  $G$  is a nut graph if and only if exactly one of the  $A_{\zeta}$  matrices has a simple eigenvalue zero, while all the others have no eigenvalue zero.  $\square$

For any  $n \in \mathbb{N}$ , the *radical* of  $n$ , denoted by  $\text{rad}(n)$ , is the largest square-free positive divisor of  $n$ . For each  $n \in \mathbb{N}$ , the *cyclotomic polynomial*  $\Phi_n(x)$  is defined as

$$\Phi_n(x) = \prod_{\zeta} (x - \zeta),$$

where  $\zeta$  ranges over the primitive  $n$ -th roots of unity. It is known that for every  $n \in \mathbb{N}$ , the polynomial  $\Phi_n(x)$  has integer coefficients and is irreducible in  $\mathbb{Q}[x]$ ; see, e.g., [29, Chapter 33]. Therefore, any  $P(x) \in \mathbb{Q}[x]$  has a root that is a primitive  $n$ -th root of unity if and only if  $\Phi_n(x) \mid P(x)$ . The following result is also well known.

**Lemma 2.5.** *Suppose that  $p^2 \mid n$ , where  $n \in \mathbb{N}$  and  $p$  is a prime. Then  $\Phi_n(x) = \Phi_{n/p}(x^p)$ .*

As an immediate consequence of Lemma 2.5, we get the next corollary.

**Corollary 2.6.** *For any  $n \in \mathbb{N}$ , we have  $\Phi_n(x) = \Phi_{\text{rad}(n)}(x^{n/\text{rad}(n)})$ .*

We will frequently use Corollary 2.6 together with the following folklore lemma.

**Lemma 2.7** ([21, Lemma 18]). *Let  $V(x), W(x) \in \mathbb{Q}[x]$ ,  $W(x) \neq 0$ , be such that  $W(x) \mid V(x)$  and the powers of all the nonzero terms of  $W(x)$  are divisible by  $\beta \in \mathbb{N}$ . Also, for any  $j \in \{0, 1, \dots, \beta - 1\}$ , let  $V^{(\beta, j)}(x)$  be the polynomial comprising the terms of  $V(x)$  whose power is congruent to  $j$  modulo  $\beta$ . Then  $W(x) \mid V^{(\beta, j)}(x)$  for every  $j \in \{0, 1, \dots, \beta - 1\}$ .*

We end the section with the next theorem by Filaseta and Schinzel on the divisibility of lacunary polynomials by cyclotomic polynomials.

**Theorem 2.8** ([23, Theorem 2]). *Let  $P(x) \in \mathbb{Z}[x]$  have  $N$  nonzero terms and suppose that  $\Phi_n(x) \mid P(x)$  for some  $n \in \mathbb{N}$ . Suppose further that  $p_1, p_2, \dots, p_k$  are distinct primes satisfying*

$$\sum_{j=1}^k (p_j - 2) > N - 2.$$

*Let  $e_j$  be the largest exponent such that  $p_j^{e_j} \mid n$ . Then for at least one  $j \in \{1, 2, \dots, k\}$ , we have  $\Phi_m(x) \mid P(x)$ , where  $m = n/p_j^{e_j}$ .*

### 3 Auxiliary polynomials

In the present section, we investigate the divisibility of four auxiliary families of polynomials by the cyclotomic polynomials. More precisely, we are interested in the polynomials

$$\begin{aligned}
Q_t(x) &:= x^{4t+7} - x^{4t+5} - x^{4t+4} + 2x^{2t+4} + x^{2t+3} + x^{2t+2} + x^{2t} - 2x^{t+3} - x^2 - 1, \\
R_t(x) &:= x^{8t+15} + x^{8t+14} + x^{8t+11} - x^{8t+10} - x^{8t+8} + 2x^{6t+9} - x^{4t+15} - x^{4t+11} \\
&\quad - x^{4t+9} + 2x^{4t+8} - 2x^{4t+7} + x^{4t+6} + x^{4t+4} + x^{4t} - 2x^{2t+6} + x^7 + x^5 - x^4 - x - 1, \\
S_t(x) &:= x^{4t+13} + x^{4t+11} + x^{4t+10} + x^{4t+9} - x^{4t+8} - x^{2t+13} - x^{2t+10} - x^{2t+9} \\
&\quad + 3x^{2t+7} + x^{2t+5} - x^{2t+4} + x^{2t+3} - x^{2t+2} + x^{2t+1} - 2x^{t+6} + x^6 - x^5 - x - 1, \\
T_t(x) &:= x^{8t+27} + x^{8t+26} + x^{8t+25} + x^{8t+22} + x^{8t+20} + x^{8t+18} + x^{8t+17} - x^{8t+16} - x^{8t+15} \\
&\quad + 2x^{6t+15} - x^{4t+26} - x^{4t+25} + x^{4t+23} - x^{4t+21} - x^{4t+20} + x^{4t+19} - x^{4t+18} - x^{4t+17} \\
&\quad + 3x^{4t+14} - 3x^{4t+13} + x^{4t+10} + x^{4t+9} - x^{4t+8} + x^{4t+7} + x^{4t+6} - x^{4t+4} + x^{4t+2} \\
&\quad + x^{4t+1} - 2x^{2t+12} + x^{12} + x^{11} - x^{10} - x^9 - x^7 - x^5 - x^2 - x - 1,
\end{aligned}$$

for each  $t \in \mathbb{N}_0$ . The four subsections of this section correspond to the four families of polynomials that are being studied.

#### 3.1 $Q_t(x)$ polynomials

In this subsection we investigate the  $Q_t(x)$  polynomials and our main result is the following lemma.

**Lemma 3.1.** *For any  $t \in \mathbb{N}_0$ , we have  $\Phi_b(x) \nmid Q_t(x)$  for each  $b \geq 2$ .*

We begin with the next claim that can be conveniently proved via computer as shown in [15].

**Claim 3.2.** *For each  $\beta \geq 6$ , there exists an element of the sequence  $4t + 7, 4t + 5, 4t + 4, 2t + 4, 2t + 3, 2t + 2, 2t, t + 3, 2, 0$  with a unique remainder modulo  $\beta$ .*

We also need the following two auxiliary results.

**Claim 3.3.** *Suppose that for some  $t \in \mathbb{N}_0$  and  $b \in \mathbb{N}$ , we have  $\Phi_b(x) \mid Q_t(x)$ . Then  $\frac{b}{\text{rad}(b)} < 6$ .*

**Proof:** By way of contradiction, suppose that  $\frac{b}{\text{rad}(b)} \geq 6$ . By Corollary 2.6, it follows that the powers of all the nonzero terms of  $\Phi_b(x)$  are divisible by  $\frac{b}{\text{rad}(b)}$ . From Lemma 2.7 and Claim 3.2, we conclude that  $\Phi_b(x)$  divides a polynomial of the form  $cx^\alpha$  for some  $c \in \mathbb{Z} \setminus \{0\}$  and  $\alpha \in \mathbb{N}_0$ , which yields a contradiction.  $\square$

**Claim 3.4.** *For any  $t \in \mathbb{N}_0$  and prime  $p \geq 11$ , we have  $\Phi_p(x) \nmid Q_t(x)$ .*

**Proof:** By way of contradiction, suppose that  $\Phi_p(x) \mid Q_t(x)$ . Then  $\Phi_p(x)$  also divides the polynomial

$$\begin{aligned}
Q_t^{\text{mod } p}(x) &:= x^{(4t+7) \bmod p} - x^{(4t+5) \bmod p} - x^{(4t+4) \bmod p} + 2x^{(2t+4) \bmod p} \\
&\quad + x^{(2t+3) \bmod p} + x^{(2t+2) \bmod p} + x^{2t \bmod p} - 2x^{(t+3) \bmod p} - x^2 - 1.
\end{aligned}$$

Since  $\Phi_p(x) = \sum_{j=0}^{p-1} x^j$ , it follows that  $\deg Q_t^{\text{mod } p}(x) \leq p-1 = \deg \Phi_p(x)$ , hence  $Q_t^{\text{mod } p}(x) \equiv 0$  or there is a  $c \in \mathbb{Q} \setminus \{0\}$  such that  $Q_t^{\text{mod } p}(x) = c \Phi_p(x)$ . In the former case, Claim 3.2 yields a contradiction. In the latter case,  $Q_t^{\text{mod } p}(x)$  has exactly  $p$  nonzero terms, which is impossible because  $p \geq 11$ .  $\square$

We are now in a position to complete the proof of Lemma 3.1.

**Proof of Lemma 3.1:** By way of contradiction, suppose that  $\Phi_b(x) \mid Q_t(x)$  holds for some  $t \in \mathbb{N}_0$  and  $b \geq 2$ . By Claim 3.3, we have  $\frac{b}{\text{rad}(b)} < 6$ . If  $b$  has no prime factor below 11, then we can use Theorem 2.8 to repeatedly cancel out distinct prime factors of  $b$  until exactly one is left. Therefore,  $\Phi_p(x) \mid Q_t(x)$  holds for some prime  $p \geq 11$ , which yields a contradiction due to Claim 3.4.

Now, suppose that  $b$  has a prime factor below 11. In this case, Theorem 2.8 can be used to cancel out all the prime factors of  $b$  above seven. Hence,  $\Phi_{b'}(x) \mid Q_t(x)$  holds for some  $b' \geq 2$  whose prime factors belong to  $\{2, 3, 5, 7\}$  and such that  $\frac{b'}{\text{rad}(b')} < 6$ . Note that there are finitely many such numbers. Besides, by Theorem 2.8, we can assume without loss of generality that the distinct prime factors  $p_1, p_2, \dots, p_k$  of  $b'$  satisfy  $\sum_{j=1}^k (p_j - 2) \leq 8$ . Observe that  $\Phi_{b'}(x) \mid Q_t(x)$  holds if and only if the polynomial

$$\begin{aligned} Q_t^{\text{mod } b'}(x) := & x^{(4t+7) \bmod b'} - x^{(4t+5) \bmod b'} - x^{(4t+4) \bmod b'} + 2x^{(2t+4) \bmod b'} \\ & + x^{(2t+3) \bmod b'} + x^{(2t+2) \bmod b'} + x^{2t \bmod b'} - 2x^{(t+3) \bmod b'} - x^2 - 1 \end{aligned}$$

is divisible by  $\Phi_{b'}(x)$ . With this in mind, we can obtain a contradiction by going through all the feasible numbers  $b'$  and then verifying that  $\Phi_{b'}(x) \nmid Q_t^{\text{mod } b'}(x)$  holds for each  $t \in \{0, 1, 2, \dots, b' - 1\}$ . This can be done, e.g., via a SageMath script, as shown in [15].  $\square$

### 3.2 $R_t(x)$ polynomials

Here, we focus on proving the next lemma.

**Lemma 3.5.** *For any  $t \in \mathbb{N}_0$ , we have  $\Phi_b(x) \nmid R_t(x)$  for each  $b \geq 3$ .*

The following result can be proved, e.g., by using a Python script, as shown in [15].

**Claim 3.6.** *For each  $\beta \geq 11$ , there exists an element of the sequence*

$$\begin{aligned} & 8t + 15, 8t + 14, 8t + 11, 8t + 10, 8t + 8, 6t + 9, 4t + 15, \\ & 4t + 11, 4t + 9, 4t + 8, 4t + 7, 4t + 6, 4t + 4, 4t, 2t + 6, 7, 5, 4, 1, 0 \end{aligned}$$

*with a unique remainder modulo  $\beta$ .*

The next claim can now be proved analogously to Claim 3.3.

**Claim 3.7.** *Suppose that for some  $t \in \mathbb{N}_0$  and  $b \in \mathbb{N}$ , we have  $\Phi_b(x) \mid R_t(x)$ . Then  $\frac{b}{\text{rad}(b)} < 11$ .*

We move to the following two auxiliary claims.

**Claim 3.8.** *Suppose that for some  $t \in \mathbb{N}_0$  and  $b \in \mathbb{N}$ , we have  $\Phi_b(x) \mid R_t(x)$ . Then  $2^2 \nmid b$ .*

**Proof:** By way of contradiction, suppose that  $2^2 \mid b$ . By Lemma 2.5, the powers of all the nonzero terms of  $\Phi_b(x)$  are even. From Lemma 2.7, we conclude that the polynomial

$$x^{8t+14} - x^{8t+10} - x^{8t+8} + 2x^{4t+8} + x^{4t+6} + x^{4t+4} + x^{4t} - 2x^{2t+6} - x^4 - 1$$

has a root that is a primitive  $b$ -th root of unity. Therefore,  $Q_t(x)$  has a primitive  $\frac{b}{2}$ -th root of unity among its roots, which yields a contradiction due to Lemma 3.1.  $\square$

**Claim 3.9.** For any  $t \in \mathbb{N}_0$  and prime  $p \geq 23$ , we have  $\Phi_p(x) \nmid R_t(x)$  and  $\Phi_{2p}(x) \nmid R_t(x)$ .

**Proof:** It can be proved analogously to Claim 3.4 that  $\Phi_p(x) \nmid R_t(x)$ . Now, by way of contradiction, suppose that  $\Phi_{2p}(x) \mid R_t(x)$ . Then  $\Phi_{2p}(x)$  divides the polynomial

$$\begin{aligned} R_t^{\text{mod } 2p}(x) := & (-1)^{\lfloor \frac{8t+15}{p} \rfloor} x^{(8t+15) \bmod p} + (-1)^{\lfloor \frac{8t+14}{p} \rfloor} x^{(8t+14) \bmod p} + (-1)^{\lfloor \frac{8t+11}{p} \rfloor} x^{(8t+11) \bmod p} \\ & - (-1)^{\lfloor \frac{8t+10}{p} \rfloor} x^{(8t+10) \bmod p} - (-1)^{\lfloor \frac{8t+8}{p} \rfloor} x^{(8t+8) \bmod p} + 2(-1)^{\lfloor \frac{6t+9}{p} \rfloor} x^{(6t+9) \bmod p} \\ & - (-1)^{\lfloor \frac{4t+15}{p} \rfloor} x^{(4t+15) \bmod p} - (-1)^{\lfloor \frac{4t+11}{p} \rfloor} x^{(4t+11) \bmod p} - (-1)^{\lfloor \frac{4t+9}{p} \rfloor} x^{(4t+9) \bmod p} \\ & + 2(-1)^{\lfloor \frac{4t+8}{p} \rfloor} x^{(4t+8) \bmod p} - 2(-1)^{\lfloor \frac{4t+7}{p} \rfloor} x^{(4t+7) \bmod p} + (-1)^{\lfloor \frac{4t+6}{p} \rfloor} x^{(4t+6) \bmod p} \\ & + (-1)^{\lfloor \frac{4t+4}{p} \rfloor} x^{(4t+4) \bmod p} + (-1)^{\lfloor \frac{4t}{p} \rfloor} x^{4t \bmod p} - 2(-1)^{\lfloor \frac{2t+6}{p} \rfloor} x^{(2t+6) \bmod p} \\ & + x^7 + x^5 - x^4 - x - 1. \end{aligned}$$

Since  $\Phi_{2p}(x) = \sum_{j=0}^{p-1} (-x)^j$ , we have  $\deg R_t^{\text{mod } 2p}(x) \leq p-1 = \deg \Phi_{2p}(x)$ , which means that  $R_t^{\text{mod } 2p}(x) \equiv 0$  or there is a  $c \in \mathbb{Q} \setminus \{0\}$  such that  $R_t^{\text{mod } 2p}(x) = c \Phi_{2p}(x)$ . In the former case, Claim 3.6 yields a contradiction, while in the latter case,  $R_t^{\text{mod } 2p}(x)$  has exactly  $p$  nonzero terms, which is not possible since  $p \geq 23$ .  $\square$

The proof of Lemma 3.5 can now be finalized.

**Proof of Lemma 3.5:** By way of contradiction, suppose that  $\Phi_b(x) \mid R_t(x)$  holds for some  $t \in \mathbb{N}_0$  and  $b \geq 3$ . Claims 3.7 and 3.8 imply that  $\frac{b}{\text{rad}(b)} < 11$  and  $2^2 \nmid b$ . If  $b$  has no prime factor from  $\{3, 5, 7, 11, 13, 17, 19\}$ , then Theorem 2.8 can be used to repeatedly cancel out distinct prime factors of  $b$  that are above 19 until only one such divisor is left. Therefore,  $\Phi_p(x) \mid R_t(x)$  or  $\Phi_{2p}(x) \mid R_t(x)$  holds for some prime  $p \geq 23$ , which is impossible due to Claim 3.9.

Now, suppose that  $b$  has a prime factor from  $\{3, 5, 7, 11, 13, 17, 19\}$ . In this case, Theorem 2.8 can be applied to cancel out all the prime factors of  $b$  above 19, which implies that  $\Phi_{b'}(x) \mid R_t(x)$  holds for some  $b' \geq 3$  whose prime factors are at most 19 and such that  $\frac{b'}{\text{rad}(b')} < 11$  and  $2^2 \nmid b'$ . Also, by Theorem 2.8, we can assume without loss of generality that the distinct prime factors  $p_1, p_2, \dots, p_k$  of  $b'$  satisfy  $\sum_{j=1}^k (p_j - 2) \leq 18$ . The rest of the proof can be carried out via computer analogously to Lemma 3.1; see [15].  $\square$

### 3.3 $S_t(x)$ polynomials

In the present subsection we study the divisibility of the  $S_t(x)$  polynomials by cyclotomic polynomials and obtain the next result.

**Lemma 3.10.** *For any  $t \in \mathbb{N}_0$ , we have  $\Phi_b(x) \nmid S_t(x)$  for each  $b \geq 2$ .*

By analogy, we start with the following claim that can be proved via a computer-assisted approach; see [15].

**Claim 3.11.** *For each  $\beta \geq 8$ , there exists an element of the sequence*

$$4t + 13, 4t + 11, 4t + 10, 4t + 9, 4t + 8, 2t + 13, 2t + 10, \\ 2t + 9, 2t + 7, 2t + 5, 2t + 4, 2t + 3, 2t + 2, 2t + 1, t + 6, 6, 5, 1, 0$$

*with a unique remainder modulo  $\beta$ .*

The next two results can be proved analogously to Claims 3.3 and 3.4, respectively.

**Claim 3.12.** *Suppose that for some  $t \in \mathbb{N}_0$  and  $b \in \mathbb{N}$ , we have  $\Phi_b(x) \mid S_t(x)$ . Then  $\frac{b}{\text{rad}(b)} < 8$ .*

**Claim 3.13.** *For any  $t \in \mathbb{N}_0$  and prime  $p \geq 23$ , we have  $\Phi_p(x) \nmid S_t(x)$ .*

We can now prove Lemma 3.10 as follows.

**Proof of Lemma 3.10:** By way of contradiction, suppose that  $\Phi_b(x) \mid S_t(x)$  holds for some  $t \in \mathbb{N}_0$  and  $b \geq 2$ . From Claim 3.12, we obtain  $\frac{b}{\text{rad}(b)} < 8$ . If  $b$  has no prime factor below 23, then by repeated use of Theorem 2.8, we conclude that  $\Phi_p(x) \mid S_t(x)$  is satisfied for some prime  $p \geq 23$ . However, by Claim 3.13, this is not possible.

Now, suppose that  $b$  has a prime factor below 23. By virtue of Theorem 2.8, we can cancel out all the prime factors of  $b$  above 19. Therefore,  $\Phi_{b'}(x) \mid S_t(x)$  holds for some  $b' \geq 2$  whose prime factors are at most 19 and such that  $\frac{b'}{\text{rad}(b')} < 8$ . By Theorem 2.8, we can also assume without loss of generality that the distinct prime factors  $p_1, p_2, \dots, p_k$  of  $b'$  satisfy  $\sum_{j=1}^k (p_j - 2) \leq 17$ . Since there are finitely many such numbers  $b'$ , the proof can be completed analogously to Lemmas 3.1 and 3.5, e.g., via a SageMath script, as shown in [15].  $\square$

### 3.4 $T_t(x)$ polynomials

We finish the section with the following lemma concerning the  $T_t(x)$  polynomials.

**Lemma 3.14.** *For any  $t \in \mathbb{N}_0$ , we have  $\Phi_b(x) \nmid T_t(x)$  for each  $b \geq 3$ .*

By analogy, we can obtain the next result, so we omit its proof.

**Claim 3.15.** *For each  $\beta \geq 20$ , there exists an element of the sequence*

$$8t + 27, 8t + 26, 8t + 25, 8t + 22, 8t + 20, 8t + 18, 8t + 17, 8t + 16, 8t + 15, 6t + 15, 4t + 26, \\ 4t + 25, 4t + 23, 4t + 21, 4t + 20, 4t + 19, 4t + 18, 4t + 17, 4t + 14, 4t + 13, 4t + 10, \quad (3) \\ 4t + 9, 4t + 8, 4t + 7, 4t + 6, 4t + 4, 4t + 2, 4t + 1, 2t + 12, 12, 11, 10, 9, 7, 5, 2, 1, 0$$

*with a unique remainder modulo  $\beta$ .*

We resume with the following two claims.

**Claim 3.16.** *Suppose that for some  $t \in \mathbb{N}_0$  and  $b \in \mathbb{N}$ , we have  $\Phi_b(x) \mid T_t(x)$ . Then  $\frac{b}{\text{rad}(b)} < 13$ .*

**Proof:** Let  $\beta := \frac{b}{\text{rad}(b)}$  and by way of contradiction, suppose that  $\beta \geq 13$ . If  $\beta \geq 20$ , then we can reach a contradiction analogously to Claims 3.3, 3.7 and 3.12. Now, suppose that  $\beta \in \{13, 14, \dots, 19\}$ . In this case, a contradiction can be obtained via computer by showing that at least one of the following five statements is true for each such  $\beta$  and any possible value of  $t \bmod \beta$ ; see [15].

**Statement 1:** There is an element of (3) with a unique remainder modulo  $\beta$ .

If this is true, then Corollary 2.6 and Lemma 2.7 imply that  $\Phi_b(x)$  divides a polynomial of the form  $cx^\alpha$  for some  $c \in \mathbb{Z} \setminus \{0\}$  and  $\alpha \in \mathbb{N}_0$ , which is impossible.

**Statement 2:** There are two elements of (3) that form an equivalence class modulo  $\beta$ , so that exactly one of them is from  $\{6t + 15, 2t + 12\}$  or exactly one of them is from  $\{4t + 14, 4t + 13\}$ .

In this case, Corollary 2.6 and Lemma 2.7 imply that  $\Phi_b(x)$  divides a polynomial of the form  $c_1x^{\alpha_1} + c_2x^{\alpha_2}$ , where  $(|c_1|, |c_2|) \in \{(3, 2), (3, 1), (2, 1)\}$  and  $\alpha_1, \alpha_2 \in \mathbb{N}_0$ . If we let  $\zeta$  be a primitive  $b$ -th root of unity, then this means that some power of  $\zeta$  equals  $\pm \frac{3}{2}$  or  $\pm 3$  or  $\pm 2$ , yielding a contradiction.

**Statement 3:** There are three elements of (3) that form an equivalence class modulo  $\beta$ , so that two of them are not from  $\{6t + 15, 4t + 14, 4t + 13, 2t + 12\}$ , while the third is from  $\{4t + 14, 4t + 13\}$ .

Here, by Corollary 2.6 and Lemma 2.7, it follows that  $\Phi_b(x)$  divides a polynomial of the form  $c_1x^{\alpha_1} + c_2x^{\alpha_2} + c_3x^{\alpha_3}$ , where  $|c_1| = |c_2| = 1$ ,  $|c_3| = 3$  and  $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}_0$ . Let  $\zeta$  be a primitive  $b$ -th root of unity and note that

$$c_1\zeta^{\alpha_1-\alpha_3} + c_2\zeta^{\alpha_2-\alpha_3} = -c_3.$$

The contradiction follows by observing that

$$|c_1\zeta^{\alpha_1-\alpha_3} + c_2\zeta^{\alpha_2-\alpha_3}| \leq |c_1\zeta^{\alpha_1-\alpha_3}| + |c_2\zeta^{\alpha_2-\alpha_3}| = 1 + 1 < 3 = |-c_3|.$$

**Statement 4:**  $\beta = 13$  and the elements  $4t + 20$  and  $4t + 7$  form an equivalence class modulo  $\beta$ .

In this case, Corollary 2.6 and Lemma 2.7 give  $\Phi_b(x) \mid -x^{4t+20} + x^{4t+7}$ , i.e.,  $\Phi_b(x) \mid x^{13} - 1$ . Hence,  $b \mid 13$ , which contradicts  $\beta = 13$ .

**Statement 5:**  $\beta = 13$  and the elements  $4t + 21$  and  $4t + 8$  form an equivalence class modulo  $\beta$ .

Here, Corollary 2.6 and Lemma 2.7 give  $\Phi_b(x) \mid -x^{4t+21} - x^{4t+8}$ , i.e.,  $\Phi_b(x) \mid x^{13} + 1$ . Therefore,  $b \mid 26$ , which contradicts  $\beta = 13$ .  $\square$

**Claim 3.17.** Suppose that for some  $t \in \mathbb{N}_0$  and  $b \in \mathbb{N}$ , we have  $\Phi_b(x) \mid T_t(x)$ . Then  $2^2 \nmid b$ .

**Proof:** By way of contradiction, suppose that  $2^2 \mid b$ . In this case, Lemma 2.5 implies that the powers of all the nonzero terms of  $\Phi_b(x)$  are even. Therefore, by Lemma 2.7, the polynomial

$$\begin{aligned} & x^{8t+26} + x^{8t+22} + x^{8t+20} + x^{8t+18} - x^{8t+16} - x^{4t+26} - x^{4t+20} - x^{4t+18} \\ & + 3x^{4t+14} + x^{4t+10} - x^{4t+8} + x^{4t+6} - x^{4t+4} + x^{4t+2} - 2x^{2t+12} + x^{12} - x^{10} - x^2 - 1 \end{aligned}$$

has a primitive  $b$ -th root of unity among its roots. This means that  $S_t(x)$  has a root that is a primitive  $\frac{b}{2}$ -th root of unity, which cannot be possible due to Lemma 3.10.  $\square$

The next claim can be proved analogously to Claim 3.9, so we omit its proof.

**Claim 3.18.** *For any  $t \in \mathbb{N}_0$  and prime  $p \geq 41$ , we have  $\Phi_p(x) \nmid T_t(x)$  and  $\Phi_{2p}(x) \nmid T_t(x)$ .*

We are now in a position to finalize the proof of Lemma 3.14.

**Proof of Lemma 3.14:** By way of contradiction, suppose that  $\Phi_b(x) \mid T_t(x)$  holds for some  $t \in \mathbb{N}_0$  and  $b \geq 3$ . From Claims 3.16 and 3.17, we get  $\frac{b}{\text{rad}(b)} < 13$  and  $2^2 \nmid b$ . If  $b$  has no odd prime factor below 41, then we can apply Theorem 2.8 to repeatedly cancel out distinct prime factors of  $b$  that are above 37 until one such divisor is left. Therefore,  $\Phi_p(x) \mid T_t(x)$  or  $\Phi_{2p}(x) \mid T_t(x)$  holds for some prime  $p \geq 41$ , yielding a contradiction by virtue of Claim 3.18.

Now, suppose that  $b$  has an odd prime factor below 41. By using Theorem 2.8, we can cancel out all the prime factors of  $b$  above 37. With this in mind,  $\Phi_{b'}(x) \mid T_t(x)$  holds for some  $b' \geq 3$  whose prime factors are at most 37 and such that  $\frac{b'}{\text{rad}(b')} < 13$  and  $2^2 \nmid b'$ . Besides, by Theorem 2.8, we can assume without loss of generality that the distinct prime factors  $p_1, p_2, \dots, p_k$  of  $b'$  satisfy  $\sum_{j=1}^k (p_j - 2) \leq 36$ . The proof can now be conveniently completed, e.g., by using a SageMath script, as shown in [15].  $\square$

## 4 Main result

In this section, we finalize the proof of Theorem 1.10. Note that from Theorem 1.6 and  $\mathfrak{N}_0^{\text{reg}} = \mathfrak{N}_1^{\text{reg}} = \mathfrak{N}_2^{\text{reg}} = \emptyset$ , it follows that  $\mathfrak{N}_d^{\text{VT}} = \mathfrak{N}_d^{\text{Cay}} = \emptyset$  holds whenever  $d$  is odd or  $d < 4$ . Besides, Theorem 1.8 determines  $\mathfrak{N}_d^{\text{VT}}$  and  $\mathfrak{N}_d^{\text{Cay}}$  for the case when  $4 \mid d$  and  $d \geq 4$ . By virtue of Theorem 1.6, to complete the proof of Theorem 1.10, it suffices to prove the existence of a  $d$ -regular Cayley nut graph of order  $n$  for any parameters  $d$  and  $n$  such that  $d \equiv 2 \pmod{4}$ ,  $d \geq 6$ ,  $4 \mid n$  and  $n \geq d + 6$ . This can be accomplished by constructing Cayley nut graphs based on dihedral groups with the desired order and degree. We begin with the following two results.

**Proposition 4.1.** *For any  $t \in \mathbb{N}_0$  and even  $m \geq 4t + 8$ , the graph*

$$\text{Cay}(\text{Dih}(m), \{r^{\pm 1}, r^{\pm 2}, r^{\pm 3}, \dots, r^{\pm(2t+1)}\} \cup \{s, rs, r^4s, r^6s\} \cup \{r^8s, r^9s, r^{10}s, \dots, r^{4t+7}s\}) \quad (4)$$

*is an  $(8t + 6)$ -regular Cayley nut graph of order  $2m$ .*

**Proof:** Let

$$A_\zeta = \begin{bmatrix} \sum_{j=1}^{2t+1} (\zeta^j + \zeta^{-j}) & 1 + \zeta^{-1} + \zeta^{-4} + \zeta^{-6} + \sum_{j=8}^{4t+7} \zeta^{-j} \\ 1 + \zeta + \zeta^4 + \zeta^6 + \sum_{j=8}^{4t+7} \zeta^j & \sum_{j=1}^{2t+1} (\zeta^j + \zeta^{-j}) \end{bmatrix}$$

for each  $m$ -th root of unity  $\zeta$ . Observe that

$$A_1 = \begin{bmatrix} 4t + 2 & 4t + 4 \\ 4t + 4 & 4t + 2 \end{bmatrix}$$

is invertible, while

$$A_{-1} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

has a simple eigenvalue zero. Therefore, by Lemma 2.4, to complete the proof, it suffices to show that

$$\left( \sum_{j=1}^{2t+1} (\zeta^j + \zeta^{-j}) \right)^2 - \left( 1 + \zeta + \zeta^4 + \zeta^6 + \sum_{j=8}^{4t+7} \zeta^j \right) \left( 1 + \zeta^{-1} + \zeta^{-4} + \zeta^{-6} + \sum_{j=8}^{4t+7} \zeta^{-j} \right) = 0 \quad (5)$$

cannot hold for any  $m$ -th root of unity  $\zeta \neq 1, -1$ .

By way of contradiction, suppose that (5) holds for some  $m$ -th root of unity  $\zeta \neq 1, -1$ . If we multiply both sides of (5) by  $(\zeta - 1)^2$ , we get

$$\begin{aligned} & (\zeta^{2t+2} - \zeta + 1 - \zeta^{-2t-1})^2 - (\zeta^{4t+8} - \zeta^8 + \zeta^7 - \zeta^6 + \zeta^5 - \zeta^4 \\ & + \zeta^2 - 1)(\zeta - \zeta^{-1} + \zeta^{-3} - \zeta^{-4} + \zeta^{-5} - \zeta^{-6} + \zeta^{-7} - \zeta^{-4t-7}) = 0. \end{aligned} \quad (6)$$

By expanding (6) and multiplying both sides by  $\zeta^{4t+7}$ , it follows that

$$\begin{aligned} & -\zeta^{8t+16} + \zeta^{8t+14} - \zeta^{8t+12} + 2\zeta^{8t+11} - \zeta^{8t+10} + \zeta^{8t+9} - \zeta^{8t+8} - 2\zeta^{6t+10} + 2\zeta^{6t+9} + \zeta^{4t+16} \\ & - \zeta^{4t+15} + \zeta^{4t+12} - \zeta^{4t+11} + \zeta^{4t+10} - 3\zeta^{4t+9} + 4\zeta^{4t+8} - 3\zeta^{4t+7} + \zeta^{4t+6} - \zeta^{4t+5} \\ & + \zeta^{4t+4} - \zeta^{4t+1} + \zeta^{4t} + 2\zeta^{2t+7} - 2\zeta^{2t+6} - \zeta^8 + \zeta^7 - \zeta^6 + 2\zeta^5 - \zeta^4 + \zeta^2 - 1 = 0. \end{aligned} \quad (7)$$

By factorizing (7) accordingly, we obtain

$$\begin{aligned} & (1 - \zeta)(\zeta^{8t+15} + \zeta^{8t+14} + \zeta^{8t+11} - \zeta^{8t+10} - \zeta^{8t+8} + 2\zeta^{6t+9} - \zeta^{4t+15} - \zeta^{4t+11} - \zeta^{4t+9} \\ & + 2\zeta^{4t+8} - 2\zeta^{4t+7} + \zeta^{4t+6} + \zeta^{4t+4} + \zeta^{4t} - 2\zeta^{2t+6} + \zeta^7 + \zeta^5 - \zeta^4 - \zeta - 1) = 0. \end{aligned}$$

Since  $\zeta \neq 1, -1$ , the desired contradiction follows from Lemma 3.5.  $\square$

**Proposition 4.2.** For any  $t \in \mathbb{N}_0$  and even  $m \geq 4t + 14$ , the graph

$$\begin{aligned} & \text{Cay}(\text{Dih}(m), \{r^{\pm 1}, r^{\pm 2}, r^{\pm 3}, \dots, r^{\pm(2t+1)}\} \\ & \cup \{s, rs, r^2s, r^5s, r^7s, r^9s, r^{10}s\} \cup \{r^{13}s, r^{14}s, r^{15}s, \dots, r^{4t+13}s\}) \end{aligned}$$

is an  $(8t + 10)$ -regular Cayley nut graph of order  $2m$ .

**Proof:** Let

$$A_\zeta = \begin{bmatrix} \sum_{j=1}^{2t+1} (\zeta^j + \zeta^{-j}) & \sum_{j \in \{0,1,2,5,7,9,10\}} \zeta^{-j} + \sum_{j=13}^{4t+13} \zeta^{-j} \\ \sum_{j \in \{0,1,2,5,7,9,10\}} \zeta^j + \sum_{j=13}^{4t+13} \zeta^j & \sum_{j=1}^{2t+1} (\zeta^j + \zeta^{-j}) \end{bmatrix}$$

for each  $m$ -th root of unity  $\zeta$ . Observe that

$$A_1 = \begin{bmatrix} 4t+2 & 4t+8 \\ 4t+8 & 4t+2 \end{bmatrix}$$

is invertible, while

$$A_{-1} = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}$$

has a simple eigenvalue zero. By virtue of Lemma 2.4, to complete the proof, it is enough to show that

$$\left( \sum_{j=1}^{2t+1} (\zeta^j + \zeta^{-j}) \right)^2 - \left( \sum_{j \in \{0,1,2,5,7,9,10\}} \zeta^j + \sum_{j=13}^{4t+13} \zeta^j \right) \left( \sum_{j \in \{0,1,2,5,7,9,10\}} \zeta^{-j} + \sum_{j=13}^{4t+13} \zeta^{-j} \right) = 0 \quad (8)$$

does not hold for any  $m$ -th root of unity  $\zeta \neq 1, -1$ .

By way of contradiction, suppose that (8) holds for some  $m$ -th root of unity  $\zeta \neq 1, -1$ . By multiplying both sides of (8) by  $(\zeta - 1)^2$ , we obtain

$$(\zeta^{2t+2} - \zeta + 1 - \zeta^{-2t-1})^2 - (\zeta^{4t+14} - \zeta^{13} + \zeta^{11} - \zeta^9 + \zeta^8 - \zeta^7 + \zeta^6 - \zeta^5 + \zeta^3 - 1)(\zeta - \zeta^{-2} + \zeta^{-4} - \zeta^{-5} + \zeta^{-6} - \zeta^{-7} + \zeta^{-8} - \zeta^{-10} + \zeta^{-12} - \zeta^{-4t-13}) = 0. \quad (9)$$

If we expand (9) and multiply both sides by  $\zeta^{4t+13}$ , it follows that

$$\begin{aligned} & -\zeta^{8t+28} + \zeta^{8t+25} - \zeta^{8t+23} + \zeta^{8t+22} - \zeta^{8t+21} + \zeta^{8t+20} - \zeta^{8t+19} + 2\zeta^{8t+17} - \zeta^{8t+15} \\ & - 2\zeta^{6t+16} + 2\zeta^{6t+15} + \zeta^{4t+27} - \zeta^{4t+25} - \zeta^{4t+24} + \zeta^{4t+23} + \zeta^{4t+22} - 2\zeta^{4t+20} \\ & + 2\zeta^{4t+19} - \zeta^{4t+17} - 3\zeta^{4t+15} + 6\zeta^{4t+14} - 3\zeta^{4t+13} - \zeta^{4t+11} + 2\zeta^{4t+9} - 2\zeta^{4t+8} \\ & + \zeta^{4t+6} + \zeta^{4t+5} - \zeta^{4t+4} - \zeta^{4t+3} + \zeta^{4t+1} + 2\zeta^{2t+13} - 2\zeta^{2t+12} - \zeta^{13} + 2\zeta^{11} \\ & - \zeta^9 + \zeta^8 - \zeta^7 + \zeta^6 - \zeta^5 + \zeta^3 - 1 = 0. \end{aligned} \quad (10)$$

Now, by factorizing (10), we reach

$$\begin{aligned} & (1 - \zeta)(\zeta^{8t+27} + \zeta^{8t+26} + \zeta^{8t+25} + \zeta^{8t+22} + \zeta^{8t+20} + \zeta^{8t+18} + \zeta^{8t+17} - \zeta^{8t+16} - \zeta^{8t+15} \\ & + 2\zeta^{6t+15} - \zeta^{4t+26} - \zeta^{4t+25} + \zeta^{4t+23} - \zeta^{4t+21} - \zeta^{4t+20} + \zeta^{4t+19} - \zeta^{4t+18} \\ & - \zeta^{4t+17} + 3\zeta^{4t+14} - 3\zeta^{4t+13} + \zeta^{4t+10} + \zeta^{4t+9} - \zeta^{4t+8} + \zeta^{4t+7} + \zeta^{4t+6} - \zeta^{4t+4} \\ & + \zeta^{4t+2} + \zeta^{4t+1} - 2\zeta^{2t+12} + \zeta^{12} + \zeta^{11} - \zeta^{10} - \zeta^9 - \zeta^7 - \zeta^5 - \zeta^2 - \zeta - 1) = 0. \end{aligned}$$

Since  $\zeta \neq 1, -1$ , a contradiction follows from Lemma 3.14.  $\square$

From Propositions 4.1 and 4.2 we obtain the next two corollaries, respectively.

**Corollary 4.3.** *Suppose that  $d \geq 6$  is such that  $d \equiv 6 \pmod{8}$ . Then for any  $n \geq d + 10$  such that  $4 \mid n$ , there exists a  $d$ -regular Cayley nut graph of order  $n$ .*

**Corollary 4.4.** *Suppose that  $d \geq 6$  is such that  $d \equiv 2 \pmod{8}$ . Then for any  $n \geq d + 18$  such that  $4 \mid n$ , there exists a  $d$ -regular Cayley nut graph of order  $n$ .*

Therefore, to complete the proof of Theorem 1.10, it remains to show the existence of a  $d$ -regular Cayley nut graph of order  $n$  for the following two cases:

- (i)  $d \geq 6$ ,  $d \equiv 6 \pmod{8}$  and  $n = d + 6$ ; and
- (ii)  $d \geq 6$ ,  $d \equiv 2 \pmod{8}$  and  $n \in \{d + 6, d + 10, d + 14\}$ .

We cover all but finitely many of the remaining  $(n, d)$  pairs through the next three propositions.

**Proposition 4.5.** *For any  $d \geq 14$  such that  $d \equiv 2 \pmod{4}$ , the graph*

$$\overline{\text{Cay}(\text{Dih}(\frac{d+6}{2}), \{r^{\pm 2}, s, r^8 s, r^9 s\})} \quad (11)$$

*is a  $d$ -regular Cayley nut graph of order  $d + 6$ .*

**Proof:** By Lemmas 2.1 and 2.2, it follows that the graph (11) is a nut graph if and only if the graph

$$\text{Cay}(\text{Dih}(\frac{d+6}{2}), \{r^{\pm 2}, s, r^8 s, r^9 s\})$$

has a simple eigenvalue  $-1$ . Let  $m = \frac{d+6}{2}$  and for each  $m$ -th root of unity  $\zeta$ , let

$$A_\zeta = \begin{bmatrix} 1 + \zeta^2 + \zeta^{-2} & 1 + \zeta^{-8} + \zeta^{-9} \\ 1 + \zeta^8 + \zeta^9 & 1 + \zeta^2 + \zeta^{-2} \end{bmatrix}.$$

Since the approach from Lemma 2.4 can also be applied to graphs where loops are allowed, it suffices to prove that exactly one of the  $A_\zeta$  matrices has a simple eigenvalue zero, while all the others are invertible.

Since

$$A_1 = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

has a simple eigenvalue zero, it remains to verify that

$$(1 + \zeta^2 + \zeta^{-2})^2 - (1 + \zeta^8 + \zeta^9)(1 + \zeta^{-8} + \zeta^{-9}) = 0 \quad (12)$$

does not hold for any  $m$ -th root of unity  $\zeta \neq 1$ . By expanding (12) and multiplying both sides by  $\zeta^9$ , we obtain

$$-\zeta^{18} - \zeta^{17} + \zeta^{13} + 2\zeta^{11} - \zeta^{10} - \zeta^8 + 2\zeta^7 + \zeta^5 - \zeta - 1 = 0.$$

The desired conclusion now follows by verifying that the polynomial

$$-x^{18} - x^{17} + x^{13} + 2x^{11} - x^{10} - x^8 + 2x^7 + x^5 - x - 1$$

is not divisible by any cyclotomic polynomial  $\Phi_b(x)$  with  $b \geq 2$ . This can easily be done by using a SageMath script; see [15].  $\square$

**Proposition 4.6.** *For any  $d \geq 22$  such that  $d \equiv 2 \pmod{4}$ , the graph*

$$\overline{\text{Cay}(\text{Dih}(\frac{d+10}{2}), \{r^{\pm 2}, r^{\pm 4}, s, r^2 s, r^6 s, r^7 s, r^{15} s\})}$$

*is a  $d$ -regular Cayley nut graph of order  $d + 10$ .*

**Proof:** Let  $m = \frac{d+10}{2}$  and for each  $m$ -th root of unity  $\zeta$ , let

$$A_\zeta = \begin{bmatrix} 1 + \zeta^2 + \zeta^{-2} + \zeta^4 + \zeta^{-4} & 1 + \zeta^{-2} + \zeta^{-6} + \zeta^{-7} + \zeta^{-15} \\ 1 + \zeta^2 + \zeta^6 + \zeta^7 + \zeta^{15} & 1 + \zeta^2 + \zeta^{-2} + \zeta^4 + \zeta^{-4} \end{bmatrix}.$$

Analogously to Proposition 4.5, it is enough to prove that exactly one of the  $A_\zeta$  matrices has a simple eigenvalue zero, while all the others are invertible. Note that

$$A_1 = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$$

has a simple eigenvalue zero. Therefore, it remains to show that

$$(1 + \zeta^2 + \zeta^{-2} + \zeta^4 + \zeta^{-4})^2 - (1 + \zeta^2 + \zeta^6 + \zeta^7 + \zeta^{15})(1 + \zeta^{-2} + \zeta^{-6} + \zeta^{-7} + \zeta^{-15}) = 0 \quad (13)$$

cannot hold for any  $m$ -th root of unity  $\zeta \neq 1$ . By expanding (13) and multiplying both sides by  $\zeta^{15}$ , we get

$$-\zeta^{30} - \zeta^{28} - \zeta^{24} - \zeta^{22} + \zeta^{21} - \zeta^{20} + 2\zeta^{19} + 3\zeta^{17} - \zeta^{16} - \zeta^{14} + 3\zeta^{13} + 2\zeta^{11} - \zeta^{10} + \zeta^9 - \zeta^8 - \zeta^6 - \zeta^2 - 1 = 0.$$

Analogously to Proposition 4.5, the result can be obtained by performing a computer-assisted verification via SageMath, as shown in [15].  $\square$

**Proposition 4.7.** *For any  $d \geq 26$  such that  $d \equiv 2 \pmod{4}$ , the graph*

$$\overline{\text{Cay}(\text{Dih}(\frac{d+14}{2}), \{r^{\pm 2}, r^{\pm 4}, r^{\pm 7}, s, r^2s, r^6s, r^7s, r^{14}s, r^{17}s, r^{19}s\})}$$

*is a  $d$ -regular Cayley nut graph of order  $d + 14$ .*

**Proof:** Let  $m = \frac{d+14}{2}$  and for each  $m$ -th root of unity, let

$$A_\zeta = \begin{bmatrix} 1 + \zeta^2 + \zeta^{-2} + \zeta^4 + \zeta^{-4} + \zeta^7 + \zeta^{-7} & 1 + \zeta^{-2} + \zeta^{-6} + \zeta^{-7} + \zeta^{-14} + \zeta^{-17} + \zeta^{-19} \\ 1 + \zeta^2 + \zeta^6 + \zeta^7 + \zeta^{14} + \zeta^{17} + \zeta^{19} & 1 + \zeta^2 + \zeta^{-2} + \zeta^4 + \zeta^{-4} + \zeta^7 + \zeta^{-7} \end{bmatrix}.$$

Analogously to Propositions 4.5 and 4.6, it suffices to verify that exactly one of the  $A_\zeta$  matrices has a simple eigenvalue zero, while all the others are invertible. Since

$$A_1 = \begin{bmatrix} 7 & 7 \\ 7 & 7 \end{bmatrix}$$

has a simple eigenvalue zero, it remains to show that

$$(1 + \zeta^2 + \zeta^{-2} + \zeta^4 + \zeta^{-4} + \zeta^7 + \zeta^{-7})^2 - (1 + \zeta^2 + \zeta^6 + \zeta^7 + \zeta^{14} + \zeta^{17} + \zeta^{19})(1 + \zeta^{-2} + \zeta^{-6} + \zeta^{-7} + \zeta^{-14} + \zeta^{-17} + \zeta^{-19}) = 0 \quad (14)$$

does not hold for any  $m$ -th root of unity  $\zeta \neq 1$ . If we expand (14) and multiply both sides by  $\zeta^{19}$ , we obtain

$$\begin{aligned} & -\zeta^{38} - 2\zeta^{36} - \zeta^{34} - \zeta^{32} - 2\zeta^{31} + \zeta^{30} - \zeta^{29} + 2\zeta^{28} + \zeta^{25} + 2\zeta^{23} + \zeta^{22} + 2\zeta^{21} - \zeta^{20} \\ & - \zeta^{18} + 2\zeta^{17} + \zeta^{16} + 2\zeta^{15} + \zeta^{13} + 2\zeta^{10} - \zeta^9 + \zeta^8 - 2\zeta^7 - \zeta^6 - \zeta^4 - 2\zeta^2 - 1 = 0. \end{aligned}$$

Similarly to Propositions 4.5 and 4.6, the proof can be completed by using a SageMath script; see [15].  $\square$

With Propositions 4.5–4.7 in mind, it remains to verify the existence of a  $d$ -regular Cayley nut graph of order  $n$  for each  $(n, d) \in \{(12, 6), (16, 10), (20, 10), (24, 10), (28, 18), (32, 18)\}$ . This is not difficult to confirm through Tables 1 and 2, which arise by performing an exhaustive search over all the vertex-transitive graphs of order below 48; see [31, 43]. Besides, by using, e.g., SageMath, we can confirm that  $\text{Cay}(\text{Dih}(6), \{r^{\pm 1}, r^3, s, r^2s, r^3s\})$  is a 6-regular Cayley nut graph of order 12, while

$$\text{Cay}(\text{Dih}(m), \{r^{\pm 1}, r^{\pm 2}, r^{\pm 3}, s, r^2s, r^3s, r^4s\})$$

is a 10-regular Cayley nut graph of order  $2m$  for each  $m \in \{8, 10, 12\}$  and

$$\text{Cay}(\text{Dih}(m), \{r^{\pm 1}, r^{\pm 2}, r^{\pm 3}, r^{\pm 4}, r^{\pm 5}, s, r^2s, r^3s, r^4s, r^5s, r^6s, r^7s, r^8s\})$$

is an 18-regular Cayley nut graph of order  $2m$  for each  $m \in \{14, 16\}$ . These observations complete the proof of Theorem 1.10.

We mention in passing that every vertex-transitive nut graph of order 8, 12, 14, 22, 38 or 46 is a Cayley graph since none of the numbers 8, 12, 14, 22, 38 and 46 is a non-Cayley number; see [34–37] and the references therein. Although there exist non-Cayley vertex-transitive graphs of orders 10, 28 and 44, none of them is a nut graph, hence the corresponding entries of Tables 1 and 2 are again the same.

The circulant graphs  $\text{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 2\})$  and  $\text{Cay}(\mathbb{Z}_{10}, \{\pm 1, \pm 2\})$  are the unique 4-regular vertex-transitive nut graph of order 8 and 10, respectively; see Figure 1. Also, the noncirculant Cayley graphs

$$\text{Cay}(\text{Dih}(6), \{r^{\pm 1}, r^3, s, r^2s, r^3s\}) \quad \text{and} \quad \overline{\text{Cay}(\text{Dih}(6), \{r^{\pm 1}, s\})} \cong \overline{C_6} \square K_2$$

are the unique 6- and 8-regular vertex-transitive nut graph of order 12, respectively; see Figure 2. Moreover, there are exactly two 10-regular vertex-transitive nut graphs of order 16, one of which is the Cayley graph

$$\text{Cay}(\text{Dih}(8), \{r^{\pm 1}, r^{\pm 2}, r^{\pm 3}, s, r^2s, r^3s, r^4s\}) \cong \overline{\text{Cay}(\text{Dih}(8), \{r^4, rs, r^5s, r^6s, r^7s\})},$$

while the other is non-Cayley; see Figure 3. Observe that the graph from Figure 3a has a Möbius ladder [24] as a spanning subgraph, while the graph from Figure 3b contains two disjoint Möbius ladders of order 8.

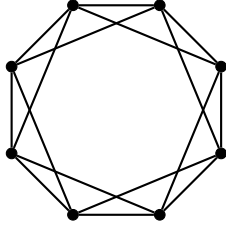
It is not difficult to prove that the graph  $\overline{\text{Cay}(\mathbb{Z}_{2m}, \{\pm 1, m\})}$ , whose complement is a Möbius ladder, is a  $(2m - 4)$ -regular nut graph of order  $2m$ , for any  $m \geq 4$  such that  $4 \mid m$ . Therefore, the graphs  $\overline{\text{Cay}(\mathbb{Z}_{16}, \{\pm 1, 8\})}$  and  $\overline{\text{Cay}(\mathbb{Z}_{32}, \{\pm 1, 16\})}$  are the unique 12-regular vertex-transitive nut graph of order 16 and 28-regular vertex-transitive nut graph of order 32, respectively, while  $\overline{\text{Cay}(\mathbb{Z}_{24}, \{\pm 1, 12\})}$  is one of the two 20-regular vertex-transitive nut graphs of order 24. The other 20-regular vertex-transitive nut graph of order 24 is also a Cayley graph and its complement is the Kronecker cover [39] of the Dürer graph [41]; see Figure 4b. Also, observe that there are exactly two 16-regular vertex-transitive nut graphs of order 20 and they are both Cayley graphs. Their complements are the prism graph  $C_{10} \square K_2$  and the cubic hamiltonian graph that can be described as  $[5, -5]^{10}$  using the exponential LCF notation [28]; see Figure 5.

$\begin{smallmatrix} n \\ d \end{smallmatrix}$	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46
4	1	1	2	2	4	2	7	4	7	5	8	6	8	7	12	8	15	9	14	10
6	0	0	1	0	4	0	15	0	39	0	55	0	72	0	105	0	224	0	265	0
8	—	0	1	3	9	14	46	38	182	92	337	180	1152	304	1501	476	3344	954	3921	1095
10	—	—	0	2	0	30	0	30	231	0	520	0	1826	0	4170	0	11046	0	17496	0
12	—	—	—	0	1	7	41	50	337	251	1042	1255	6353	2736	12971	7051	52394	23428	76948	34140
14	—	—	—	—	0	0	9	0	201	0	1085	0	8284	0	27709	0	116986	0	225700	0
16	—	—	—	—	—	0	2	9	104	147	1134	1293	16569	6607	53738	28713	298855	133196	673180	324689
18	—	—	—	—	—	—	0	0	18	0	418	0	7178	0	40935	0	320178	0	909468	0
20	—	—	—	—	—	—	—	0	2	13	164	389	4650	4192	37632	34901	381977	278017	1277372	1054239
22	—	—	—	—	—	—	—	—	0	0	27	0	1232	0	21720	0	287618	0	1379665	0
24	—	—	—	—	—	—	—	—	—	0	3	23	417	696	10411	12764	222069	211740	1417958	1244165
26	—	—	—	—	—	—	—	—	—	—	0	0	28	0	2862	0	81731	0	781098	0
28	—	—	—	—	—	—	—	—	—	—	—	0	1	23	566	1292	31537	43212	388524	537870
30	—	—	—	—	—	—	—	—	—	—	—	—	0	0	55	0	5809	0	159876	0
32	—	—	—	—	—	—	—	—	—	—	—	—	—	0	4	29	1198	2806	53256	81165
34	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0	0	80	0	9844	0
36	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0	3	54	1378	3790
38	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0	0	113	0
40	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0	5	43
42	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0	0
44	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
$\Sigma$	1	1	4	5	20	23	150	101	1121	508	4793	3146	47770	14565	214391	85234	1815064	693416	7376081	3281206

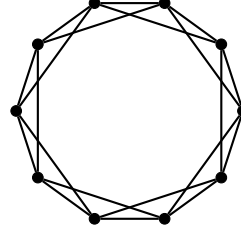
Tab. 1: The number of vertex-transitive nut graphs of a given order  $n$  and degree  $d$ .

$\begin{smallmatrix} n \\ d \end{smallmatrix}$	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46
4	1	1	2	2	3	2	7	4	7	5	8	5	7	7	12	8	15	9	14	10
6	0	0	1	0	3	0	14	0	37	0	55	0	66	0	102	0	216	0	265	0
8	—	0	1	3	9	13	44	38	181	86	337	179	1090	292	1485	476	3303	954	3921	1095
10	—	—	0	0	1	0	29	0	230	0	520	0	1764	0	4142	0	10938	0	17496	0
12	—	—	—	0	1	7	40	50	337	251	1042	1251	6211	2736	12948	7051	52194	23428	76948	34140
14	—	—	—	—	0	0	8	0	200	0	1085	0	8068	0	27663	0	116598	0	225700	0
16	—	—	—	—	—	0	2	9	103	141	1134	1289	16198	6557	53667	28713	298282	133196	673180	324689
18	—	—	—	—	—	—	0	0	18	0	418	0	6974	0	40913	0	319663	0	909468	0
20	—	—	—	—	—	—	—	0	2	13	164	386	4519	4192	37578	34901	381495	278017	1277372	1054239
22	—	—	—	—	—	—	—	—	0	0	27	0	1192	0	21677	0	287139	0	1379665	0
24	—	—	—	—	—	—	—	—	—	0	3	22	371	684	10396	12764	221593	211740	1417958	1244165
26	—	—	—	—	—	—	—	—	—	—	0	0	24	0	2833	0	81421	0	781098	0
28	—	—	—	—	—	—	—	—	—	—	—	0	1	23	560	1292	31405	43210	388524	537870
30	—	—	—	—	—	—	—	—	—	—	—	—	0	0	52	0	5755	0	159876	0
32	—	—	—	—	—	—	—	—	—	—	—	—	—	0	4	29	1179	2806	53256	81165
34	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0	0	77	0	9844	0
36	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0	3	54	1378	3790
38	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0	0	113	0
40	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0	5	43
42	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0	0
44	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0
$\Sigma$	1	1	4	5	17	22	144	101	1115	496	4793	3132	46485	14491	214032	85234	1811276	693414	7376081	3281206

Tab. 2: The number of Cayley nut graphs of a given order  $n$  and degree  $d$ .

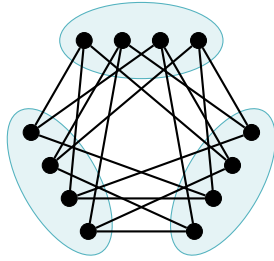


(a) The unique 4-regular vertex-transitive nut graph of order 8, which is isomorphic to  $\text{Cay}(\mathbb{Z}_8, \{\pm 1, \pm 2\})$ .

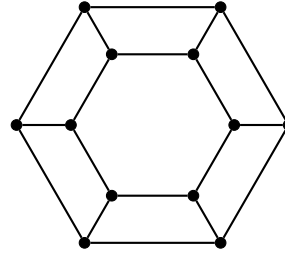


(b) The unique 4-regular vertex-transitive nut graph of order 10, which is isomorphic to  $\text{Cay}(\mathbb{Z}_{10}, \{\pm 1, \pm 2\})$ .

Fig. 1: The unique 4-regular vertex-transitive nut graph of order 8 and 10.

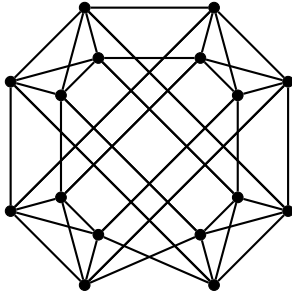


(a) The unique 6-regular vertex-transitive nut graph of order 12, which is isomorphic to  $\text{Cay}(\text{Dih}(6), \{r^{\pm 1}, r^3, s, r^2s, r^3s\})$ . The graph contains three cliques represented by shaded regions; edges within cliques are not drawn. Source: [6, Figure 11].

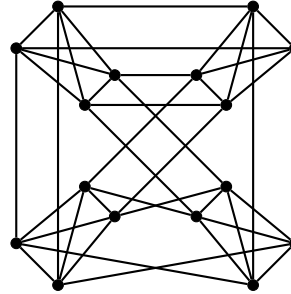


(b) The complement of the unique 8-regular vertex-transitive nut graph of order 12, which is isomorphic to  $\text{Cay}(\text{Dih}(6), \{r^{\pm 1}, s\})$ , i.e., the prism graph  $C_6 \square K_2$ . Source: [19, Figure 1].

Fig. 2: The unique 6- and 8-regular vertex-transitive nut graph of order 12 (drawn as the graph or its complement).



(a) The complement of the unique 10-regular Cayley nut graph of order 16, which is isomorphic to  $\text{Cay}(\text{Dih}(8), \{r^4, rs, r^5s, r^6s, r^7s\})$ .



(b) The complement of the unique 10-regular non-Cayley vertex-transitive nut graph of order 16.

Fig. 3: The complements of the only two 10-regular vertex-transitive nut graphs of order 16.

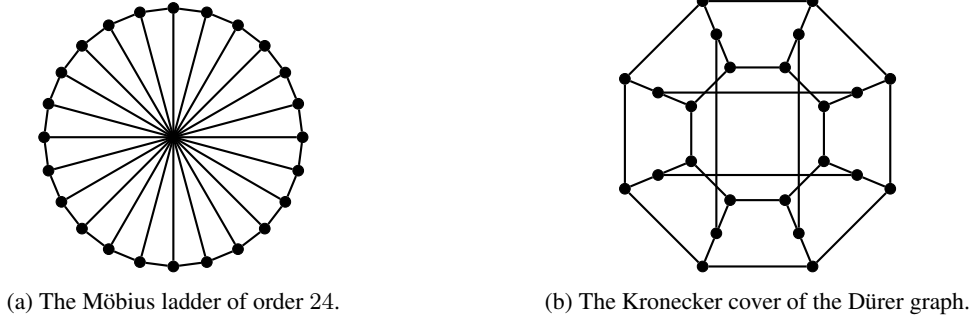


Fig. 4: The complements of the only two 20-regular vertex-transitive nut graphs of order 24, each of which is a Cayley graph.

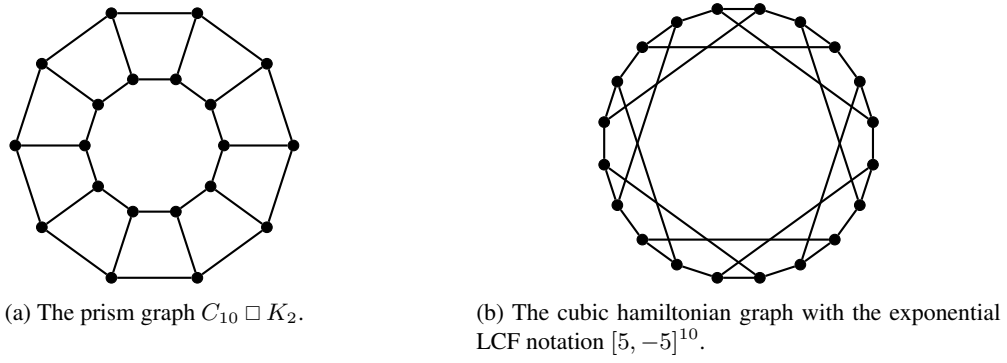


Fig. 5: The complements of the only two 16-regular vertex-transitive nut graphs of order 20, each of which is a Cayley graph.

## 5 Conclusion

Theorem 1.10 completely resolves the vertex-transitive (resp. Cayley) nut graph order–degree existence problem, thus providing the solution to Problem 1.5 and an inverse result for Theorem 1.6. Its results can be alternatively stated as follows.

**Corollary 5.1.** *For any  $n \in \mathbb{N}$  and  $d \in \mathbb{N}_0$ , there exists a  $d$ -regular vertex-transitive nut graph of order  $n$  if and only if:*

- (i)  $n$  and  $d$  are both even, with at least one of them divisible by four; and
- (ii)  $d \geq 4$  and  $n \geq d + 4$ .

**Corollary 5.2.** *For any  $n \in \mathbb{N}$  and  $d \in \mathbb{N}_0$ , there exists a  $d$ -regular Cayley nut graph of order  $n$  if and only if:*

- (i)  $n$  and  $d$  are both even, with at least one of them divisible by four; and
- (ii)  $d \geq 4$  and  $n \geq d + 4$ .

All the constructions used in Section 4 relied on Cayley graphs based on dihedral groups. In [19], it was shown that for any  $d \geq 8$  such that  $8 \mid d$ , the graph  $\overline{C_{\frac{d+4}{2}} \square K_2} \cong \text{Cay}(\text{Dih}(\frac{d+4}{2}), \{r^{\pm 1}, s\})$  is a  $d$ -regular Cayley nut graph of order  $d + 4$ . Besides, it is not difficult to verify by using, e.g., SageMath, that  $\text{Cay}(\text{Dih}(8), \{r^{\pm 1}, r^{\pm 2}, r^{\pm 3}, s, r^2 s\})$  is an 8-regular Cayley nut graph of order 16. With all of this in mind together with Theorem 1.7, we reach the next result.

**Theorem 5.3.** *Suppose that  $n \in \mathbb{N}$  and  $d \in \mathbb{N}_0$  are such that:*

- (i)  $n$  and  $d$  are both even, with at least one of them divisible by four; and
- (ii)  $d \geq 4$  and  $n \geq d + 4$ .

*Then there is a  $d$ -regular Cayley nut graph of order  $n$  for the cyclic or dihedral group.*

From here, we obtain the following conclusion.

**Corollary 5.4.** *The cyclic and dihedral groups are sufficient to construct Cayley nut graphs that cover all the possible combinations of orders and degrees attainable by a vertex-transitive nut graph.*

A *bicirculant graph* is a graph that has an automorphism with two orbits of equal size. These graphs are the derived graphs of  $\mathbb{Z}_m$ -voltage pregraphs of order two; see [40, Section 3.5] and [32, 33, 38, 42]. As shown in Lemma 2.4, the Cayley graphs for dihedral groups are a subclass of the bicirculant graphs. Therefore, it is natural to extend the investigation of the nut property to bicirculant graphs. To this end, we need the next proposition.

**Proposition 5.5.** *For any  $d$ -regular bicirculant nut graph of order  $n$ , the following holds:*

- (i)  $n$  and  $d$  are both even, with at least one of them divisible by four; and
- (ii)  $d \geq 4$  and  $n \geq d + 4$ .

**Proof:** Let  $G$  be a  $d$ -regular bicirculant nut graph of order  $n$ . Observe that  $A(G)$  has the form

$$\begin{bmatrix} C_0 & C_1^T \\ C_1 & C_2 \end{bmatrix},$$

where  $C_1$  is a binary circulant matrix, while  $C_0$  and  $C_2$  are binary symmetric circulant matrices with zero diagonal. Let  $n = 2m$ , so that  $C_0, C_1, C_2 \in \mathbb{R}^{m \times m}$ , and let  $S_0, S_1$  and  $S_2$  be the connection sets of  $C_0, C_1$  and  $C_2$ , respectively. Now, for each  $m$ -th root of unity  $\zeta$ , let

$$A_\zeta = \begin{bmatrix} \sum_{j \in S_0} \zeta^j & \sum_{j \in S_1} \zeta^{-j} \\ \sum_{j \in S_1} \zeta^j & \sum_{j \in S_2} \zeta^j \end{bmatrix}.$$

By arguing analogously to Lemma 2.4, it follows that exactly one of the  $A_\zeta$  matrices has a simple eigenvalue zero, while all the others are invertible.

Let  $\zeta_0$  be the unique  $m$ -th root of unity such that  $A_{\zeta_0}$  has an eigenvalue zero. We trivially observe that  $\zeta_0$  must be real, since otherwise both  $A_{\zeta_0}$  and  $A_{\bar{\zeta}_0}$  would have an eigenvalue zero. Note that  $d = |S_0| + |S_1| = |S_2| + |S_1|$  and

$$A_1 = \begin{bmatrix} |S_0| & |S_1| \\ |S_1| & |S_0| \end{bmatrix}.$$

Regardless of whether  $\zeta_0 = 1$  or  $\zeta_0 = -1$ , it is not difficult to see that  $|S_0|$  and  $|S_1|$  are of the same parity, which means that  $d$  is even. Since  $\mathfrak{N}_0^{\text{reg}} = \mathfrak{N}_2^{\text{reg}} = \emptyset$ , we get  $d \geq 4$ . Also, the only  $d$ -regular graph of order  $d + 2$  is  $\frac{d+2}{2}K_2$ , hence

$$\eta\left(\frac{d+2}{2}K_2\right) = \frac{d+2}{2} > 1$$

implies that  $n \geq d + 4$ .

It remains to prove that  $4 \mid n$  or  $4 \mid d$ . By way of contradiction, suppose that  $m$  is odd and  $d \equiv 2 \pmod{4}$ . In this case,  $-1$  is not an  $m$ -th root of unity, hence  $\zeta_0 = 1$ . Since  $A_1$  is noninvertible, we have  $|S_0| = |S_1|$ , which implies that  $|S_0|$  is odd. This yields a contradiction because a (cyclic) group of odd order has no self-inverse element apart from the identity.  $\square$

As an immediate corollary to Theorem 5.3 and Proposition 5.5, we obtain the following result.

**Corollary 5.6.** *For any  $n \in \mathbb{N}$  and  $d \in \mathbb{N}_0$ , there exists a  $d$ -regular nut graph of order  $n$  that is a circulant or bicirculant graph if and only if:*

- (i)  $n$  and  $d$  are both even, with at least one of them divisible by four; and
- (ii)  $d \geq 4$  and  $n \geq d + 4$ .

Note that bicirculant nut graphs need not be regular. For example, the graph with the adjacency matrix

$$\begin{bmatrix} C_0 & C_1^\top \\ C_1 & C_2 \end{bmatrix},$$

with  $C_0, C_1, C_2 \in \mathbb{R}^{18}$  having the connection sets

$$\{1\}, \quad \{0, 2\} \quad \text{and} \quad \{1, 2, 3\},$$

respectively, is a bicirculant nut graph of order 36 where the vertices from one orbit are of degree four, while the vertices from the other orbit are of degree eight. Let  $\mathfrak{N}_{d_1, d_2}^{\text{bicirc}}$  be the set of all the orders attainable by a bicirculant nut graph where the vertices from the two orbits have degrees  $d_1$  and  $d_2$ , respectively. It is natural to pose the following problem.

**Problem 5.7.** For any  $d_1, d_2 \in \mathbb{N}_0$ , determine the set  $\mathfrak{N}_{d_1, d_2}^{\text{bicirc}}$ .

We end the paper with two more corollaries of Theorem 1.10.

**Corollary 5.8.** *For any  $d \geq 4$  such that  $4 \mid d$ , we have*

$$\mathfrak{N}_d^{\text{reg}} \supseteq \{n \in \mathbb{N} : n \text{ is even and } n \geq d + 4\}.$$

**Corollary 5.9.** *For any  $d \geq 6$  such that  $d \equiv 2 \pmod{4}$ , we have*

$$\mathfrak{N}_d^{\text{reg}} \supseteq \{n \in \mathbb{N} : 4 \mid n \text{ and } n \geq d + 6\}.$$

Although Corollaries 5.8 and 5.9 give a partial solution to Problem 1.1 for the case when  $d$  is even, the regular nut graph order–degree existence problem seems much more difficult to solve. Corollary 5.6 justifies this claim and implies that different constructions not relying on circulant or bicirculant graphs would need to be used to further investigate Problem 1.1.

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