Discrete Mathematics and Theoretical Computer Science ....
.... **vol. 28:2 #3 (2026)**

Planar-Toroidal Decomposition of K_{12}

Allan Bickle[1] Russell Campbell[2]

[1] *Department of Mathematics, Purdue University, West Lafayette, USA*
[2] *School of Computing, University of the Fraser Valley, Abbotsford, Canada*

revisions 24th July 2025; accepted 28th Dec. 2025.

In 1978, Anderson and White asked whether there is a decomposition of K_{12} into two graphs, one planar and one toroidal. Using theoretical arguments and a computer search of all maximal planar graphs of order 12, we show that no such decomposition exists. We further show that if G is planar of order 12 and $H \subseteq \overline{G}$ is toroidal, then H has at least two fewer edges than $\overline{G}$. A computer search found all 123 unique pairs (G, H) that make this an equality.

Keywords: planar, torus, decomposition, embedding

1 Introduction

Many researchers have worked on problems involving splitting complete graphs into subgraphs with various topological properties.

Definition 1.1. *A **decomposition** of G is a set of nonempty subgraphs, called **factors**, whose edge sets partition $E(G)$. The subgraphs are said to **decompose** G. The **thickness** $\theta(G)$ of a graph G is the minimum number of planar graphs that decompose G.*

Theorem 1.2. *We have $\theta(K_n) = \left\lfloor \frac{n+7}{6} \right\rfloor$ unless $n \in \{9, 10\}$, and $\theta(K_9) = \theta(K_{10}) = 3$.*

The case of K_9 was settled by Harary, Battle and Kodoma [5] and Tutte [25]. Tutte checked all 50 maximal planar graphs of order 9, showing that none of their complements are planar. A general construction showing $\theta(K_n) = \left\lfloor \frac{n+7}{6} \right\rfloor$ was found by Beineke and Harary [8] for $n \not\equiv 4 \mod 6$. Finally, Vasok [26] and Alekseev and Gonchakov [1] completed the proof of the theorem for K_{6r+4}.

Definition 1.3. *A **torus** is a surface with one handle (or hole). A graph that can be drawn on a torus with no crossings is **toroidal**.*

*The **genus of a surface** is the number of handles (or holes) it has. The surface with k handles is denoted S_k. An **embedding** of a graph is a drawing with no crossings. The **genus of a graph** $\gamma(G)$ is the minimum genus so that it has an embedding on a surface with this genus. A region is a **2-cell** if any closed curve within that region can be continuously contracted to a point within that region.*

*The S_k-**thickness** $\theta_k(G)$ of a graph G is the minimum number of S_k-embeddable graphs that decompose G.*

Beineke [6] has shown that $\theta_1(K_n) = \left\lfloor \frac{n+4}{6} \right\rfloor$ and $\theta_2(K_n) = \left\lfloor \frac{n+3}{6} \right\rfloor$. Thus the **toroidal thickness** $\theta_1(K_{12}) = 2$ [6], meaning that K_{12} can be decomposed into two toroidal graphs.

Definition 1.4. *A graph G is (γ, γ') **bi-embeddable** if G can be embedded in S_γ, a sphere with γ handles, and $\overline{G}$ can be embedded in $S_{\gamma'}$. Let $N(\gamma, \gamma')$ be the size of the smallest complete graph which cannot be edge-partitioned into two parts embeddable in S_γ, and $S_{\gamma'}$, respectively.*

ISSN 1365–8050

The problem of finding upper and lower bounds for $N(\gamma, \gamma')$ was first studied in 1974 by Anderson and Cook in [3]. Results on thickness imply that $N(0,0) = 9$. The work of Ringel [20] and Beineke [6] on toroidal thickness showed that $N(1,1) = 14$, and Beineke showed that $N(2,2) = 15$. Bi-embeddings were studied further by Anderson [2], Cabaniss and Jackson [12] and Sun [24]. Cabaniss [11] surveyed bi-embeddings, while Beineke [7] surveyed biplanar graphs.

In 1978, Anderson and White [4] were the first to explicitly raise the question of determining $N(0,1)$. This can be accomplished by determining whether there is a maximal planar graph of order 12 whose complement is maximal toroidal. We will show that this is not possible. Along with the known decomposition of K_{11} into $\left\{C_9 + 2K_1, \overline{C}_9 \cup K_2\right\}$, this shows that $N(0,1) = 12$.

In 2013, Bickle and White [10] found bounds on $\gamma(G) + \gamma\left(\overline{G}\right)$ and $\gamma(G) \cdot \gamma\left(\overline{G}\right)$. In particular, they showed that $\gamma(G) + \gamma\left(\overline{G}\right) \geq \left\lceil \frac{1}{12}\left(n^2 - 13n + 24\right)\right\rceil$, and that this is attained for order $n = 12s + 11$ and for $n \in \{13, 25, 37, 49\}$. Sun [24] further showed this is attained for $n = 24s + 13$.

Conjecture 1.5. *(Bickle/White [10]) For all $n \geq 11$, the lower bound $\left\lceil \frac{1}{12}\left(n^2 - 13n + 24\right)\right\rceil$ of $\gamma(G) + \gamma\left(\overline{G}\right)$ is attained by some graph G.*

For $n = 12$, this says that there is a graph G for which $\gamma(G) + \gamma\left(\overline{G}\right) = 1$. That is, there is a decomposition of K_{12} into two graphs, one planar and one toroidal. Thus we have disproved the conjecture when $n = 12$.

Definitions of terms and notation not defined here appear in [9]. In particular, $n(G)$ is the number of vertices of a graph G. The degree of a vertex v is denoted $d_G(v)$, or $d(v)$ when the graph in question is clear. The neighborhood of a vertex v is denoted $N(v)$, and the closed neighborhood is denoted $N[v]$. The join of graphs G and H is denoted $G + H$. A **separating set** of a connected graph G is a set S of vertices so that $G - S$ is disconnected.

2 Theoretical Approaches

We describe three theoretical approaches to checking whether a maximal planar graph of order 12 has a complement that embeds on the torus. Each approach eliminates many cases, but even together, they do not eliminate all possibilities. We hope to eventually use the descriptions to both form the basis of a non-computer proof, and provide yet another way for collections of graphs that can be used as input to help design efficient algorithms, specifically, using graphs whose factors embed on the plane and the torus. These approaches may also be useful in related problems, such as other values of $N(\gamma, \gamma')$.

2.1 Vertex Degrees

Any planar graph has size $m \leq 3n - 6$, and any toroidal graph has size $m \leq 3n$. A complete graph has size $\binom{n}{2}$. For $n = 12$, these numbers are 30, 36, and 66. Thus if a plane-torus decomposition existed, both factors have the maximum number of edges. For planar graphs being maximal and being a triangulation are equivalent. Note however that a maximal toroidal graph need not be a triangulation [17].

Definition 2.1. *We say an ordered pair of graphs $\left(G, \overline{G}\right)$ are **PT12** if G is planar, $\overline{G}$ is toroidal, and $\left\{G, \overline{G}\right\}$ decomposes K_{12}.*

Lemma 2.2. *If $\left(G, \overline{G}\right)$ are PT12, then $3 \leq \delta(G) \leq 5 \leq \Delta(G) \leq 8$ and $3 \leq \delta\left(\overline{G}\right) \leq 6 \leq \Delta\left(\overline{G}\right) \leq 8$.*

Proof: In a triangulation, every vertex of G has degree at least 3 when $n \geq 4$. If $d_G(v) = d$, $d_{\overline{G}}(v) = n - 1 - d$, Thus $\Delta\left(\overline{G}\right) \leq 8$ and $\Delta(G) \leq 8$. It is well-known that for any planar graph, $\delta(G) \leq 5$. Thus $\Delta\left(\overline{G}\right) \geq 6$. □

There are 7595 maximal planar graphs with order 12, which are listed on the Combinatorial Object Server [15]. Of these graphs, there are 3476 with $\Delta > 8$.

Lemma 2.3. *If $\left(G, \overline{G}\right)$ are PT12, and $d_G(v) = 8$, then the 3 vertices not in $N_G[v]$ form an independent set. The degree 8 vertices of G are all adjacent.*

Proof: If $d_G(v) = 8$, $d_{\overline{G}}(v) = 3$. Since $\overline{G}$ is a triangulation, the neighbors of v in $\overline{G}$ induce a triangle. These vertices must form an independent set in G. The degree 3 vertices in $\overline{G}$ must all be nonadjacent, so the degree 8 vertices of G are all adjacent. □

2.2 Counting Triangles

A maximal planar graph with order n has $2n-4$ triangular regions. A toroidal triangulation with order n has $2n$ triangular regions. If $\left(G, \overline{G}\right)$ are PT12, then $\overline{G}$ has 24 triangular regions. Thus if $\overline{G}$ has fewer than 24 non-separating triangles, it can be discarded.

For example, the icosahedron IC is maximal planar with order 12. It has 20 independent sets of size 3, so $\overline{IC}$ has 20 triangles. Thus $\overline{IC}$ is not toroidal (this is stated without proof in [4]).

Note that a maximal planar or toroidal graph may have other triangles that are not the boundaries of regions. Any vertex of degree 3 has its neighbors induce a triangle. More generally, identifying two maximal planar graphs on a triangle produces another maximal planar graph, and identifying a maximal planar graph and a toroidal triangulation produces a toroidal triangulation. A **separating triangle** forms a cutset of G or $\overline{G}$.

A toroidal triangulation may also have triangles that are not 2-cell. A **handle triangle** cannot be contracted to a point in the torus. For example, the toroidal embedding of K_7 has $\binom{7}{3} = 35$ triangles, 14 regions and 21 handle triangles.

Intuitively, it appears that graphs that are close to regular have fewer triangles in their complements.

Theorem 2.4. *(Goodman [16], Sauvé [21]) Let $t(G)$ be the number of triangles of G. Then*

$$t(G) + t\left(\overline{G}\right) = \binom{n}{3} - (n-2)\,m + \sum_v \binom{d(v)}{2}.$$

A short proof of this is due to Schwenk [22]. Since $t(G) + t\left(\overline{G}\right)$ depends only on the degree sequence of G, some degree sequences can be ruled out immediately. We use d^r to indicate r vertices of degree d.

Corollary 2.5. *If $\left(G, \overline{G}\right)$ are PT12, then G does not have degree sequence 5^{12}, $4^1 5^{10} 6^1$, $4^2 5^8 6^2$, $4^3 5^6 6^3$, $3^1 5^9 6^2$, $4^2 5^9 7^1$, or $3^1 5^{10} 7^1$.*

Proof: Since G is maximal planar it has size $m = 3n-6$. When $n = 12$, Goodman's formula becomes

$$t(G) + t\left(\overline{G}\right) = -80 + \sum_v \binom{d(v)}{2}.$$

Since $t(G) \geq 20$ and $t\left(\overline{G}\right) \geq 24$, we need $\sum_v \binom{d(v)}{2} \geq 124$. The first six sequences all have $\sum_v \binom{d(v)}{2} < 124$. If G has degree sequence $3^1 5^{10} 7^1$, $\sum_v \binom{d(v)}{2} = 124$, but $t(G) \geq 21$ since G must also have a separating triangle due to the degree 3 vertex. □

2.3 Separating Sets

A separating set in G implies that $\overline{G}$ contains a complete bipartite graph. This observation provides a reasonable starting point to eliminate large numbers of cases.

Lemma 2.6. *If $(G, \overline{G})$ are PT12, then G has no separating triangle whose deletion leaves subgraphs with orders 4 and 5.*

Proof: If not, then $\overline{G}$ contains $K_{4,5}$, which has genus 2 [28]. □

Theorem 2.7. *If $(G, \overline{G})$ are PT12, then G has no separating triangle whose deletion leaves subgraphs with orders 3 and 6.*

Proof: Assume to the contrary that $(G, \overline{G})$ are PT12 and G has a separating triangle whose deletion leaves subgraphs with orders 3 and 6. Let A be the set of vertices of the separating triangle, B be the set of 3 vertices, and C be the set of 6 vertices. Now B and C induce $K_{3,6}$ in $\overline{G}$, which has an embedding on the torus with every region a 4-cycle.

Now A and B induce a maximal planar subgraph H of G with order 6. This must be either $K_{2,2,2}$ or $K_2 + P_4$. There are three different nonequivalent regions in $K_2 + P_4$ that can be A, but in each case some vertex v of A is adjacent to all vertices in B. Then in $\overline{G}$, v is adjacent to no vertex of B. But this is impossible, since every region of $K_{3,6}$, even when triangulated, contains a vertex of B. Thus $H = K_{2,2,2}$. Thus any vertex $v \in A$ is adjacent to exactly one vertex of B in $\overline{G}$.

Since the vertices of A are all adjacent in G, they are all nonadjacent in $\overline{G}$. Since $\overline{G}$ is a triangulation, each (4-cycle) region of $K_{3,6}$ on the torus must have an edge between two vertices of C. Adding these edges thus produces $K_{3,3,3}$.

Now in $\overline{G}$, each vertex in A is adjacent to exactly one vertex of B and exactly two vertices of C. Each pair of vertices in C adjacent to a vertex of A are also adjacent to each other in $\overline{G}$, so nonadjacent in G.

Let $A = \{u, v, w\}$. Each vertex of A is adjacent in G to exactly four vertices of C. We seek to show that there is no planar graph on $A \cup C$ satisfying the established conditions.

If any pair of vertices of A (say u and v) has at least three common neighbors in C, then two of them must be adjacent, and not adjacent to w, contradicting Lemma 2.3.

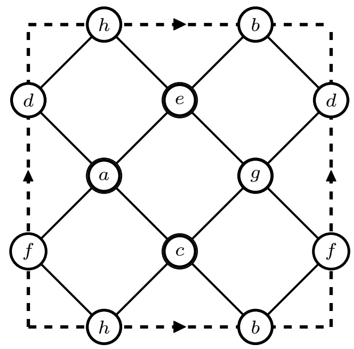
Suppose some pair of vertices of A (say u and v) has exactly two common neighbors in C, which must be adjacent. Then one of them (call it x) must be adjacent to w. Now the triangles uwx and vwx have three neighbors of w in their interiors. Thus one (say uwx) contains at least two such neighbors, and hence two adjacent vertices both not adjacent to v. This contradicts Lemma 2.3.

If each pair of vertices in A has exactly one common neighbor in C, then C contains at least 9 vertices, a contradiction. □

Next we consider deleting four vertices to produce two disjoint sets of four vertices.

Theorem 2.8. *If $(G, \overline{G})$ are PT12, then G has no separating 4-cycle whose deletion leaves two subgraphs with order 4.*

Proof: Assume to the contrary that $(G, \overline{G})$ are PT12 and G has a separating 4-cycle $C = uvwxu$ whose deletion leaves two (not necessarily connected) subgraphs with order 4. Let $A = \{a, b, g, h\}$ and $B = \{c, d, e, f\}$ be the two sets of 4 vertices. Now A and B induce $K_{4,4}$ in $\overline{G}$, which can only be embedded on the torus with 8 regions of length 4 (see the figure below).



Assume that C has at most one chord, and say uw is not in G. Then uw must be in $\overline{G}$, so it is contained in a region of $K_{4,4}$, say $acgea$. Thus u and w are both adjacent to d, f, b, and h in G.

Assume C has chord vx. Now in G, d and f must be the on opposite side of cycle C from b and h. But then edge vx creates a crossing in G, contradicting planarity.

Assume vx is not a chord of C, so it is in $\overline{G}$. If vx is in a region of $\overline{G}$ not containing one of c or e (say c) , then v and x are both adjacent to c in G. If vx is in another region of $\overline{G}$ containing both c and e, then all vertices of C are adjacent to d and f in G. Either way, this creates a crossing in G, since all vertices of B are on the same side of cycle C.

Assume C has two chords, so its vertices induce K_4. Theorem 2.7 and Lemma 2.6 show that no region of the K_4 contains 3 or 4 vertices. Thus all four regions contain exactly two vertices. Each adjacent pair in $V(G-C)$ and its surrounding triangle in K_4 induce $P_3+\overline{K}_2$. Since $\Delta(G)\leq 8$, summing degrees shows that each vertex in $V(C)$ has degree 8. Thus there are two adjacent vertices inside vwx, not adjacent to u, contracting Lemma 2.3. □

3 Computer Search

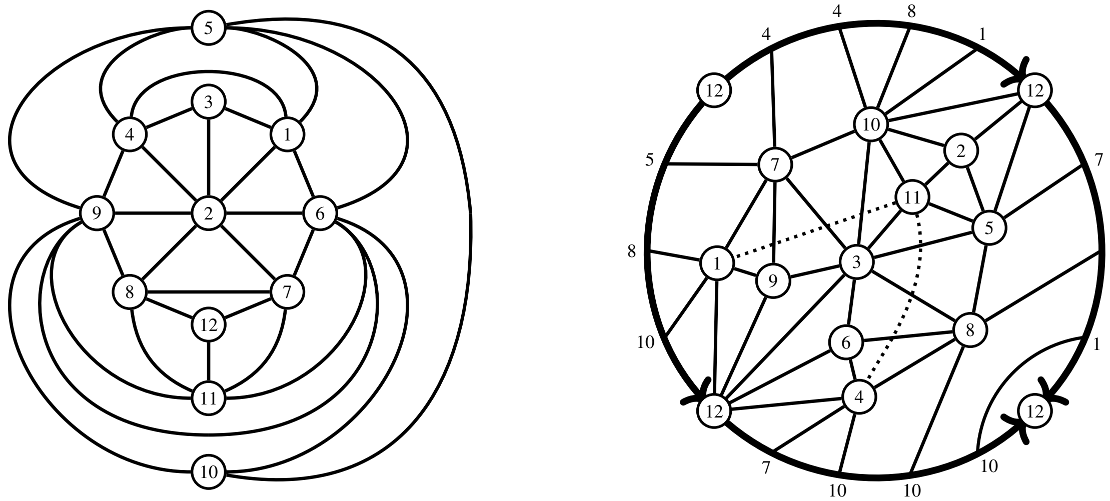
Embedding algorithms for the torus are complex. Planar graphs are characterized by only two forbidden minors, K_5 and $K_{3,3}$ [27]. However, for the torus there are many more–Myrvold and Woodcock found over 250000 minors by computational search [19]. Some previous attempts to devise algorithms for toroidal embedding have been erroneous, as shown by Myrvold [18]. There is currently no error-free implementation of a linear-time toroidal embedding algorithm.

Among the 7595 plane triangulations, there are 3476 graphs with maximum degree greater than 8 corresponding to Lemma 7. There are 256 graphs with an isolated set of three vertices all not adjacent to a vertex of degree 8. The filtered set of embeddings are listed in a text file named `tri_isolated_3.txt` included in the GitHub repository [14].

The first computational search would have completed in about 30 minutes on a laptop with Intel `i7-9750H` processor with full CPU utilization. We filtered out plane triangulations that have $\Delta\geq 9$ and checked for any complement that had a torus embedding; there were none.

Next, we considered all possible ways to remove exactly one edge from the complement of a plane triangulation with 12 vertices. The search took about 4 days 8 hours and 24 minutes, with none found. For each complement, one edge was chosen to be removed, then the program searched for a torus embedding, and output any found.

The computational search for removing all possible pairs of edges from each complement of each plane triangulation with 12 vertices took about 59 days. For each complement, two edges were chosen to be removed, then the program searched for a torus embedding, and output any found. Duplicates were logged



(a) The 5525th plane triangulation listed in the Combinatorial Object Server.

(b) A complement embedding listed on page 72 from [14] found by combinatorial search. Opposite sides of the region boundary are identified to form the torus.

Fig. 1: A planar triangulation and its complement embedded on the torus with two omitted edges shown as dotted curves.

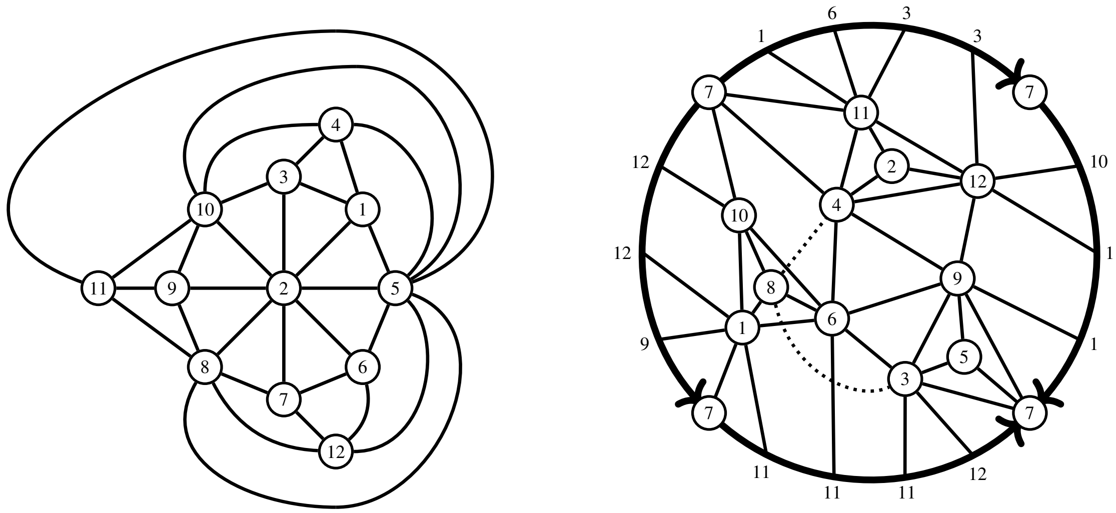
in a text file of output. Once isomorphic embeddings were removed, there were 123 unique maximal torus embeddings with 2 edges removed from the complement of plane triangulations with 12 vertices. Through use of `planedraw` by Gunnar Brinkmann, the list of 123 torus embeddings can be seen as figures in the PDF compiled document available in our GitHub repository [14] along with a subfolder of images giving corresponding embeddings of the triangulations of the plane on 12 vertices. The images were created using personal `C/C++` code by Campbell. Two examples of a triangulation with 12 vertices and its complement so that two edges can be deleted to embed on the torus are given in Figures 1 and 2.

For the search algorithm, we did not program it to avoid constructing duplicate embeddings. Two combinatorial embeddings G and H on an oriented surface can be considered equivalent (or flip-isomorphic) if the circular adjacency lists for each vertex of G are reversed and then equal to the circular adjacency lists of H. The reversal of all adjacency lists represents flipping a clockwise ordering to a counterclockwise ordering on the surface and vice versa. To reduce the number of equivalent embeddings found in an algorithm, one can canonically choose in the circular adjacency list which vertices will be the first two in any ordering during combinatorial searches. Although allowing equivalent duplicates increases runtime, it helps to check correctness where any recursive construction of an embedding matches at least one duplicate.

For more specific algorithm descriptions that can be applied to either orientable or nonorientable surfaces, refer to Campbell's dissertation [13] available online through UVic DSpace (URI link provided in reference section).

The list of plane triangulations of order 12 can also be generated by Sulanke's `Surftri` [23], and once compiled, with the following command (`-a` option for ascii format; `0` option required to specify genus):

```
./surftri -a 12 0
```



(a) The 5557[th] planar triangulation listed in the Combinatorial Object Server.

(b) A complement embedding listed on page 91 from [14] found by combinatorial search. Opposite sides of the region boundary are identified to form the torus.

Fig. 2: A planar triangulation and its complement embedded on the torus with two omitted edges shown as dotted curves.

To execute the command, one also needs the corresponding `genus0.alpha` file giving the one irreducible triangulation of the sphere K_4 that can either be downloaded from Sulanke's website or just create the file with its one-line contents: "`4 bcd,adc,abd,acb`." The program will print one graph on each line of output starting with each graph's order, followed by adjacency lists separated by comma given as alphabetic labels of the vertex neighbors. The adjacency lists can either be treated as all in counterclockwise or all clockwise order, as needed. `Surftri` outputs the 7595 triangulation plane embeddings in less than a second on a basic modern computer. As isomorphism is a computationally complex task, we do not check which triangulation matches among those listed in the Combinatorial Object Server, but we wish to encourage use of multiple resources to foster a more resilient research community.

Lastly, to aid our research efforts and to double-check any results, an image collection of the 7595 plane triangulations was generated procedurally with a drawing algorithm written in C/C++. Note that individual images procedurally generated are not currently copyrightable in Canadian law, and not likely to be copyrightable in United States law. However, collections of such images can be protected. The collection is available in GitHub repository and license described therein [14].

Acknowledgement. *This paper is dedicated to Art White, who informed the first author of this problem a decade ago.*

Campbell wishes to thank Allan Bickle for awareness of the problem and their collaboration.

References

[1] V. B. Alekseev and V. S. Gonchakov, Thickness of arbitrary complete graphs (Russian) *Mat. Sbornik* 101 (1976), 212-230.

[2] I. Anderson, Infinite families of biembedding numbers, *J. Graph Theory* 3 (1979), 263-268.

[3] I. Anderson and R. J. Cook, Bi-embeddings of graphs. *Glasgow Math. J.* 15 (1974), 162-165.

[4] I. Anderson, and A. T. White, Current graphs and bi-embeddings. *J. Graph Theory* 2 (1978), 231-239.

[5] J. Battle, F. Harary and Y. Kodoma, Every planar graph with nine points has a nonplanar complement, *Bull. Amer. Math Soc.* 68 (1962), 569-571.

[6] L. Beineke, Minimal decompositions of complete graphs into subgraphs with embeddability properties, *Canad. J. Math.* 21 (1969), 992-1000.

[7] L. W. Beineke, Biplanar graphs: A survey, *Computers Math. Appl.* 34 (1997), 1-8.

[8] L. W. Beineke and F. Harary, The thickness of the complete graph, *Canad. J. Math.* 17 (1965), 850-859.

[9] A. Bickle, *Fundamentals of Graph Theory*, AMS (2020).

[10] A. Bickle and A. White, Nordhaus-Gaddum results for genus, *Discrete Math.* 313 6 (2013), 824-829.

[11] S. Cabaniss, A survey of biembeddings of graphs, *Congr. Numer.* 70 (1990), 171-180.

[12] S. Cabaniss and B. Jackson, Infinite families of bi-embeddings, *Discrete Math.* 82 2 (1990), 127-141.

[13] R. Campbell, Finding Obstructions within Irreducible Triangulations. Dissertation, University of Victoria, 2017. http://hdl.handle.net/1828/8212.

[14] R. Campbell, Collection of 123 torus embedding figures, and image collection of the plane triangulations of order 12, Feb 2025. https://github.com/ctrain79/K12-split.

[15] The Combinatorial Object Server, http://combos.org/index

[16] A. W. Goodman, On sets of acquaintances and strangers at any party, *Amer. Math. Monthly* 66 (1959), 778-783.

[17] F. Harary, P. C. Kainen, A. J. Schwenk, and A. T. White, A maximal toroidal graph which is not a triangulation, *Math. Scand.* 33 1 (1974), 108-112.

[18] W. Myrvold and W. Kocay, Errors in graph embedding algorithms, *J. Comput. System Sci.* 77, 2 (2011), 430-438.

[19] W. Myrvold and J. Woodcock, A large set of torus obstructions and how they were discovered, *Journal of Combinatorics* 25 1 (2018), 1-16.

[20] G. Ringel, Die toroidale dicke des vollstandigen graphen, *Math. Z.* 87 (1965), 19-26.

[21] L. Sauvé, On chromatic graphs, *Amer. Math. Monthly* 68 (1961), 107-111.

[22] A. J. Schwenk, Acquaintance party problem, *Amer. Math. Monthly* 79 (1972), 1113-1117.

[23] T. Sulanke, Generating triangulations of surfaces. https://tsulanke.pages.iu.edu/graphs/surftri/index.html, 2006. Accessed: 15 Aug 2024.

[24] T. Sun, On the bigenus of the complete graphs, *Austral. J. Combin.* 84 1 (2022), 212-219.

[25] W. T. Tutte, The non-biplanar character of the complete 9-graph, *Canad. Math. Bull.* 6 (1963), 319-330.

[26] J. M. Vasak, The thickness of the complete graph, *Notices Amer. Math. Soc.* 23 (1976), A-479.

[27] K. Wagner, Uber eine eigenschaft des ebenen komplexes, *Mathematische Annalen* 114 (1937), 570-590.

[28] A. T. White, *Graphs of Groups on Surfaces, Interactions and Models*, North-Holland, Amsterdam, 2001.