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Pattern Avoidance for Fibonacci Sequences using k-Regular Words

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Two k-ary Fibonacci recurrences are $a_k(n) = a_k(n-1) + k \cdot a_k(n-2)$ and $b_k(n) = k \cdot b_k(n-1) + b_k(n-2)$. We provide a simple and direct proof that $a_k(n)$ is the number of k-regular words over $[n] = \{1, 2, \ldots, n\}$ that avoid patterns $\{121, 123, 132, 213\}$ when using base cases $a_k(0) = a_k(1) = 1$ for any $k \geq 1$. This was previously proven by Kuba and Panholzer in the context of Wilf-equivalence for restricted Stirling permutations, and it creates Simion and Schmidt's classic result on the Fibonacci sequence when $k = 1$, and the Jacobsthal sequence when $k = 2$. We complement this theorem by proving that $b_k(n)$ is the number of k-regular words over $[n]$ that avoid $\{122, 213\}$ with $b_k(0) = b_k(1) = 1$ for any $k \geq 2$. Finally, we prove that $|\mathsf{Av}_n^2(\overline{121}, 123, 132, 213)| = a_1(n)^2$ for $n \geq 0$. That is, vincularizing the Stirling pattern in Kuba and Panholzer's Jacobsthal result gives the Fibonacci-squared numbers.

Keywords: permutation patterns, pattern avoidance, regular words, uniform permutations, Stirling permutations, Fibonacci sequence, Jacobsthal sequence

1 Introduction

The Fibonacci sequence is arguably the most famous integer sequence in mathematics, and the term *generalized Fibonacci sequence* has been used to describe an increasingly wide variety of related sequences. Here we consider two families of generalizations involving a second parameter k.

Definition 1. *The* Fibonacci-k numbers *are* $a_k(n) = a_k(n-1) + k \cdot a_k(n-2)$ *with* $a_k(0) = a_k(1) = 1$.

Definition 2. *The* k-Fibonacci numbers *are* $b_k(n) = k \cdot b_k(n-1) + b_k(n-2)$ *with* $b_k(0) = b_k(1) = 1$.

We provide pattern avoidance results for the sequences $\{a_k(n)\}_{n \geq 0}$ and $\{b_k(n)\}_{n \geq 0}$. The objects are *k-regular words* meaning that each symbol in $[n] = \{1, 2, \ldots, n\}$ appears k times, and no subword can have the same relative order as any of the patterns. We let S_n^k be the set of k-regular words over $[n]$ and $\mathsf{Av}_n^k(\pi_1, \pi_2, \ldots, \pi_m) \subseteq S_n^k$ be the subset that avoids all m patterns. We focus on two families of words.

Definition 3. *The* Fibonacci-k words *of length kn are the words in* $\mathsf{Av}_n^k(121, 123, 132, 213) \subseteq S_n^k$.

Definition 4. *The* k-Fibonacci words *of length kn are the words in* $\mathsf{Av}_n^k(122, 213) \subseteq S_n^k$.

We provide a simple proof that the Fibonacci-k words are counted by the Fibonacci-k numbers for $k \geq 1$. This was previously proven by Kuba and Panholzer (see the $\overline{C_2}$ case of Theorem 3 in the citation).

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Theorem 1 (Kuba and Panholzer (2012)). $a_k(n) = |\mathsf{Av}_n^k(121, 123, 132, 213)|$ *for all* $k \geq 1$ *and* $n \geq 0$.

For example, the Fibonacci-2 numbers $a_2(n)$ create OEIS[(i)] A001045 given by $1, 1, 3, 5, 11, 21, 43, \ldots$, and also known as the *Jacobsthal sequence*. Its first four terms count the Fibonacci-2 words in (1)–(4).

$$1 = |\mathsf{Av}_0^2(121, 123, 132, 213)| = |\{\epsilon\}| \tag{1}$$

$$1 = |\mathsf{Av}_1^2(121, 123, 132, 213)| = |\{11\}| \tag{2}$$

$$3 = |\mathsf{Av}_2^2(121, 123, 132, 213)| = |\{1122, 2112, 2211\}| \tag{3}$$

$$5 = |\mathsf{Av}_3^2(121, 123, 132, 213)| = |\{223311, 322311, 331122, 332112, 332211\}| \tag{4}$$

Note that $233211 \in S_3^2$ since it is a 2-regular word over [3]. However, $\underline{2}\underline{3}3\underline{2}11 \notin \mathsf{Av}_3^2(121, 123, 132, 213)$ since its underlined subword 232 is order isomorphic to the pattern 121. Hence, it does not appear in (4).

Kuba and Panholzer previously proved Theorem 1 in a much broader context involving Stirling permutations (see Section 1.4). Here we provide a simple and direct combinatorial proof. We also complement their result for Fibonacci-k words with a new pattern avoidance result for k-Fibonacci words. More specifically, we prove that the k-Fibonacci words are counted by the k-Fibonacci numbers for $k \geq 2$.

Theorem 2. $b_k(n) = |\mathsf{Av}_n^k(122, 213)|$ *for all* $k \geq 2$ *and* $n \geq 0$.

For example, the 2-Fibonacci number sequence $\{b_n\}_{n \geq 0}$ is $1, 1, 3, 7, 17, 41, 99, 239, 577, \ldots$ (A001333). Its first four terms count the 2-Fibonacci words over $[n]$ for $n = 0, 1, 2, 3$ as shown in (5)–(8).

$$1 = |\mathsf{Av}_0^2(122, 213)| = |\{\epsilon\}| \tag{5}$$

$$1 = |\mathsf{Av}_1^2(122, 213)| = |\{11\}| \tag{6}$$

$$3 = |\mathsf{Av}_2^2(122, 213)| = |\{2112, 2121, 2211\}| \tag{7}$$

$$7 = |\mathsf{Av}_3^2(122, 213)| = |\{322311, 332112, 323112, 332211, 323211, 332121, 323121\}| \tag{8}$$

Figure 1 provides Fibonacci words of both types for $k = 3$, and illustrates why Theorems 1 and 2 hold. Readers who are ready to delve into the proofs of our two main results can safely skip ahead to Section 2. The remainder of this section further contextualizes Theorems 1 and 2 and introduces one more result.

1.1 Classic Pattern Avoidance Results: Fibonacci and Catalan

Theorem 1 provides a k-ary generalization of the classic pattern avoidance result by Simion and Schmidt involving permutations and the Fibonacci numbers. Their statement of the result is provided below.

Theorem 3 (Proposition 15 in Simion and Schmidt (1985)). *For every* $n \geq 1$,

$$|\mathsf{Av}_n(123, 132, 213)| = F_{n+1} \tag{9}$$

where $\{F_n\}_{n \geq 0}$ *is the Fibonacci sequence, initialized by* $F_0 = 0$, $F_1 = 1$.

When comparing Theorems 1 and 3, note that permutations are 1-regular words, and they all avoid 121. Thus, $\mathsf{Av}_n(123, 132, 213) = \mathsf{Av}_n^1(123, 132, 213) = \mathsf{Av}_n^1(121, 123, 132, 213)$. In other words, the 121 pattern in Theorem 1 is hidden in the special case of $k = 1$ in Theorem 3. Also note that (9) has off-by-one indexing (i.e., subscript n versus $n + 1$) and it holds for $n = 0$ despite the stated $n \geq 1$ bound.

An even earlier result on pattern avoiding permutations is stated below.

(i) See the Online Encyclopedia of Integer Sequences (OEIS) OEIS Foundation Inc. (2025) for all sequence references.

Theorem 4 (MacMahon (1915) and Knuth (1968)). *For every $n \geq 0$,*

$$|\mathsf{Av}_n(123)| = \mathcal{C}_n \qquad \textit{and} \qquad |\mathsf{Av}_n(213)| = \mathcal{C}_n \tag{10}$$

where $\{\mathcal{C}_n\}_{n\geq 0}$ is the Catalan sequence starting with $\mathcal{C}_0 = 1$ and $\mathcal{C}_1 = 1$.

Permutations avoid 122, so $\mathsf{Av}_n(213) = \mathsf{Av}^1_n(122, 213)$. Thus, the patterns in Theorem 2 are equivalent to those in Theorem 4 when $k = 1$. But Theorem 4 is not a special case of Theorem 2, as our new result only holds when $k \geq 2$. This gap makes sense as the Catalan numbers do not follow a simple two term recurrence like $a_k(n)$ or $b_k(n)$, and the 1-Fibonacci words $\mathsf{Av}^1_n(122, 213) = \mathsf{Av}_n(213)$ are Catalan words.

1.2 Base Cases: $(0, 1)$-Based or $(1, 1)$-Based

Simion and Schmidt's result uses the customary base cases of $F_0 = 0$ and $F_1 = 1$ for Fibonacci numbers. However, we use base cases of $a_k(0) = b_k(0) = 1$ and $a_k(1) = b_k(1) = 1$ in our k-ary generalizations.

The distinction between $(0, 1)$-*based* and $(1, 1)$-*based* sequences can be dismissed as cosmetic for the Fibonacci-k recurrence $a_k(n) = a_k(n-1) + k \cdot a_k(n-2)$ since the resulting sequences $0, 1, 1, k, \ldots$ and $1, 1, k, \ldots$ coincide from the first 1. The resulting sequences are indeed *shifted* by one index relative to each other, which is important if we want to be sensitive to the off-by-one indexing issue found in (9).

In contrast, the $n = 0$ term is critical to the k-Fibonacci recurrence $b_k(n) = k \cdot b_k(n-1) + b_k(n-2)$ since the $(0, 1)$-based sequence $0, 1, k, k^2+1, \ldots$ and the $(1, 1)$-based sequence $1, 1, k+1, k^2+k+1, \ldots$ diverge if $k \geq 2$. In particular, the well-known *Pell sequence* $\{P(n)\}_{n\geq 0}$ (A000129) follows our $b_2(n)$ recurrence with $P(0) = 0$ and $P(1) = 1$, so it is not covered by our Theorem 2 as it is $(0, 1)$-based. Hartung and Williams (2024) found pattern avoidance results for $(1, b)$-based Pell sequences with $b \geq 2$.

With apologies to the Pell numbers, we suggest that $(1, 1)$-based sequences are more natural, at least in the context of pattern avoidance. This is due to the unique word of length $n = 0$ that avoids all patterns, namely the empty word ϵ. So an $n = 0$ term of 0 mistakenly counts a singleton set $\{\epsilon\}$ as an empty set $\emptyset$.

1.3 Four Parameter Generalizations beyond k-Fibonacci and Fibonacci-k

As mentioned earlier, there are many different notions of *generalized Fibonacci numbers*. To better discuss similar sequences, it is helpful to define the (b_0, b_1)-*based* k_1-*Fibonacci*-k_2 *recurrence* as follows,

$$f(n) = k_1 \cdot f(n-1) + k_2 \cdot f(n-2) \text{ with } f(0) = b_0 \text{ and } f(1) = b_1. \tag{11}$$

The resulting sequence $\{f(n)\}_{n\geq 0}$ is the (b_0, b_1)-*based* k_1-*Fibonacci*-k_2 *sequence*. When using these terms, we omit k_1 and/or k_2 when they are equal to 1. Thus, our Fibonacci-k numbers can be described as the $(1, 1)$-based Fibonacci-k numbers, or as shifted $(0, 1)$-based Fibonacci-k numbers (as per Section 1.2). Likewise, our k-Fibonacci numbers can be described as $(1, 1)$-based k-Fibonacci numbers outside of this paper, and they are not equivalent to $(0, 1)$-based k-Fibonacci numbers when $k \geq 2$.

Table 1 collects previously studied (b_0, b_1)-based k_1-Fibonacci-k_2 sequences that are not covered by our work. For example, the aforementioned Pell sequence is the $(0, 1)$-based 2-Fibonacci sequence using our terminology. More generally, the *kth-Fibonacci sequences*[(ii)] in Falcón and Plaza (2007) are the $(0, 1)$-based k-Fibonacci sequences from the previous paragraph. When preparing Table 1 we found the summary in Panwar (2021) to be very helpful. Also note that the same four parameters are used in Gupta et al. (2012) with their *generalized Fibonacci sequences* being (a, b)-based p-Fibonacci-q sequences.

(ii) Somewhat confusingly, the title of this well-cited paper is *On the Fibonacci k-numbers* and it contains a section titled *k-Fibonacci numbers*, but the term used throughout the paper is *kth Fibonacci sequence*. We use the latter in Table 1.

Name	Bases	k_1	k_2	Numbers	OEIS	Pattern Avoidance
Pell (2nd Fibonacci)	$(0,1)$	2	1	$0,1,2,5,12,29,70,169,\ldots$	A000129	$\mathsf{Av}_n(123,2143,3214)$ or $\mathsf{Av}_n(2\dot{1},231,4321)$
3rd Fibonacci	$(0,1)$	3	1	$0,1,3,10,33,109,360,1189,\ldots$	A006190	
4th Fibonacci	$(0,1)$	4	1	$0,1,4,17,72,305,1292,5473,\ldots$	A001076	
5th Fibonacci	$(0,1)$	5	1	$0,1,5,26,135,701,3640,18901,\ldots$		
6th Fibonacci	$(0,1)$	6	1	$0,1,6,37,228,1405,8658,53353,\ldots$	A005668	
7th Fibonacci	$(0,1)$	7	1	$0,1,7,50,357,2549,18200,129949,\ldots$		
8th Fibonacci	$(0,1)$	8	1	$0,1,8,65,528,4289,34840,283009,\ldots$		
9th Fibonacci	$(0,1)$	9	1	$0,1,9,82,747,6805,61992,564733,\ldots$	A099371	
Mersenne	$(0,1)$	3	-2	$0,1,3,7,15,31,63,127,\ldots$	A000225	$\mathsf{Av}_n(2\dot{1},132,213)$
Lucas	$(2,1)$	1	1	$2,1,3,4,7,11,18,29,\ldots$	A000032	$d_n(2413)$
Jacobsthal-Lucas	$(2,1)$	1	2	$2,1,5,7,17,31,65,127,\ldots$	A014551	
Pell-Lucas	$(2,2)$	2	1	$2,2,6,14,34,82,198,478,\ldots$	A002203	

Tab. 1: Previously studied (b_0,b_1)-based k_1-Fibonacci-k_2 sequences that are not $a_k(n)$ or $b_k(n)$. Each row contains a sequence that is not covered by our results, along with its alternate name. For example, the Mersenne numbers can also be described as the $(0,1)$-based 3-Fibonacci-(-2) numbers. Avoidance results are due to Barcucci et al. (2006), Baril (2011) (dotted), and Cratty et al. (2016) (double list d_n).

1.4 Pattern Avoidance with Stirling Words

A *Stirling permutation* is typically defined as a word with two copies of each value in $[n]$ and the property that for each $i \in [n]$, the values between the two copies of i are larger than i. In other words, it is a 2-regular word over $[n]$ that avoids 212. The Stirling permutations with $n=3$ are given in (12).

$$\begin{aligned}\mathsf{Av}_3^2(212) = \{&112233, 112332, 113322, 122133, 122331, 123321, 133122, 133221,\\ &221133, 221331, 223311, 233211, 331122, 331221, 332211\}\end{aligned} \tag{12}$$

Famously, Gessel and Stanley (1978) proved that $|\mathsf{Av}_n^2(212)| = (2n-1)!!$. For example, (12) verifies that there are $5!! = 5\cdot 3\cdot 1 = 15$ such words for $n=3$. Generalized k-*Stirling words* are the words in $\mathsf{Av}_n^k(212)$ using our notation, and were studied under the name r-*multipermutations* in Park (1994).

Kuba and Panholzer (2012) investigated Wilf-equivalence for k-Stirling words that avoid a subset of patterns in S_3. They proved that there are five $\mathbb{N}$-Wilf classes[(iii)] that avoid three such patterns. That is, there are five counting functions parameterized by k for $|\mathsf{Av}_n^k(212,\alpha,\beta,\gamma)|$ with distinct $\alpha,\beta,\gamma \in S_3$. In particular, their class $\overline{C}_2$ avoids $\lambda = \{312,231,321\}$ and is identical to $\mathsf{Av}_k^2(121,123,132,213)$ used in our Theorem 1. This is because k-regular word classes respect the symmetries of the square, and complementing our patterns results in 212 and λ. Their proof of this case also uses the same recursive decomposition as our proof of Theorem 1. However, its scope and approach obfuscate the simplicity of this important special case. For example, the proof of their three-pattern result uses their two-pattern result, so readers must process the special case in stages. The use of generating functions and partial fraction expansions also belie the simple structure of the strings. Their representative patterns for $\overline{C}_2$ also requires scaling-up subwords before prefixing 1s and 2s (cf., Figure 1a). For these reasons, our proof of Theorem 1 is somewhat simpler, even though the same structure is used.

(iii) $\mathbb{N}$-*Wilf-equivalence* means Wilf-equivalence for all $k \geq 1$. The other four classes are 0, $n+k-1$, $1+k\cdot(n-1)$, and $\binom{n+k-1}{k}$.

Another result in Kuba and Panholzer (2012) involves the *k-Catalan numbers* $\mathcal{C}_{k,n}$, which enumerate k-ary trees with n nodes and generalize the standard Catalan numbers $\mathcal{C}_n = \mathcal{C}_{2,n}$ from Section 1.1. The authors connect these generalized Catalan numbers to Stirling permutations by proving that $|\mathsf{Av}_n^k(212, 312)| = \mathcal{C}_{k+1,n}$, with the $k = 1$ case providing one generalization of Knuth's contribution to Theorem 4 (also see Section 1.5).

The aforementioned k-Catalan result was also obtained independently within the order theory community. In particular, the elements of the m-Tamari lattice introduced by Bergeron and Préville-Ratelle (2012) are m-Dyck paths which are enumerated by $\mathcal{C}_{m+1,n}$. As shown by Novelli and Thibon (2020), the elements can also be viewed as equivalence classes of $(m + 1)$-Stirling words (using their 121-avoiding representation) under a generalization of the sylvester congruence described by Hivert et al. (2005); representatives of these classes are precisely those that avoid the appropriate pattern from S_3. We also found Pons (2015) and Ceballos and Pons (2024) to be helpful towards understanding these results.

Archer et al. (2018) introduced quasi-Stirling permutations: 2-regular words on $[n]$ that avoid the patterns 1212 and 2121. The paper shows a bijection between ordered rooted labeled trees and the set of quasi-Stirling permutations, and finds the number of quasi-Stirling permutations that avoid between two and five patterns of length 3. Elizalde and Luo (2024) consider nonnesting permutations: 2-regular words on $[n]$ that avoid 1221 and 2112, and gives enumeration results for the number of nonnesting permutations that avoid at least two permutations of length 3.

1.5 Pattern Avoidance with Regular Words

There has been a relative lack of pattern avoidance results on non-Stirling regular words. Said another way, most pattern avoidance results on regular words have (implicitly) assumed that 121 is avoided. Omitting this assumption allowed us to discover Theorem 2 — which uses the non-Stirling pattern 122 — as a natural companion to Theorem 1.

A significant pattern avoidance result for (non-Stirling) regular words involves the *k-ary Catalan numbers* $\mathcal{C}_{k,n}$ which count k-ary trees with n nodes. While Theorem 4 shows that there are two distinct patterns for $\mathcal{C}_n = \mathcal{C}_{2,n}$ (up to symmetries of the square), there are three distinct pairs for $k > 2$.

- Kuba and Panholzer (2012) proved that $|\mathsf{Av}_n^{k-1}(212, 312)| = \mathcal{C}_{k,n}$.
- Defant and Kravitz (2020) proved that $|\mathsf{Av}_n^{k-1}(221, 231)| = \mathcal{C}_{k,n}$.
- Williams (2023) proved that $|\mathsf{Av}_n^{k-1}(112, 123)| = \mathcal{C}_{k,n}$.

Note that the first two cases collapse into one case when $k = 2$ since complementing and reversing 312 gives 231, while the other pattern is hidden. Williams also observed that the three results are collectively characterized by choosing a single pattern from S_3 and a pattern of 1s and 2s that is consistent with it[(iv)].

We mention that k-regular words arise elsewhere under a variety of names, including *uniform permutations*, m-permutations, and *fixed-frequency multiset permutations*. For example, the middle levels theorem by Mütze (2016, 2023) is conjectured to have a generalization to these words (see Shen and Williams (2021)) and significant partial results were proven in a broader context by Gregor et al. (2023).

(iv) The author was not initially aware of the Kuba and Panholzer result. The omission will be corrected in its extended version.

1.5.1 Regular Words and Non-Classical Patterns

To our knowledge, regular words have not been combined with other well-known non-classical pattern avoidance concepts[v] (e.g., barred, dotted, mesh, etc). We argue that such combinations are promising by contributing one more result involving (squared) Fibonacci numbers.

Definition 5. *The* Fibonacci-squared numbers *are* $c(n) = a_1(n)^2$.

In a *fully vincularized pattern* (also known as a *consecutive pattern*) all of the matching symbols must have consecutive positions within the string (see Babson and Steingrímsson (2000)). In particular, $\overline{121}$ is present in a string if and only if it contains three consecutive symbols of the form xyx with $y > x$.

Definition 6. *The* Fibonacci-squared words *of length* $2n$ *are the words in* $\mathsf{Av}_n^2(\overline{121}, 123, 132, 213) \subseteq S_n^2$.

Informally, the following theorem states that fully vincularizing the Stirling pattern in Theorem 1 changes the Jacobsthal sequence into the Fibonacci-squared sequence.

Theorem 5. $c(n) = \mathsf{Av}_n^2(\overline{121}, 123, 132, 213)$ *for all* $n \geq 0$.

For example, the Fibonacci-squared sequence $\{c_n\}_{n\geq 0}$ is $1, 1, 4, 9, 25, 64, 169, 441, \ldots$ (A007598). Its first four terms count the Fibonacci-squared words over $[n]$ for $n = 0, 1, 2, 3$ as shown in (13)–(16).

$$1 = |\mathsf{Av}_0^2(\overline{121}, 123, 132, 213)| = |\{\epsilon\}| \tag{13}$$

$$1 = |\mathsf{Av}_1^2(\overline{121}, 123, 132, 213)| = |\{11\}| \tag{14}$$

$$4 = |\mathsf{Av}_2^2(\overline{121}, 123, 132, 213)| = |\{1122, 1221, 2112, 2211\}| \tag{15}$$

$$\begin{aligned} 9 = |\mathsf{Av}_3^2(\overline{121}, 123, 132, 213)| = |\{&223311, 233112, 233211, 322311, 323112, \\ &331122, 331221, 332112, 332211\}| \end{aligned} \tag{16}$$

For example, note that 233112 appears in (16) but not in (4). This is because it contains the pattern 121 but not the consecutive pattern $\overline{121}$. Figure 2 provides Fibonacci-squared words up to $n = 5$, and illustrates why Theorem 5 holds. We prove this result in Section 4 using a new recurrence for A007598.

1.6 Outline

Sections 2 and 3 prove Theorems 1 and 2, respectively. The proofs involve simple bijections which could be suitable exercises for students and researchers interested in regular word pattern avoidance. Section 4 gives our vincular result on Fibonacci-squared numbers. Section 5 closes with final remarks.

As previously mentioned, Theorem 1 was proven in a broader context by Kuba and Panholzer (2012). We encourage readers to investigate related results by Janson et al. (2011) and Kuba and Panholzer (2019).

2 Pattern Avoidance for Fibonacci-k Words

The Fibonacci-k sequences up to $k = 9$ are illustrated in Table 2.

Lemma 1. *Suppose* $\alpha \in S_n^k$ *where* α *avoids the patterns* 123, 132, 213 *and let* $y \in \{2, 3, \ldots, n\}$. *Then the smallest symbol that can occur before* y *in* α *is* $y{-}1$.

Proof: Let $\alpha \in S_n^k$ where α avoids 123, 132, 213 and suppose for contradiction that the symbol x occurs before the symbol y in α, where $x < y{-}1$. Consider each of the relative positions of the symbol $y{-}1$:

[v] For further background on pattern avoidance concepts we recommend Bevan (2015).

$Av_4^3(121,123,132,213)$	$Av_2^3(121,123,132,213)$
333444 111222	111222
333444 211122	211122
333444 221112	221112
333444 222111	222111
433344 111222	total: $a_3(2)=4$
433344 211122	
433344 221112	
433344 222111	
443334 111222	
443334 211122	
443334 221112	
443334 222111	$Av_3^3(121,123,132,213)$
444 222333111	222333111
444 322233111	322233111
444 332223111	332223111
444 333111222	333111222
444 333211122	333211122
444 333221112	333221112
444 333222111	333222111
total: $a_3(4)=19$	total: $a_3(3)=7$

(a) Fibonacci-k words for $k=3$ (in lexicographic order). These words contain k copies of $[n]$ and avoid the patterns 121, 123, 132, and 213. Theorem 1 proves that they are enumerated by the $(1,1)$-based Fibonacci-k numbers: $a_k(n)=a(n-1)+k\cdot a(n-2)$ with base cases $a_k(0)=a_k(1)=1$. The lists above show how the $n=4$ words are constructed from $k=3$ copies of the $n=2$ words (blue) and one copy of the $n=3$ words (red). The totals match $a_3(4)=a_3(3)+3\cdot a_3(2)=7+3\cdot 4=19$. The special case of $k=1$ is a classic result for Fibonacci numbers: $\mathsf{Av}_n(123,132,213)=F_{n+1}$ (see Section 1.1).

$Av_4^3(122,213)$	$Av_2^3(122,213)$
44 333 4 221112	221112
44 333 4 221121	221121
44 333 4 221211	221211
44 333 4 222111	222111
44 33 4 2223111	total: $b_3(2)=4$
44 33 4 2231112	
44 33 4 2231121	
44 33 4 2231211	
44 33 4 2232111	
44 33 4 2321112	
44 33 4 2321121	
44 33 4 2321211	
44 33 4 2322111	
44 33 4 3221112	
44 33 4 3221121	
44 33 4 3221211	
44 33 4 3222111	
44 3 4 32223111	
44 3 4 32231112	
44 3 4 32231121	
44 3 4 32231211	
44 3 4 32232111	
44 3 4 32321112	
44 3 4 32321121	
44 3 4 32321211	
44 3 4 32322111	
44 3 4 33221112	
44 3 4 33221121	
44 3 4 33221211	
44 3 4 33222111	$Av_3^3(122,213)$
44 4 332223111	332223111
44 4 332231112	332231112
44 4 332231121	332231121
44 4 332231211	332231211
44 4 332232111	332232111
44 4 332321112	332321112
44 4 332321121	332321121
44 4 332321211	332321211
44 4 332322111	332322111
44 4 333221112	333221112
44 4 333221121	333221121
44 4 333221211	333221211
44 4 333222111	333222111
total: $b_3(4)=43$	total: $b_3(3)=13$

(b) k-Fibonacci words for $k=3$ (in lexicographic order). These words contain k copies of $[n]$ and avoid the patterns 122 and 213. Theorem 2 proves that they are enumerated by the $(1,1)$-based k-Fibonacci numbers when $k\geq 2$: $b_k(n)=k\cdot b(n-1)+b(n-2)$ with base cases $b_k(0)=b_k(1)=1$. The lists to the right show how the $n=4$ words are constructed from one copy of the $n=2$ words (blue) and $k=3$ copies of the $n=3$ words (red). The totals match $b_3(4)=3\cdot b_3(3)+b_3(2)=3\cdot 13+4=43$.

Fig. 1: Illustrating Theorems 1 and 2 and their proofs. (a) The Fibonacci-k words are k-regular words avoiding $\{121,123,132,213\}$ and they are enumerated by the Fibonacci-k numbers $a_k(n)$ for any $k\geq 1$. (b) The k-Fibonacci words are k-regular words avoiding $\{122,213\}$ and they are enumerated by the k-Fibonacci numbers $b_k(n)$ for any $k\geq 2$.

Name	Recurrence $a(n)$	Numbers	OEIS	Pattern Avoidance
Fibonacci	$a(n-1) + a(n-2)$	$1, 1, 2, 3, 5, 8, 13, 21, \ldots$	A000045†	$\mathsf{Av}_n(123, 132, 213)$
Jacobsthal	$a(n-1) + 2 \cdot a(n-2)$	$1, 1, 3, 5, 11, 21, 43, 85, \ldots$	A001045†	$\mathsf{Av}_n^2(121, 123, 132, 213)$
Fibonacci-3	$a(n-1) + 3 \cdot a(n-2)$	$1, 1, 4, 7, 19, 40, 97, 217, \ldots$	A006130	$\mathsf{Av}_n^3(121, 123, 132, 213)$
Fibonacci-4	$a(n-1) + 4 \cdot a(n-2)$	$1, 1, 5, 9, 29, 65, 181, 441, \ldots$	A006131	$\mathsf{Av}_n^4(121, 123, 132, 213)$
Fibonacci-5	$a(n-1) + 5 \cdot a(n-2)$	$1, 1, 6, 11, 41, 96, 301, 781, \ldots$	A015440	$\mathsf{Av}_n^5(121, 123, 132, 213)$
Fibonacci-6	$a(n-1) + 6 \cdot a(n-2)$	$1, 1, 7, 13, 55, 133, 463, 1261, \ldots$	A015441†	$\mathsf{Av}_n^6(121, 123, 132, 213)$
Fibonacci-7	$a(n-1) + 7 \cdot a(n-2)$	$1, 1, 8, 15, 71, 176, 673, 1905, \ldots$	A015442†	$\mathsf{Av}_n^7(121, 123, 132, 213)$
Fibonacci-8	$a(n-1) + 8 \cdot a(n-2)$	$1, 1, 9, 17, 89, 225, 937, 2737, \ldots$	A015443	$\mathsf{Av}_n^8(121, 123, 132, 213)$
Fibonacci-9	$a(n-1) + 9 \cdot a(n-2)$	$1, 1, 10, 19, 109, 280, 1261, 3781, \ldots$	A015445	$\mathsf{Av}_n^9(121, 123, 132, 213)$

Tab. 2: The Fibonacci-k numbers $a_k(n) = a_k(n-1) + k \cdot a_k(n-2)$ with $a_k(0) = 1$ and $a_k(1) = 1$ for $1 \leq k \leq 9$ with their sequences and our pattern avoidance results from Theorem 1. Previous pattern-avoiding results are by Simion and Schmidt (1985) ($k = 1$) (see also Baril (2011) and Baril et al. (2019)), Mansour and Robertson (2002) ($k = 2$), and Sun (2024) ($k = 3$). †OEIS entry starts with 0.

- $(y-1) \cdots x \cdots y$ contains the pattern 213.
- $x \cdots (y-1) \cdots y$ contains the pattern 123.
- $x \cdots y \cdots (y-1)$ contains the pattern 132.

Since each of these patterns must be avoided, we see that x cannot occur before y in α. □

Proposition 1. *Let $n \geq 2$. Then $\alpha \in \mathsf{Av}_n^k(121, 123, 132, 213)$ if and only if one of the following is true:*

(1) $\alpha = n^k \gamma$ where $\gamma \in \mathsf{Av}_{n-1}^k(121, 123, 132, 213)$;

(2) $\alpha = n^b(n-1)^k n^{k-b} \gamma$ where $0 \leq b < k$ and $\gamma \in \mathsf{Av}_{n-2}^k(121, 123, 132, 213)$.

Proof: We start by proving that any element of $\mathsf{Av}_n^k(121, 123, 132, 213)$ is of form (1) or (2) above. Suppose $\alpha = a_1 a_2 \cdots a_m \in \mathsf{Av}_n^k(121, 123, 132, 213)$. If the first k symbols are not all n, let a_i be the first symbol of α that is not n, where $0 \leq i < k$. Since at least one copy of n must follow a_i, a_i can only be equal to $n-1$ by Lemma 1. We claim that all $k-1$ remaining copies of $n-1$ must follow immediately. Suppose for contradiction that a different symbol, t, appears before one of the copies of $n-1$. Then t cannot be equal to n, because more copies of $n-1$ must follow, creating the sub-permutation $(n-1)n(n-1)$, which is the forbidden pattern 121. Thus all copies of $n-1$ must precede the remaining copies of n. The symbol t cannot be less than $n-1$, because there are remaining copies of n to the right, and by Lemma 1, the smallest symbol that can occur before n is $n-1$. Therefore, if α does not begin with n^k, then α must begin with b copies of n, followed by k copies of $n-1$, followed by $k-b$ copies of n, where $0 \leq b < k-1$.

We now show that any element described by (1) or (2) is a member of $\mathsf{Av}_n^k(121, 123, 132, 213)$. Observe first that $\alpha = n^k \gamma \in \mathsf{Av}_n^k(121, 123, 132, 213)$ if $\gamma \in \mathsf{Av}_{n-1}^k(121, 123, 132, 213)$, since none of the patterns we wish to avoid start with a largest symbol. Suppose that α is a word of the form $n^b(n-1)^k n^{k-b} \gamma$, where $\gamma \in \mathsf{Av}_{n-2}^k(121, 123, 132, 213)$ and $0 \leq b < k$. Observe that $n^b(n-1)^k n^{k-b}$ does not contain the pattern 121, and given that γ also did not contain this pattern, α also cannot, since the symbols in γ are all smaller than n and $n-1$. Similarly, the patterns 123, 132, and 213 also cannot occur in α, since the prefix contains the largest values n and $n-1$, and the forbidden patterns did not occur in γ. □

The following remark and corollary complete our proof of Theorem 1 for Fibonacci-k words.

Remark 1. *The set* $\mathsf{Av}_0^k(121, 123, 132, 213) = \{\epsilon\}$*; the set* $\mathsf{Av}_1^k(121, 123, 132, 213) = \{1^k\}$.

Corollary 1. *The following holds for all* $k \geq 1$ *and* $n \geq 2$ *where* $|\mathsf{Av}_0^k(121, 123, 132, 213)| = 1$ *and* $|\mathsf{Av}_1^k(121, 123, 132, 213)| = 1$*,*

$$|\mathsf{Av}_n^k(121, 123, 132, 213)| = |\mathsf{Av}_{n-1}^k(121, 123, 132, 213)| + k \cdot |\mathsf{Av}_{n-2}^k(121, 123, 132, 213)|. \tag{17}$$

Proof: By Proposition 1, the elements of $\mathsf{Av}_n^k(121, 123, 132, 213)$ can be constructed as follows:

- Create $|\mathsf{Av}_{n-1}^k(121, 123, 132, 213)|$ elements of $\mathsf{Av}_n^k(121, 123, 132, 213)$ by inserting n^k in front of each element of $\mathsf{Av}_{n-1}^k(121, 123, 132, 213)$;
- Create $k \cdot |\mathsf{Av}_{n-2}^k(121, 123, 132, 213)|$ elements of $\mathsf{Av}_n^k(121, 123, 132, 213)$ by inserting a prefix of the form $n^b(n-1)^k n^{k-b}$ in front of each element of $\mathsf{Av}_{n-2}^k(121, 123, 132, 213)$ where $0 \leq b < k$. There are k such prefixes.

Therefore, with Remark 1 as a base case, we have our result. Note that in the second step of constructing $\mathsf{Av}_2^k(121, 123, 132, 213) = \{2^b 1^k 2^{k-b} : 0 \leq b \leq k\}$, we create k elements by placing $2^b 1^k 2^{k-b}$ in front of the empty word, to obtain $2^b 1^k 2^{k-b} \cdot \epsilon = 2^b 1^k 2^{k-b}$, for each $0 \leq b < k$. □

3 Pattern Avoidance for k-Fibonacci Words

The k-Fibonacci sequences up to $k = 9$ are illustrated in Table 3.

Lemma 2. *Suppose* $\beta \in \mathsf{Av}_n^k(122, 213)$ *with* $n \geq 2$ *and* $k \geq 2$*. For any* $x, y \in \{1, 2, \ldots, n\}$ *with* $y > x$*, at most one copy of* y *can occur to the right of* x *in* β*.*

Proof: Suppose to the contrary that $\beta \in \mathsf{Av}_n^k(122, 213)$, and in β, some symbol $x < y$ occurs with more than one copy of y to the right of x. Then β contains the pattern 122, which contradicts that $\beta \in \mathsf{Av}_n^k(122, 213)$. □

Lemma 3. *Suppose* $\beta \in \mathsf{Av}_n^k(122, 213)$ *with* $n \geq 2$ *and* $k \geq 2$ *and* $y \in \{2, 3, \ldots, n\}$*. Then the smallest symbol that can occur before* y *in* β *is* $y-1$*.*

Proof: Let $\beta \in \mathsf{Av}_n^k(122, 213)$ and suppose that x occurs before y in β, where $x < y-1$. By Lemma 2, at most one copy of $y-1$ occurs to the right of x, and thus at least $k-1$ copies of $y-1$ occur to the left of x. But then β contains $(y-1) \cdots x \cdots y$, which is 213. □

Let $\alpha = a_1 a_2 \cdots a_n$ be a word. We define $\alpha' = \text{insert}(\alpha, x, i)$ to be the result of inserting the symbol x into position i of α. That is, $\text{insert}(\alpha, x, i) = a_1 a_2 \cdots a_{i-1} x a_i \cdots a_n$.

Proposition 2. *Let* $n \geq 2$ *and* $k \geq 2$*. Then* $\beta \in \mathsf{Av}_n^k(122, 213)$ *if and only if one of the following is true:*

(1) $\beta = n^{k-1}\alpha'$ *for some* $\alpha' = \textit{insert}(\alpha, n, i+1)$ *with* $0 \leq i \leq k-1$ *and* $\alpha \in \mathsf{Av}_{n-1}^k(122, 213)$*;*

Name	Recurrence $b(n)$	Numbers	OEIS	Pattern Avoidance
Catalan		$1, 1, 2, 5, 14, 42, 132, 429, \ldots$	A000108	$\mathsf{Av}_n(132)$
2-Fibonacci	$2 \cdot b(n{-}1) + b(n{-}2)$	$1, 1, 3, 7, 17, 41, 99, 239, \ldots$	A001333	$\mathsf{Av}_n^2(112, 132)$
3-Fibonacci	$3 \cdot b(n{-}1) + b(n{-}2)$	$1, 1, 4, 13, 43, 142, 469, 1549, \ldots$	A003688	$\mathsf{Av}_n^3(112, 132)$
4-Fibonacci	$4 \cdot b(n{-}1) + b(n{-}2)$	$1, 1, 5, 21, 89, 377, 1597, 6765, \ldots$	A015448	$\mathsf{Av}_n^4(112, 132)$
5-Fibonacci	$5 \cdot b(n{-}1) + b(n{-}2)$	$1, 1, 6, 31, 161, 836, 4341, 22541, \ldots$	A015449	$\mathsf{Av}_n^5(112, 132)$
6-Fibonacci	$6 \cdot b(n{-}1) + b(n{-}2)$	$1, 1, 7, 43, 265, 1633, 10063, 62011, \ldots$	A015451	$\mathsf{Av}_n^6(112, 132)$
7-Fibonacci	$7 \cdot b(n{-}1) + b(n{-}2)$	$1, 1, 8, 57, 407, 2906, 20749, 148149, \ldots$	A015453	$\mathsf{Av}_n^7(112, 132)$
8-Fibonacci	$8 \cdot b(n{-}1) + b(n{-}2)$	$1, 1, 9, 73, 593, 4817, 39129, 317849, \ldots$	A015454	$\mathsf{Av}_n^8(112, 132)$
9-Fibonacci	$9 \cdot b(n{-}1) + b(n{-}2)$	$1, 1, 10, 91, 829, 7552, 68797, 626725, \ldots$	A015455	$\mathsf{Av}_n^9(112, 132)$

Tab. 3: The k-Fibonacci numbers $b_k(n) = k \cdot b_k(n-1) + b_k(n-2)$ with $b_k(0) = 1$ and $b_k(1) = 1$ for $2 \leq k \leq 9$ with their sequences and our pattern avoidance results from Theorem 2. Previous pattern-avoiding results for $k = 2$ are by Gao and Niederhausen (2011), Kuba and Panholzer (2012), and Gao and Chen (2014). The Catalan numbers do not follow a Fibonacci-style recurrence, and are not covered by Theorem 2. However, they are included in the table as they count 1-regular words avoiding 112 and 132 (with the former hidden). Many pattern-avoiding results exist for the Catalan numbers, including Knuth (1968), MacMahon (1915), and Baril (2011).

(2) $\beta = n^{k-1}(n{-}1)^k n\gamma$ *where* $\gamma \in \mathsf{Av}_{n-2}^k(122, 213)$.

Proof: We start by proving that any element of $\mathsf{Av}_n^k(122, 213)$ is of form (1) or (2) above. Let $\beta \in \mathsf{Av}_n^k(122, 213)$ where $k \geq 2$. By Lemma 2, for any $x < n$, at most one copy of n occurs to the right of x, and thus at least $k-1$ copies of n occur to the left of x. Therefore, β must begin with n^{k-1}. Consider the position of the kth copy of n. The smallest symbol that can be to its left is $n{-}1$ by Lemma 3. If all k copies of $n{-}1$ occur before the kth copy of n, then $\beta = n^{k-1}(n{-}1)^k n\gamma$ where $\gamma \in \mathsf{Av}_{n-2}^k(122, 213)$. Otherwise, suppose $i < k$ copies of $n{-}1$ occur before the kth copy of n. Then β must begin with $n^{k-1}(n{-}1)^i n(n{-}1)^{k-i-1}$, since at least $k{-}1$ copies of $n{-}1$ must precede all smaller symbols by Lemma 2. Note that any element α of $\mathsf{Av}_{n-1}^k(122, 213)$ begins with $(n{-}1)^{k-1}$, again by Lemma 2. Therefore, β is of the form $n^{k-1}\alpha'$ for some $\alpha' = \text{insert}(\alpha, n, i+1)$ with $0 \leq i \leq k-1$ and $\alpha \in \mathsf{Av}_{n-1}^k(122, 213)$.

We now show that any element described by (1) or (2) is a member of $\mathsf{Av}_n^k(122, 213)$. Consider a word of the form $n^{k-1}(n{-}1)^k n\gamma$ where $\gamma \in \mathsf{Av}_{n-2}^k(122, 213)$. The prefix $n^{k-1}(n{-}1)^k n$ does not contain 122 or 213, and since only smaller symbols occur in γ, $n^{k-1}(n{-}1)^k n\gamma$ also avoids these patterns. Therefore, $n^{k-1}(n{-}1)^k n\gamma \in \mathsf{Av}_n^k(122, 213)$. Similarly, suppose α is in $\mathsf{Av}_{n-1}^k(122, 213)$, and $\alpha' = \text{insert}(\alpha, n, i+1)$, where $0 \leq i \leq k-1$. Consider $n^{k-1}\alpha'$. Since $\alpha \in \mathsf{Av}_{n-1}^k(122, 213)$, α avoids the patterns 122 and 213. Inserting a copy of n within $(n{-}1)^{k-1}$ to create α' will not create either of these patterns, nor will inserting n^{k-1} in front of α'. Thus, $n^{k-1}\alpha' \in \mathsf{Av}_n^k(122, 213)$. □

The following remark and corollary complete our proof of Theorem 2 for k-Fibonacci words.

Remark 2. *The set* $\mathsf{Av}_0^k(122, 213) = \{\epsilon\}$, *where* ϵ *is the empty word. The set* $\mathsf{Av}_1^k(122, 213) = \{1^k\}$.

Corollary 2. *This holds for* $n \geq 2$ *and* $k \geq 2$ *where* $|\mathsf{Av}_0^k(122, 213)| = 1$ *and* $|\mathsf{Av}_1^k(122, 213)| = 1$,

$$|\mathsf{Av}_n^k(122, 213)| = k \cdot |\mathsf{Av}_{n-1}^k(122, 213)| + |\mathsf{Av}_{n-2}^k(122, 213)|. \tag{18}$$

Proof: By Proposition 2, the elements of $\mathsf{Av}_n^k(122, 213)$ can be constructed as follows:

- Create $k \cdot |\mathsf{Av}^k_{n-1}(122, 213)|$ elements of $\mathsf{Av}^k_n(122, 213)$ by placing n^{k-1} in front of each element of $\alpha' = \text{insert}(\alpha, n, i+1)$, for each $\alpha \in \mathsf{Av}^k_{n-1}(122, 213)$ and each $0 \le i \le k-1$;

- Create $|\mathsf{Av}^k_{n-2}(122, 213)|$ elements of $\mathsf{Av}^k_n(122, 213)$ by placing a prefix of the form $n^{k-1}(n-1)^k n$ in front of each element of $\mathsf{Av}^k_{n-2}(122, 213)$.

Therefore, with Remark 2 as a base case, we have our result. Note that in the second step of constructing $\mathsf{Av}^k_2(122, 213) = \{2^{k-1}1^b21^{k-b} : 0 \le b \le k\}$, we create $|\mathsf{Av}^k_0(122, 213)| = 1$ element by placing $2^{k-1}1^k2$ in front of the empty word, to obtain $2^{k-1}1^k2 \cdot \epsilon = 2^{k-1}1^k2$. □

4 Pattern Avoidance for the Fibonacci-Squared Sequence

One motivation of this paper is to promote the use of k-regular words in the pattern avoidance community. To further this goal, we demonstrate here that k-regular words can be combined with non-classical patterns to produce interesting enumeration results.

A *vincular pattern* allows pairs of adjacent symbols in the pattern to be specified as being *attached*, meaning that a subword only contains the pattern if its corresponding pair of symbols are adjacent. Theorem 5 (see Section 1.5.1) modifies Theorem 1 by forcing the 121 pattern in $\mathsf{Av}^2_n(121, 123, 132, 213)$ to be fully attached or consecutive (i.e., both 12 and 21 are attached), and states that the resulting Fibonacci-squared words are counted by the Fibonacci-squared sequence $c(n) = a_1(n)^2$.

The Fibonacci-squared sequence is known to satisfy a number of different formulae, including the following three term recurrence (see A007598).

$$c'(n) = 2 \cdot c'(n-1) + 2 \cdot c'(n-2) - c'(n-3) \text{ with } c'(0) = c'(1) = 1 \text{ and } c'(2) = 4. \tag{19}$$

We begin this section by proving that the Fibonacci-squared sequence also obeys the following recurrence.

$$c''(n) = c''(n-1) + 3 \cdot c''(n-2) + 2 \cdot \sum_{i=3}^{n} c''(n-i) \text{ with } c''(0) = c''(1) = 1. \tag{20}$$

Lemma 4. *The Fibonacci-squared numbers follow the recurrences in* (19) *and* (20)*. That is,* $c(n) = c'(n) = c''(n)$ *for all* $n \ge 0$.

Proof: As previously mentioned, $c(n) = c'(n)$ is known (see A007598) and we prove the second equality. First observe that $c'(0) = 1 = c''(0)$ and $c'(1) = 1 = c''(1)$, and that $c''(2) = c''(1) + 3c''(0) = 4$, which is equal to $c'(2)$. Suppose that $c'(k) = c''(k)$ for all k with $3 \le k \le n-1$ and consider $c'(n) = 2 \cdot c'(n-1) + 2 \cdot c'(n-2) - c'(n-3)$. This is equal to

$c'(n-1) + 2 \cdot c'(n-2) - c'(n-3) + c'(n-1)$, which by the inductive hypothesis,

$$= c''(n-1) + 2 \cdot c''(n-2) - c''(n-3) + c''(n-1)$$
$$= c''(n-1) + 2 \cdot c''(n-2) - c''(n-3) + c''(n-2) + 3 \cdot c''(n-3) + 2 \cdot \sum_{i=3}^{n-1} c''(n-1-i)$$
$$= c''(n-1) + 3 \cdot c''(n-2) + 2 \cdot c''(n-3) + 2 \cdot \sum_{i=3}^{n-1} c''(n-1-i)$$
$$= c''(n-1) + 3 \cdot c''(n-2) + 2 \cdot \sum_{i=3}^{n} c''(n-i) = c''(n). \quad \square$$

Ultimately, we will prove that Fibonacci-squared words can be decomposed into smaller words according to (20). Towards this decomposition, we make several definitions involving regular words below.

Given a non-empty k-regular word over $[n]$, its *base* is its longest suffix that is a k-regular word over $[m]$ for some $m < n$. That is, if $\gamma \in S_n^k$ is non-empty, then its base is the word $\beta \in S_m^k$ that maximizes m subject to $\gamma = \alpha\beta$ and $0 \leq m < n$. The assumption that $\gamma \in S_n^k$ is non-empty is equivalent to $n > 0$ (with the implicit assumption that $k \geq 1$). The base is well-defined for all non-empty k-regular words. This is because all such γ have a shortest suitable suffix, namely $\beta = \epsilon$. Our definition also ensures that the base is a *strict suffix* (i.e., it is not equal to entire word). As a result, the remaining prefix is non-empty, and we refer to it as the *annex*. Equivalently, the annex is the shortest non-empty prefix consisting of k copies of each symbol in $\{n, n-1, \ldots, m+1\}$ for some $m < n$. Given a non-empty regular word $\gamma \in S_n^k$, its *standard partition* is $\gamma = \alpha\beta$, where β is its base and α is its annex.

Standard partitions are illustrated for several regular words γ below. When considering these examples, remember that the base β is chosen to be as long as possible without equalling γ and that $\beta = \epsilon$ is allowed.

$$\gamma = \underbrace{656556}_{\alpha}\underbrace{443234322111}_{\beta} \in S_6^3 \qquad \gamma = \underbrace{123456}_{\alpha}\underbrace{}_{\beta} \in S_6^1 \qquad \gamma = \underbrace{333}_{\alpha}\underbrace{222111}_{\beta} \in S_3^3$$

Now we return our attention to Fibonacci-squared words, which are a subset of the 2-regular words. Since a Fibonacci-squared word avoids $\overline{121}$, 123, 132, 213, so too does every prefix and suffix of it. In particular, the both the annex and the base of any Fibonacci-squared word avoid all four patterns. Due to the structure of the patterns, the converse is also true. $\square$

Lemma 5. *If a non-empty* 2*-regular word* $\gamma \in S_n^2$ *has standard partition* $\gamma = \alpha\beta$ *and both* α *and* β *avoid the patterns* $\overline{121}$, 123, 132, 213, *then* γ *is a Fibonacci-squared word. That is,* $\gamma \in \mathsf{Av}_n^2(\overline{121}, 123, 132, 213)$.

Proof: Each of the four patterns has a smaller symbol followed by a larger symbol. If α and β each avoid these patterns, then it is impossible for α followed by β to contain any of these patterns, since the symbols in α are all larger than the symbols in β. $\square$

Now we make final preparations before proving the main result of this section. Our proof considers an arbitrary Fibonacci-squared word $\gamma \in \mathsf{Av}_n^2(\overline{121}, 123, 132, 213)$ from left-to-right. We argue that its annex is either one of four special cases, or can be described as a path within a particular tree that we now define. The *prefix-tree* T_n is a bi-rooted tree with $4(n-1)$ vertices with labels taken from $[n]$. It contains a primary path that follows a *two steps forward one step back* pattern from n down to 1.

$$n, n-2, n-1, n-3, n-2, n-4, n-3, \ldots, 5, 6, 4, 5, 3, 4, 2, 3, 1. \tag{21}$$

Every second node i on the primary path (i.e., $n-2, n-3, n-4, \ldots, 2, 1$) has an edge to another copy of i which has an edge to another copy of $i+1$. Finally, one root is labeled n and has an edge to another copy of $n-1$, while a second root is labeled $n-1$ and has an edge to another copy of n, and both of these non-root nodes have an edge to the start of the primary path. In the prefix tree, the label 1 appears twice, the labels 2 and n appear three times, and all other labels appear 4 times. Thus the tree contains $2 + 2 \cdot 3 + 4(n-3) = 8 + 4n - 12 = 4(n-1)$ vertices. Figure 2 illustrates T_n and T_4.

Theorem 6. *Any word $\gamma \in \mathsf{Av}_n^2(\overline{121}, 123, 132, 213)$ has the standard partition $\alpha\beta$ where α is either a root-to-leaf path on the prefix tree, or is one of the four special cases nn, $(n-1)(n-1)nn$, $(n-1)nn(n-1)$, $n(n-1)(n-1)n$.*

Proof: We describe all possible annexes for a word $\gamma \in \mathsf{Av}_n^2(\overline{121}, 123, 132, 213)$. Observe first that nn is an annex. Next suppose $\gamma \in \mathsf{Av}_n^2(\overline{121}, 123, 132, 213)$ and γ does not begin with nn. By Lemma 1, the only element that can precede n is $n-1$. We first consider length-4 prefixes where all symbols are from $\{n-1, n\}$. There are $\frac{4!}{2!2!} = 6$ possible such prefixes; 5 that do not start with nn. Of the remaining possibilities, both $(n-1)n(n-1)n$ and $n(n-1)n(n-1)$ contain $\overline{121}$, and thus are excluded. This leaves three possibilities, $(n-1)(n-1)nn$, $(n-1)nn(n-1)$, and $n(n-1)(n-1)n$, which are all annexes.

The annexes described so far are the four special cases not generated by the prefix tree. Next suppose $\gamma \in \mathsf{Av}_n^2(\overline{121}, 123, 132, 213)$ and the first four symbols are not all from $\{n-1, n\}$. Both copies of n must occur in the first four positions since the only symbol that can precede n is $n-1$. The only symbol that can precede $n-1$ is $n-2$. Therefore, if γ is not in one of the special cases, the first four symbols of γ consist of two copies of n, one copy of $n-1$, and one copy of $n-2$. Since $n-2$ cannot precede n, $n-2$ cannot occur in the first two positions. Further, $n-2$ in the third position results in $nn(n-2)(n-1)$, the already considered annex nn. Thus, $n-2$ can only be in the fourth position. There are two ways to place n, n, $n-1$ in the first three positions that avoid the annex nn. Thus, if γ does not begin with one of the four special cases, then γ begins with $(n-1)nn(n-2)$ or $n(n-1)n(n-2)$. Observe that the only difference between these two strings is whether the first two symbols are $(n-1)n$ or $n(n-1)$, the two roots of the prefix tree.

Continuing from $(n-1)n$ or $n(n-1)$, the remaining copy of $n-1$ can only be preceded by $n-2$. Thus, there are only two ways for γ to continue: $(n-2)(n-1)$ or $(n-1)$. The first case results in the annexes $(n-1)nn(n-2)(n-2)(n-1)$ and $n(n-1)n(n-2)(n-2)(n-1)$, corresponding with the first leaf on the prefix tree. In the second case, $(n-1)nn(n-2)$ or $n(n-1)n(n-2)$ continues with $(n-1)$. The symbol $n-2$ is not yet paired, and another $n-2$ cannot follow immediately, because this would create the forbidden $\overline{121}$. All copies of symbols greater than $n-2$ have already occurred, and the smallest symbol that can proceed $n-2$ is $n-3$, so $n-3$ must be next. It can either be that both copies of $n-3$ precede the second copy of $n-2$, or a single copy precedes $n-2$. The first case creates the annexes $(n-1)nn(n-2)(n-1)(n-3)(n-3)(n-2)$ and $n(n-1)n(n-2)(n-1)(n-3)(n-3)(n-2)$, corresponding with the second leaf on the prefix tree. Otherwise, we continue down the primary path with a single copy of $n-3$ preceding $n-2$.

We are now in the same position as in the previous step. The $n-3$ is unpaired, but placing $n-3$ immediately would create $\overline{121}$. We can either continue to the next leaf with $(n-4)(n-4)(n-3)$, creating two annexes, or continue down the primary path with $(n-4)(n-3)$, in which case we are in an equivalent position.

At each step, root $(n-1)nn\cdots$ or $n(n-1)n\cdots$ ends with $(n-k+1)(n-k+2)$, and can either continue with the leaf $(n-k)(n-k)(n-k+1)$, or down the primary path with $(n-k)(n-k+1)$. The first case creates two annexes. The second case, continues down the primary path and encounters an equivalent pair of choices. This pattern continues down to $k = n-1$, where the annex is now a word in $\mathsf{Av}_n^2(\overline{121}, 123, 132, 213)$, and the base is the empty word. We have found all possible annexes for γ. $\square$

Corollary 3 restates Theorem 5 using Lemma 4.

Corollary 3. $c''(n) = |\mathsf{Av}_n^2(\overline{121}, 123, 132, 213)|$ *for all* $n \geq 0$.

Proof: We proceed by strong induction on n. The base cases of $n = 0$ and $n = 1$ hold by the following.

- $|\mathsf{Av}_0^2(\overline{121}, 123, 132, 213)| = |\{\epsilon\}| = 1 = c''(0)$.
- $|\mathsf{Av}_0^2(\overline{121}, 123, 132, 213)| = |\{11\}| = 1 = c''(1)$.

Now we must prove the following.

$$|\mathsf{Av}_n^2(\overline{121}, 123, 132, 213)| = c''(n) = c''(n-1) + 3 \cdot c''(n-2) + 2\sum_{i=3}^{n} c''(n-i). \tag{22}$$

Consider each of the possible annexes for $\gamma \in \mathsf{Av}_n^2(\overline{121}, 123, 132, 213)|$. Words of the form $nn\beta$ where $\beta \in \mathsf{Av}_{n-1}^2(\overline{121}, 123, 132, 213)$ give $|\mathsf{Av}_{n-1}^2(\overline{121}, 123, 132, 213)|$ elements to $\mathsf{Av}_n^2(\overline{121}, 123, 132, 213)$. This accounts for the $c''(n-1)$ term in (22). The three annexes $(n-1)(n-1)nn$, $(n-1)nn(n-1)$, and $n(n-1)(n-1)n$ each contribute $|\mathsf{Av}_{n-2}^2(\overline{121}, 123, 132, 213)|$ elements to $\mathsf{Av}_n^2(\overline{121}, 123, 132, 213)$. This accounts for the $3c''(n-2)$ term in (22).

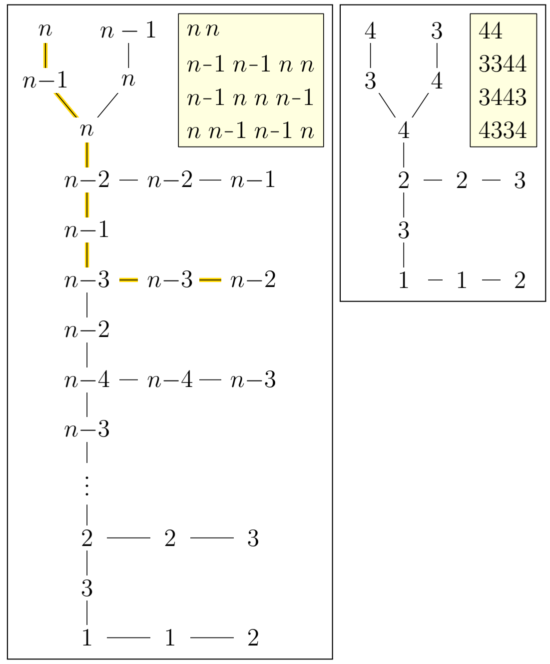
For each $2 \leq k \leq n-1$, there are two root-to-leaf paths in the prefix tree. These are the annexes $(n-1)nn\cdots(n-k)(n-k)(n-k+1)$ and $n(n-1)n\cdots(n-k)(n-k)(n-k+1)$, each of which composes with elements of $\mathsf{Av}_{n-k-1}^2(\overline{121}, 123, 132, 213)$. These are the root-to-leaf paths in the prefix tree, and together, these contribute $2 \cdot \sum_{i=3}^{n} c''(n-i)$ to (22). $\square$

5 Final Remarks

We considered three pattern avoiding results involving k-regular words and Fibonacci sequences. This includes a simple proof of a known result for Fibonacci-k words, a new result for k-Fibonacci words, and a new result for the Fibonacci-squared sequence. We hope that these results, together with those for k-Catalan sequences (see Section 1.5) and new results on $(1, b)$-based Pell numbers by Hartung and Williams (2024) will serve as inspiration for further study of pattern avoidance in regular words or even non-regular words (see Burstein (1998)). We conclude with some additional comments and thoughts.

$Av_3^2(\overline{121}, 123, 132, 213)$	$Av_0^2(\overline{121}, 123, 132, 213)$
3344 1122	ϵ
3344 1221	total: $c(0) = 1$
3344 2112	
3344 2211	$Av_1^2(\overline{121}, 123, 132, 213)$
344223 11	11
34423112	total: $c(1) = 1$
3443 1122	
3443 1221	
3443 2112	
3443 2211	$Av_2^2(\overline{121}, 123, 132, 213)$
4334 1122	1122
4334 1221	1221
4334 2112	2112
4334 2211	2211
434223 11	total: $c(2) = 4$
43423112	$Av_3^2(\overline{121}, 123, 132, 213)$
44 223311	223311
44 233112	233112
44 233211	233211
44 322311	322311
44 323112	323112
44 331122	331122
44 331221	331221
44 332112	332112
44 332211	332211
total: $c(4) = 25$	total: $c(3) = 9$

(a) Fibonacci-squared words (in lexicographic order). These words contain 2 copies of $[n]$ and avoid the consecutive pattern $\overline{121}$ and the classic patterns 123, 132, and 213. Theorem 5 proves that they are enumerated by the Fibonacci-squared numbers $c(n) = a_1(n)^2$, which satisfy the recurrence $c''(n) = c''(n-1) + 3 \cdot c''(n-2) + 2\sum_{i=3}^{n} c''(n-i)$ as proven in Lemma 4. The lists above show how the $n = 4$ words are constructed from two copies of the $n-4 = 0$ words (grey), two copies of the $n-3 = 1$ words (green), three copies of the $n-2 = 2$ words (blue), and one copy of the $n-1 = 3$ words (red). The totals match $c(4) = c(3) + 3 \cdot c(2) + 2 \cdot c(1) + 2 \cdot c(0) = 9 + 3 \cdot 4 + 2 \cdot 1 + 2 \cdot 1 = 25$.



(b) The prefix-tree T_n (left). It is drawn with its two roots (n and $n-1$) at the top, and its $n-2$ leaves on the right. The tree's root-to-leaf paths encode annexes. For example, the highlighted path encodes the annex $\alpha' = n\,n{-}1\,n\,n{-}2\,n{-}1\,n{-}3\,n{-}3\,n{-}2$. Thus, $\alpha'\beta' \in \mathsf{Av}_n^2(\overline{121}, 123, 132, 213)$ for any $\beta' \in \mathsf{Av}_{n-4}^2(\overline{121}, 123, 132, 213)$ by Lemma 5. Starting at the second copy of n the vertical edges follow a *two steps forward one step back* pattern downward, and horizontal edges complete the prefix with the two smallest and yet-to-be paired values. The inset shows the four short annexes (e.g., $n\ n$). The prefix-tree T_n for $n = 4$ (right). The four root-to-leaf paths give the annexes 434223, 344223, 43423112, and 34423112 found in (a). The remaining annexes are the special cases 44 with 3344, 3443, and 4334.

Fig. 2: Illustrating Theorem 5 and its proof. (a) The Fibonacci-squared words are 2-regular words avoiding $\{\overline{121}, 123, 132, 213\}$ and they are enumerated by the Fibonacci-squared numbers $c(n)$ (or equivalently, $c''(n)$). Each word is written in its standard partition $\gamma = \alpha\beta$ where α is the annex and β is the base. The bases are smaller Fibonacci-squared words, while each annex is one of four special cases or can be visualized as a labeled root-to-leaf path in the prefix-tree (b).

5.1 Other Fibonacci Sequences with Parameter k

Pattern avoidance with k-regular words is particularly promising for sequence families with one parameter. Other Fibonacci sequences with this property include those using base cases $(1, k)$ or $(k, 1)$ or (k, k). Another natural target is the $(1, 1)$-based k-Fibonacci-k sequence, which satisfies the following formula.

$$d_k(n) = k \cdot d_k(n-1) + k \cdot d_k(n-2) \text{ with } d_k(0) = 1 \text{ and } d_k(1) = 1. \tag{23}$$

Interestingly, a pattern avoidance result for $d_2(n)$ was established using d-permutations by Sun (2024). More specifically, $d_2(n)$ is the number of 3-permutations of $[n]$ avoiding 231 and 312. See Bonichon and Morel (2022) for further information on d-permutations (which are not d-regular permutations). These sequences are summarized in Table 4. Note $d_k(n)$ does not refer to the double list notion in Table 1.

Another single parameter generalization is the *k-generalized Fibonacci numbers* which involve summing the previous k entries in the sequence. Pattern avoidance results involving these numbers have been considered by Egge and Mansour (2004).

Recurrence $d(n)$	Numbers	OEIS	Pattern Avoidance
$d(n-1) + d(n-2)$	$1, 1, 2, 3, 5, 8, 13, 21, \ldots$	A000045†	$\mathsf{Av}_n(123, 132, 213)$
$2 \cdot d(n-1) + 2 \cdot d(n-2)$	$1, 1, 4, 10, 28, 76, 208, 568, \ldots$	A026150	$S_n^2(231, 312)$
$3 \cdot d(n-1) + 3 \cdot d(n-2)$	$1, 1, 6, 21, 81, 306, 1161, 4401, \ldots$	A134927	
$4 \cdot d(n-1) + 4 \cdot d(n-2)$	$1, 1, 8, 36, 176, 848, 4096, 19776, \ldots$	A164545‡	
$5 \cdot d(n-1) + 5 \cdot d(n-2)$	$1, 1, 10, 55, 325, 1900, 11125, 65125, \ldots$	A188168	
$6 \cdot d(n-1) + 6 \cdot d(n-2)$	$1, 1, 12, 78, 540, 3708, 25488, 175176, \ldots$		
$7 \cdot d(n-1) + 7 \cdot d(n-2)$	$1, 1, 14, 105, 833, 6566, 51793, 408513, \ldots$		
$8 \cdot d(n-1) + 8 \cdot d(n-2)$	$1, 1, 16, 136, 1216, 10816, 96256, 856576, \ldots$		
$9 \cdot d(n-1) + 9 \cdot d(n-2)$	$1, 1, 18, 171, 1701, 16848, 166941, 1654101, \ldots$		

Tab. 4: The k-Fibonacci-k sequences $d_k(n) = k \cdot d_k(n-1) + k \cdot d_k(n-2)$ with $d_k(0) = 1$ and $d_k(1) = 1$ provide an alternate way of parameterizing the Fibonacci numbers. A previous pattern-avoiding result for $k = 2$ is by Sun (2024) where S_n^2 denotes pattern avoidance in 3-permutations. (Note that a 3-permutation is not a 3-regular permutation but rather a pair of permutations and their 3-dimensional diagram Bonichon and Morel (2022).) †OEIS entry starts with 0. ‡OEIS entry omits an initial 1.

5.2 Avoiding Patterns and Multi-Patterns

When working with regular words it is natural to avoid *multi-patterns* (i.e., patterns that have repeated symbols). In particular, our three main theorems each included the avoidance of one multi-pattern. More broadly, we can ask for regular word avoidance results for any subset of patterns and multi-patterns. To aid in this discussion, let M_n^m be the set of strings of length n containing each symbol in $[m]$ at least once. For example, $M_3^2 = \{112, 121, 122, 211, 212, 221\}$ is the set of multi-patterns with 2 distinct symbols and length 3. Standard patterns arise when $m = n$ (i.e., $M_n^n = S_n$).

One challenge in this direction of inquiry is the sheer number of possibilities. For example, if we consider standard patterns and multi-patterns of length $n = 3$, then there are $2^{12} = 4096$ different subsets of $S_3 \cup M_3^2$. Moreover, each of these subsets must be considered for regular words of various frequencies. For this reason we must be conservative when trying to summarize all possible results of this type, even when restricted to patterns of length three.

α, β	Parameterized Sequence	Proof	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$
$123, 112$ $132, 121$ $132, 122$	$(r+1)$-Catalan	Williams (2023) Kuba and Panholzer (2012) Defant and Kravitz (2020)	A001764	A002293	A002294	A002295	A002296	A007556	A059967
$132, 112$	r-Fibonacci	Theorem 2	A001333	A003688	A015448	A015449	A015451	A015453	A015454
$132, 221$	$(1, r+1)$-based Pell	Hartung and Williams (2024)	A001333	A048654	A048655	A048693	A048694	A048695	A048696
$132, 211$	$r(n-1)+1$	Hartung and Williams (2024)	A005408	A016777	A016813	A016861	A016921	A016993	A017077
$123, 211$	$1, r+1, 0, 0, 0, \ldots$	Hartung and Williams (2024)							
$123, 121$ $132, 212$	$\sum_{i=0}^{n} \frac{\binom{n}{i}}{n-i+1} \binom{n+(r-1)\cdot i-1}{n-i}$	Conjecture 1	A109081	A161797	A321798	A321799			

Tab. 5: Avoiding $\alpha \in S_3$ and $\beta \in M_3^2$ in r-regular words: known results, new results, and a conjecture. The $|S_3| \cdot |M_3^2| = 36$ pairs of (α, β) partition into the $\frac{36}{4} = 9$ equivalence classes listed above based on the symmetries of the square (e.g., $\{(123, 121), (321, 121), (123, 212), (321, 212)\}$ is represented by $(123, 121)$ above).

One manageable goal is to understand all possibilities of avoiding one standard pattern $\alpha \in S_3$ and one multi-pattern $\beta \in M_3^2$. There are $|S_3| \cdot |M_3^2|/4 = \frac{6 \cdot 6}{4} = 9$ distinct (α, β) pairs to considewr up to the symmetries of the square. Table 5 summarizes the known results, including our new result in Theorem 2.

When creating Table 5 we observed that there are two non-isomorphic case that are currently unresolved. Based on computational experiments involving generating and testing regular words (see Williams (2009)), we feel comfortable in stating the following conjecture for the missing cases. While formula (24) in Conjecture 1 is fairly complicated, Table 5 shows that there are existing OEIS entries for $r = 2, 3, 4, 5$.

Conjecture 1. *The number of r-regular words over $[n]$ that avoid 123 and 121, or 132 and 212, is below.*

$$|\mathsf{Av}_n^r(123, 121)| = |\mathsf{Av}_n^r(132, 212)| = \sum_{i=0}^{n} \frac{\binom{n}{i}}{n-i+1} \binom{n+(r-1) \cdot i-1}{n-i} \tag{24}$$

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