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Pin classes II: Small pin classes

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Pin permutations play an important role in the structural study of permutation classes, most notably in relation to simple permutations and well-quasi-ordering, and in enumerative consequences arising from these. In this paper, we continue our study of *pin classes*, which are permutation classes that comprise all the finite subpermutations contained in an infinite pin permutation. We show that there is a phase transition at $\mu \approx 3.28277$: there are uncountably many different pin classes whose growth rate is equal to μ, yet only countably many below μ. Furthermore, by showing that all pin classes with growth rate less than μ are essentially defined by pin permutations that possess a periodic structure, we classify the set of growth rates of pin classes up to μ.

Keywords: permutation classes, pin permutation, growth rate, generating function

1 Introduction

This paper continues the study of pin classes started in Jarvis (2024). Here, we investigate the spectrum of 'small' growth rates of such classes, up to growth rate $\mu \approx 3.28277$. This work follows the line of enquiry undertaken by several papers – for example Huczynska and Vatter (2006); Kaiser and Klazar (2003) and Vatter (2010, 2011, 2019) – which explored the spectrum of growth rates in general permutation classes. Of special note are Pantone and Vatter (2020) who established a complete characterisation of the real numbers up to $\xi \approx 2.30522$ that are permutation class growth rates, and Bevan (2018) who showed that every real number greater than $\lambda \approx 2.35526$ is the growth rate of some permutation class.

Many of the features of the spectrum of growth rates up to ξ and from λ are driven by a single underpinning object: the *oscillations*. These objects govern several phase transitions in the possible growth rates and number of distinct classes, most notably at $\kappa \approx 2.20557$ (the largest real root of $z^3 - 2z^2 - 1$) which represents the smallest growth rate for which there exist uncountably many distinct permutation classes – see Vatter (2011).

Oscillations are one type of *pin permutation* (for full definitions, see Section 2), which were originally introduced by Brignall, Huczynska, and Vatter (2008a) as a tool in a Ramsey-theoretic study of simple permutations. Since then, they have been studied from a number of viewpoints: Bassino, Bouvel, Pierrot, and Rossin (2015) describe an efficient decision procedure to determine whether a given permutation class contains arbitrarily long pin permutations, while Bassino, Bouvel, and Rossin (2011) showed that

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the permutation class formed of all pin permutations and their subpermutations has a rational generating function. More recently, Brignall and Vatter (2026) used pin permutations to construct uncountably many well-quasi-ordered permutation classes with distinct enumerations, thereby disproving a conjecture of Vatter (2015). Finally, Chudnovsky, Kim, Oum, and Seymour (2016) use the term *chain* to describe an analogous object in the study of hereditary classes of graphs.

Given an (infinite) pin permutation, the *pin class* is formed by taking all finite subpermutations that are contained in it. In the previous paper by Jarvis (2024), it was shown (among other results) that pin classes always have growth rates. In this paper, we consider the pin classes with smallest growth rate, and establish a phase transition at growth rate $\mu \approx 3.28277$, the largest real zero of $z^5-4z^4+3z^3-2z^2-z+2$. Specifically:

Theorem 5.2 *There are uncountably many pin classes $\mathcal{C}$ with growth rate equal to μ.*

Theorem 6.3 *Any pin class $\mathcal{C}$ for which $\mathrm{gr}(\mathcal{C}) < \mu$ is defined by a pin permutation whose structure is ultimately periodic, and has growth rate equal either to κ or to ν_ℓ for some $\ell \geqslant 1$, defined to be the largest real zero of*

$$z^{2\ell} - 4z^{2\ell-1} + 3z^{2\ell-2} - 2z^{2\ell-3} - z^{2\ell-4} + 2z^{2\ell-5} + 1.$$

It has been known for some time that κ is the least growth rate of a pin class (thanks to the appearance of infinite oscillations established by Vatter (2011)). One interesting observation arising from our work is that the second least growth rate of a pin class is not reached until $\nu = \nu_1 \approx 3.06918$. Also of note is that, among the uncountable collection of pin classes identified that have growth rate $\mu \approx 3.28277$, there exist classes whose defining pin sequence is not eventually recurrent (Proposition 5.3), even though all pin classes with smaller growth rates must be defined by a periodic or eventually periodic sequence.

The rest of this paper is organised as follows. In Section 2, we cover some basic definitions and results, including those concerning growth rates, the 'box sum' decomposition, pin sequences and pin permutations, and the combinatorics of words. Section 3 develops some theory for pin permutations and begins the process of restricting which pin permutations need to be considered when working below growth rate μ. Sections 4 and 6 together fully classify the pin classes whose growth rate is less than μ, while the intervening Section 5 exhibits various pin classes that occur at growth rate μ.

2 Preliminaries

We refer the reader to Bevan (2015) for background information and basic definitions concerning permutations and permutation classes.

The following definition is derived from the study of permutation classes, but here note that we present it in fuller generality. Let (s_n) be a sequence of non-negative integers. The *upper* and *lower growth rates* are defined, respectively, to be

$$\overline{\mathrm{gr}}((s_n)) = \limsup_{n\to\infty} \sqrt[n]{s_n}, \quad \text{and} \quad \underline{\mathrm{gr}}((s_n)) = \liminf_{n\to\infty} \sqrt[n]{s_n},$$

when these quantities exist. If $\overline{\mathrm{gr}}((s_n)) = \underline{\mathrm{gr}}((s_n))$, then the sequence has an actual *growth rate*, typically denoted $\mathrm{gr}((s_n))$.

We will frequently abuse this notation, for example if we are given some collection $\mathcal{C}$ of combinatorial objects, then $\overline{\mathrm{gr}}(\mathcal{C})$ and $\underline{\mathrm{gr}}(\mathcal{C})$ are defined using the sequence $(|\mathcal{C}_n|)$ that counts the number of objects in $\mathcal{C}$ of each size n. If $\mathcal{C}$ is a proper permutation class, for example, then this recovers the standard notions of growth rates for permutation classes. Similarly, if we are given a generating function $f(z)$, then we can also use the term $\mathrm{gr}(f(z))$ to refer to the growth rate of the sequence of coefficients in the Taylor expansion of $f(z)$, or, equivalently (by standard analytic combinatorics), the reciprocal of the radius of convergence of the formal power series $f(z)$.

Recall that the *plot* of a permutation π is the points $(i, \pi(i))$ in the Euclidean plane. Consider a set $\mathcal{L}$ of horizontal and vertical lines drawn in this plane – we refer to this as a *gridded plane*. For a permutation π and a collection $\mathcal{L}$ of lines defining a gridded plane, the *gridded permutation* $\pi^\sharp$ is the pair $(\pi, \mathcal{L})$. Such an object is typically regarded as a division of the plot of π into a grid of rectangular cells.

For enumerative purposes, the length of a gridded permutation is the same as its underlying ungridded version. However, two griddings $\pi^\sharp$ and $\pi^\natural$ of the same underlying permutation π are only regarded as equal if they have the same number of vertical and horizontal lines, and if each rectangular cell of one gridding contains exactly the same points of π as the corresponding rectangular cell of the other.

Given a permutation class $\mathcal{C}$ and fixed non-negative integers k, ℓ, we then let

$$\mathcal{C}^\sharp = \{(\pi, \mathcal{L}) \;:\; \pi \in \mathcal{C} \text{ and } \mathcal{L} \text{ has exactly } k \text{ horizontal and } \ell \text{ vertical lines}\},$$

that is, the collection of all permutations in $\mathcal{C}$, each divided in every possible way using the prescribed number of lines. Clearly every permutation in $\mathcal{C}$ gives rise to multiple elements of $\mathcal{C}^\sharp$, but in the context of growth rates this presents no problem, as the following result shows.

Lemma 2.1 (C.f. Vatter (2011) Proposition 2.1)**.** *Let $\mathcal{C}$ be a permutation class, and let $\mathcal{C}^\sharp$ denote the set of gridded permutations with some fixed numbers k of horizontal and ℓ of vertical lines. Then $\overline{\mathrm{gr}}(\mathcal{C}) = \overline{\mathrm{gr}}(\mathcal{C}^\sharp)$ and $\underline{\mathrm{gr}}(\mathcal{C}) = \underline{\mathrm{gr}}(\mathcal{C}^\sharp)$.*

Since the wording of the above lemma differs somewhat from its other appearances in the literature, we provide the short proof.

Proof: For a permutation of length n, there are precisely $\binom{n+k}{k}$ distinct ways to place the k horizontal lines, and $\binom{n+\ell}{\ell}$ ways to place the vertical ones. Thus, each $\pi \in \mathcal{C}$ gives rise to precisely $P(n) = \binom{n+k}{k}\binom{n+\ell}{\ell}$ different gridded permutations in $\mathcal{C}^\sharp$, and so

$$|\mathcal{C}_n^\sharp| = P(n)|\mathcal{C}_n|,$$

where $\mathcal{C}_n$ denotes the set of permutations of length n in $\mathcal{C}$. The lemma now follows by noting that the function $P(n)$ is a polynomial of degree $k + \ell$, and hence $\sqrt[n]{P(n)} \to 1$ as $n \to \infty$. □

The above lemma guarantees that, as far as asymptotic enumeration is concerned, we can move freely between gridded and ungridded permutations. We would like to work *exclusively* with gridded permutations, and for this we need to extend the standard notion of permutation containment (see Bevan Bevan (2015) for a formal definition): given two gridded permutations $\sigma^\sharp$ and $\pi^\sharp$ which possess the same number of horizontal and vertical lines, we say that $\sigma^\sharp \leqslant \pi^\sharp$ if there is an injection f that embeds σ into π so that $\sigma^\sharp$ is equal (as a gridded permutation) to the image $f(\sigma)$ in the gridded plane containing π.

While we have defined the set $\mathcal{C}^\sharp$ to comprise *all* possible griddings of permutations in $\mathcal{C}$, we are typically only interested in sets made up of particular griddings: providing every $\pi \in \mathcal{C}$ possesses at least one gridded version in this set, then the upper and lower growth rates of the set will again coincide with those of $\mathcal{C}$.

Pin permutations The *rectangular hull* of a set of points in the plot of a permutation (or gridded permutation) is the smallest axis-parallel rectangle that contains them. Let $(p_1, \dots, p_k)$ be a sequence of points in the plane (or in a gridded plane). A *proper pin* for $(p_1, \dots, p_k)$ is a point p that lies outside the rectangular hull of $\{p_1, \dots, p_k\}$, but which lies horizontally or vertically between $\{p_1, \dots, p_{k-1}\}$ and p_k. The position of the pin p relative to the rectangular hull of $\{p_1, \dots, p_k\}$ naturally leads us to describe a proper pin as *left*, *right*, *up* or *down*.

A *pin permutation* is a permutation defined by a sequence of points $(p_1, \dots, p_k)$ for which each p_i ($i \geqslant 3$) is a proper pin for $(p_1, \dots, p_{i-1})$. This definition implicitly assumes that the first two points p_1 and p_2 are specified in advance, and for our purposes it suffices to consider pin permutations initiated in the following way. We work in the 2×2 gridded plane – that is, the plane equipped with one horizontal and one vertical line, and we can take these lines to be the x- and y-axes, respectively (so that the intersection of the two lines is at the origin). Each of the four regions of our grid, therefore, corresponds to a quadrant, and these are numbered 1 to 4 in the usual manner:

$$\begin{array}{c|c} 2 & 1 \\ \hline 3 & 4 \end{array}.$$

From now on, we will be working exclusively with permutations that are gridded in this 2×2 grid. Note that Jarvis (2024) works instead with the notion of a *centered* permutation, which is a permutation equipped with an additional coordinate. There is a clear equivalence between these and 2×2 gridded permutations, in which the additional coordinate of a centered permutation corresponds to the origin of a 2×2 gridded permutation. In this paper, either viewpoint can be taken, and in any case we will drop the $\sharp$ superscript.

To construct a pin permutation $(p_1, p_2, \dots)$ in the 2×2 grid, we use the following procedure: point p_1 can be placed in any of the four quadrants, and point p_2 is then placed so that it is a proper pin for $\{(0, 0), p_1\}$. We can thus assign the point p_2 with one of the same four directions as subsequent pins. Subsequent points are then added according to the rules defined above, with the additional requirement that the origin $(0, 0)$ belongs to each rectangular hull. In other words, we treat the origin $(0, 0)$ as a '0th' pin.

Pin permutations necessarily alternate their direction: if $(p_1, p_2, \dots)$ is a pin permutation and p_i is an up or down (respectively, left or right) pin, then p_{i+1} must be a left or right (respectively, up or down) pin in order to ensure that it separates p_i from all earlier pins. Note further that for $i > 2$ the quadrant containing p_i is determined by the directions of p_{i-1} and p_i, whereas the quadrant containing p_2 is determined by the direction of p_2, and the quadrant containing p_1.

We note here two specific families of pin permutations: an *oscillation* is a pin permutation for which each pin has one of only two directions: if these directions are right/up or down/left, then we have an *increasing oscillation*, otherwise we have a *decreasing oscillation*.

Given the way that a pin's direction defines its role in the construction of a pin permutation, it is natural to encode pin permutations using words over a finite alphabet. We will call the encoding that has been used most frequently in the past the *basic encoding*; this encoding uses the alphabet $\{\mathsf{l}, \mathsf{r}, \mathsf{u}, \mathsf{d}\}$ to encode each pin p_i for $i \geqslant 2$, together with a *quadrant letter* from $\{1, 2, 3, 4\}$ to encode the position of p_1. In this way, a pin permutation π of length n corresponds to a word $e(\pi)$ of length n in the language defined by the following regular expression:

$$\{1, 2, 3, 4\}\{\varepsilon, \mathsf{u}, \mathsf{d}\}(\{\mathsf{l}, \mathsf{r}\}\{\mathsf{u}, \mathsf{d}\})^* \cup \{1, 2, 3, 4\}\{\varepsilon, \mathsf{l}, \mathsf{r}\}(\{\mathsf{u}, \mathsf{d}\}\{\mathsf{l}, \mathsf{r}\})^*$$

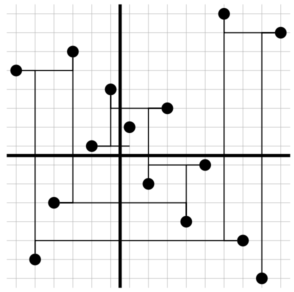
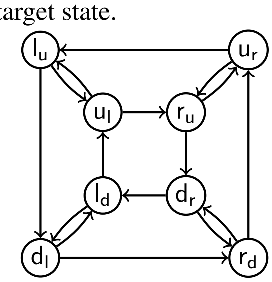


Fig. 1: A pin permutation whose basic encoding is the word 1lurdrdlduldrurd, and whose memory encoding is $\mathsf{u_r l_u u_l r_u d_r r_d d_r l_d u_l l_u d_l r_d u_r r_u d_r}$.

whee ε denotes the empty word.

In this paper we find a slightly different encoding to be more convenient. This encoding, which we call the *memory encoding*, has its origins in Brignall and Vatter (2026) and also uses an eight-letter alphabet. However, in the memory encoding each letter also records the direction of the *previous* pin. The alphabet is $\mathcal{A}_m = \{\mathsf{l_u}, \mathsf{l_d}, \mathsf{r_u}, \mathsf{r_d}, \mathsf{u_l}, \mathsf{u_r}, \mathsf{d_l}, \mathsf{d_r}\}$, with the convention that the letter $\mathsf{r_u}$, for example, encodes a right pin that follows an up pin. Note too that the pin p_1 can also be encoded with precisely one of these eight letters depending upon which quadrant contains p_1, and in which direction (left/right or up/down) the next pin p_2 lies. For example, if p_1 is in quadrant 1 and p_2 is an up or down pin, then p_1 is encoded by the letter $\mathsf{r_u}$. With the memory encoding, a pin permutation π corresponds to a word $m(\pi)$ over $\mathcal{A}_m$ that is accepted by the following finite state automaton. In this automaton all states are both start and accept states, and all transitions inherit the label of their target state.



We call any word w accepted by this automaton a *pin word*. There is a one-to-one correspondence between pin words and pin permutations: one direction is provided by the memory encoding, while the other direction follows by placing pins according to the directions dictated in the memory encoding. Much of our later work will involve constructing the (gridded) pin permutation π_w from a given pin word w.

For an example of a pin permutation and its two encodings, see Figure 1.

The memory encoding is more amenable than the basic encoding to handling subpermutations of pin permutations. For example, consider the permutation $\pi = 4731526$, with basic encoding $e(\pi) = 1\mathsf{urulur}$ and memory encoding $m(\pi) = \mathsf{r_u u_r r_u u_r l_u u_l r_u}$. The permutation 25314 can be obtained by deleting the first two points of the sequence, and this can be witnessed in the memory encoding simply by deleting the first two letters of $m(\pi)$. On the other hand, to obtain the basic encoding of 25314 from $e(\pi)$, one must delete the first two letters and replace the third with an appropriate numeral. This complication can be handled by a transducer (as used in Brignall, Ruškuc, and Vatter (2008b)), but here we have chosen to avoid this.

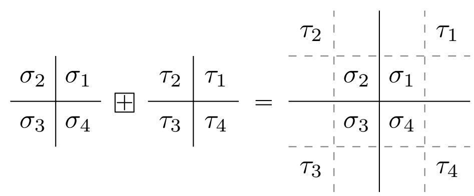
Note that in the memory encoding, if the predecessor letter of some given letter is known, then the subscript on the given letter is redundant. For ease of presentation, we will frequently exploit this fact and suppress the subscripts used in the memory encoding except where this redundancy does not occur.

Now let $w = w_1 w_2 \cdots$ be an *infinite* pin word (or a *pin sequence*). The infinite (gridded) permutation π_w can be constructed in the same way as for finite permutations. The *pin class* corresponding to π_w consists of all the finite subpermutations of π_w:

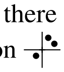
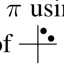
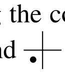
$$\mathcal{C}_w = \{\sigma : \sigma \leqslant \pi_w\}.$$

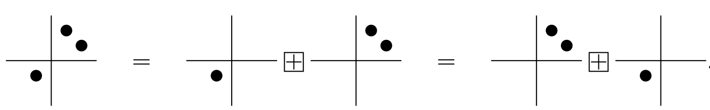
Note here that we are considering containment $\leqslant$ of *gridded* permutations, as defined earlier. Note further that, although we will be working exclusively with gridded classes $\mathcal{C}_w$, the 'ungridded' versions of such classes form permutation classes, and of course both the gridded and un-gridded versions have the same growth rate by Lemma 2.1.

Box sums A widely-used tool in the study of permutation classes is the notion of direct (and skew) sums. Here, we will use the same generalisation of this notion to the 2×2 grid as introduced by Jarvis (2024). Let σ and τ be two permutations in the 2×2 gridded plane. The *box sum* of σ and τ, denoted $\sigma \boxplus \tau$, is the gridded permutation of length $|\sigma| + |\tau|$ formed by inserting a copy of σ at the origin of a copy of τ. If we denote the points of σ and τ in quadrant $i = 1, 2, 3, 4$ by σ_i and τ_i, then the box sum can be viewed graphically as follows.



A gridded permutation is *box decomposable* or $\boxplus$-decomposable if it can be expressed as the box sum of two smaller permutations, and *box indecomposable* (or $\boxplus$-indecomposable) otherwise. Graphically, box decomposability can be recognised by a set of points whose rectangular hull includes the origin and has the property that there are no points in the regions above, below, to the left or to the right of this hull.

As with direct and skew sums, one can decompose a permutation π into a $\boxplus$-sum of box indecomposable permutations, and the collection S of indecomposable permutations in this sum is unique. However, unlike direct and skew sums, it is possible for there to be more than one expression for π using the collection S. For example, the gridded permutation  has two expressions as a box sum of  and  as follows:



It is clear that a sum of two $\boxplus$-indecomposable permutations, each of which is wholly contained in opposing quadrants, can always be taken in either order. We say that such a pair of $\boxplus$-indecomposable permutations *commute*. In fact, we have:

Observation 2.2 (See Jarvis (2024))**.** *Let σ and τ be a pair of $\boxplus$-indecomposable gridded permutations. Then σ and τ commute if and only if $\sigma = \tau$, or σ and τ are wholly contained in opposing quadrants.*

To see this, note that if σ and τ are not wholly contained in opposing quadrants, then in $\sigma \boxplus \tau$ there is a point p of σ that lies horizontally or vertically between the origin and some point q of τ. Similarly, in $\tau \boxplus \sigma$ the points corresponding to p and q must be in the other order: that is, q separates p from the origin. Since both σ and τ are $\boxplus$-indecomposable, we can only have $\sigma \boxplus \tau = \tau \boxplus \sigma$ if σ can be embedded in τ, and vice versa. That is, $\sigma = \tau$.

Box closed classes and enumeration A (gridded) class $\mathcal{C}$ is *box closed* (or $\boxplus$-closed) if $\sigma \boxplus \tau \in \mathcal{C}$ for every $\sigma, \tau \in \mathcal{C}$, and the *box closure* of a set $\mathcal{D}$ is the smallest box-closed class containing $\mathcal{D}$, and is denoted $\boxplus \mathcal{D}$. These concepts are natural extensions of similar ideas for sum and skew closed (ungridded) classes, but whereas for (say) a sum-closed class we can write $f(z) = 1/(1-g(z))$ for the relationship between the generating function of the class f and that of the sum-indecomposable permutations g, the corresponding expression for box-closed classes would result in overcounting permutations whose decomposition into box indecomposable permutations involves adjacent commuting pairs.

To handle commuting pairs, we require a result originally due to Cartier and Foata (1969), and first applied to permutation classes by (Bevan, 2014, Lemma 7). Given a finite alphabet $\mathcal{A}$ and a collection of *commutation rules* (that is, unordered pairs of letters that may be read in either order), the *trace monoid* $\mathcal{M}(\mathcal{A})$ is the set of equivalence classes of words over $\mathcal{A}$, under the equivalence relation determined by the commutation rules.

Lemma 2.3 (Cartier and Foata (1969); see also (Flajolet and Sedgewick, 2009, Note V.10))**.** *Let $\mathcal{A}$ be a finite alphabet and let C be a set of commutation rules on elements of $\mathcal{A}$. Then the trace monoid $\mathcal{M}(\mathcal{A})$ has generating function*

$$M(z) = \left(\sum_F (-1)^{|F|} z^{|F|} \right)^{-1}$$

where the sum is over all sets F composed of distinct letters that commute pairwise.

It is possible to associate each letter of $\mathcal{A}$ with a polynomial or power series, from which we obtain the following corollary.

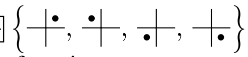
Corollary 2.4. *Let $\mathcal{A}$ be a finite alphabet and let $g : \mathcal{A} \to \mathbb{Z}[\![z]\!]$ be any mapping. Then the trace monoid $\mathcal{M}(\mathcal{A})$ has weighted generating function*

$$M_g(z) = \left(\sum_F (-1)^{|F|} \prod_{a \in F} g(a) \right)^{-1}$$

where the sum is over all sets F of distinct letters that commute pairwise.

In the case of the four-quadrant grids in this paper, we use a five letter alphabet $\{a, \ell_1, \ell_2, \ell_3, \ell_4\}$, and two commutation relations, $\ell_1\ell_3 = \ell_3\ell_1$ and $\ell_2\ell_4 = \ell_4\ell_2$. For a $\boxplus$-closed class $\mathcal{C}$, define the function $g : \{a, \ell_1, \ell_2, \ell_3, \ell_4\} \to \mathbb{Q}(z)$ so that $g(\ell_i)$ is the generating function for the $\boxplus$-indecomposable permutations wholly contained in quadrant i, and $g(a)$ is the generating function for all the other $\boxplus$-indecomposable permutations. By Corollary 2.4, the generating function for $\mathcal{C}$ is

$$f_{\mathcal{C}}(z) = \frac{1}{1 - (g(a) + g(\ell_1) + g(\ell_2) + g(\ell_3) + g(\ell_4)) + g(\ell_1)g(\ell_3) + g(\ell_2)g(\ell_4)}.$$

By way of an example, consider $\mathcal{X}^\sharp = \boxplus$ . We take $g(a) = 0$ and $g(\ell_i) = z$ for $i = 1, 2, 3, 4$, which gives us the generating function

$$f_{\mathcal{X}^\sharp}(z) = \frac{1}{1 - 4z + 2z^2}.$$

This class is precisely the class of gridded permutations that can be drawn on a geometric 'X'. The generating function for the ungridded version of $\mathcal{X}$ is (see Waton (2007) or Elizalde (2011))

$$\frac{1 - 3z}{1 - 4z + 2z^2}.$$

Note that the denominator agrees with that for $\mathcal{X}^\sharp$, confirming that $\mathcal{X}$ and $\mathcal{X}^\sharp$ have the same growth rate as required by Lemma 2.1.

We now briefly turn our attention to general theory regarding the growth rates of classes. First, for a permutation class $\mathcal{C}$ that is sum- or skew-closed, Arratia (1999) observed that the sequence $(|\mathcal{C}_n|)$ is *supermultiplicative* (that is, $|\mathcal{C}_m||\mathcal{C}_n| \leqslant |\mathcal{C}_{m+n}|$), and therefore Fekete's lemma implies that $\mathrm{gr}(\mathcal{C})$ exists. This fact can alternatively be derived from the *supercritical sequence schema* described in Section V.2 of Flajolet and Sedgewick (2009).

Neither approach works directly for $\boxplus$-closed classes, because of the expressions provided by Lemma 2.3: for example, the counting sequence for the $\mathcal{X}$ class is not supermultiplicative, nor does it satisfy the positivity condition for the supercritical sequence schema. Nevertheless, it is of course still the case that the geometric 'X' has growth rate $1/\rho$, where $z = \rho$ is the smallest real positive solution to $4z - 2z^2 = 1$, since ρ is the unique dominant singularity of $f_{\mathcal{X}^\sharp}(z)$.

Restricting our attention to pin classes, the previous paper established the following result.

Theorem 2.5 (Jarvis (2024))**.** *For any pin class $\mathcal{C}$, the growth rate $\mathrm{gr}(\mathcal{C})$ exists.*

Combinatorics of Words We have already discussed several examples of words over finite alphabets, but here we briefly survey some definitions and results from the standard literature on the combinatorics of words that we will require. Further details about the concepts here can be found in the survey article by Cassaigne and Nicolas (2010), and references therein.

Given a (finite or infinite) word w, a *factor* of w is a contiguous subsequence of w. An infinite word w over a finite alphabet is said to be:

- *periodic* if there exists k such that $w_i = w_{k+i}$ for all $i \geqslant 1$. The smallest such value of k for which this holds is called the *period* of w,
- *eventually periodic* if $w = uw'$ for some finite word u and periodic word w'. The *period* of w is defined to be equal to the period of w'.
- *recurrent* if every finite factor of w appears infinitely often in w, and
- *eventually recurrent* if $w = uw'$ for some finite word u and recurrent word w'.

Clearly, every periodic (respectively, eventually periodic) word w is recurrent (resp., eventually recurrent).

Given an infinite word w, the *complexity* of w is the function $p_w(n)$ that counts the number of distinct factors of length n that appear in w. Since every factor of w of length n can be extended in at least one way by adding a letter on the right hand end, there is a natural injection from factors of length n to factors of length $n + 1$, for all n. Consequently, we have.

Proposition 2.6 (See Cassaigne and Nicolas (2010)). *For an infinite word w, the complexity p_w is increasing.*

In the case of words that are not periodic, we have the following result which dates back to the 1930s.

Theorem 2.7 (Morse and Hedlund (1938)). *For an infinite word w that is not periodic or eventually periodic, p_w is strictly increasing.*

By contrast, words that are periodic or eventually periodic must have a complexity function that is constant for suitably large n. In fact, closer analysis yields the following.

Lemma 2.8 (Essentially due to Morse and Hedlund (1938)). *For an infinite word w over a finite alphabet, if there exists N such that $p_w(N) = p_w(N+1)$, then $p_w(n) = p_w(N)$ for all $n \geqslant N$.*

Proof: For each $n \geqslant 1$, let ρ_n denote any natural injection from the factors of length n in w to the factors of length $n+1$ in w, in which a single letter is added to the right hand end of each factor. We will be done if we can show that ρ_n is a bijection between the factors of lengths n and $n+1$, for all $n \geqslant N$, and we show this by induction.

The base case is clear, so fix $k \geqslant N$ for ρ_k. Take a factor $a = a_1 \cdots a_{k+1}$ of w of length $k+1$, and let a_{k+2} be the letter such that $\rho_{k+1}(a) = a_1 \cdots a_{k+2}$. Suppose that b is any letter for which $a_1 \cdots a_{k+1}b$ is a factor of w.

Let $a' = a_2 \cdots a_{k+1}$. The words $a'a_{k+2}$ and $a'b$ are both factors of w of length $k+1$ formed by extending a' to its right. Since ρ_k is a bijection, there is only one letter that can be used to extend a' to the right. Thus $\rho_k(a') = a'a_{k+2} = a'b$ so $a_{k+2} = b$. This shows that a extends uniquely to the right, and so ρ_{k+1} is a bijection between the factors of lengths $k+1$ and $k+2$, as required. □

Let w be an infinite word over a finite alphabet A. If a finite factor u of w appears infinitely often, we say that u is a *recurrent factor*, otherwise it is a *non-recurrent factor*. The *recurrent complexity* is the function $q_w(n)$ that counts the number of distinct recurrent factors of length n in w. Although this notion seems natural enough, it does not seem to have appeared in the literature previously. Clearly $p_w(n) \geqslant q_w(n)$ for all n, and $p_w = q_w$ if and only if w is recurrent. We observe the following.

Theorem 2.9. *For an infinite word w, q_w is increasing. Furthermore, if w is not eventually periodic, then q_w is strictly increasing.*

Proof: For the first part, let n be arbitrary, and let a be any recurrent factor of length n. Since w is a word over a finite alphabet A, say, there exists at least one symbol $x \in A$ such that the letter after infinitely many instances of the word a is x. Thus, ax is a recurrent factor of length $n+1$, which establishes an injection from the set of recurrent factors of length n to those of length $n+1$.

For the second part, suppose that $q_w(n) = q_w(n+1)$ for some n. Write $w = uw'$, where u is a finite prefix that has been chosen so that w' contains none of the non-recurrent factors of length up to $n+1$ that appeared in w. Note that $q_w(k) = q_{w'}(k)$ for all k. Furthermore, by construction we have $q_{w'}(k) = p_{w'}(k)$ for all $k \leqslant n+1$. Thus

$$p_{w'}(n) = q_{w'}(n) = q_w(n) = q_w(n+1) = q_{w'}(n+1) = p_{w'}(n+1),$$

which shows that w' is eventually periodic by Theorem 2.7. However, since $w = uw'$, it follows that w is also eventually periodic. □

Lemma 2.10. *For an infinite word w over a finite alphabet, if there exists N such that $q_w(N) = q_w(N+1)$, then $q_w(n) = q_w(N)$ for all $n \geqslant N$.*

We omit the proof of this lemma, since it is essentially the same as that given in Lemma 2.8.

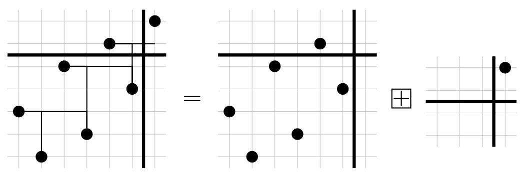
3 Pin class structure

We are now in a position to begin to describe the structure of permutations in pin classes. By definition, a pin permutation $(p_1, \ldots, p_n)$ has the property that the set of points $\{p_1, \ldots, p_i\}$ for any $i \geqslant 2$ cannot form a component for a box decomposition. On the other hand, the pin p_i is the only point of the pin permutation that cuts the rectangular hull of $\{p_1, \ldots, p_{i-1}\}$, thus its removal renders a box decomposition.

Observation 3.1. *For $n \geqslant 3$, let $u = u_1 \cdots u_n$ be a pin word and $\pi_u = (p_1, \ldots, p_n)$ the associated pin permutation. For any $1 < i < n$, the subpermutation π' formed by deleting the point p_i from π_u satisfies*

$$\pi' = \pi_{u_1 \cdots u_{i-1}} \boxplus \pi_{u_{i+1} \cdots u_n}.$$

It is natural to wonder whether every pin permutation gives rise to a $\boxplus$-indecomposable permutation. Unfortunately, this is not quite true, as the following example illustrates.



Jarvis (2024) establishes a complete classification of the box-decomposable pin permutations. We will return to this issue in Section 4, but for now it suffices to note the following.

Proposition 3.2. *Let w be an infinite pin word, and let $\pi \in \mathcal{C}_w$ be a box-indecomposable permutation. Then $\pi = \pi_{w'}$ for some factor w' of w.*

Proof: Take any box indecomposable permutation $\pi \in \mathcal{C}_w$. We can witness the membership of π in $\mathcal{C}_w$ by embedding its points in that of some pin permutation $(p_1, \ldots, p_N)$ for some $N \geqslant |\pi|$, which we may take to be as small as possible. In turn, this pin permutation corresponds to a factor $w_M \cdots w_{M+N-1}$ of w, for some M.

If $N > |\pi|$, then since N was taken to be as small as possible, this implies that any given embedding of π in $(p_1, \ldots, p_N)$ must use both p_1 and p_N, and hence there is some p_i, with $1 < i < N$, that is *not* used in the embedding. Observation 3.1 now implies that π embeds in $\pi_{w_M \cdots w_{M+i-2}} \boxplus \pi_{w_{M+i} \cdots w_{M+N-1}}$, with at least one point from each of the two box sum components used in the embedding. This implies that π is box decomposable, but this is a contradiction. Hence $N = |\pi|$ and the proof is complete. $\square$

In the context of pin words, we can now derive the following useful result.

Proposition 3.3. *If w is a recurrent pin word, then C_w is $\boxplus$-closed.*

Proof: It suffices to show that if $\sigma, \tau \in \mathcal{C}_w$ with the property that τ is $\boxplus$-indecomposable, then $\sigma \boxplus \tau \in \mathcal{C}_w$.

Let u be any finite prefix of w such that σ embeds in π_u. Then we can write $w = uw'$. By Proposition 3.2, the $\boxplus$-indecomposable permutation τ in $\mathcal{C}$ can be constructed using a factor of w. Since w is recurrent, this factor appears infinitely often. In particular, we can find this factor in the word w' in such a way as it does not use the first letter of w', and the embedding of $\sigma \boxplus \tau$ in π_w now follows. $\square$

In cases where the defining word w is not recurrent, we can recover a subclass of $\mathcal{C}_w$ that *is* $\boxplus$-closed. The *box-interior* of the class $\mathcal{C}_w$ is defined by

$$\mathcal{C}_w^{\boxplus} = \{\sigma \in \mathcal{C}_w : \sigma \leqslant \pi_u \text{ for some recurrent factor } u\}.$$

Clearly $\mathcal{C}_w^{\boxplus} \subseteq \mathcal{C}_w$ is a permutation class, and $\mathcal{C}_w^{\boxplus} = \mathcal{C}_w$ if and only if w is recurrent (this follows by an argument similar to that given in the proof of Proposition 3.3). However, even more is true:

Theorem 3.4 (Jarvis (2024))**.** *For any pin word w, we have* $\mathrm{gr}(\mathcal{C}_w) = \mathrm{gr}(\mathcal{C}_w^{\boxplus})$.

3.1 Visits

Given a pin word w, a *visit* to a quadrant q is a finite contiguous subsequence $w_{i+1} \cdots w_{i+k}$ of w such that the k corresponding points of π_w are placed in quadrant q, while neither the point corresponding to w_i (if it exists) nor w_{i+k+1} belongs to quadrant q. Trivially, we have:

Proposition 3.5. *The permutation corresponding to any visit forms an oscillation, which is increasing if it belongs to quadrants 1 or 3, and decreasing otherwise.*

Each visit $w_{i+1} \cdots w_{i+k}$ is initiated by an *arrival*, being the first point of the visit (corresponding to w_{i+1}), and a *departure*, being the last point (corresponding to w_{i+k}). Our work on establishing growth rates will, in part, rely on the analysis of the number and length of visits to different quadrants.

Note the requirement that a visit is a *finite* subsequence. Thus, it is possible for an infinite pin sequence to contain only finitely many visits, or possibly none at all. The number of quadrants which a pin sequence visits infinitely often strongly controls the range of growth rates of the corresponding pin class. Before we demonstrate this, we first show that pin sequences can be freely truncated at the start without impacting the growth rate.

Proposition 3.6. *For any pin sequence w, and any expression $w = vw'$ where v is a finite prefix, we have* $\mathrm{gr}(\mathcal{C}_w) = \mathrm{gr}(\mathcal{C}_{w'})$.

Proof: Since v is finite, the pin classes $\mathcal{C}_w$ and $\mathcal{C}_{w'}$ have the property that their box interiors are identical. The statement follows by Theorem 3.4. $\square$

A particularly useful application of this proposition is the following.

Proposition 3.7. *For any pin sequence w that visits quadrant q only finitely many times, we have* $\mathrm{gr}(\mathcal{C}_w) = \mathrm{gr}(\mathcal{C}_{w'})$ *where $w = vw'$ and w' never visits quadrant q.*

We now begin our classification of small growth rates.

Theorem 3.8. *Let w be an infinite pin word that contains only finitely many visits. Then* $\mathrm{gr}(\mathcal{C}_w) = \kappa \approx 2.20557$*, the largest real root of* $z^3 - 2z^2 - 1$.

Proof: By Proposition 3.7, we can, by truncating the word if necessary, assume that w in fact contains *no* visits. This means that w belongs to a single quadrant and is thus periodic.

Without loss generality we have $w = (\mathsf{u_r r_u})^\infty$, and the pin class $\mathcal{C}_w$ is $\boxplus$-closed by Proposition 3.3. By inspection, the number of $\boxplus$-indecomposable permutations in $\mathcal{C}_w$ of lengths $1, 2, 3, \ldots$ are $1, 1, 2, 2, \ldots$, and thus $\mathcal{C}_w$ has generating function

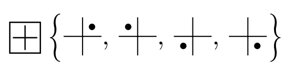
$$f_{C_w}(z) = \frac{1}{1 - \frac{z+z^3}{1-z}} = \frac{1-z}{1-2z-z^3}.$$

Thus $\mathrm{gr}(\mathcal{C}_w) = \kappa \approx 2.20557$, the largest real root of $z^3 - 2z^2 - 1$. □

Any pin word that contains arbitrarily many visits necessarily must visit at least *two* quadrants infinitely often. Our next task is to establish that any pin class whose defining sequence visits more than two quadrants infinitely often has growth rate well above our target growth rate of $\mu \approx 3.28277$.

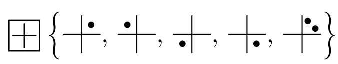
Proposition 3.9. *Let w be an infinite pin word, and suppose that $\mathcal{C}_w^\sharp$ visits every quadrant infinitely often. Then* $\mathrm{gr}(\mathcal{C}_w^\sharp) \geqslant 2 + \sqrt{2} \approx 3.414$.

Proof: Any class that visits every quadrant infinitely often must contain the box closure



which is the class $\mathcal{X}$ considered at the end of Section 2. The generating function of $\mathcal{X}^\sharp$ has denominator $1 - 4z + 2z^2$, and thus $\mathrm{gr}(\mathcal{X}) = 2 + \sqrt{2}$, from which the proposition follows. □

Note that, in fact, the *smallest* growth rate of a pin class in all four quadrants is approximately 3.48806 (being the largest real root of $z^5 - 5z^4 + 6z^3 - 2z^2 - z - 3$) and is achieved by the 'Widdershins spiral' defined by the word $w = (\mathsf{u_r ldr})^\infty$. To see this, consider the following class:



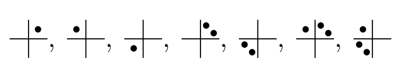
The generating function for this class has denominator $1 - 4z + z^2 + z^3$, and thus has growth rate approximately equal to 3.69109. Consequently, by considering all symmetries of the above class, the smallest pin class in all four quadrants must be defined by a pin sequence which has the property that, except for finitely many exceptions, every visit is formed of a single letter. It is now straightforward to argue that the only pin sequences that are permissible are $w = (\mathsf{u_r ldr})^\infty$ and its 'reverse', $w = (\mathsf{u_l rdl})^\infty$.

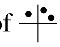
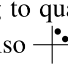
We now establish a lower bound for the growth rates of pin classes that visit three quadrants. Let $\lambda \approx 3.28481$ be the largest positive zero of $z^5 - 3z^4 - z^3 + z - 1$.

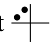
Proposition 3.10. *Let w be an infinite pin sequence, such that $\mathcal{C}_w^\sharp$ visits precisely three quadrants infinitely often. Then* $\mathrm{gr}(\mathcal{C}_w^\sharp) \geqslant \lambda$.

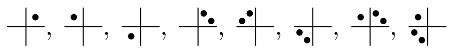
Proof: Without loss of generality, suppose that $\mathcal{C}_w^\sharp$ visits quadrants 1–3 infinitely often. By truncating and appealing to Proposition 3.7 as necessary, without loss of generality we can assume that w never visits quadrant 4.

We first claim that the following $\boxplus$-indecomposable permutations belong to the box-interior, $\mathcal{C}_w^{\boxplus}$:



This claim is clear for the three singletons (since w visits all of quadrants 1–3 infinitely often), so now consider any index k such that w_k is an arrival in quadrant 1. This arrival must be from quadrant 2, so $w_{k-1} = \mathsf{u_l}$ and $w_k = \mathsf{r_u}$. Furthermore, the next letter w_{k+1} must also belong to quadrant 1, hence $w_{k+1} = \mathsf{u_r}$. Thus, every arrival into quadrant 1 defines a copy of  (and hence also ), and there are arbitrarily many such arrivals so these patterns belong to $\mathcal{C}_w^{\boxplus}$. A similar argument applies in quadrant 3 for the other two patterns.

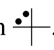
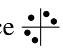
Next, suppose that  $\in \mathcal{C}_w^{\boxplus}$. Then $\mathcal{C}_w^{\boxplus}$ contains the $\boxplus$-closure of the following eight permutations:



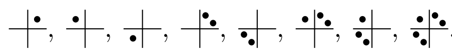
To find the generating function of the $\boxplus$-closure of these permutations, we note that the box indecomposables in quadrant 1 have generating function $z + z^2$, as do those in quadrant 3. By Corollary 2.4, the generating function of this $\boxplus$-closure is therefore

$$\frac{1}{1 - (3z + 3z^2 + 2z^3) + (z + z^2)^2} = \frac{1}{1 - 3z - 2z^2 + z^4},$$

from which we can see that $\mathrm{gr}(\mathcal{C}_w) \geqslant 3.542$ (the largest real root of $z^4 - 3z^3 - 2z^2 + 1$).

Thus, we may now assume that $\mathcal{C}_w^{\boxplus}$ does not contain . By a similar argument to that used earlier, we can find arbitrarily many indices k such that $w_{k-2}w_{k-1}w_kw_{k+1}w_{k+2} = \mathsf{d_l l_d u_l r_u u_r}$, and hence  $\in \mathcal{C}_w^{\boxplus}$.

Thus, in this case $\mathcal{C}_w^{\boxplus}$ contains the following eight permutations.



The generating function of this $\boxplus$-closure is

$$\frac{1}{1 - (3z + 2z^2 + 2z^3 + z^5) + (z + z^2)^2} = \frac{1}{1 - 3z - z^2 + z^4 - z^5},$$

and therefore $\mathrm{gr}(\mathcal{C}_w) \geqslant \lambda$ (the largest real root of $z^5 - 3z^4 - z^3 + z - 1$). $\square$

In fact, the *smallest* pin class in three quadrants is given by the word $w = (\mathsf{u_r ldlur})^\infty$ and has growth rate approximately 3.36132 (equal to the largest real root of $z^5 - 4z^4 + 2z^3 + z^2 - 2z + 1$). The details of the argument that establishes this are similar to, but more intricate than, that given above, and requires some additional theory presented in Jarvis (2024).

4 Visits to two quadrants and periodic words

We have seen that when a pin sequence contains only finitely many visits, or contains arbitrarily many visits to three or four quadrants, then the growth rate of the corresponding pin class must equal κ or exceed $\lambda \approx 3.28481$, respectively.

It remains to consider the possible growth rates for pin classes whose sequences visit precisely two quadrants infinitely often. By symmetry and Proposition 3.7, we can restrict our attention to pin classes whose sequences only ever visit quadrants 1 and 2, and do so infinitely often.

Before we go further, we need to make the relationship between factors of a pin sequence and the $\boxplus$-indecomposable permutations more precise. First, as mentioned earlier there exist pin words that define $\boxplus$-decomposable permutations. This presents no problem for sufficiently long pin words in two quadrants:

Lemma 4.1 (Lemma 6.1 of Brignall and Vatter (2026), see also Jarvis (2024))**.** *Let w be a pin sequence that visits precisely two quadrants, and let f be a finite factor of w of length at least 3. Then π_f is $\boxplus$-indecomposable.*

From an enumerative perspective, we would also like there to be a one-to-one correspondence between pin words appearing as factors in a pin sequence w, and box-indecomposable permutations in the pin class $\mathcal{C}_w$. In general this is not true – for a full classification, see Jarvis (2024) – but in two quadrants it again holds for sufficiently long pin words.

Lemma 4.2 (Proposition 6.3 of Brignall and Vatter (2026))**.** *Let w be a pin sequence that visits precisely two quadrants, and let f and f' be finite factors of w of length at least 4. If $\pi_f = \pi_{f'}$, then $f = f'$.*

4.1 Pin sequences from binary words

As we are now restricting our attention to pin sequences that visit precisely two quadrants, we can introduce a method to construct these pin sequences from binary words. This construction is essentially the same as that used by Brignall and Vatter (2026), but here we undertake a slightly more careful enumerative analysis.

A pin sequence that is wholly contained in quadrants 1 and 2 must comprise a series of left and right steps, interleaved with up steps. Given an infinite binary sequence b, there is therefore a natural injection ϕ to a two-quadrant pin sequence $w = \phi(b)$, via $\phi(0) = \mathsf{l}\mathsf{u}_\mathsf{l}$ and $\phi(1) = \mathsf{r}\mathsf{u}_\mathsf{r}$. Note that ϕ can never produce a pin sequence that begins with u_l or u_r, and thus in general it is not surjective.

However, since we are primarily concerned with growth rates of pin classes, note that for any pin sequence w that begins with u_l or u_r, we can simply remove the first letter to form the pin sequence w', and then $\mathrm{gr}(\mathcal{C}_w) = \mathrm{gr}(\mathcal{C}_{w'})$ by Proposition 3.6. Consequently, we can without loss of generality restrict our attention to pin sequences that are the images under ϕ of binary sequences.

We now evaluate the complexity and recurrent complexity of $w = \phi(b)$, given the complexity p_b and recurrent complexity q_b of b, and thus count the number of $\boxplus$-indecomposable permutations of each length in $\mathcal{C}_w$ and $\mathcal{C}_w^{\boxplus}$. This argument is essentially the same as that given by Jarvis (2024), but we include it here for completeness.

For a binary factor a of length n appearing in b, $\phi(a)$ is a pin word of length $2n$ appearing as a factor of $w = \phi(b)$. In addition, if a pin word x is formed by removing the first and/or last letters of $\phi(a)$, then (trivially) x is also a factor of $\phi(b)$. Furthermore, the only occurrences of x as a factor in $\phi(b)$ are those that embed in an occurrence of $\phi(a)$. Thus, each factor a of length $n \geqslant 2$ in the binary word b accounts for one word $\phi(a)$ of length $2n$, two words of length $2n-1$ (delete the first or last letter of $\phi(a)$), and one word of length $2n-2$ (delete both the first and last letters of $\phi(a)$). These considerations yield:

Proposition 4.3. *Let b be an infinite binary sequence, and let $w = \phi(b)$. Then, for $n \geqslant 1$,*

$$p_w(n) = \begin{cases} p_b(k) + p_b(k+1) & n = 2k \\ 2p_b(k+1) & n = 2k+1 \end{cases}$$

$$q_w(n) = \begin{cases} q_b(k) + q_b(k+1) & n = 2k \\ 2q_b(k+1) & n = 2k+1. \end{cases}$$

By Lemmas 4.1 and 4.2, every factor of length $n \geqslant 4$ uniquely defines a $\boxplus$-indecomposable permutation, and thus we have:

Proposition 4.4. *Let b be an infinite binary sequence, and let $w = \phi(b)$. The number of $\boxplus$-indecomposable permutations in $\mathcal{C}_w$ of each length $n \geqslant 4$ is $p_b(k)+p_b(k+1)$ if $n = 2k$, and $2p_b(k+1)$ if $n = 2k+1$. Furthermore, if w visits two quadrants infinitely often, then there are 2 $\boxplus$-indecomposable permutations of each length 1 and 2, and $2p_b(2) - 2$ of length 3.*

Similarly, the number of $\boxplus$-indecomposable permutations in $\mathcal{C}_w^{\boxplus}$ of each length $n \geqslant 4$ is $q_b(k)+q_b(k+1)$ if $n = 2k$, and $2q_b(k+1)$ if $n = 2k+1$. Furthermore, if w visits two quadrants infinitely often, then there are 2 $\boxplus$-indecomposable permutations of each length 1 and 2, and $2q_b(2) - 2$ of length 3.

4.2 A spectrum of growth rates

Let $v = (\mathsf{r_u u l u})^\infty = \phi((10)^\infty)$. In the next two propositions, we show that v is essentially the unique pin sequence in two quadrants for which $\mathcal{C}_v$ has the smallest possible growth rate.

Proposition 4.5. $\mathrm{gr}(\mathcal{C}_v^\sharp) = \nu$*, where* $\nu \approx 3.06918$ *is the largest real root of* $z^4 - 3z^3 - 2$.

Proof: Since v is periodic, $\mathcal{C}_v$ is $\boxplus$-closed by Proposition 3.3. By inspection, there are 2 $\boxplus$-indecomposable permutations of each length $n = 1$, 2 and 3 in $\mathcal{C}_v$. Since $v = \phi((01)^\infty)$ and $p_{(01)^\infty}(n) = 2$ for all n, it follows by Proposition 4.4 that there are precisely 4 $\boxplus$-indecomposable permutations of each length $n \geqslant 4$ in $\mathcal{C}_v$. Thus the generating function for $\mathcal{C}_w$ is

$$f(z) = \frac{1}{1 - \frac{2z+2z^4}{1-z}} = \frac{1-z}{1-3z-2z^4}$$

and the proposition follows. □

Proposition 4.6 (Jarvis (2024))**.** *Suppose that w visits exactly two quadrants infinitely often. Then $\mathrm{gr}(\mathcal{C}_w^\sharp) \geqslant \nu$, with equality if and only if up to symmetry there exists a finite word w' such that $w = w'(\mathsf{luru})^\infty$.*

Proof: By symmetry and since $\mathcal{C}_w = \mathcal{C}_{w'}$ for any infinite suffix w' of w (by Proposition 3.6), it suffices to consider pin sequences w that only visit quadrants 1 and 2. Thus w is a word over the alphabet $\{\mathsf{u_r}, \mathsf{u_l}, \mathsf{l_u}, \mathsf{r_u}\}$, and indeed we may also assume that $w = \phi(b)$ for some binary sequence b (by removing the first letter of w if necessary).

We consider the recurrent complexity of b, and therefore that of w. Each arrival of w into a quadrant is witnessed in b by an occurrence of 01 or 10. Thus $q_b(1) = 2$ and by Theorem 2.9 therefore $q_b(n) \geqslant 2$ for all n. Proposition 4.4 now implies that the number of $\boxplus$-indecomposable permutations in $\mathcal{C}_w^{\boxplus}$ of lengths $n = 1, 2, 3, 4, \ldots$ is at least $2, 2, 2, 4, \ldots$. The proof of Proposition 4.5 now shows therefore that $\mathrm{gr}(\mathcal{C}_w) = \mathrm{gr}(\mathcal{C}_w^{\boxplus}) \geqslant \nu$.

Now suppose that b contains 00 as a recurrent factor. Then $q_b(3) \geqslant 3$, and so the sequence of $\boxplus$-indecomposable permutations in $\mathcal{C}_w^{\boxplus}$ of lengths $n = 1, 2, 3, 4, \ldots$ is at least $2, 2, 4, 6, \ldots$. This sequence has generating function

$$\frac{2z + 2z^3 + 2z^4}{1-z}$$

and hence

$$\mathrm{gr}(\mathcal{C}_w^{\boxplus}) \geqslant \mathrm{gr}\left(\frac{1-z}{1-3z-2z^3-2z^4}\right) > 3.24.$$

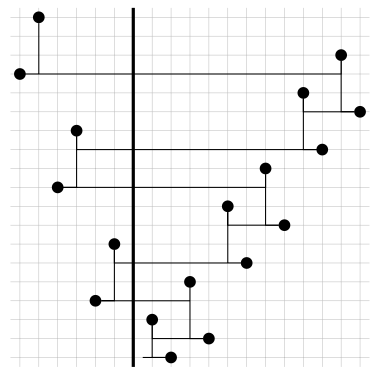


Fig. 2: The start of the infinite permutation corresponding to the word $w_{1,2}$.

(Recall that the growth rate of a generating function refers to the growth rate of the sequence of coefficients of the formal power series.)

Similarly, if b contains 11 as a recurrent factor, then again we have $\mathrm{gr}(\mathcal{C}_w^{\boxplus}) > 3.24$.

Thus, if $\mathrm{gr}(\mathcal{C}_w) = \nu$, then b cannot contain infinitely many copies of 00 or 11, and hence $b = b'(01)^\infty$ for some finite word b'. We now take $w' = \phi(b')$, so that $w = w'(\mathsf{luru})^\infty$, and the proof follows by Proposition 3.6 and Proposition 4.5. $\square$

As well as showing that the pin sequence v is essentially the unique pin sequence of smallest growth rate, the above proof in fact also shows that if either of the binary factors 00 or 11 appears as recurrent factors in a binary word b, then $\mathrm{gr}(\mathcal{C}_{\phi(b)}) > 3.24$. Our next task is to investigate the two-quadrant classes that contain one or both of these factors infinitely often.

For positive integers k and ℓ, Let $b_{k,\ell} = (0^k 1^\ell)^\infty$, and let $w_{k,\ell} = \phi(b_{k,\ell}) = ((\mathsf{lu})^k(\mathsf{ru})^\ell)^\infty$ denote the periodic pin sequence of period $2(k+\ell)$. Set $\nu_{k,\ell} = \mathrm{gr}(\mathcal{C}_{w_{k,\ell}})$, and in the case where $k = 1$, we suppress the '1' and write simply ν_ℓ. See Figure 2 for an example.

We begin with the cases in which a pin sequence visits both quadrants 1 and 2 infinitely often, and infinitely many of those visits in each quadrant contain at least three points.

Proposition 4.7. *Let w be a pin sequence which visits only quadrants 1 and 2, and in which both* rur *and* lul *are recurrent factors. Then $\mathrm{gr}(\mathcal{C}_w) \geqslant \nu_{2,2} \approx 3.39752$, with equality if and only if w is eventually equal to $w_{2,2} = \phi(b_{2,2}) = (\mathsf{lulururu})^\infty$.*

Proof: First, $b_{2,2} = (0011)^\infty$ has complexity function $p_b(1) = 2$ and $p_b(n) = 4$ for $n \geqslant 2$. Thus $w_{2,2}$ has complexity function $p_w(1) = p_w(2) = 2$, $p_w(3) = 6$ and $p_w(n) = 8$ for $n \geqslant 4$. This sequence has generating function $(2z + 4z^3 + 2z^4)/(1-z)$. Since $w_{2,2}$ is periodic, it follows that

$$\nu_{2,2} = \mathrm{gr}(\mathcal{C}_{w_{2,2}}) = \mathrm{gr}\left(\frac{1-z}{1-3z-4z^3-2z^4}\right) \approx 3.39752.$$

For reference $\nu_{2,2}$ is the largest real zero of $z^4 - 3z^3 - 4z - 2$.

Now consider a pin sequence w as in the statement of the proposition. As before, we may assume that $w = \phi(b)$ for some binary sequence b.

Since w contains both rur and lul as recurrent factors, b must contain both 00 and 11 as recurrent factors. Furthermore, b must also contain 01 and 10, and hence $p_b(2) \geqslant 4$. Therefore, the counting sequence of $\boxplus$-indecomposable permutations in $\mathcal{C}_w^{\boxplus}$ is at least $2, 2, 6, 8, 8, \ldots$. This is the same sequence as the complexity of $w_{2,2}$, hence $\mathrm{gr}(\mathcal{C}_w) = \mathrm{gr}(\mathcal{C}_w^{\boxplus}) \geqslant \nu_{2,2}$.

Now suppose that in addition, b contains 000 as a recurrent factor. In this case, $q_b(3) \geqslant 5$, and this implies that the counting sequence of $\boxplus$-indecomposable permutations in $\mathcal{C}_w^{\boxplus}$ is at least $2, 2, 6, 9, 10, 10, \ldots$. This has generating function $(2z + 4z^3 + 3z^4 + z^5)/(1 - z)$ from which we find that $\mathrm{gr}(\mathcal{C}_w^{\boxplus}) \geqslant 3.423$.

Similar analysis applies if b contains any of 111, 101 or 010 as recurrent factors. Consequently, if $\mathrm{gr}(\mathcal{C}_w) = \nu_{2,2}$, then b cannot contain any of 000, 111, 101 or 010 as recurrent factors, which implies that $b = b'(0011)^\infty$ for some finite word b'. The result now follows by the same arguments as used previously. □

We now investigate pin classes whose corresponding sequences visit quadrants 1 and 2 and whose growth rates are below $\nu_{2,2}$. We begin by considering the classes generated by the periodic sequences $w_{k,\ell}$. Since $\nu_{k,\ell} < \nu_{2,2}$, it follows from Proposition 4.7 that $k = 1$ or $\ell = 1$.

Lemma 4.8. *We have $\nu_2 = \nu_{1,2} \approx 3.24796$ (the largest real zero of $z^4 - 3z^3 - 2z - 2$), while for $\ell > 2$ the growth rate ν_ℓ is equal to the largest real zero of*

$$z^{2\ell} - 4z^{2\ell-1} + 3z^{2\ell-2} - 2z^{2\ell-3} - z^{2\ell-4} + 2z^{2\ell-5} + 1.$$

Proof: By inspection, $b_{1,\ell} = (01^\ell)^\infty$ has complexity function

$$p_{b_{1,\ell}}(n) = \begin{cases} n+1 & 1 \leqslant n \leqslant \ell \\ \ell+1 & n > \ell. \end{cases}$$

Thus, the generating function for the $\boxplus$-indecomposable permutations in $\mathcal{C}_{w_{1,\ell}}$ is

$$g_{1,\ell}(z) = \frac{2z + 2z^3 + 3z^4 + z^5 + z^6 + \cdots + z^{2\ell-1}}{1 - z} = \frac{2z - 2z^2 + 2z^3 + z^4 - 2z^5 - z^{2\ell}}{(1 - z)^2}$$

if $\ell > 2$, and if $\ell = 2$ then the generating function is

$$g_{1,2}(z) = \frac{2z + 2z^3 + 2z^4}{1 - z}.$$

The result in the lemma now follows by standard combinatorial analysis. □

The following table gives the first few growth rates for these classes.

ℓ	Sequence of box decomposables	$\nu_\ell = \mathrm{gr}(\mathcal{C}_{w_{1,\ell}})$
1	$2, 2, 2, 4, \ldots$	3.06918
2	$2, 2, 4, 6, 6, \ldots$	3.24796
3	$2, 2, 4, 7, 8, 8, \ldots$	3.27963
4	$2, 2, 4, 7, 8, 9, 10, 10, \ldots$	3.28248
5	$2, 2, 4, 7, 8, 9, 10, 11, 12, 12, \ldots$	3.28274
6	$2, 2, 4, 7, 8, 9, 10, 11, 12, 13, 14, 14, \ldots$	3.28277

5 Non-periodic classes in two quadrants

As $\ell \to \infty$, the family of counting sequences in the table at the end of the previous section converges to the sequence $2, 2, 4, 7, 8, 9, \ldots$ with generating function

$$g_\star(z) = \frac{2z - 2z^2 + 2z^3 + z^4 - 2z^5}{(1-z)^2}.$$

The growth rate of $1/(1 - g_\star(z))$ is $\mu \approx 3.28277$, the largest real zero of $z^5 - 4z^4 + 3z^3 - 2z^2 - z + 2$. It is this growth rate that features in our two main theorems (Theorems 5.2 and 6.3), although note that a priori, we have not yet exhibited a single pin class with growth rate μ. In fact, certainly there cannot exist a periodic pin sequence w that generates a set of box-indecomposable permutations with generating function $g_\star(z)$, since the number of box-indecomposables generated by such a word must be eventually constant.

In this section we establish Theorem 5.2 by constructing uncountably many pin classes of growth rate μ, each defined by a recurrent sequence. Additionally, we exhibit a pin class defined by a sequence that is neither recurrent nor eventually recurrent, also of growth rate μ. In Section 6, we go on to prove that the only possible growth rates of pin classes below μ are those in the table at the end of the preceding section.

A *Sturmian sequence* is an infinite binary sequence s that has complexity function $p_s(n) = n + 1$ for all n. One example of such a sequence is the *Fibonacci sequence*,

$$0100101001001\cdots$$

which is most easily defined as the fixed point of the substitution in which $0 \mapsto 01$ and $1 \mapsto 0$.

Here, we state only a minimal number of properties of these sequences, and refer the reader to Chapter 6 of Fogg (2002) for further information.

(1) There are uncountably many Sturmian sequences with pairwise distinct sets of factors.

(2) All Sturmian sequences are recurrent.

(3) Sturmian sequences are the minimal sequences that are not periodic, in the sense that any sequence for which there exists n such that the sequence contains the same number of factors of length $n+1$ as of length n is necessarily eventually periodic.

The following result follows by Proposition 4.3.

Proposition 5.1. *For a Sturmian sequence s, $\phi(s)$ contains $n + 3$ distinct factors of length $n \geqslant 1$.*

We can now prove the following.

Theorem 5.2. *There exist uncountably many distinct pin sequences w with $\mathrm{gr}(\mathcal{C}_w) = \mu$.*

Proof: Given a Sturmian sequence s, Proposition 5.1 tells us that $\phi(s)$ contains precisely $n + 3$ distinct factors of each length n. By considering small lengths separately, and noting that each pin factor of length 4 or more gives rise to a distinct box-indecomposable permutation (by Lemmas 4.1 and 4.2), we see that the sequence of box indecomposables in $\mathcal{C}_{\phi(s)}$ is $2, 2, 4, 7, 8, 9, \ldots$. Thus the box indecomposables in $\mathcal{C}_{\phi(s)}$ have generating function $g_\star(z)$.

Since the Sturmian sequence s is recurrent, it follows that $\mathcal{C}_{\phi(s)}$ is $\boxplus$-closed, and thus the generating function for $\mathcal{C}_s$ is $1/(1-g_\star(z))$, and hence $\mathrm{gr}(\mathcal{C}_{\phi(s)}) = \mu$.

Finally, given two distinct Sturmian sequences s and t, the fact that s and t have distinct sets of factors means that $\phi(s)$ and $\phi(t)$ also have distinct sets of factors, and thus by Lemma 4.2 $\mathcal{C}_{\phi(s)}$ and $\mathcal{C}_{\phi(t)}$ have distinct sets of box-indecomposable permutations. The result now follows by property (1) above. □

Our second task in this section is to exhibit one further pin class with growth rate μ, which is *not* box-closed. Define $b^{(1)} = 10$, and for $i \geqslant 2$ iteratively define

$$b^{(i)} = b^{(i-1)}1^i0.$$

Define $b^\star = \lim_{i\to\infty} b^{(i)} = 10\,110\,1110\,11110\,111110\,\cdots$, and let $w^\star = \phi(b^\star)$. Note that $b^\star$ (and hence also $w^\star$) is not periodic or recurrent, since any factor of the form 01^i0 appears precisely once in $b^\star$. Consequently, the class $\mathcal{C}_{w^\star}$ is not box-closed, but we can appeal to Theorem 3.4

Proposition 5.3. $\mathrm{gr}(\mathcal{C}_{w^\star}) = \mu$.

Proof: We claim that

$$\mathcal{C}^{\boxplus}_{w^\star} = \boxplus\left\{\pi^\sharp_w : w \text{ is a factor of } \phi(1^k01^\ell) = (\mathsf{ru})^k\mathsf{lu}(\mathsf{ru})^\ell \text{ for some } k,\ell\right\}.$$

That the left hand side contains the right is clear, since every sequence of the form 1^k01^ℓ is a recurrent factor of $b^\star$. Conversely, note that any factor of $w^\star$ which contains two or more left steps appears precisely once in $w^\star$. Thus, the only factors of $w^\star$ that are recurrent are those that contain at most one left step, and these are precisely the factors that appear in the right hand side of the claimed equality.

It is now straightforward to check that $\mathcal{C}^{\boxplus}_{w^\star}$ contains $n+3$ box-indecomposable permutations of each length $n \geqslant 4$, while for $n = 1, 2, 3$ there are $2, 2, 4$ box-indecomposable permutations, respectively. By the earlier calculations, this tells us that $\mathrm{gr}(\mathcal{C}^{\boxplus}_{w^\star}) = \mu$, and hence $\mathrm{gr}(\mathcal{C}_{w^\star}) = \mu$ by Theorem 3.4. □

6 Classification of pin classes below μ

We now complete a full classification of the pin classes whose growth rates are less than μ. Such pin classes must be defined by sequences that visit at most two quadrants infinitely often (by Propositions 3.9 and 3.10), and Proposition 3.7 allows us to restrict our attention to pin sequences that visit at most two quadrants. Consequently, we can now proceed by studying infinite binary sequences, and utilise the injection ϕ described in Subsection 4.1.

Lemma 6.1. *For any infinite binary sequence b, if $\mathrm{gr}(\mathcal{C}_{\phi(b)}) < \mu$, then b is eventually periodic.*

Proof: Since $\mathrm{gr}(\mathcal{C}_{\phi(b)}) < \mu$ and μ is the growth rate of the pin class associated with Sturmian sequences, the recurrent complexity function q_b must satisfy, for some n, $q_b(n) < n+1$.

Pick the smallest such n, so that $q_b(n-1) \geqslant n$. Since $q_b(n) \geqslant q_b(n-1)$ (by the first part of Theorem 2.9), it follows that $q_b(n) = q_b(n-1) = n$. Now, by the second part of Theorem 2.9 we see that b is eventually periodic. □

With this established, we now narrow the range of potential binary sequences that give rise to classes with growth rate below μ.

Proposition 6.2. *Let b be a binary sequence for which $q_b(n) > n+1$ for some n. Then $\mathrm{gr}(\mathcal{C}_{\phi(w)}) \geqslant \mu$.*

Proof: First, recall that the sequence of $\boxplus$-indecomposable permutations generated by a Sturmian sequence via the mapping ϕ has the generating function

$$g_\star(z) = \frac{2z - 2z^2 + 2z^3 + z^4 - 2z^5}{(1-z)^2}.$$

Note that $g_\star(1/\mu) = 1$, and $z = 1/\mu \approx 0.30462$ is the smallest positive real solution of $g_\star(z) = 1$.

For each $k > 1$, set

$$s_k(n) = \begin{cases} n+1 & n < k \\ k+2 & n \geqslant k. \end{cases}$$

Consider any binary sequence b which satisfies the hypothesis of the proposition. Note that b must have recurrent complexity satisfying $q_b(n) \geqslant s_k(n)$ for some $k > 1$ and for all n. (The case $k = 1$ can be excluded, since $q(1) \leqslant 2$ for any binary sequence.)

Suppose first that $k \geqslant 4$ (we will handle small cases separately later). Let $g_b(z)$ denote the generating function for the non-empty $\boxplus$-indecomposable permutations in $\mathcal{C}^{\boxplus}_{\phi(b)}$. By applying Proposition 4.4 to the sequence $s_k(n)$, we see that

$$[z^n]g_b(z) \geqslant \begin{cases} 2 & n \leqslant 2 \\ 4 & n = 3 \\ n+3 & 4 \leqslant n \leqslant 2k-3 \\ 2k+2 & n = 2k-2 \\ 2k+4 & n \geqslant 2k-1. \end{cases}$$

The generating function for the coefficients on the right hand side of the above expression is

$$g_s(z) = \frac{2z - 2z^2 + 2z^3 + z^4 - 2z^5 + z^{2k-2} - 2z^{2k}}{(1-z)^2}.$$

Importantly, we have that

$$g_s(z) - g_\star(z) = \frac{z^{2k-2} - 2z^{2k}}{(1-z)^2} = z^{2k-2}\frac{1-2z^2}{(1-z)^2} \geqslant 0 \text{ for } 0 \leqslant z \leqslant \tfrac{1}{2}.$$

Thus for $0 \leqslant z \leqslant \frac{1}{2}$ we see that $g_b(z) \geqslant g_s(z) \geqslant g_\star(z)$. Now $z = 1/\mu \approx 0.30562$ is the smallest positive real solution to $g_\star(z) = 1$, and since $g_b(z) \geqslant g_\star(z)$ for $0 \leqslant z \leqslant \frac{1}{2}$ it follows that the smallest positive real solution to $g_b(z) = 1$ is at most $1/\mu$. This value of z is the reciprocal of $\mathrm{gr}\left(\mathcal{C}^{\boxplus}_{\phi(b)}\right)$, from which we see that

$$\mathrm{gr}(\mathcal{C}_{\phi(b)}) = \mathrm{gr}\left(\mathcal{C}^{\boxplus}_{\phi(b)}\right) \geqslant \mu.$$

We now briefly cover the cases $k = 2$ and $k = 3$. When $k = 2$, the sequence of $\boxplus$-indecomposable permutations in $\mathcal{C}^{\boxplus}_{\phi(b)}$ dominates the sequence $2, 2, 6, 8, 8, 8, \ldots$, which has generating function $(2z +$

$4z^3 + 2z^4)/(1-z)$. The solution to $(2z + 4z^3 + 2z^4)/(1-z) = 1$ is $z \approx 0.29433$, and therefore $\mathrm{gr}\left(\mathcal{C}^{\boxplus}_{\phi(b)}\right) \geqslant 1/0.29434 \approx 3.397 > \mu$.

Similarly, when $k = 3$, the sequence of $\boxplus$-indecomposables is at least $2, 2, 4, 8, 10, 10, \ldots$, and this yields a class whose growth rate is at least $3.310 > \mu$. □

The following theorem now completes the classification of small pin sequences.

Theorem 6.3. *Let w be an infinite pin sequence such that $\mathrm{gr}(\mathcal{C}_w) < \mu$. Then w is eventually periodic, and either $\mathrm{gr}(C_w) = \kappa$, or $\mathrm{gr}(C_w) = \nu_\ell$ for some $\ell \geqslant 1$.*

Proof: By the remarks at the beginning of this section, we can restrict our attention to pin sequences w that visit at most two quadrants, and thus $w = \phi(b)$ for some binary word b. Lemma 6.1 shows that b is eventually periodic, and note that b is eventually periodic if and only if w is eventually periodic.

To establish the growth rate of $\mathcal{C}_w$, note that by Proposition 3.6 we can assume that w is in fact periodic. Thus $p_b(n) = q_b(n)$ for all n, and the resulting class $\mathcal{C}_{\phi(b)}$ is $\boxplus$-closed. Proposition 6.2 shows that a binary sequence b for which $\mathrm{gr}(\mathcal{C}_{\phi(b)}) < \mu$ has a complexity function that satisfies $p_b(n) \leqslant n+1$ for all n, and Lemma 2.8 now implies that there exists $\ell \geqslant 2$ such that

$$p_b(n) = \begin{cases} n+1 & n < \ell \\ \ell + 1 & n \geqslant \ell. \end{cases}$$

The statement and proof of Lemma 4.8 now complete the proof: the generating function for the $\boxplus$-indecomposable permutations in $\mathcal{C}_{\phi(b)} = \mathcal{C}^{\boxplus}_{\phi(b)}$ is $g_{\phi(b)} = g_{1,\ell}(z)$, and consequently $\mathrm{gr}(\mathcal{C}_{\phi(b)}) = \nu_\ell$. □

7 Concluding remarks

At first sight, the phase transition we have established at μ might appear to bear some resemblance to the one at κ (as observed by Vatter (2011)). For example, $\kappa \approx 2.20557$ is the first growth rate of uncountably many classes, while μ is the first growth rate of uncountably many *pin* classes. However, they do also display significant differences; note that for any $\epsilon > 0$, there are only finitely many growth rates of pin classes less than $\mu - \epsilon$, yet the first accumulation point of permutation classes occurs at growth rate 2 (which is the limit of the growth rates of permutation classes enumerated by the generalised Fibonacci numbers), and in fact κ is an accumulation point of accumulation points.

In fact, μ has one further curious property: it is also an accumulation point of growth rates *from above* as the following family of pin classes shows. For $k \geqslant 2$, consider the pin sequence $w_k = \left((\mathsf{ru})^k \mathsf{lu} (\mathsf{ru})^{2k}\right)^\infty$. This generates a pin class whose sequence of $\boxplus$-indecomposable permutations has generating function

$$g_k(z) = \frac{2z - 2z^2 + 2z^3 + z^4 - 2z^5 + z^{2k+2} - z^{4k} - z^{4k+4}}{(1-z)^2}$$

It is clear that the series expansion of $g_k(z)$ approaches that of $g_\star(z)$ as $k \to \infty$, and straightforward to verify that the solutions to $g_k(z) = 1$ are smaller than $1/\mu$, and monotone increasing with k. Thus, the growth rate of the class $\mathcal{C}_{w_k}$ exceeds μ, but approaches μ as $k \to \infty$.

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